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$N=1$ Superconformal Tensor Calculus

—— *Multiplets with External Lorentz Indices and Derivative Operations**) ——

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In $N=1$ conformal supergravity, we completely determine the transformation laws of the most general multiplets with arbitrary external Lorentz indices, and then define the spinor derivative operations. Unconstrained (general type) multiplets exist for arbitrary external indices while chiral and linear multiplets exist only for purely undotted spinor indices. This comes from the particular fact that the superconformal spinor derivative is covariant only on multiplets satisfying some restrictive conditions. We define also a new class of spinor derivative operators each of which depends on the choice of a multiplet $\boldsymbol{u}$ playing the role of covariantization but is applicable covariantly to any multiplets. New chiral and linear multiplets defined through these spinor derivatives exist for arbitrary external Lorentz indices when $\bar{\boldsymbol{u}}$ is a (real or complex) linear multiplet. The connections of these spinor derivatives and constrained multiplets to those in superspace formulations of Poincaré supergravities are clarified.

§ 1. Introduction and summary

Supergravity has recently become of increasing interest, in particular, in connection with realistic model building of grand unified theories. Indeed it has been recognized that supergravity gives essential (and welcome) effects, for example, in spontaneous supersymmetry breaking (through super-Higgs phenomenon), mass splitting between particles and their superpartners and gauge hierarchy.[1),2)]

In spite of such increasing interest, the explicit calculations of, e.g., Lagrangian in supergravity are not so simple enough for the non-experts to follow easily. Therefore it would be much desirable to develop a simple practical calculational framework of supergravity.

The superconformal framework is presumably the simplest and most convenient one. The conformal supergravity has *larger* local-symmetries but closes with *fewer* fields than Poincaré supergravity, hence being much easier to handle. Poincaré supergravity theories (i.e., various auxiliary field formulations of Poincaré supergravity) are systematically derivable in this superconformal framework. The variety of Poincaré supergravities simply comes from various possibilities of the choice of the so-called *compensating multiplet*, the component fields of which are used to fix the extraneous gauge freedoms of superconformal theory. Tensor calculi of Poincaré supergravities also result from the unique superconformal tensor calculus according to those particular gauge-fixing conditions.

At first sight one might regard the Poincaré tensor calculus as more practical than the superconformal one since the extraneous gauge freedoms have to be fixed sooner or later, and hence the necessary gauge-fixing procedure in the latter is an extra task. But the fact is contrary. The very existence of those extraneous gauge-fixing freedoms makes it

*) This is a revised version of our previous report with the same title, CERN Preprint TH-3672.

possible to simplify the practical calculations greatly. Indeed the particular set of gauge-fixing conditions corresponding to Poincaré tensor calculus is chosen (and fixed) so as to fit only pure supergravity system. Therefore once the matters are coupled, it becomes a bizzare choice of gauge; for instance, the dilatation gauge is such one that the Einstein term does not take a canonical form $-(1/2)R$ but the form multiplied by a function of the matter fields. The price to be paid in order to cure this ridiculous gauge is laborious tasks such as Weyl rescaling and chiral transformation in the component field level. These are in fact tedious and complicated processes necessarily contained in Poincaré tensor calculus calculations. But they are simply bypassed in superconformal tensor calculus framework in which one retains the gauge freedoms to impose a "natural" set of gauge conditions for each considered system. These points were demonstrated explicitly by the present authors[3] for the Yang-Mills-matter-supergravity system which had been first discussed by Cremmer, Ferrara, Girardello and Van Proyen[2] in Poincaré tensor calculus framework.

A further superiority of superconformal calculus to the particular Poincaré one resides in its universal nature as a framework. For instance, the interrelation between different auxiliary field formulations of Poincaré supergravity can be discussed only in superconformal framework, since there is no way in a given Poincaré framework to describe another Poincaré formulation. It was shown in Ref. 4) by using superconformal framework that the class of interactions which can be described in the new minimal and (Breitenlohner's) non-minimal auxiliary field formulations are only particular cases of the interactions describable in the old minimal one.[*]

The structure of $N=1$ and 2 conformal supergravities is now completely understood through the long efforts by many authors including Kaku, Townsend, van Niewenhuizen, Ferrara, Grisaru, de Wit, van Holten and Van Proeyen.[5)~8)] For $N=1$ superconformal tensor calculus, a rather complete list of formulae can be found in paper 9) by the present authors, also in the excellent review articles by de Wit and by Van Proyen.[10)] (For $N=2$ superconformal tensor calculus, see Refs.11) and 12)). It is, however, very strange that there has appeared no literature in which the *spinor derivative* operation $\mathscr{D}_{\tilde{\alpha}}(=\mathscr{D}_\alpha, \mathscr{D}_{\dot{\alpha}})$ is discussed in the superconformal framework. In order to define the spinor derivative of supermultiplet, we need first of all clarify the properties of *supermultiplets with external Lorentz indices*, which has not been discussed either up to now. In view of the increasing importance of superconformal tensor calculus as a practical calculational tool, it is absolutely necessary to do these tasks. This is the primary subject of this paper. The spinor derivatives would immediately appear if we want to investigate the supergravity effects for the matter-Yang-Mills system with "non-minimal" kinetic terms or in the higher order quantum corrections. Further the knowledge of supermultiplets with external indices would be necessary, for instance, to investigate the higher N supergravity in terms of $N=1$ superfields, or to consider massive multiplets containing spin $3/2$ particle.

Now we briefly recapitulate the basic points of conformal supergravity.[5),9)13)] Conformal supergravity is essentially the gauge theory of superconformal algebra $SU(2,2|1)$, the generators X_A of which are Poincaré generators P_m and M_{mn}, dilatation D, conformal boost K_m, Q and S supersymmetries and chiral $U(1)$ symmetry A. The gauge fields $h_\mu{}^A$ and transformation parameters ε^A, as well as the curvatures (field strength) $R^A_{\mu\nu}$,

[*] The proof of this statement is, however, restricted to the classical level and to the interaction types not containing spinor derivatives of matter supermultiplets.

are denoted/defined as

$$h_\mu{}^A X_A = e_\mu{}^m P_m + \frac{1}{2}\omega_\mu{}^{mn} M_{mn} + \bar{\psi}_\mu Q + b_\mu D + A_\mu A + \bar{\varphi}_\mu S + f_\mu{}^m K_m ,$$

$$\epsilon^A X_A = \xi^m P_m + \frac{1}{2}\lambda^{mn} M_{mn} + \bar{\varepsilon} Q + \rho D + \theta A + \bar{\zeta} S + \xi_K{}^m K_m ,$$

$$R^A_{\mu\nu} \equiv R_{\mu\nu}(X^A) = \partial_\nu h_\mu{}^A - \partial_\mu h_\nu{}^A + h_\nu{}^B h_\mu{}^C f_{CB}{}^A , \tag{1·1}$$

where $f_{AB}{}^C$ is the structure constant of superconformal algebra $[X_A, X_B\} = f_{AB}{}^C X_C$. In order to relate the P_m-transformation to the general coordinate transformation, there are imposed suitable constraints [Eq. (B·1)] on $R^m_{\mu\nu}(P)$, $R_{\mu\nu}(Q)$ and $R^{mn}_{\mu\nu}(M)$, by which gauge fields $\omega_\mu{}^{mn}$, φ_μ and $f_\mu{}^m$ become dependent variables expressed in terms of other independent fields $e_\mu{}^m$, ψ_μ, A_μ and b_μ. Then the gauge transformations originally given by

$$\delta^0_{X^B}(\varepsilon^B) h_\mu{}^A = \partial_\mu \varepsilon^A + h_\mu{}^B \varepsilon^C f_{CB}{}^A , \tag{1·2}$$

and the algebra receive some modifications: First the changes of gauge transformation occur only on Q and P_m transformations. All the modifications of Q-transformation are the addition of following pieces to the original group law (1·2) only for the dependent variables:*),[5]

$$\delta'_Q(\varepsilon)\omega_\mu{}^{mn} = \frac{1}{2} R^{mn}(Q)\gamma_\mu \varepsilon , \tag{1·3a}$$

$$\delta'_Q(\varepsilon)\varphi_\mu = \frac{1}{4} i\gamma^\nu \{\gamma_5 R_{\nu\mu}(A) + \tilde{R}_{\nu\mu}(A)\}\varepsilon , \tag{1·3b}$$

$$\delta'_Q(\varepsilon) f_\mu{}^m = -\frac{1}{2} R^{\rm cov}_{\nu\mu}(S)\sigma^{m\nu}\varepsilon - \frac{1}{4} e^{m\nu} \tilde{R}^{\rm cov}_{\nu\mu}(S)\gamma_5 \varepsilon . \tag{1·3c}$$

The P_m transformation is replaced by the following $\tilde{P}_m$ transformation

$$\delta_{\tilde{P}}(\xi^m) = \delta_{GC}(\xi^m e_m{}^\nu) - \sum_{A\neq P} \delta_A(\xi^m h_m{}^A), \tag{1·4}$$

where δ_{GC} is the general coordinate transformation and the summation in the second term runs over all the transformations other than P_m transformation. With this replacement $P_m \to \tilde{P}_m$ always understood when the group index A becomes P_m, the commutator algebra of the same form as the original group rule

$$[\delta_A(\varepsilon^A), \delta_B(\varepsilon^B)] = \sum_C \delta_C(\varepsilon^A \varepsilon^B f_{BA}{}^C), \tag{1·5}$$

holds except for $[\delta_{\tilde{P}}, \delta_Q]$ and $[\delta_{\tilde{P}}, \delta_{\tilde{P}}]$. These exceptional commutators are given by[9]

$$[\delta_{\tilde{P}}(\xi^m), \delta_Q(\varepsilon)] = \sum_{A=M,S,K} \delta_A(\xi^m \delta'_Q(\varepsilon) h_m{}^A), \tag{1·6a}$$

$$[\delta_{\tilde{P}}(\xi_1{}^m), \delta_{\tilde{P}}(\xi_2{}^m)] = \sum_{A\neq \tilde{P}} \delta_A(\xi_1{}^m \xi_2{}^n R^{{\rm cov}\,A}_{mn}). \tag{1·6b}$$

*) The superscript "cov" attached to the curvatures $R^A_{\mu\nu}$ represents the additional Q-covariantization corresponding to this modification (1·3) of Q-transformation law, which is hence necessary only for $A = M^{mn}$, S, K^m [Eq. (B·3) in Appendix B].

For the convenience of practical use, we cite here the explicit form of (1·5)

$$[\delta_M(\lambda_1{}^{ab}),\ \delta_M(\lambda_2{}^{ab})]=\delta_M(\lambda_2{}^{ac}\lambda_1{}^{cb}-\lambda_1{}^{ac}\lambda_2{}^{cb}),$$

$$\left[\begin{pmatrix}\delta_{\tilde{P}}(\xi^m)\\ \delta_K(\xi_K{}^m)\end{pmatrix},\ \delta_M(\lambda^{ab})+\delta_D(\rho)\right]=\begin{pmatrix}\delta_{\tilde{P}}(\lambda^{mn}\xi^n-\rho\xi^m)\\ \delta_K(\lambda^{mn}\xi_K{}^n+\rho\xi_K{}^m)\end{pmatrix},$$

$$[\delta_K(\xi_K{}^m),\ \delta_{\tilde{P}}(\xi^m)]=2\delta_D(\xi_K\cdot\xi)+2\delta_M(\xi_K{}^a\xi^b-\xi_K{}^b\xi^a),$$

$$[\delta_Q(\varepsilon),\ \delta_K(\xi_K{}^m)]=\delta_S(-\xi_K{}^m\gamma_m\varepsilon),\qquad [\delta_S(\zeta),\ \delta_{\tilde{P}}(\xi^m)]=\delta_Q(\xi^m\gamma_m\zeta),$$

$$\left[\begin{pmatrix}\delta_Q(\varepsilon)\\ \delta_S(\zeta)\end{pmatrix},\ \delta_M(\lambda^{ab})+\delta_D(\rho)+\delta_A(\theta)\right]=\begin{pmatrix}\delta_Q\left(\frac{1}{2}\lambda^{ab}\sigma_{ab}\varepsilon-\frac{1}{2}\rho\varepsilon+\frac{3}{4}\theta i\gamma_5\varepsilon\right)\\ \delta_S\left(\frac{1}{2}\lambda^{ab}\sigma_{ab}\zeta+\frac{1}{2}\rho\zeta-\frac{3}{4}\theta i\gamma_5\zeta\right)\end{pmatrix},$$

$$[\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)]=\delta_{\tilde{P}}\left(\frac{1}{2}\bar{\varepsilon}_2\gamma^m\varepsilon_1\right),\qquad [\delta_S(\zeta_1),\ \delta_S(\zeta_2)]=\delta_K\left(\frac{1}{2}\bar{\zeta}_1\gamma^m\zeta_2\right),$$

$$[\delta_Q(\varepsilon),\ \delta_S(\zeta)]=\delta_D\left(-\frac{1}{2}\bar{\varepsilon}\zeta\right)+\delta_M(\bar{\varepsilon}\sigma^{ab}\zeta)+\delta_A(\bar{\varepsilon}i\gamma_5\zeta).\tag{1·7}$$

All the other commutators than appearing in (1·7) and (1·6) vanish. Since the transformation law of matter fields is determined such that the superconformal algebra holds also on them, the whole commutation relations (1·6) and (1·7) hold on any fields. The conformally covariant derivative D_m on the fields ϕ carrying only flat Lorentz indices, which will be used frequently, is defined through the $\tilde{P}_m$ transformation as

$$\delta_{\tilde{P}}(\xi^m)\phi=\xi^m(\partial_m\phi-\sum_{A\neq\tilde{P}}\delta_A(h_m{}^A)\phi)\equiv\xi^m D_m\phi\,.\tag{1·8}$$

For more details, we refer the reader to Refs. 5), 9)~13) in particular to the previous paper[9] of the present authors. For the notations and conventions we follow van Nieuwenhuizen's review article[13] throughout this paper, except for some conventions about two component spinor notations and the dual of anti-symmetric tensors which are summarized in Appendix A.

This paper is organized as follows. In §2, we discuss the most general (unconstrained-) type of superconformal multiplets with arbitrary external Lorentz indices as well as arbitrary Weyl and chiral weights. They are found to exist with no restrictions. Their full superconformal transformation laws are given in §2.A. We present a basic theorem in §2.B which is very useful in constructing a new multiplet from a given (generally reducible) multiplet through any operations (e.g., multiplication, differentiation). On the basis of those, we define spinor derivatives $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot{\alpha}}$ as covariant operations on superconformal multiplets in §2.C. We shall see that some restrictive conditions on Weyl and chiral weights of the multiplet have to be satisfied in order for the spinor derivative operations to be superconformally covariant. This is a remarkable fact particular to the superconformal theory; indeed, in any Poincaré versions, the spinor derivative operations are applicable to any multiplets.

Section 3 is devoted to the discussion of constrained-type multiplets and tensor calculus. It will be shown in §3.A, B that chiral multiplets exist only for the case of purely undotted spinor external Lorentz indices. The same is true for the linear multiplets. These facts come from the above mentioned restrictions for the superconformal spinor derivatives. The chiral projection operator Π is also defined there, and various identities for multiple operations of Π and $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot\alpha}$ are derived. The constrained-type multiplets are noticed to be expressible in terms of unconstrained multiplet (i.e., prepotential) also in this superconformal case (§3.C). Multiplication rules and explicit component formulae for arbitrary functions of multiplets are given in §3.D.

In §4, we introduce yet other spinor derivative operators $\mathcal{D}_\alpha{}^{(u)}$, $\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}$ which we call "$\boldsymbol{u}$-associated spinor derivatives". This new class of derivatives can be defined only when we prepare a Lorentz scalar multiplet and use its component fields to covariantize the operation in addition to the usual conformal gauge fields. Therefore these spinor derivatives become dependent on the choice of the $\boldsymbol{u}$ multiplet but instead become applicable (i.e., covariant) to any superconformal multiplets. On these points, the $\boldsymbol{u}$-associated spinor derivatives bear much resemblance to those in Poincaré supergravities. In fact, in the last subsection 4.D, $\mathcal{D}_\alpha{}^{(u)}$ and $\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}$ are shown to coincide essentially with the Poincaré spinor derivatives $\mathcal{D}_\alpha{}^{\mathrm{P}}$ and $\bar{\mathcal{D}}_{\dot\alpha}{}^{\mathrm{P}}$ when $\boldsymbol{u}$ is chosen to be the compensating multiplet of the corresponding Poincaré supergravity version (i.e., $\boldsymbol{u}=\boldsymbol{\Sigma}_0$ (chiral), $\boldsymbol{L}_0$ (real linear) and $\underset{\sim}{\mathcal{L}}_0$(complex anti-linear) for the old minimal, new minimal and non-minimal versions, respectively.

But before that, purely in superconformal framework, we first clarify the relation between the previous conformal spinor derivative $\mathcal{D}_{\tilde\alpha}$ and the $\boldsymbol{u}$-associated one $\mathcal{D}_{\tilde\alpha}{}^{(u)}$ (§4.A). Next, defining the vector derivative $\mathcal{D}_m{}^{(u)}$ also, we demonstrate how the commutation relations of those $\boldsymbol{u}$-associated derivatives (and hence "curvatures" and "torsions") are calculated from superconformal algebra (§4.B). These commutators show remarkable facts: the "$\boldsymbol{u}$-chiral multiplets" $\boldsymbol{\Sigma}_A{}^{(u)}$, defined by the constraint $\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}\boldsymbol{\Sigma}_A{}^{(u)}=0$, exist for *arbitrary* external Lorentz indices A when $\bar{\boldsymbol{u}}$ is a linear multiplet, $\boldsymbol{L}$ (real) or $\underset{\sim}{\mathcal{L}}$ (complex). Otherwise, e.g., when $\boldsymbol{u}=\boldsymbol{\Sigma}$(chiral), the $\boldsymbol{u}$-chiral multiplets exist only for purely undotted spinor indices. (These correspond to the facts which are well-known in the context of the old minimal ($\boldsymbol{u}=\boldsymbol{\Sigma}_0$) and non-minimal ($\boldsymbol{u}=\underset{\sim}{\mathcal{L}}_0$) Poincaré supergravities,[14),15)] but are less known in the new minimal case ($\boldsymbol{u}=\boldsymbol{L}_0$).) All the $\boldsymbol{u}$-chiral multiplets reduce essentially to the usual superconformal chiral multiplets for purely undotted spinor index cases. Quite the same results hold also for the "$\boldsymbol{u}$-linear multiplets" (§4.C). It should be noticed that the presence of the $\boldsymbol{u}$-chiral multiplets in superconformal framework implies that even in old minimal Poincaré supergravity framework one can have chiral and linear multiplets with arbitrary external Lorentz indices if a matter multiplet $\boldsymbol{L}$ or $\underset{\sim}{\mathcal{L}}$ is prepared as $\boldsymbol{u}$.

Appendix B gives a collection of identities and transformation laws of superconformal curvatures $R_{\mu\nu}(X^A)$ which are necessary to check the validity of superconformal algebra on general-type multiplets.

§ 2. Conformal multiplets of general type; $\underset{\sim}{\mathcal{V}}_A$

2. A. *Superconformal transformation laws*

Now in this section we concentrate on the multiplets of most general type in superconformal framework and determine their superconformal transformation laws.

The *general multiplet* $\underset{\sim}{\mathcal{V}}_A$ carries an arbitrary external Lorentz index A representing a set of undotted and dotted spinor indices $(\alpha_1\cdots\alpha_m;\ \dot\beta_1\cdots\dot\beta_n)$. (If it is totally symmetric both with respect to the undotted indices and to the dotted indices, it is an irreducible representation of Lorentz group. We, however, do not require this irreducibility property unless stated otherwise). $\mathcal{V}_A$ has $(8+8)\times\dim A(\dim A=$dimension of Lorentz-group repr. $A)$ complex components of fields which we denote by*)

$$\mathcal{V}_A=[\mathcal{C}_A,\ \mathcal{Z}_{\hat\alpha A},\ \mathcal{H}_A,\ \mathcal{K}_A,\ \mathcal{B}_{mA},\ \Lambda_{\hat\alpha A},\ \mathcal{D}_A], \tag{2·1}$$

where $\mathcal{Z}_{\hat\alpha A}$ and $\Lambda_{\hat\alpha A}$ are spinors with respect to the internal index $\hat\alpha$ (4-component-spinor) which we shall often omit hereafter. To the first component $\mathcal{C}_A$, which is defined to have the lowest Weyl weight in the multiplet, we assign the most general superconformal transformation law:

$$\delta_Q(\varepsilon)\mathcal{C}_A=\frac{1}{2}\bar\varepsilon i\gamma_5\mathcal{Z}_A\,, \tag{2·2a}$$

$$\delta_M(\lambda^{ab})\mathcal{C}_A=\frac{1}{2}\lambda^{ab}(\Sigma^{ab})_A{}^B\mathcal{C}_B\equiv\frac{1}{2}\lambda^{ab}(\Sigma^{ab}\mathcal{C})_A\,, \tag{2·2b}$$

$$\delta_D(\rho)\mathcal{C}_A=w\rho\,\mathcal{C}_A\,, \tag{2·2c}$$

$$\delta_A(\theta)\mathcal{C}_A=\frac{1}{2}in\theta\,\mathcal{C}_A\,, \tag{2·2d}$$

$$\delta_S(\zeta)\mathcal{C}_A=\delta_K(\xi_K{}^m)\mathcal{C}_A=0\,. \tag{2·2e}$$

Here Σ^{ab} is the representation matrix of the Lorentz generators and the (real) parameters w and n in (2·2c and d) define the *Weyl* and *chiral* weight of $\mathcal{C}_A$ fields. Properties (2·2e) are enforced by the fact that the S-supersymmetry and conformal boost K_m transformations lower the Weyl weight while $\mathcal{C}_A$ is the lowest Weyl-weight component. Since $\mathcal{Z}_A$ in (2·2a) stands for a general spinor (with respect to internal index) imposed no constraint, equation (2·2a) may be regarded as defining the second component $\mathcal{Z}_A$ rather than specifying the property of $\mathcal{C}_A$.

Once the transformation law is settled for the first component $\mathcal{C}_A$, the superconformal algebra (1·5), (1·6) determines the full transformation laws of the whole multiplet *uniquely* (up to the definition convention of higher components) as will be explained shortly. Therefore we obtain the following transformation laws for the general multiplet $\mathcal{V}_A$:

i) *Q-transformations*

$$\delta_Q(\varepsilon)\mathcal{C}_A=\frac{1}{2}\bar\varepsilon i\gamma_5\mathcal{Z}_A\,, \tag{2·3a}$$

$$\delta_Q(\varepsilon)\mathcal{Z}_A=(-)^A\frac{1}{2}(i\gamma_5\mathcal{H}_A-\mathcal{K}_A-\mathcal{B}_A+\not{D}\mathcal{C}_Ai\gamma_5)\varepsilon\,, \tag{2·3b}$$

$$\delta_Q(\varepsilon)\mathcal{H}_A=\frac{1}{2}\bar\varepsilon i\gamma_5(\not{D}\mathcal{Z}_A+\Lambda_A), \tag{2·3c}$$

*) We denote complex component fields generally by script letters $\mathcal{C}$, $\mathcal{Z}$, except for the Λ-component for which we use capital greek Λ. Real component fields are denoted by ordinary letters such as C, Z, or by lowercase Greek λ.

$$\delta_Q(\varepsilon)\mathcal{K}_A=-\frac{1}{2}\bar{\varepsilon}(\not{D}\mathcal{Z}_A+\Lambda_A), \tag{2·3d}$$

$$\delta_Q(\varepsilon)\mathcal{B}_{mA}=-\frac{1}{2}\bar{\varepsilon}(D_m\mathcal{Z}_A+\gamma_m\Lambda_A)-\frac{1}{4}R_{ab}(Q)i\gamma_5\gamma_m\varepsilon(\Sigma^{ab}\mathcal{C})_A, \tag{2·3e}$$

$$\delta_Q(\varepsilon)\Lambda_A=(-)^A\frac{1}{2}(\sigma\cdot\mathcal{F}_A+i\gamma_5\mathcal{D}_A)+\frac{1}{8}\{\gamma_m\varepsilon R_{ab}(Q)\gamma_m(\Sigma^{ab}\mathcal{Z})_A$$
$$+\gamma_5\gamma_m\varepsilon R_{ab}(Q)\gamma_5\gamma_m(\Sigma^{ab}\mathcal{Z})_A\} \tag{2·3f}$$

$$\delta_Q(\varepsilon)\mathcal{D}_A=\frac{1}{2}\bar{\varepsilon}i\gamma_5\not{D}\Lambda_A-\frac{1}{4}\bar{\varepsilon}(R_{ab}(A)+\gamma_5\tilde{R}_{ab}(A))(\Sigma^{ab}\mathcal{Z})_A$$
$$+(-)^A\frac{1}{4}\bar{\varepsilon}\{i\gamma_5(\Sigma^{ab}\mathcal{B})_A-(\Sigma^{ab}\not{D}\mathcal{C})_A\}\hat{R}_{ab}(Q), \tag{2·3g}$$

where

$$\mathcal{F}_{mnA}\equiv D_m\mathcal{B}_{nA}-D_n\mathcal{B}_{mA}+\frac{1}{2}i\varepsilon_{mnkl}[D_k, D_l]\mathcal{C}_A$$
$$=D_m\mathcal{B}_{nA}-D_n\mathcal{B}_{mA}+\frac{1}{4}i\varepsilon_{mnkl}R_{kl}^{\text{cov}\,ab}(M)(\Sigma^{ab}\mathcal{C})_A$$
$$+\frac{1}{2}R_{ab}(Q)\mathcal{Z}_A+\frac{1}{2}w\,\mathcal{C}_AR_{mn}(A)-\frac{1}{2}n\,\mathcal{C}_A\tilde{R}_{mn}(A), \tag{2·4}$$

and the sign factor $(-)^A$ denotes the even-odd of the number of spinor indices contained in A. (The sign factors all disappear when the component fields $\mathcal{C}_A$, $\mathcal{Z}_{\dot{a}A}$, …are always kept on the right most among the factors in each term.)

ii) *S-transformation*:

$$\delta_S(\zeta)\mathcal{C}_A=0,$$
$$\delta_S(\zeta)\mathcal{Z}_A=-i(n+w\gamma_5)\zeta\,\mathcal{C}_A+i\gamma_5\sigma_{ab}\zeta(\Sigma^{ab}\mathcal{C})_A,$$
$$\delta_S(\zeta)\mathcal{H}_A=\frac{1}{2}i\bar{\zeta}\{(w-2)\gamma_5+n\}\mathcal{Z}_A+\frac{1}{2}\bar{\zeta}i\gamma_5\sigma_{ab}(\Sigma^{ab}\mathcal{Z})_A,$$
$$\delta_S(\zeta)\mathcal{K}_A=\frac{1}{2}\bar{\zeta}(w-2+n\gamma_5)\mathcal{Z}_A+\frac{1}{2}\bar{\zeta}\sigma_{ab}(\Sigma^{ab}\mathcal{Z})_A,$$
$$\delta_S(\zeta)\mathcal{B}_{mA}=\frac{1}{2}\bar{\zeta}(w+1+n\gamma_5)\gamma_m\mathcal{Z}_A+\frac{1}{2}\bar{\zeta}\sigma_{ab}\gamma_m(\Sigma^{ab}\mathcal{Z})_A,$$
$$\delta_S(\zeta)\Lambda_A=(-)^{A+1}\frac{1}{2}(i\gamma_5\mathcal{H}_A+\mathcal{K}_A+\mathcal{B}_A-\not{D}\mathcal{C}_Ai\gamma_5)(w+n\gamma_5)\zeta$$
$$+(-)^A\frac{1}{2}\{\Sigma^{ab}(i\gamma_5\mathcal{H}+\mathcal{K}+\mathcal{B}-\not{D}\mathcal{C}\,i\gamma_5)\}_A\sigma_{ab}\zeta,$$
$$\delta_S(\zeta)\mathcal{D}_A=i\bar{\zeta}(n+w\gamma_5)(\Lambda_A+\frac{1}{2}\not{D}\mathcal{Z}_A)+\bar{\zeta}i\gamma_5\sigma_{ab}\left\{\Sigma^{ab}(\Lambda+\frac{1}{2}\not{D}\mathcal{Z})\right\}_A. \tag{2·5}$$

iii) *D-, A- and K_m-transformations;* $\delta_{DAK}\equiv\delta_D(\rho)+\delta_A(\theta)+\delta_K(\xi^m)$

$$\delta_{DAK}\mathcal{C}_A=w\mathcal{C}_A\rho+\frac{1}{2}in\mathcal{C}_A\theta\,,$$

$$\delta_{DAK}\mathcal{Z}_A=\left(w+\frac{1}{2}\right)\mathcal{Z}_A\rho+\left(\frac{1}{2}in-\frac{3}{4}i\gamma_5\right)\mathcal{Z}_A\theta\,,$$

$$\delta_{DAK}\mathcal{H}_A=(w+1)\mathcal{H}_A\rho+\left(\frac{1}{2}in\mathcal{H}_A+\frac{3}{2}\mathcal{K}_A\right)\theta\,,$$

$$\delta_{DAK}\mathcal{K}_A=(w+1)\mathcal{K}_A\rho+\left(\frac{1}{2}in\mathcal{K}_A-\frac{3}{2}\mathcal{H}_A\right)\theta\,,$$

$$\delta_{DAK}\mathcal{B}_{mA}=(w+1)\mathcal{B}_{mA}\rho+\frac{1}{2}in\mathcal{B}_{mA}\theta+\{2in\delta_{mn}\mathcal{C}_A+i\varepsilon_{mnab}(\Sigma^{ab}\mathcal{C})_A\}\xi^n\,,$$

$$\delta_{DAK}\Lambda_A=(w+\frac{3}{2})\Lambda_A\rho+\left(\frac{1}{2}in+\frac{3}{4}i\gamma_5\right)\Lambda_A\theta$$
$$+\{-(w+n\gamma_5)\gamma_m\mathcal{Z}_A+\sigma_{ab}\gamma_m(\Sigma^{ab}\mathcal{Z})_A\}\xi^m\,,$$

$$\delta_{DAK}\mathcal{D}_A=(w+2)\mathcal{D}_A\rho+\frac{1}{2}in\mathcal{D}_A\theta$$
$$-\{2wD_m\mathcal{C}_A+2in\mathcal{B}_{mA}+2(\Sigma_{mn}D_n\mathcal{C})_A-i\varepsilon_{mnab}(\Sigma^{ab}\mathcal{B}_n)_A\}\xi^m\,. \qquad (2\cdot6)$$

Here the obvious transformation laws under local Lorentz and general coordinate transformations are omitted. D_m stands for conformally covariant derivative defined in (1·8). The explicit appearance of the curvatures $R_{mn}(Q)$ and $R_{mn}(A)$ in the Q-transformation law (2·3) is a new feature particular to the multiplets with external Lorentz indices.

We now explain how uniquely these full conformal transformation laws (2·3)~(2·6) are determined from the first component ($\mathcal{C}_A$) one (2·2). First Eq. (2·2a) defines the second component $\mathcal{Z}_A$ (with Weyl weight $w+1/2$), as remarked before; namely, in the form

$$\delta_Q\mathcal{C}_A\sim\mathcal{Z}_A\,. \qquad (2\cdot7)$$

This determines uniquely the transformation law of $\mathcal{Z}_A$ under the transformations other than Q, i.e. M^{ab}, D, A, S and K_m which we denote by X' generically. Indeed, applying the $\delta_{X'}$ to (2·7), we obtain

$$\delta_{X'}\mathcal{Z}_A\sim[\delta_{X'},\delta_Q]\mathcal{C}_A+\delta_Q(\delta_{X'}\mathcal{C}_A). \qquad (2\cdot8)$$

Since the transformation X' does not raise the Weyl weight of fields (it is only the Q-transformation that can raise the Weyl weight), $\delta_{X'}\mathcal{C}_A$ in the second term is given in terms of $\mathcal{C}_A$ alone as is seen in (2·2b~e). Therefore if we require the superconformal algebra (1·7) of the form

$$[\delta_{X'},\delta_Q]=f_{QX'}{}^Y\delta_Y \qquad (Y=Q,M,D,A,S,K) \qquad (2\cdot9)$$

to hold, then, by (2·8), $\delta_{X'}\mathcal{Z}_A$ is uniquely determined from the first component transformation law (2·2).

The remaining Q-transformation $\delta_Q(\varepsilon)\mathcal{Z}_A$ is determined by requiring the Q-Q algebra

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]=\frac{1}{2}(\bar{\varepsilon}_2\gamma^m\varepsilon_1)D_m \tag{2·10}$$

to hold on the *previous* $\mathcal{C}_A$ *field;* namely, for a general form

$$\delta_Q(\varepsilon)\mathcal{Z}_A=\frac{1}{2}(\mathcal{S}_A+i\gamma_5\mathcal{P}_A+\mathcal{V}_{mA}\gamma^m+\mathcal{A}_{mA}i\gamma_m\gamma_5+\mathcal{T}_{mnA}\sigma^{mn})\varepsilon\,,$$

this requirement fixes uniquely $\mathcal{A}_{mA}$ and $\mathcal{T}_{mnA}$ parts as

$$\mathcal{A}_{mA}=D_m\mathcal{C}_A\,,\qquad \mathcal{T}_{mnA}=0\,, \tag{2·11}$$

and leaves the others $\mathcal{S}_A$, $\mathcal{P}_A$ and $\mathcal{V}_{mA}$ arbitrary. The latter undetermined parts *define* the third three components $-\mathcal{K}_A$, $\mathcal{H}_A$ and $-\mathcal{B}_{mA}$ (with Weyl weight $w+1$) and hence we obtain (2·3b).

Clearly this procedure can be repeated. The $(k+1)$-th component field(s) $\phi^{(k+1)}$ (with Weyl weight $w+k/2$) is (are) defined through the Q-transformation of the previous k-th component $\phi^{(k)}$. This defining equation uniquely determines the transformation law $\delta_{X'}\phi^{(k+1)}$ other than δ_Q with the requirement of superconformal algebra (2·9). The Q-transformation $\delta_Q\phi^{(k+1)}$ is determined in such a way that the Q-Q algebra (2·10) holds on the previous component $\phi^{(k)}$. The undetermined part there defines the next higher component $\phi^{(k+2)}$. This procedure ends with the determination of $\delta_Q(\varepsilon)\mathcal{D}_A$ in which no undetermined parts are left to satisfy the Q-Q algebra on Λ_A, thus completing the full transformation laws given in (2·3)~(2·6).

Some comments are in order:

i) As a defining equation of the fourth component Λ_A, we can choose either one of $\delta_Q\mathcal{H}_A$, $\delta_Q\mathcal{K}_A$ and $\delta_Q\mathcal{B}_{mA}$. Probably the choice $\delta_Q\mathcal{H}_A$ would be simplest. Thus we can take the following four Q-transformations as a complete set of the defining equations for higher components

a) $\delta_Q\mathcal{C}_A$ (2·3a) defines $\mathcal{Z}_A$,

b) $\delta_Q\mathcal{Z}_A$ (2·3b) defines $\mathcal{H}_A$, $\mathcal{K}_A$ and $\mathcal{B}_{mA}$,

c) $\delta_Q\mathcal{H}_A$ (2·3c) defines Λ_A,

d) $\delta_Q\Lambda_A$ (2·3f) defines $\mathcal{D}_A$. (2·12)

ii) Although we have derived the full transformation laws (2·3)~(2·6), it should be noted that *not all* of commutation relations of superconformal algebra have been used in the above procedure. Therefore it is still quite a non-trivial matter to check whether the full algebra actually holds on all the component fields. What we have still to check is the unused commutation relations, Q-X' algebra (2·9) on $\mathcal{K}_A$, $\mathcal{B}_{mA}$ and $\mathcal{D}_A$, Q-Q algebra (2·10) on $\mathcal{D}_A$ and the algebra $[\delta_{X_1'}, \delta_{X_2'}]$ $(X_1', X_2'\neq Q)$ on all components. The authors have calculated these explicitly and confirmed that the full conformal algebra is consistently satisfied by the transformation laws (2·3)~(2·6). For the reader's convenience we collect some useful identities necessary for these calculations in Appendix B.

iii) It should be emphasized again that the full transformation laws (2·3)~(2·6) are *uniquely* determined from the first component transformation law (2·2) if we demand the superconformal algebra. This is of course up to arbitrariness in defining the higher component fields; for instance we could have defined the fourth component by

$$\Lambda_A + \alpha i\gamma_5 \hat{R}_{ab}(Q)(\Sigma^{ab}C)_A \qquad (\alpha;\ \text{arbitrary parameter})$$

instead of our Λ_A. This type of arbitrariness is inessential and we have fixed our convention by (2·3a, b, c and f). We call the transformation laws (2·3)~(2·6) "standard form" and the above procedure to define higher components by (2·12) "standard procedure".

2. B. *Construction of multiplet $\mathcal{V}_A$ from another $\boldsymbol{\phi}$*

We will often have a necessity to construct a general multiplet $\mathcal{V}_A$ from another given multiplet $\boldsymbol{\phi}$ through some operations such as differential operations D_α, $\bar{D}D$, multiplication. We present in this subsection a theorem which is very useful generally for such cases.

Let us start with the following probably well-known Lemma:[13]

Lemma

Let $\boldsymbol{\phi}=[\phi_1, \phi_2, \cdots \phi_n]$ be any conformal multiplet (reducible, generally); that is, we assume that a full superconformal algebra holds on ϕ_i's. Then the algebra holds also on arbitrary function $f(\phi)$ of ϕ_i's which may contain derivatives ($\partial/\partial x^\mu$) in the coefficients.

Proof. It is sufficient if we can prove the validity of the algebra on $\partial_\mu A$ and on $A\cdot B$ for arbitrary functions $A(\phi)$ and $B(\phi)$ on which the algebra is assumed to hold. This is a trivial excercise and is omitted here. □

Now we can construct a new general multiplet $\mathcal{V}_A(\boldsymbol{\phi})$ rather easily from $\boldsymbol{\phi}$. Let us suppose we have succeeded in constructing a suitable functions $C_A(\phi)$ such that its superconformal transformation law (other than δ_Q) coincides with the standard first component law given in (2·2). Then, starting with $C_A(\phi)$, we can determine the higher components of the multiplet by performing the δ_Q-transformation successively, i.e., by following the standard procedure (2·12) described in the preceding subsection. The following theorem guarantees that the set of fields $[C_A(\phi), \mathcal{Z}_A(\phi), \cdots, \mathcal{D}_A(\phi)]$ constructed in this way actually becomes a conformal general multiplet $\mathcal{V}_A$:

THEOREM

Let $C_A(\phi)$ be a function of ϕ_i's, the component fields of a (generally, reducible) multiplet $\boldsymbol{\phi}$. Then a necessary and sufficient condition for a superconformal (general) multiplet $\mathcal{V}_A(\boldsymbol{\phi})$ containing $C_A(\phi)$ as its first component to exist, is that the superconformal transformations $\delta_{X'}$ (other than δ_Q) of $C_A(\phi)$ satisfy the standard form of first component transformation laws (2·2b~e).

Proof. Necessity of the condition is obvious since the form of the first component transformation laws (2·2b~e) was a most general one. To prove it sufficient, we recall the fact that the transformation laws of whole components of a multiplet $\mathcal{V}_A$ are uniquely determined from the first component one by the requirement of superconformal algebra.

On the other hand, the superconformal algebra is assured to hold on any function of ϕ by the previous Lemma. Therefore if the transformation law of the first component $\mathcal{C}_A(\phi)$ coincides with the standard form one (2·2) and if the higher components are defined through the standard procedure given in (2·12), the full transformation laws of the whole multiplet coincide with those (2·3)~(2·6) of standard form. This guarantees for the set of fields $[\mathcal{C}_A(\phi), \mathcal{Z}_A(\phi), \cdots, \mathcal{D}_A(\phi)]$ to be really a consistent conformal multiplet. □

Remarks. This theorem is very useful. (It may be implicitly known to the authors of Ref. 8)). We have only to examine $\delta_{X'}$ transformation law of $\mathcal{C}_A(\phi)$. Further, in almost all cases, it is trivial to check δ_M, δ_D, δ_A, δ_K transformations and the only non-trivial point is to check whether

$$\delta_S \mathcal{C}_A(\phi)=0$$

is satisfied or not. With this theorem, it is sufficient to calculate four Q-transformations (2·12) to determine higher components, and we need no tedious check of consistency such as:

i) Does $\Lambda_A(\phi)$, which is calculated from $\delta_Q \mathcal{H}_A(\phi)$ in our procedure, coincide with the ones that would be obtained also from $\delta_Q \mathcal{K}_A(\phi)$ or $\delta_Q \mathcal{B}_{mA}(\phi)$?

ii) Does the Q-transformation of $\mathcal{D}_A(\phi)$ take the desired form (2·3g), $\delta_Q(\varepsilon)\mathcal{D}_A(\phi) \sim \bar{\varepsilon} D\!\!\!/ \Lambda_A(\phi)+\cdots$?

These are automatically satisfied.

Before concluding this subsection we introduce a convenient notation to denote a conformal multiplet $\mathcal{V}_A$. Since the first component $\mathcal{C}_A$ uniquely specifies the whole multiplet, we can use such a symbol as

$$\mathcal{V}_A=[\![\mathcal{C}_A]\!] \ . \tag{2·13}$$

2. C. *Spinor derivative operations $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$*

The theorem in the preceding subsection can be widely made use of for any type of derivation of a multiplet from given multiplet(s). Here we discuss *spinor derivative* operations $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot\alpha}$, superconformal analogue of D_α, $\bar{D}_{\dot\alpha}$ of rigid supersymmetry.

As in rigid case, the multiplet $\mathcal{D}_\alpha \mathcal{V}_A$, if any definable, would be a multiplet whose first component is $\mathcal{Z}_{\alpha A}$,

$$\mathcal{D}_\alpha \mathcal{V}_A=[\![\mathcal{Z}_{\alpha A}]\!] \ . \tag{2·14}$$

But, in contrast to the rigid supersymmetry case, this can be a multiplet only when a special condition is satisfied. Indeed the theorem says that it can be a conformal multiplet if and only if the first component $\mathcal{Z}_{\alpha A}$ satisfies the standard form transformation laws (2·2b~e). The only non-trivial condition is the vanishing of S-supersymmetry transformation law:

$$0=\delta_S(\zeta)\mathcal{Z}_{\alpha A}=-i(w+n)\zeta_\alpha \mathcal{C}_A+i(\sigma^{ab}\zeta)_\alpha(\Sigma^{ab}\mathcal{C})_A \,. \tag{2·15}$$

In order to find the most general solution to (2·15), let us write the external index A more explicitly as

$$A=(\beta_1, \beta_2, \cdots, \beta_m\,;\, \dot\gamma_1, \dot\gamma_2, \cdots, \dot\gamma_l)\equiv(\beta_1, \cdots, \beta_m\,;\, \dot\Gamma)$$

and assume only for the undotted spinor part $(\beta_1, \cdots, \beta_m)$ to be totally symmetric (i.e., irreducible repr. of Lorentz group only with respect to undotted spinor indices). Then the second term of (2·15) is rewritten as

$$i(\sigma^{ab}\zeta)_\alpha(\Sigma^{ab}C)_A=i(\sigma^{ab})_\alpha{}^\gamma\zeta_\gamma\sum_{i=1}^m(\sigma^{ab})_{\beta_i}{}^\delta C_{\beta_1\cdots\overset{i}{\underset{\vee}{\delta}}\cdots\beta_m\dot\Gamma}+i(\sigma^{ab})_\alpha{}^\gamma\zeta_\gamma(\Sigma_+{}^{ab})_{\dot\Gamma}{}^{\dot\Delta}C_{\beta_1\cdots\beta_m\dot\Delta}\,, \quad (2\cdot16)$$

where $\underset{\vee}{i}$ is the reminder of the place sitted originally by β_i. The last term of (2·16) vanishes because $\Sigma_+{}^{ab}$ is selfdual while $(\sigma^{ab})_\alpha{}^\gamma=(\sigma_-{}^{ab})_\alpha{}^\gamma$ is anti-selfdual (cf., Appendix A). We now need the following formulae: One is the completeness relation

$$(\sigma^{ab})_\alpha{}^\gamma(\sigma_{ab})_\beta{}^\delta=\delta_\alpha{}^\gamma\delta_\beta{}^\delta-2\delta_\alpha{}^\delta\delta_\beta{}^\gamma\,, \quad (2\cdot17)$$

and the others are[16)]

$$\mathscr{S}^{(m+1)}_{(\alpha\beta_1\beta_2\cdots\beta_m)}=\frac{1}{m+1}[1+\sum_{i=1}^m(\alpha,\beta_i)]\mathscr{S}^{(m)}_{(\beta_1\beta_2\cdots\beta_m)}\,,$$

$$\mathscr{A}^{(2)}_{(\alpha\beta_i)}=\frac{1}{2}[1-(\alpha,\beta_i)], \quad (2\cdot18)$$

where $\mathscr{S}^{(m)}[\mathscr{A}^{(m)}]$ is the symmetrizer (anti-symmetrizer) with respect to the indicated m indices (normalized with "unit strength"), and (α,β_i) denotes the transposition operator between α and β_i. By the help of (2·16)~(2·18), the RHS of (2·15) is rewritten as

$$\begin{aligned}\delta_S(\zeta)\mathscr{Z}_{\alpha A}&=-i[(w+n-m)\zeta_\alpha C_{\beta_1\cdots\beta_m\dot\Gamma}+\sum_{i=1}^m 2\zeta_{\beta_i}C_{\beta_1\cdots\overset{i}{\underset{\vee}{\alpha}}\cdots\beta_m\dot\Gamma}]\\&=-i[(w+n-m)+2\sum_{i=1}^m(\alpha,\beta_i)]\zeta_\alpha C_{\beta_1\cdots\beta_m\dot\Gamma}\\&=-i\Big[\frac{w+n+m}{m+1}(1+\sum_{i=1}^m(\alpha,\beta_i))+\frac{w+n-(m+2)}{m+1}\sum_{i=1}^m(1-(\alpha,\beta_i))\Big]\\&\quad\times\mathscr{S}^{(m)}_{(\beta_1\cdots\beta_m)}\zeta_\alpha C_{\beta_1\cdots\beta_m\dot\Gamma}\\&=-i\Big[(w+n+m)\mathscr{S}^{(m+1)}_{(\alpha\beta_1\cdots\beta_m)}+2\frac{w+n-(m+2)}{m+1}\sum_{i=1}^m\mathscr{A}^{(2)}_{(\alpha\beta_i)}\Big]\zeta_\alpha C_{\beta_1\cdots\beta_m\dot\Gamma}\,.\end{aligned} \quad (2\cdot19)$$

We can perform a similar decomposition to $\mathscr{Z}_{\alpha A}$

$$\mathscr{Z}_{\alpha\beta_1\cdots\beta_m\dot\Gamma}=\Big[\mathscr{S}^{(m+1)}_{(\alpha\beta_1\cdots\beta_m)}+\frac{2}{m+1}\sum_{i=1}^m\mathscr{A}^{(2)}_{(\alpha,\beta_i)}\Big]\mathscr{Z}_{\alpha\beta_1\cdots\beta_m\dot\Gamma}\,, \quad (2\cdot20)$$

by using the identity derivable similarly from (2·18)

$$\mathscr{S}^{(m)}_{(\beta_1\beta_2\cdots\beta_m)}=\mathscr{S}^{(m+1)}_{(\alpha\beta_1\cdots\beta_m)}+\frac{2}{m+1}\sum_{i=1}^m\mathscr{A}^{(2)}_{(\alpha\beta_i)}\mathscr{S}^{(m)}_{(\beta_1\cdots\beta_m)}\,. \quad (2\cdot21)$$

Then, noticing that the terms with different symmetry properties are linearly independent of each other, we obtain from (2·19) and (2·20)

$$\delta_S(\zeta)\mathscr{Z}_{(\alpha\beta_1\cdots\beta_m)\dot\Gamma}=-i(w+n+m)\zeta_{(\alpha}C_{\beta_1\cdots\beta_m)\dot\Gamma}\,,$$

$$\delta_S(\zeta)\mathscr{Z}^\alpha{}_{\alpha\beta_2\cdots\beta_m\dot\Gamma}=-i\{w+n-(m+2)\}\zeta^\alpha C_{\alpha\beta_2\cdots\beta_m\dot\Gamma}\,, \quad (2\cdot22)$$

where we have used the fact $\mathscr{A}^{(2)}_{(\alpha\beta)}X_{\alpha\beta}\equiv X_{[\alpha\beta]}=(1/2)\varepsilon_{\alpha\beta}X^\gamma{}_\gamma$ and the notation $X_{(\alpha\beta_1\cdots\beta_m)}\equiv$

$\mathcal{S}^{(m+1)}_{(\alpha\beta_1\cdots\beta_m)}X_{\alpha\beta_1\cdots\beta_m}$. From (2·22) we see when condition (2·15) is satisfied and obtain the following result:

i) $\mathcal{D}_{(\alpha}\underset{\sim}{\mathcal{V}}_{\beta_1\cdots\beta_m)\dot{\Gamma}}=[\![\mathcal{Z}_{(\alpha\beta_1\cdots\beta_m)\dot{\Gamma}}]\!]$ is a conformal multiplet if and only if $w+n=-m$, (2·23a)

ii) $\mathcal{D}^{\alpha}\underset{\sim}{\mathcal{V}}_{\alpha\beta_2\cdots\beta_m\dot{\Gamma}}=[\![\mathcal{Z}^{\alpha}_{\alpha\beta_2\cdots\beta_m\dot{\Gamma}}]\!]$ is a conformal multiplet if and only if $w+n=m+2$. (2·23b)

(Remember that m is the number of external undotted spinor indices of $\underset{\sim}{\mathcal{V}}_A$). These are the multiplets with Weyl weight $w+1/2$ and chiral weight $n-3/2$, and therefore the operator $\mathcal{D}_\alpha$ carries Weyl and chiral weights $(1/2, -3/2)$.

The condition for $\bar{\mathcal{D}}_{\dot{\alpha}}$ operation is also found similarly or simply by taking complex conjugate of (2·23) and changing the sign of chiral weight n:

i) $\bar{\mathcal{D}}_{(\dot{\alpha}}\underset{\sim}{\mathcal{V}}_{\dot{\beta}_1\cdots\dot{\beta}_m)\Gamma}$ is a conformal multiplet if and only if $w-n=-m$, (2·24a)

ii) $\bar{\mathcal{D}}^{\dot{\alpha}}\underset{\sim}{\mathcal{V}}_{\dot{\alpha}\dot{\beta}_2\cdots\dot{\beta}_m\Gamma}$ is a conformal multiplet if and only if $w-n=m+2$, (2·24b)

with m now being the number of dotted spinor indices of $\underset{\sim}{\mathcal{V}}_A$.

The results (2·23) and (2·24) state the desired most general conditions and otherwise $\mathcal{D}_{\tilde{\alpha}}\underset{\sim}{\mathcal{V}}_A$ cannot be a conformal multiplet. In other words, the spinor derivatives $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot{\alpha}}$ are superconformally *covariant* only when special weight-conditions (2·23) and (2·24) are satisfied. (What would be worth mentioning here is the fact that quite the same is true even in *rigid superconformal* case. This is an interesting fact that is not well known.) This situation is in sharp contrast with the rigid (non-conformal) supersymmetry case in which D_α and $\bar{D}_{\dot{\alpha}}$ are literally covariant operations for any multiplets $\underset{\sim}{\mathcal{V}}_A$ with arbitrary external indices.

The particularly restrictive nature of the weight condition (2·23) or (2·24) would become clearer if we try to operate $\mathcal{D}_\alpha$ or $\bar{\mathcal{D}}_{\dot{\alpha}}$ twice on a multiplet $\underset{\sim}{\mathcal{V}}_A$. Indeed, recalling that the multiplet $\mathcal{D}_\alpha\underset{\sim}{\mathcal{V}}_A$, if definable, carries Weyl and chiral weights $(w+1/2, n-3/2)$ and has $(m+1)$ undotted spinor indices when $\underset{\sim}{\mathcal{V}}_A$ has those numbers (w, n) and m, respectively, we easily see that

$$\mathcal{D}^{\beta_1}\mathcal{D}_{(\beta_0}\underset{\sim}{\mathcal{V}}_{\beta_1\beta_2\cdots\beta_m)\dot{\Gamma}}\,, \qquad \mathcal{D}_{(\beta_0}\mathcal{D}^{\beta_m}\underset{\sim}{\mathcal{V}}_{\beta_1\cdots\beta_{m-1})\beta_m\dot{\Gamma}}\,,$$

for instance, cannot be defined since the weight condition (2·23) for the second $\mathcal{D}$ operation is not satisfied whenever it is met for the first $\mathcal{D}$ operation. The only definable second-order differentiations are the following two, but they in fact turn out to be zero unfortunately:

$$\mathcal{D}_{(\alpha_1}\mathcal{D}_{\alpha_2}\underset{\sim}{\mathcal{V}}_{\beta_1\cdots\beta_m)\dot{\Gamma}}=0\,,$$

$$\mathcal{D}^{\beta_1}\mathcal{D}^{\beta_2}\underset{\sim}{\mathcal{V}}_{\beta_1\beta_2\beta_3\cdots\beta_m\dot{\Gamma}}=0\,. \tag{2·25}$$

The vanishing property of these multiplets, which is easily understood by examining their first components, is reminiscent of the anti-commutative nature of two $\mathcal{D}_\alpha$'s just like the relation $\{D_\alpha, D_\beta\}=0$ in rigid supersymmetry case. Nevertheless we have no such relation as $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\}=0$ here in superconformal case (rigid or local) since $\mathcal{D}_\alpha$ is definable only in some special cases (2·23) and furthermore no nontrivial second-order $\mathcal{D}_\alpha$ operations exist. These particular properties implies presumably the *impossibility* of superspace superfield formalism of local superconformal theory in which *both* dilatation and chiral $U(1)$ are gauged as the tangent group generators on superspace,*) as is necessitated to incorporate two free parameters (w, n) of Weyl and chiral weights of multiplets (cf., Refs. 17)~19)).

We give here the explicit full component expression of $\mathscr{D}_\alpha \underset{\sim}{\mathscr{V}}_A$, which can be calculated by following the standard procedure:

$$\mathcal{C}_{\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=\mathscr{Z}_{\alpha A}\,,$$

$$\mathscr{Z}_{\dot{\alpha}\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=(\mathscr{P}_{\mathrm{R}}C^{-1})_{\dot{\alpha}\alpha}(\mathscr{H}_A+i\mathscr{K}_A)+(i\gamma_m\mathscr{P}_{\mathrm{R}}C^{-1})_{\dot{\alpha}\alpha}(\mathscr{B}_{mA}+iD_m\mathcal{C}_A),$$

$$\mathscr{H}_{\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=-i\mathscr{K}_{\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=\Lambda_{\alpha A}-i(\hat{R}^{\mathrm{R}}_{ab}(Q))_\alpha(\Sigma_-{}^{ab}\mathcal{C})_A\,,$$

$$\mathscr{B}_{m\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=i(\gamma_m\Lambda_A+2\sigma_{mn}D_n\mathscr{Z}_A)_\alpha,$$

$$\begin{aligned}\Lambda_{\dot{\alpha}\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=&(\sigma^{mn}\mathscr{P}_{\mathrm{R}}C^{-1})_{\dot{\alpha}\alpha}\Big\{4D_mD_n\mathcal{C}_A-(\Sigma_-{}^{ab}\mathcal{C})_AR^{\mathrm{cov}}_{ab}(M^{mn})\\&-\frac{1}{2}iR_{ab}{}^{\mathrm{R}}(Q)\sigma_{mn}(\Sigma_-{}^{ab}\mathscr{Z})_A\Big\}-(\gamma_m\mathscr{P}_{\mathrm{R}}C^{-1})_{\dot{\alpha}\alpha}D_m(\mathscr{H}_A+i\mathscr{K}_A)\\&+(\mathscr{P}_{\mathrm{R}}C^{-1})_{\dot{\alpha}\alpha}\Big\{\mathscr{D}_A-iD_m\mathscr{B}_A{}^m+\Box\mathcal{C}_A+i(\Sigma_-{}^{ab}\mathcal{C})_AR^-_{ab}(A)\\&+\frac{1}{2}iR^{\mathrm{R}}_{ab}(Q)(\Sigma_-{}^{ab}\mathscr{Z})_A\Big\},\end{aligned}$$

$$\begin{aligned}\mathscr{D}_{\alpha A}(\mathscr{D}\underset{\sim}{\mathscr{V}})=&-(\not{D}\Lambda_A+\Box\mathscr{Z}_A-2\sigma_{mn}D_mD_n\mathscr{Z}_A)_\alpha-i(\Sigma_-{}^{ab}\mathscr{Z}_\alpha)_AR^-_{ab}(A)\\&-\{\sigma_{mn}(\Sigma_-{}^{ab}\mathscr{Z})_A\}_\alpha R^{\mathrm{cov}}_{ab}(M^{mn})+4i(\hat{R}^{\mathrm{covR}}_{ab}(S))_\alpha(\Sigma_-{}^{ab}\mathcal{C})_A\\&-i(\hat{R}^{\mathrm{R}}_{ab}(Q))_\alpha\{\Sigma_-{}^{ab}(\mathscr{H}+i\mathscr{K})\}_A\,,\end{aligned} \tag{2·26}$$

where C is charge conjugation matrix, $\mathscr{P}_{\mathrm{R}}\equiv(1+\gamma_5)/2$, and the hat $\wedge$ and superscript R attached to $R_{ab}(Q)$ and $R^{\mathrm{cov}}_{ab}(S)$ implies $\hat{R}_{ab}(Q)\equiv(R_{ab}(Q)C^{-1})^T$ and $\hat{R}^{\mathrm{R}}_{ab}(Q)\equiv\mathscr{P}_{\mathrm{R}}\hat{R}_{ab}(Q)$, $\Box$ being conformally covariant d'Alambertian D_mD_m. Here, of course, in conformity with restriction (2·23), formula (2·26) should be understood to be multiplied by $\mathscr{S}^{(m+1)}_{(\alpha\beta_1\cdots\beta_m)}$ or by $\varepsilon^{\alpha\beta_i}$ depending on which condition $w+n=-m$, or $w+n=m+2$ is satisfied by the multiplet $\underset{\sim}{\mathscr{V}}_A=\underset{\sim}{\mathscr{V}}_{\beta_1\cdots\beta_m\dot{\gamma}}$.

To conclude this section we mention the fact that "partial integration" is possible with respect to our spinor derivative $\mathscr{D}_\alpha$ or $\bar{\mathscr{D}}_{\dot{\alpha}}$. The basis of this fact is the identities:

$$[\mathscr{D}^\alpha\underset{\sim}{\mathscr{V}}_\alpha]_D\simeq0\,,\qquad[\bar{\mathscr{D}}_{\dot{\alpha}}\underset{\sim}{\mathscr{V}}^{\dot{\alpha}}]_D\simeq0\,, \tag{2·27}$$

with $\simeq$ indicating the equality up to total derivative terms. Here $[\cdots]_D$ denotes D-type density formula, introduced in the previous paper,[9] which gives superconformally invariant density for any (general-type) multiplet carrying Weyl weight 2 and chiral weight 0.**) So $\underset{\sim}{\mathscr{V}}_\alpha$ and $\underset{\sim}{\mathscr{V}}^{\dot{\alpha}}$ here should carry Weyl and chiral weights $(3/2, \pm3/2)$, respectively. The proof of (2·27) goes similarly to the rigid supersymmetry case.

*) Indeed no one has ever carried such a programme out explicitly, aside from mentioning merely the possibility.

**) In Ref. 9), the D-type density formula $[V]_D$ was mentioned to only for *real* (hence zero chiral weight) multiplet V with Weyl weight 2. However, the reality requirement was necessary only for the hermiticity of the density but not for the conformal invariance.

Recalling the relation between D-type and F-type density formulae,[9] we have, for instance,

$$[\mathscr{D}_{\dot{\alpha}} \underset{\sim}{\mathcal{V}}^{\dot{\alpha}}]_D \simeq [\Pi(\mathscr{D}_{\dot{\alpha}} \underset{\sim}{\mathcal{V}}^{\dot{\alpha}})]_F \qquad (2\cdot28)$$

with Π being the chiral projection (or embedding) which was discussed previously[9] for scalar case and will be extended below to general non-scalar case. The identity $\Pi \mathscr{D}_{\dot{\alpha}} \underset{\sim}{\mathcal{V}}^{\dot{\alpha}} = 0$ shown in (3·9), proves the desired equation (2·27). Now applying (2·27) to $\underset{\sim}{\mathcal{V}}_\alpha = \underset{\sim}{\mathcal{V}}_{\alpha\dot{r}} \underset{\sim}{\mathcal{V}}^{(2)\dot{r}}$ we directly obtain partial integration formula

$$0 = [(\mathscr{D}^\alpha \underset{\sim}{\mathcal{V}}^{(1)}_{\alpha\dot{r}}) \underset{\sim}{\mathcal{V}}^{(2)\dot{r}}]_D + [(-)^{\dot{r}+1} \underset{\sim}{\mathcal{V}}^{(1)}_{\alpha\dot{r}} (\mathscr{D}^\alpha \underset{\sim}{\mathcal{V}}^{(2)\dot{r}})]_D\,, \qquad (2\cdot29)$$

which is in fact covariant when $\underset{\sim}{\mathcal{V}}^{(1)}_{\alpha\dot{r}}$ and $\underset{\sim}{\mathcal{V}}^{(2)\dot{r}}$ carry weights $((3/2)-x, (3/2)+x)$ and $(x, -x)$, respectively, with arbitrary number x.

§ 3. Multiplets of constrained types and tensor calculus

3. A. *Chiral multiplets Σ_A and chiral projection operation*

There are smaller multiplets than the general type one $\underset{\sim}{\mathcal{V}}_A$ considered in the previous section. Among them the most familiar one would be the chiral multiplet Σ_A which is obtained from the general one $\underset{\sim}{\mathcal{V}}_A$ by imposing a constraint:

$$\mathscr{D}_{\dot{\alpha}} \underset{\sim}{\mathcal{V}}_A = 0\,. \qquad (3\cdot1)$$

(Here no symmetry between the indices $\dot{\alpha}$ and A is implied). This constraint is superconformally covariant only when

i) Weyl and chiral weights (w,n) of $\underset{\sim}{\mathcal{V}}_A$ satisfy $w=n$

and

ii) the index A is purely undotted spinors; $A=(\alpha_1 \cdots \alpha_l)$. (3·2)

This is because the two weight-conditions in (2·24a and b) can never be satisfied simultaneously unless $m=0$ [for which condition (2·24b) becomes empty]. We have encountered the first weight condition i) $w=n$ already for the chiral multiplet without external Lorentz indices.[9] The second constraint ii) for the property of external Lorentz index was previously found in the Poincaré supergravity context by Fishler[14] and probably by many authors working in superspace superfield approach. Fishler, however, claimed that further constraints have to be imposed on the component fields for the chiral multiplet $\Sigma_{\alpha_1 \cdots \alpha_2}$ to exist, but it is *not* true; only conditions (3·2) are sufficient for the existence of Σ_A (in Poincaré as well as conformal cases).

The chiral multiplet, defined by (3·1), has $(2+2)\times \dim A$ (complex) independent components denoted by

$$\Sigma^{(w=n)}_{A=(\alpha_1 \cdots \alpha_l)} = [\mathcal{A}_A,\ \chi_{RA} \equiv \mathcal{P}_R \chi_A,\ \mathcal{F}_A], \qquad (3\cdot3)$$

in terms of which solution of (3·1) is given by

$$\underset{\sim}{\mathcal{V}}(\Sigma_A) = [\mathcal{A}_A,\ -i\chi_{RA},\ -\mathcal{F}_A,\ i\mathcal{F}_A,\ iD_m \mathcal{A}_A,\ 0,\ 0]. \qquad (3\cdot4)$$

This equation (3·4) may be viewed as an embedding formula of chiral multiplet Σ_A (3·3) into a general multiplet $\mathcal{V}_A$, and takes the same form as was previously found for scalar chiral multiplet case.[9] Owing to (3·4), the transformation law of the chiral multiplet is readable from that of $\mathcal{V}_A$. In particular the Q- and S-transformations are given explicitly by

$$\delta_{QS}\mathcal{A}_A \equiv (\delta_Q(\varepsilon)+\delta_S(\zeta))\mathcal{A}_A = \frac{1}{2}\bar{\varepsilon}_R \chi_{RA},$$

$$\delta_{QS}\chi_{RA} = (-)^A[\not{D}\mathcal{A}_A \varepsilon_L + \mathcal{F}_A \varepsilon_R + \{2w\mathcal{A}_A - (\Sigma^{ab}\mathcal{A})_A \sigma_{ab}\}\zeta_R],$$

$$\delta_{QS}\mathcal{F}_A = \frac{1}{2}\bar{\varepsilon}_L \not{D}\chi_{RA} + \bar{\zeta}_R\left\{(1-w)\chi_{RA} - \frac{1}{2}\sigma_{ab}(\Sigma^{ab}\chi_R)_A\right\}. \qquad (3\cdot5)$$

Let us now discuss a projection operation (or embedding) of general multiplet $\mathcal{V}_A$ into a chiral one $\Sigma_A = \Pi\mathcal{V}_A$, which is an analogue of the operation $\bar{D}\bar{D}\mathcal{V}_A$ of the rigid supersymmetry case. Therefore we look for the chiral multiplet $\Pi\mathcal{V}_A$ whose first component is given by $(\mathcal{H}_A - i\mathcal{K}_A)/2$. But the Theorem says that such a conformal multiplet exists if and only if $(\mathcal{H}_A - i\mathcal{K}_A)/2$ is S-inert. Thus we find from (2·5) that the chiral projection $\Pi\mathcal{V}_A$ exists only when $\mathcal{V}_A$ satisfies the conditions

$$\text{i) } w = n+2 \quad \text{and} \quad \text{ii) } A\text{: purely undotted spinor indices}. \qquad (3\cdot6)$$

Then, following the standard procedure we obtain*)

$$\Pi\mathcal{V}_A = \left[\frac{1}{2}(\mathcal{H}_A - i\mathcal{K}_A),\ i\mathcal{P}_R(\not{D}\mathcal{Z}_A + \Lambda_A),\ -\frac{1}{2}(\mathcal{D}_A + \Box C_A + iD_m\mathcal{B}_{mA})\right]. \qquad (3\cdot7)$$

Again this takes the same form as the previous formula for the multiplet without external indices. The chiral multiplet (3·7) carries Weyl and chiral weights $(w+1, n-3) = (w+1, w+1)$. It should be noted that the chiral projection $\Pi\mathcal{V}_A$ here has nothing to do with the twice spinor derivative $\mathcal{D}^{\dot\alpha}(\mathcal{D}_{\dot\alpha}\mathcal{V}_A)$ since the latter is not definable as noticed in the previous section.

A successive operation of $\mathcal{D}_\alpha$(or $\mathcal{D}_{\dot\alpha}$) after Π, or vice versa, may however be consistent and non-trivial. The following is the (complete) list of such operations definable when the indicated conditions are satisfied by the operand multiplet $\mathcal{V}_A$:

$$\begin{array}{lll} \text{i)} & \Pi\mathcal{D}_{(\alpha}\mathcal{V}_{\beta_1\cdots\beta_m)}\,; & w=n=-m/2\,, \\ & \Pi\mathcal{D}^{\beta_1}\mathcal{V}_{\beta_1\beta_2\cdots\beta_m}\,; & w=n=m/2+1\,, \\ \text{ii)} & \mathcal{D}_{(\alpha}\Pi\mathcal{V}_{\beta_1\cdots\beta_m)}\,; & w=n+2=-(m/2+1), \\ & \mathcal{D}^{\beta_1}\Pi\mathcal{V}_{\beta_1\beta_2\cdots\beta_m}\,; & w=n+2=m/2\,, \end{array} \qquad (3\cdot8)$$

where $\mathcal{V}_A$ should not have dotted spinor indices. These operations raise Weyl and chiral weights by the amount (3/2, 3/2). The other consistent operations with the same weights lead to zero:

$$\left.\begin{array}{lll} \text{iii)} & \Pi\mathcal{D}^{\dot\alpha}\mathcal{V}_{\dot\alpha\Gamma}=0\,; & w=n+3 \\ \text{iv)} & \mathcal{D}_{\dot\alpha}\Pi\mathcal{V}_{\Gamma}=0\,; & w=n+2 \end{array}\right\} \quad \Gamma\text{: undotted spinor indices}. \qquad (3\cdot9)$$

*) We are using notation $\Pi\mathcal{V}_A$ to denote the chiral embedded multiplet in place of the previous notation $\Sigma(\mathcal{V}_A)$ of Ref. 9). This is in order to emphasize the *operator* property of chiral projection Π in distinction from a mere chiral multiplet notation Σ.

The only definable operation with weights (2, 0) is

v) $\mathcal{D}^{\alpha}\Pi\mathcal{D}_{\alpha}\underset{\sim}{\mathcal{V}}$; no external indices & $w=n=0$, (3·10)

and the following operators with weights (5/2, 9/2) are covariant but lead to zero:

vi) $\Pi\mathcal{D}_{(\alpha}\Pi\underset{\sim}{\mathcal{V}}_{\beta_1\cdots\beta_m)}=0$; $w=n+2=-(m/2+1)$,

$\Pi\mathcal{D}^{\beta_1}\Pi\underset{\sim}{\mathcal{V}}_{\beta_1\beta_2\cdots\beta_m}=0$; $w=n+2=m/2$. (3·11)

The formulae for the anti-chiral multiplet $\bar{\Sigma}_A$ and anti-chiral projection $\bar{\Pi}\underset{\sim}{\mathcal{V}}_A$ are easily obtained by taking complex conjugation in all the above formulae for Σ_A and $\Pi\underset{\sim}{\mathcal{V}}_A$. [Notice that complex conjugation implies also the replacements: R (right-handed)→L, n (chiral weight)→$-n$.] An interesting identity concerning (3·10) is

$$(\mathcal{D}^{\alpha}\Pi\mathcal{D}_{\alpha}+\mathcal{D}_{\dot{\alpha}}\bar{\Pi}\mathcal{D}^{\dot{\alpha}})\underset{\sim}{\mathcal{V}}=0 \qquad (3\cdot12)$$

valid on scalar general multiplet with $w=n=0$. Operations $\Pi\Pi$ and $\bar{\Pi}\bar{\Pi}$ are not covariant but

vii) $\Pi\bar{\Pi}\underset{\sim}{\mathcal{V}}$; no external indices & $w=0$, $n=2$

$\bar{\Pi}\Pi\underset{\sim}{\mathcal{V}}$; no external indices & $w=0$, $n=-2$,

are covariant. These are weight (2, 0) operations different from that in (3·10). Identities (3·9) and (3·11) are of course analogues to $(\bar{D}\bar{D})\bar{D}_{\dot{\alpha}}=\bar{D}_{\dot{\alpha}}(\bar{D}\bar{D})=0$ and $D^{\alpha}\bar{D}^2D_{\alpha}+\bar{D}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}}=0$ of rigid supersymmetry.

3. B. *Linear multiplet* $\underset{\sim}{\mathcal{L}}_A$

The linear multiplet $\underset{\sim}{\mathcal{L}}_A$ is a multiplet subject to a constraint such that its chiral projection vanishes

$$\Pi\underset{\sim}{\mathcal{L}}_A=0 \qquad (3\cdot13)$$

in analogy with $\bar{D}\bar{D}\underset{\sim}{\mathcal{L}}_A=0$ of rigid supersymmetry. Owing to conditions (3·6) for the chiral projection to be definable, the linear multiplet exists only when its weight satisfies $w=n+2$ and the external index A is purely undotted spinors. The solution of (3·13) takes the form

$$\underset{\sim}{\mathcal{L}}^{(w,n=w-2)}_{A=(\alpha_1\cdots\alpha_l)}=[\mathcal{C}_A,\ \mathcal{Z}_A,\ \mathcal{H}_A,\ -i\mathcal{H}_A,\ \mathcal{B}_{mA},\ \Lambda_{LA}+\not{D}\mathcal{Z}_{LA},\ -\Box\mathcal{C}_A-iD_m\mathcal{B}_{mA}] \qquad (3\cdot14)$$

with the suffix L implying left-handed (internal) spinor; $\mathcal{Z}_{LA}\equiv\mathcal{P}_L\mathcal{Z}_A$. The (6+6)×dim A(complex) independent components are $[\mathcal{C}_A,\ \mathcal{Z}_A,\ \mathcal{H}_A,\ \beta_{mA},\ \Lambda_{LA}]$.*) Their transformation laws are clear from the embedding form (3·14) into a general multiplet.

The multiplet $\boldsymbol{L}$ subject to a further constraint that the anti-chiral projection also vanishes: $\Pi\boldsymbol{L}=\bar{\Pi}\boldsymbol{L}=0$, was called real linear multiplet in the previous paper.[9] Such a stringent constraint, however, can be imposed only to the multiplet carrying *no* external indices and the Weyl and chiral weights (2, 0), as is seen from condition (3·6) and its complex conjugate. Therefore there is no non-scalar real linear multiplet.

In the absence of external Lorentz indices, all the meaningful constrained-type

*) In the previous paper,[9] we have chosen the real part (Majorana) $\lambda\equiv\mathrm{Re}\Lambda$, instead of the present choice Λ_L, as a parametrization of the last independent component of $\underset{\sim}{\mathcal{L}}$. In the presence of external Lorentz index A, however, it becomes impossible in general to take such a Majorana field parametrization.

multiplets are chiral and linear ones. In the present case with external indices, however, it is clearly possible to have a more variety of constraints leading to new types of multiplets. "Pseudo-chiral" multiplets mentioned in the next subsection is an example. More important examples are the "$\boldsymbol{u}$-chiral" and "$\boldsymbol{u}$-linear" multiplets which will be found and discussed in §4.

3. C. *Constrained type multiplets as field strength of gauge fields*

It is well known in the case of rigid supersymmetry that the chiral and linear multiplets can be viewed as field strength of gauge field multiplets called prepotential usually.[20] Interestingly, despite the stringent conditions on spinor derivative and chiral projection operations, such a view still holds in this superconformal framework for the constrained multiplets with arbitrary (of course allowed) external Lorentz indices.

First and simplest is the chiral multiplet $\boldsymbol{\Sigma}_A$. It is written in a "field strength" form

$$\boldsymbol{\Sigma}^{(w,n=w)}_{\alpha_1\cdots\alpha_l} = \Pi \underset{\sim}{\mathcal{V}}^{(w-1,n-3)}_{\alpha_1\cdots\alpha_l} \tag{3·15}$$

in terms of a prepotential $\underset{\sim}{\mathcal{V}}_{\alpha_1\cdots\alpha_l}$ (general type multiplet). Indeed, because of iii) of (3·9) the RHS is invariant under a pregauge transformation

$$\delta\underset{\sim}{\mathcal{V}}^{(w-1,n-3)}_{\alpha_1\cdots\alpha_l} = \mathcal{D}^{\dot\beta}\underset{\sim}{\mathcal{U}}^{(w-3/2,n-9/2)}_{\dot\beta\alpha_1\cdots\alpha_l} \tag{3·16}$$

with a general-type multiplet parameter $\underset{\sim}{\mathcal{U}}_{\dot\beta\alpha_1\cdots\alpha_l}$, and further the original defining constraint $\mathcal{D}_{\dot\alpha}\boldsymbol{\Sigma}_A=0$ is viewed as a "Bianchi identity"

$$\mathcal{D}_{\dot\alpha}\Pi\underset{\sim}{\mathcal{V}}^{(w-1,n-3)}_{\alpha_1\cdots\alpha_l}=0\,, \tag{3·17}$$

being identical with iv) of (3·9). Notice that these equations (3·15)~(3·17) are consistent with the specific weight conditions required in superconformal case provided that the multiplets appearing there carry Weyl and chiral weights indicated on their shoulder.

Second is the linear multiplet $\underset{\sim}{\mathcal{L}}_A$, to which is assignable a field strength form:

$$\underset{\sim}{\mathcal{L}}^{(w,w-2)}_{\alpha_1\cdots\alpha_l} = \mathcal{D}^{\dot\beta}\underset{\sim}{\mathcal{V}}^{(w-1/2,w-7/2)}_{\dot\beta\alpha_1\cdots\alpha_l}\,. \tag{3·18}$$

This is invariant under the following transformation of the prepotential $\underset{\sim}{\mathcal{V}}_{\dot\beta\alpha_1\cdots\alpha_l}$:

$$\delta\underset{\sim}{\mathcal{V}}^{(w-1/2,w-7/2)}_{\dot\beta\alpha_1\cdots\alpha_l} = \mathcal{D}^{\dot\gamma}\underset{\sim}{\mathcal{U}}^{(w-1,w-5)}_{\dot\beta\dot\gamma\alpha_1\cdots\alpha_l} \tag{3·19}$$

because of identity (2·25). Equation iii) of (3·9) plays the role of the Bianchi identity assuring that $\underset{\sim}{\mathcal{L}}_A$ of the form (3·18) satisfies the linear multiplet constraint $\Pi\underset{\sim}{\mathcal{L}}_A=0$.

A more interesting example is the real linear multiplet $\boldsymbol{L}$, to which Siegel[21] was the first to give such a view in the rigid supersymmetry context. The field strength form of $\boldsymbol{L}$ is now given in terms of chiral multiplet prepotential $\boldsymbol{\Sigma}_\alpha$ and its complex conjugate $\bar{\boldsymbol{\Sigma}}_{\dot\alpha}$:

$$\boldsymbol{L} = \mathcal{D}^\alpha\boldsymbol{\Sigma}_\alpha^{(3/2,3/2)} + \mathcal{D}_{\dot\alpha}\bar{\boldsymbol{\Sigma}}^{\dot\alpha(3/2,3/2)}\,. \tag{3·20}$$

This is indeed invariant owing to (3·12) under the pregauge transformation

$$\delta\boldsymbol{\Sigma}_\alpha = \Pi\,\mathcal{D}_\alpha\boldsymbol{U}^{(0,0)} \tag{3·21}$$

with a *real* (general-type) multiplet parameter $\boldsymbol{U}$. The Bianchi identities $\Pi\boldsymbol{L}=\bar\Pi\boldsymbol{L}=0$ for (3·20) are understandable from Eq. iii) of (3·9) and the identity

$$\Pi\, \mathcal{D}^\alpha \Sigma_\alpha^{(3/2,3/2)}=0\,, \tag{3·22}$$

holding as a special case of (3·11).

This view presumably persists in any type of constrained multiplets. For example, we can conceive new-type multiplets $\boldsymbol{\Phi}^I{}_{\dot\beta_1\cdots\dot\beta_m\Gamma}$ [with weights $(w,\ w+m)$ and $\boldsymbol{\Phi}^{II}{}_{\dot\beta_1\cdots\dot\beta_m\Gamma}$ with weights $(w,\ w-m-2)$] subject to a constraints

$$\mathcal{D}_{(\dot\alpha}\boldsymbol{\Phi}^I{}_{\dot\beta_1\cdots\dot\beta_m)\Gamma}=0\,,\qquad \mathcal{D}^{\dot\beta_1}\boldsymbol{\Phi}^{II}{}_{\dot\beta_1\dot\beta_2\cdots\dot\beta_m\Gamma}=0 \tag{3·23}$$

respectively, both of which may be called "pseudo-chiral". Their field strength forms and pregauge transformations are given by

$$\boldsymbol{\Phi}^I{}_{\dot\beta_1\cdots\dot\beta_m\Gamma}=\mathcal{D}_{(\dot\beta_1}\underset{\sim}{\mathcal{V}}^I{}_{\dot\beta_2\cdots\dot\beta_m)\Gamma}\,,\quad \boldsymbol{\Phi}^{II}{}_{\dot\beta_1\cdots\dot\beta_m\Gamma}=\mathcal{D}^{\dot\alpha}\underset{\sim}{\mathcal{V}}^{II}{}_{\dot\alpha\dot\beta_1\cdots\dot\beta_m\Gamma}\,. \tag{3·24}$$

$$\delta\underset{\sim}{\mathcal{V}}^I{}_{\dot\beta_2\cdots\dot\beta_m\Gamma}=\begin{cases}\mathcal{D}_{(\dot\beta_2}\underset{\sim}{\mathcal{U}}^I{}_{\dot\beta_3\cdots\dot\beta_m)\Gamma} & \text{for } m\geqq 2\,,\\ \Pi\underset{\sim}{\mathcal{U}}^I_\Gamma & \text{for } m=1\,,\end{cases}$$

$$\delta\underset{\sim}{\mathcal{V}}^{II}{}_{\dot\alpha\dot\beta_1\cdots\dot\beta_m\Gamma}=\mathcal{D}^{\dot\gamma}\underset{\sim}{\mathcal{U}}^{II}{}_{\dot\gamma\dot\alpha\dot\beta_1\cdots\dot\beta_m\Gamma}\,. \tag{3·25}$$

The Bianchi identities and gauge invariance result from (2·25). One can also convince oneself of the consistency of these equations with the various weight conditions.

3. D. *Multiplication rules and component formulae for functions of multiplets*

The multiplication rules are easily derived by the help of the Theorem. Consider a set of general multiplets $\{\underset{\sim}{\mathcal{V}}_{A_i}=[\![\, \mathcal{C}_{A_i}]\!]\}_{i=1,2,\cdots n}$. Then we can define generally a function $\Phi_A(\underset{\sim}{\mathcal{V}}_{A_i})\equiv\boldsymbol{\Phi}_A$ so as to give a general multiplet having the first component $\Phi_A(\mathcal{C}_{A_i})$: $\boldsymbol{\Phi}_A=[\![\,\Phi_A(\mathcal{C}_{A_i})]\!]$. This indeed can define a conformal multiplet because $\Phi_A(\mathcal{C}_{A_i})$ satisfies the standard first component transformation law (2·2) for any function Φ_A. The higher components are calculated by the standard procedure and are given in terms of the function $\Phi_A\equiv\Phi_A(\mathcal{C}_{A_i})$ of the first components $\mathcal{C}_{A_i}$:

$$\mathcal{C}(\boldsymbol{\Phi}_A)=\Phi_A(\mathcal{C}_{A_i}),$$

$$\mathcal{Z}(\boldsymbol{\Phi}_A)=\mathcal{Z}'\Phi_A\,,$$

$$\begin{pmatrix}\mathcal{H}(\boldsymbol{\Phi}_A)\\ \mathcal{K}(\boldsymbol{\Phi}_A)\\ \mathcal{B}_m(\boldsymbol{\Phi}_A)\end{pmatrix}=\left[\begin{pmatrix}\mathcal{H}'\\ \mathcal{K}'\\ \mathcal{B}_m{}'\end{pmatrix}+\frac{1}{4}\bar{\mathcal{Z}}'\begin{pmatrix}-1\\ i\gamma_5\\ i\gamma_5\gamma_m\end{pmatrix}\mathcal{Z}'\right]\Phi_A\,, \tag{3·26}$$

$$\Lambda(\boldsymbol{\Phi}_A)=\left[\Lambda'+\frac{1}{2}(\mathcal{H}'-i\gamma_5\mathcal{K}'+i\gamma_5\not{\mathcal{B}}'-\not{D}\mathcal{C}')\mathcal{Z}'-\frac{1}{4}\mathcal{Z}'(\bar{\mathcal{Z}}'\mathcal{Z}')\right]\Phi_A\,,$$

$$\mathcal{D}(\boldsymbol{\Phi}_A)=\Big[\mathcal{D}'+\frac{1}{2}(\mathcal{H}'\mathcal{H}'+\mathcal{K}'\mathcal{K}'-\mathcal{B}_m{}'\mathcal{B}_m{}'-D_m\mathcal{C}'D_m\mathcal{C}')-\bar{\Lambda}'\mathcal{Z}'-\frac{1}{2}\bar{\mathcal{Z}}'\not{D}\mathcal{Z}'$$
$$-\frac{1}{4}\bar{\mathcal{Z}}'(\mathcal{H}'-i\gamma_5\mathcal{K}'+i\gamma_5\not{\mathcal{B}}')\mathcal{Z}'+\frac{1}{16}(\bar{\mathcal{Z}}'\mathcal{Z}')(\bar{\mathcal{Z}}'\mathcal{Z}')\Big]\Phi_A\,,$$

where the primed fields on the RHS's denote differential operators

$$\mathcal{Z}'=\sum_{i=1}^n\mathcal{Z}_{A_i}\frac{\partial}{\partial\mathcal{C}_{A_i}}\,,\qquad \mathcal{H}'=\sum_{i=1}^n\mathcal{H}_{A_i}\frac{\partial}{\partial\mathcal{C}_{A_i}}\,,\quad \text{etc.,} \tag{3·27}$$

operating on the function $\Phi_A(\mathcal{C}_{A_i})$. Notice there that $(\partial/\partial\mathcal{C}_{A_i})$ may be fermionic and then the order should be taken care of, e.g.,

$$\overline{\mathcal{Z}}'\mathcal{Z}' \equiv \sum_{i,j}\left(\overline{\mathcal{Z}}_{A_i}\frac{\partial}{\partial \mathcal{C}_{A_i}}\right)\left(\mathcal{Z}_{A_j}\frac{\partial}{\partial \mathcal{C}_{A_j}}\right)$$

$$= \sum_{i,j}(-)^{A_i(A_j+1)}(\overline{\mathcal{Z}}_{A_i}\mathcal{Z}_{A_j})\frac{\partial^2}{\partial \mathcal{C}_{A_i}\partial \mathcal{C}_{A_j}} = \sum_{i,j}(-)^{A_i}(\overline{\mathcal{Z}}_{A_i}\mathcal{Z}_{A_j})\frac{\partial^2}{\partial \mathcal{C}_{A_j}\partial \mathcal{C}_{A_i}}. \quad (3\cdot 28)$$

For the case of a bilinear function $\Phi_A(\mathcal{C}_{A_i}) = \mathcal{C}_{A_1}\mathcal{C}_{A_2}$, this formula (3·26) of course represents the simplest multiplication law, giving $\mathcal{V}_{A_1}\cdot\mathcal{V}_{A_2}$. Formula (3·26) takes quite the same form as the one previously obtained for Lorentz-scalar multiplet case in Ref. 2).

Similarly the arbitrary function $g_A(\Sigma_{A_i}) \equiv \boldsymbol{g}_A$ of a set of chiral multiplets $\{\Sigma_{A_i} = [\mathcal{A}_{A_i}, \chi_{RA_i}, \mathcal{F}_{A_i}]\}_{i=1,2,\cdots,n}$ defines again a chiral multiplet. The components of this multiplet are given by

$$\boldsymbol{g}_A = \left[g_A(\mathcal{A}_{A_i}) \equiv g_A, \quad \chi_R' g_A, \quad \left(\mathcal{F}' - \frac{1}{4}\bar{\chi}_R'\chi_R'\right)g_A\right] \quad (3\cdot 29)$$

with similar notations to (3·27), e.g., $\mathcal{F}' = \sum_i \mathcal{F}_{A_i}(\partial/\partial\mathcal{A}_{A_i})$. This type of formula for Lorentz scalar multiplet case was first given in Ref. 22) in Poincaré supergravity, and was given in superconformal context in Refs. 2) and 9). If we know the embedding formula (3·4) of chiral into general, formula (3·29) is a special case of (3·26).

§ 4. *u*-associated spinor derivatives and connection with Poincaré supergaravity versions

4. A. *u-associated spinor derivative*

As we have noted in §2.C, successive operations of the spinor derivatives $\mathcal{D}_{\tilde{\alpha}}$ are not superconformally covariant in general and hence the (anti-) commutation relations like $\{\mathcal{D}_{\tilde{\alpha}}, \mathcal{D}_{\tilde{\beta}}\}$ cannot be discussed at all. Nevertheless that spinor derivative operation $\mathcal{D}_{\tilde{\alpha}}$ was the only definable one as far as the "covariantization" of the operation is done solely by using the original superconformal gauge fields ($e_\mu{}^m$, ψ_μ,$\cdots$in (1·1)). But, here, if we prepare a Lorentz-scalar (matter) multiplet, $\boldsymbol{u}$, and use the component fields of $\boldsymbol{u}$ to covariantize the operation, we can define yet other spinor derivative operators $\mathcal{D}_\alpha{}^{(u)}$, $\bar{\mathcal{D}}_{\dot{\alpha}}{}^{(u)}$ in such a way that their arbitrary successive operations become superconformally covariant on any multiplets and hence their anti-commutation relations can be discussed.

Let us choose a multiplet $\boldsymbol{u}$ and denote the component fields of $\boldsymbol{u}$ by

$$\boldsymbol{u} = [\mathcal{C}_u, \mathcal{Z}_u, \mathcal{H}_u, \mathcal{K}_u, \mathcal{B}_u{}^m, \Lambda_u, \mathcal{D}_u],$$

generically, although not all of them may be independent when $\boldsymbol{u}$ is a constrained type multiplet. Assuming that $\boldsymbol{u}$ carries Weyl and chiral weights (w_0, n_0) with $w_0 + n_0 \neq 0$, we see from (2·5) the S-transformation laws of the first and second component fields as

$$\delta_S(\zeta)\mathcal{C}_u = 0, \qquad \delta_S(\zeta)\mathcal{Z}_{u\alpha} = -i(w_0 + n_0)\zeta_\alpha\mathcal{C}_u. \quad (4\cdot 1)$$

Therefore a spinor $\lambda_\alpha{}^S$ defined by

$$\lambda_\alpha{}^S \equiv i\mathcal{Z}_{u\alpha}/(w_0 + n_0)\mathcal{C}_u \quad (4\cdot 2)$$

yields just the transformation parameter ζ under the S-transformation,

$$\delta_S(\zeta)\lambda_\alpha{}^S=\zeta_\alpha\,, \tag{4·3}$$

and hence can be utilized for the "S-covariantization" (or "S-invariantization") in defining spinor derivative operations.

We define a new spinor derivative operation $\mathcal{D}_\alpha{}^{(u)}$ which we call "$\boldsymbol{u}$-associated spinor derivative", on an arbitrary multiplet $\underset{\sim}{\mathcal{V}}_A=[\mathcal{C}_A, \mathcal{Z}_A, \cdots]$ with weights (w, n) by

$$\mathcal{D}_\alpha{}^{(u)}\underset{\sim}{\mathcal{V}}_A=[\![\mathcal{Z}_{\alpha A}+i(n+w)\lambda_\alpha{}^S\mathcal{C}_A-i(\sigma_{ab})_\alpha{}^\beta\lambda_\beta{}^S(\Sigma^{ab}\mathcal{C})_A]\!]\,. \tag{4·4}$$

Here we are using the notation $[\![\mathcal{C}_A]\!]$ introduced in (2·13) which denotes the multiplet with its first component $\mathcal{C}_A$. Notice that the first component of this multiplet is S-inert (and K-inert, of course) since the second and third terms of the RHS are just "S-covariantization" for $\mathcal{Z}_{\alpha A}$ which transforms

$$\delta_S(\zeta)\mathcal{Z}_{\alpha A}=-i(w+n)\zeta_\alpha\mathcal{C}_A+i(\sigma_{ab})_\alpha{}^\beta\zeta_\beta(\Sigma^{ab}\mathcal{C})_A\,. \tag{4·5}$$

Therefore, by the Theorem, the RHS of (4·4) gives a superconformal multiplet for any $\underset{\sim}{\mathcal{V}}_A$; in other words, the $\boldsymbol{u}$-associated spinor derivative $\mathcal{D}_\alpha{}^{(u)}$ is superconformally covariant on $^\forall\underset{\sim}{\mathcal{V}}_A$ carrying arbitrary weights and Lorentz index. $\mathcal{D}_\alpha{}^{(u)}$ carries Weyl and chiral weights $(1/2, -3/2)$ just as previous $\mathcal{D}_\alpha$.

It is not difficult to relate $\mathcal{D}_\alpha{}^{(u)}$ with the previous spinor derivative $\mathcal{D}_\alpha$ defined in §2.C. For a multiplet $\underset{\sim}{\mathcal{V}}_A=\underset{\sim}{\mathcal{V}}^{(w,n)}_{\beta_1\cdots\beta_m\dot\Gamma}$ (totally symmetric with respect to the m undotted spinor indices $\beta_1, \cdots, \beta_m$) carrying weight (w, n), the relation is given by

$$\mathcal{D}_\alpha{}^{(u)}\underset{\sim}{\mathcal{V}}_A=\boldsymbol{u}^{(m+w+n)/(w_0+n_0)}\Big[\mathcal{D}_{(\alpha}\boldsymbol{u}^{-(m+w+n)/(w_0+n_0)}\underset{\sim}{\mathcal{V}}^{(w,n)}_{\beta_1\cdots\beta_m)\dot\Gamma}$$
$$-\frac{1}{m+1}\boldsymbol{u}^{-2(m+1)/(w_0+n_0)}\sum_{i=1}^{m}\varepsilon_{\alpha\beta_i}\mathcal{D}^\delta\boldsymbol{u}^{(m+2-w-n)/(w_0+n_0)}\underset{\sim}{\mathcal{V}}^{(w,n)}_{\beta_1\cdots\underset{\delta}{\overset{i}{\vee}}\cdots\beta_m\dot\Gamma}\Big]. \tag{4·6}$$

This is understandable as follows: In the RHS the various powers of $\boldsymbol{u}$ are multiplied so that the weight conditions (2·23) for the applicability of the spinor derivative $\mathcal{D}_\alpha$ are satisfied and hence the RHS gives a superconformal multiplet with weight $(w+1/2, n+3/2)$ in accordance with the LHS. Therefore it is sufficient to check that the first components of the both sides agree with each other. (Identity (2·21) is used there.)

The dotted spinor derivative $\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}$ is defined through the complex conjugation of $\mathcal{D}_\alpha{}^{(u)}$:

$$\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}\underset{\sim}{\mathcal{V}}_A\equiv(\mathcal{D}_\alpha{}^{(u)}(\underset{\sim}{\mathcal{V}}_A)^*)^*\,. \tag{4·7}$$

From expression (4·6) (or (4·4) directly), we notice an important property of $\mathcal{D}_\alpha{}^{(u)}$; on the multiplets $\underset{\sim}{\mathcal{V}}_A$ carrying purely dotted spinor indices $A=\dot\Gamma$ and satisfying the weight relation $w+n=0$, the $\boldsymbol{u}$-associated derivative $\mathcal{D}_\alpha{}^{(u)}$ reduces to the ordinary spinor derivative $\mathcal{D}_\alpha$ *independently* of $\boldsymbol{u}$:

$$\mathcal{D}_\alpha{}^{(u)}\underset{\sim}{\mathcal{V}}_{\dot\Gamma}{}^{(w=-n)}=\mathcal{D}_\alpha\underset{\sim}{\mathcal{V}}_{\dot\Gamma}{}^{(w=-n)}\,. \tag{4·8}$$

Similarly, for $\underset{\sim}{\mathcal{V}}_A$ with $w-n=0$ and purely undotted spinor index Γ,

$$\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}\underset{\sim}{\mathcal{V}}_\Gamma{}^{(w=n)}=\bar{\mathcal{D}}_{\dot\alpha}\underset{\sim}{\mathcal{V}}_\Gamma{}^{(w=n)}\,. \tag{4·9}$$

Another interesting property of $\mathcal{D}_\alpha{}^{(u)}$ is

$$\mathcal{D}_\alpha{}^{(u)}\boldsymbol{u}=0 \quad \text{or} \quad \bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}\bar{\boldsymbol{u}}=0\,, \tag{4·10}$$

which also follows from (4·4) or (4·6) immediately.

4. B. *Commutation relations of* ***u****-associated derivatives*

Expression (4·4) for $\mathcal{D}_\alpha{}^{(u)}$ and a similar one for $\bar{\mathcal{D}}_{\dot\alpha}{}^{(u)}$ can be rewritten in a more convenient form if we suppress the spinor indices α and $\dot\alpha$ by multiplying a (4-component) dummy spinor $\bar\eta=(\eta^\alpha, \bar\eta_{\dot\alpha})$:

$$\bar\eta i\gamma_5\mathcal{D}^{(u)}\underset{\sim}{\mathcal{V}}_A\equiv i(\eta^\alpha\mathcal{D}_\alpha{}^{(u)}-\bar\eta_{\dot\alpha}\bar{\mathcal{D}}^{(u)\dot\alpha})\underset{\sim}{\mathcal{V}}_A=[\![\tilde\delta_Q(2\eta)\mathcal{C}_A]\!]\,,$$

$$\tilde\delta_Q(2\eta)=\delta_Q(2\eta)+\delta_D(-\bar\eta\lambda^S)+\delta_A(2\bar\eta i\gamma_5\lambda^S)+\delta_M(2\bar\eta\sigma_{ab}\lambda^S). \tag{4·11}$$

Here λ^S is a 4-component Majorana $\lambda_{\bar a}{}^S\equiv(\lambda_\alpha{}^S, \bar\lambda^{S\dot\alpha})^T$ with $\lambda_\alpha{}^S$ of (4·2) and $\bar\lambda_{\dot\alpha}{}^S=(\lambda_\alpha{}^S)^*$, i. e., in 4-component notation,

$$\lambda^S=\frac{1}{w_0+n_0}i\gamma_5\left(\frac{\mathcal{P}_R\mathcal{Z}}{\mathcal{C}}+\frac{\mathcal{P}_L\mathcal{Z}^C}{\mathcal{C}^*}\right) \tag{4·12}$$

with $\mathcal{P}_{R,L}\equiv(1\pm\gamma_5)/2$ and $\mathcal{Z}^C$ being the charge conjugation of $\mathcal{Z}$.

We now can define "***u***-associated *vector* derivative" $\mathcal{D}_m{}^{(u)}$ in a very similar way. An obvious candidate for the first component of $\mathcal{D}_m{}^{(u)}\underset{\sim}{\mathcal{V}}_A$ would be the conformally covariant derivative $D_m\mathcal{C}_A$ defined in (1·8), but $D_m\mathcal{C}_A$ unfortunately cannot be a first component of conformal multiplet since it is neither invariant under S-transformation nor under K-transformation:

$$\delta_K(\xi_n)D_m\mathcal{C}_A=2w\,\mathcal{C}_A\xi_m-2\xi_n(\Sigma^{mn}\mathcal{C})_A\,,$$

$$\delta_S(\zeta)D_m\mathcal{C}_A=\frac{1}{2}\bar{\mathcal{Z}}_A i\gamma_5\gamma_m\zeta\,. \tag{4·13}$$

Therefore the "invariantizations under K- and S-transformations" should be performed by the help of ***u***-multiplet component fields as before, and we are led to the definition:

$$\mathcal{D}_m{}^{(u)}\underset{\sim}{\mathcal{V}}_A=[\![D_m\mathcal{C}_A-2wV_m{}^K\mathcal{C}_A+2V_n{}^K(\Sigma^{mn}\mathcal{C})_A$$

$$+\frac{1}{2}\bar\chi^S\gamma_m i\gamma_5\mathcal{Z}_A+\frac{1}{4}(\bar\chi^S\gamma_5\gamma_m\chi^S)\{\delta_{mn}n\,\mathcal{C}_A+(\tilde\Sigma_{mn}\mathcal{C})_A\}]\!]\,. \tag{4·14}$$

Here $V_m{}^K$ is the following "K-covariantization vector field", characterized by $\delta_K(\xi_n)V_m{}^K=\xi_m$, which is made from ***u***-multiplet components $\mathcal{C}_u$ and $\mathcal{Z}_u$:*)

$$V_m{}^K=(4w_0)^{-1}(\mathcal{C}_u{}^{-1}D_m\mathcal{C}_u+\mathcal{C}_u{}^{*-1}D_m\mathcal{C}_u{}^*). \tag{4·15}$$

The S-transformation of this $V_m{}^K$ is non-trivial and defines χ^S in (4·14):

$$\delta_S(\zeta)V_m{}^K=-\frac{1}{4}\bar\zeta\gamma_m\chi^S\,,$$

$$\chi^S=(2w_0)^{-1}i\gamma_5(\mathcal{C}_u{}^{-1}\mathcal{Z}_u+\mathcal{C}_u{}^{*-1}\mathcal{Z}_u{}^C). \tag{4·16}$$

Interestingly this χ^S also gives an "S-covariantization spinor field" which is generally

*) The $V_m{}^K$ in (4·15) is chosen to be a real field so as to be consistent with the "reality" of vector derivative $\mathcal{D}_m{}^{(u)}$.

different from the previous λ^S given in (4·12) as far as $\mathscr{Z}\neq\mathscr{Z}^C$. It indeed transforms as

$$\delta_S(\zeta)\chi^S=\zeta\,, \qquad \delta_K(\xi)\chi^S=0 \tag{4·17}$$

just as λ^S did, and is actually invariantizing quantity (4·14) under S-transformations.

Equation (4·14) is also rewritten in terms of superconformal transformations:

$$\xi^m\mathscr{D}_m{}^{(u)}\underset{\sim}{\mathcal{V}}_A=\Big[\!\!\Big[\{\delta_{\tilde{P}}(\xi)+\delta_M[2(\xi_a V_b{}^K-\xi_b V_a{}^K)+\frac{1}{4}\varepsilon_{abmn}\xi_m(\bar{\chi}^S\gamma_5\gamma_m\chi^S)]$$
$$+\delta_D(-2\xi\cdot V^K)+\delta_A(-\frac{1}{2}\xi_m\bar{\chi}^S i\gamma_5\gamma_m\chi^S)+\delta_Q(-\xi^m\gamma_m\chi^S)\}\,\mathcal{C}_A\Big]\!\!\Big]. \tag{4·18}$$

If we use expressions (4·11) and (4·18), we can easily calculate the commutation relations between the $\boldsymbol{u}$-associated derivatives $\mathscr{D}_{\hat{A}}{}^{(u)}\equiv(\mathscr{D}_\alpha{}^{(u)},\mathscr{D}_{\dot{\alpha}}{}^{(u)},\mathscr{D}_m{}^{(u)})$ directly from the superconformal algebra (1·7). Generally we obtain the commutation relations of the form (cf., Refs. 23), 24))

$$[\mathscr{D}_{\hat{A}}{}^{(u)},\mathscr{D}_{\hat{B}}{}^{(u)}\}=-T_{\hat{A}\hat{B}}{}^{\hat{C}}\mathscr{D}_{\hat{C}}{}^{(u)}-\frac{1}{2}R_{\hat{A}\hat{B}}{}^{ab}\mathbf{M}_{ab}-F_{\hat{A}\hat{B}}\mathbf{A}-G_{\hat{A}\hat{B}}\mathbf{D}\,, \tag{4·19}$$

where $\mathbf{M}_{ab}$, $\mathbf{A}$ and $\mathbf{D}$ are Lorentz, chiral and Weyl transformation generators which "count" the corresponding *quantum numbers of multiplets*, i.e., those of first components; e.g., with a parameter λ^{ab},

$$\frac{1}{2}\lambda^{ab}\mathbf{M}_{ab}\underset{\sim}{\mathcal{V}}_A=\frac{1}{2}\lambda^{ab}(\Sigma_{ab}\underset{\sim}{\mathcal{V}})_A=[\![\delta_M(\lambda^{ab})\,\mathcal{C}_A]\!]\;. \tag{4·20}$$

The torsion $T_{\hat{A}\hat{B}}{}^{\hat{C}}$ and curvatures $R_{\hat{A}\hat{B}}$, $F_{\hat{A}\hat{B}}$, $G_{\hat{A}\hat{B}}$ are now all superconformal multiplets (dependent on $\boldsymbol{u}$). Therefore we now have a simple algorithm to obtain such geometrical quantities as torsion and curvatures from the superconformal algebra (1·7).*)

For instance, the commutator of the spinor derivatives is reduced to the commutator of a superconformal transformation $\tilde{\delta}_Q(2\eta)$ of (4·11):

$$[\bar{\eta}_1 i\gamma_5\mathscr{D}^{(u)},\bar{\eta}_2 i\gamma_5\mathscr{D}^{(u)}]=[\![[\tilde{\delta}_Q(2\eta_1),\tilde{\delta}_Q(2\eta_2)]\,\mathcal{C}_A]\!]\;. \tag{4·21}$$

Let us calculate, in particular, the anti-cummutator $\{\mathscr{D}_\alpha{}^{(u)},\mathscr{D}_\beta{}^{(u)}\}$ corresponding to simplest case in which both η_1 and η_2 are taken right-handed. We calculate the RHS commutator by using the algebra (1·7) and by taking into account that the λ^S field in the arguments of $\delta_{D,A,M}$ terms in $\tilde{\delta}_Q$ (4·11) should also be transformed:

$$[\tilde{\delta}_Q(2\eta_{1\mathrm{R}}),\tilde{\delta}_Q(2\eta_{2\mathrm{R}})]=-\delta_M(\mathscr{R}\,\bar{\eta}_{2\mathrm{R}}\sigma_{ab}\eta_{1\mathrm{R}}),$$
$$\mathscr{R}\equiv4(n_0+w_0)^{-1}(\mathscr{H}_u+i\mathscr{K}_u)\,\mathcal{C}_u{}^{-1}-2(n_0+w_0-2)(\bar{\lambda}_\mathrm{R}{}^S\lambda_\mathrm{R}{}^S). \tag{4·22}$$

Noticing also that

$$[\![\delta_M(\mathscr{R}\,\bar{\eta}_{2\mathrm{R}}\sigma_{ab}\eta_{1\mathrm{R}})\,\mathcal{C}_A]\!]\;=[\![\mathscr{R}]\!]\;\frac{1}{2}\bar{\eta}_{2\mathrm{R}}\sigma_{ab}\eta_{1\mathrm{R}}\mathbf{M}_{ab}\underset{\sim}{\mathcal{V}}_A\,, \tag{4·23}$$

since $\mathscr{R}$ given by (4·22) is S- and K-invariant as it should be, we obtain

*) This kind of connection between supersymmetry algebra and geometrical quantities has been already known to Sohnius and West[25] in a different context.

$$\{\mathcal{D}_\alpha{}^{(u)}, \mathcal{D}_\beta{}^{(u)}\} = -\frac{1}{2} [\![\mathcal{R}]\!] (\sigma^{ab})_{\alpha\beta}\mathbf{M}_{ab} \tag{4·24}$$

with $(\sigma^{ab})_\alpha{}^\gamma \varepsilon_{\gamma\beta} \equiv (\sigma^{ab})_{\alpha\beta}$, implying

$$T_{\alpha\beta}{}^\gamma = T_{\alpha\beta}{}^{\dot\gamma} = T_{\alpha\beta}{}^m = F_{\alpha\beta} = G_{\alpha\beta} = 0\,,$$
$$R_{\alpha\beta}^{ab} = (\sigma^{ab})_{\alpha\beta}\ [\![\mathcal{R}]\!]\,. \tag{4·25}$$

The calculations of other commutators such as $\{\mathcal{D}_\alpha{}^{(u)}, \mathcal{D}_{\dot\alpha}{}^{(u)}\}$ are similar (although become slightly more complicated) and are omitted here. [Such calculations for $\boldsymbol{u}=\boldsymbol{\Sigma}$, $\boldsymbol{L}$ and $\underset{\sim}{\mathcal{L}}$ can be found in the original version of this paper.[26]]

4. C. ***u**-chiral and **u**-linear multiplets*

We call conformal multiplets defined by a constraint

$$\mathcal{D}_{\dot\alpha}{}^{(u)}\underset{\sim}{\mathcal{V}}_A = 0 \tag{4·26}$$

"$\boldsymbol{u}$-chiral multiplets" and denote them by $\boldsymbol{\Sigma}_A{}^{(u)}$. As is well-known,[15),23)] constraint (4·26) and commutation relation (4·24) imply

$$0 = \{\mathcal{D}_{\dot\alpha}{}^{(u)}, \mathcal{D}_{\dot\beta}{}^{(u)}\}\boldsymbol{\Sigma}_A{}^{(u)} = -\frac{1}{2} [\![\mathcal{R}^*]\!] (\bar\sigma^{ab})_{\dot\alpha\dot\beta}\mathbf{M}_{ab}\boldsymbol{\Sigma}_A{}^{(u)}\,, \tag{4·27}$$

and so it is necessary for the existence of non-trivial solutions $\boldsymbol{\Sigma}_A{}^{(u)}$ that either the condition that

$$\text{i)}\quad [\![\mathcal{R}]\!] = 0\,,\quad \text{i.e.,}\quad \mathcal{R} = 0 \tag{4·28}$$

or

ii) the Lorentz index A is purely undotted spinor; $A=(\alpha_1, \cdots, \alpha_l)$

holds. The latter condition ii) is because $(\bar\sigma^{ab})_{\dot\alpha\dot\beta}$ is self-dual, $\bar\sigma^{ab}=\bar\sigma_+{}^{ab}$, and the properties $\mathbf{M}_{ab}\boldsymbol{\Sigma}_A = (\Sigma_{\bar{ab}})_A^B\boldsymbol{\Sigma}_B$ (i.e., anti-self-dual) for purely undotted spinor indices A and $\bar\sigma_+{}^{ab}\Sigma_-{}^{ab}=0$ (see Appendix A).

As a matter of fact, Eq. (4·27) is the "integrability condition" of the "differential" equation (4·26) and so the above condition i) or ii) is a sufficient condition for $\boldsymbol{\Sigma}_A{}^{(u)} \neq 0$ to exist. Indeed it is easy to see from the commutation relation (4·24) that the following operators satisfy the chiral projection property $\mathcal{D}_{\dot\alpha}{}^{(u)}\Pi^{(u)}\underset{\sim}{\mathcal{V}}_A=0$:

$$\text{i)}\quad \Pi^{(u)} = -\frac{1}{4}\mathcal{D}_{\dot\alpha}{}^{(u)}\mathcal{D}^{(u)\dot\alpha}\quad (\text{when } \mathcal{R}=0), \tag{4·29a}$$

$$\text{ii)}\quad \Pi^{(u)} = -\frac{1}{4}\{\mathcal{D}_{\dot\alpha}{}^{(u)}\mathcal{D}^{(u)\dot\alpha} - [\![\mathcal{R}^*]\!]\}$$

(only on $\underset{\sim}{\mathcal{V}}_A$ with purely undotted spinor indices A) (4·29b)*)

*) In deriving (4·29b) we need a property $\mathcal{D}_{\dot\alpha}{}^{(u)}\ [\![\mathcal{R}^*]\!] = 0$ which follow from the definitions (4·22) and (4·4). This property is made more manifest if we notice the equation

$$[\![\mathcal{R}^*]\!] = 4\bar{\boldsymbol{u}}^{-2/(w_0+n_0)}\Pi\bar{\boldsymbol{u}}^{2/(w_0+n_0)}\,,$$

where Π is the chiral projection (3·7). This equation, also makes it manifest that $[\![\mathcal{R}^*]\!]$ vanishes when $\bar{\boldsymbol{u}}$ is a linear multiplet $\boldsymbol{L}$ or $\underset{\sim}{\mathcal{L}}$.

for the above two cases, respectively, and hence $\Pi^{(u)}\underset{\sim}{\mathcal{V}}_A$ gives a non-trivial solution of Eq. (4·26).

From expression (4·22), we see $[\![\mathcal{R}^*]\!]=0$ when $\boldsymbol{u}$ is a complex anti-linear multiplet $\underset{\sim}{\bar{\mathcal{L}}}$ or a real linear multiplet $\bar{\boldsymbol{L}}=\boldsymbol{L}$. Therefore the above result implies that $\underset{\sim}{\bar{\mathcal{L}}}$-chiral and $\boldsymbol{L}$-chiral multiplets exist for *arbitrary* external Lorentz indices A and with arbitrary Weyl and chiral weights, in sharp contrast to the usual conformal chiral multiplets in the previous section. For the case $\mathcal{R}\neq 0$ as is the case when $\boldsymbol{u}=\boldsymbol{\Sigma}$(chiral), on the other hand, the $\boldsymbol{u}$-chiral multiplets exist only for purely undotted spinor indices A.

The fact that here appears no restriction on Weyl and chiral weights is not surprising since multiplication of powers of $\bar{\boldsymbol{u}}$ to $\boldsymbol{\Sigma}_A{}^{(u)}$ can change those weights without violating the chirality constraint $\bar{\mathcal{D}}_{\dot{\alpha}}{}^{(u)}\boldsymbol{\Sigma}_A{}^{(u)}=0$ because $\bar{\mathcal{D}}_{\dot{\alpha}}{}^{(u)}\bar{\boldsymbol{u}}=0$ [(4·10)]. Therefore we can conveniently assume without loss of generality that $\boldsymbol{\Sigma}_A{}^{(u)}$ carry equal Weyl and chiral weights $w=n$. Then, for the cases when external indices A are purely undotted spinor ones, all the $\boldsymbol{u}$-chiral multiplets become identical with the usual conformal chiral multiplets $\boldsymbol{\Sigma}_A$ in §3 *independently* of $\boldsymbol{u}$ since the $\boldsymbol{u}$-associated spinor derivative $\bar{\mathcal{D}}_{\dot{\alpha}}{}^{(u)}$ reduces to the ordinary spinor derivative $\bar{\mathcal{D}}_{\dot{\alpha}}$ in the case $m=0$ and $w=n$ [see Eq. (4·9)].

Quite a similar discussion can be made on "$\boldsymbol{u}$-linear multiplets" $\underset{\sim}{\mathcal{L}}_A{}^{(u)}$, which are defined by a constraint

$$\Pi^{(u)}\underset{\sim}{\mathcal{L}}_A{}^{(u)}=0 \tag{4·30}$$

with the $\boldsymbol{u}$-chiral projection operator $\Pi^{(u)}$ of (4·29). They exist for arbitrary Lorentz indices when $\boldsymbol{u}=\underset{\sim}{\bar{\mathcal{L}}}$ or $\boldsymbol{L}$ while exist only for purely undotted spinor case when, e.g., $\boldsymbol{u}=\boldsymbol{\Sigma}$.

4.D. *Connection with Poincaré supergravity versions*

As is well-known,[5),7)~9)] various versions of Poincaré supergravity come from different choices of *compensating multiplets*, by the component fields of which one fixes the extraneous gauge freedoms of superconformal theory such as dilatation D, conformal supersymmetry S.

The Poincaré supergravity versions, (I) old minimal,[27)] (II) new minimal[28)] and (III) non-minimal,[29)] correspond to the choices of (I) a chiral $\boldsymbol{\Sigma}_0{}^{(1,1)}$,[5)] (II) a real linear $\boldsymbol{L}_0{}^{(2,0)}$[7)] and (III) a complex linear $\underset{\sim}{\mathcal{L}}_0{}^{(w_0,w_0-2)}$[7),30)] compensating multiplet, respectively (with Weyl and chiral weights indicated on the shoulders).

Taking into account the role of compensating fields, one can convince oneself that the usual covariant derivatives $\mathcal{D}_{\tilde{A}}{}^{\mathrm{P}}=E_{\tilde{A}}{}^M\mathcal{D}_M{}^{\mathrm{P}}(\tilde{A}=\alpha,\,\dot{\alpha},\,m)$ in superspace formulation of Poincaré supergravity[23)~25),31),32)] should be identified (aside from trivial weight adjustment) with the present $\boldsymbol{u}$-associated spinor and vector derivatives $\mathcal{D}_\alpha{}^{(u)}$, $\bar{\mathcal{D}}_{\dot{\alpha}}{}^{(u)}$ and $\mathcal{D}_m{}^{(u)}$ with the choices $\boldsymbol{u}=\boldsymbol{\Sigma}_0$, $\boldsymbol{L}_0$ and $\underset{\sim}{\bar{\mathcal{L}}}_0$ for those three Poincaré versions, respectively. [A suitable covariance argument would suffice for deriving this identification, although there remains some arbitrariness in the convention of the choice of gauge fields in the vector derivative $\mathcal{D}_m{}^{(u)}$. More detailed discussion was given in Ref. 26).] More precisely,

$$\mathcal{D}_\alpha{}^{\mathrm{P}}=\boldsymbol{C}\mathcal{D}_\alpha{}^{(u)},\qquad \mathcal{D}_m{}^{\mathrm{P}}=\boldsymbol{C}\bar{\boldsymbol{C}}\mathcal{D}_m{}^{(u)}, \tag{4·31a}$$

where, respectively for each versions,

(I) $\quad \boldsymbol{u}=\boldsymbol{\Sigma}_0,\quad \boldsymbol{C}=\bar{\boldsymbol{\Sigma}}_0{}^{-1}\boldsymbol{\Sigma}_0{}^{1/2}$,

(II) $\quad \boldsymbol{u}=\boldsymbol{L}_0,\quad \boldsymbol{C}=\boldsymbol{L}_0{}^{-1/4}=\bar{\boldsymbol{C}}$,

(III) $\boldsymbol{u}=\underset{\sim}{\mathcal{L}}_0, \quad \boldsymbol{C}=\underset{\sim}{\mathcal{L}}_0{}^u\underset{\sim}{\mathcal{L}}_0{}^v$

with

$$\begin{cases} u=\dfrac{1}{4}\left(\dfrac{3}{w_0-2}-\dfrac{1}{w_0}\right), \\ v=-\dfrac{1}{4}\left(\dfrac{3}{w_0-2}+\dfrac{1}{w_0}\right). \end{cases} \tag{4·31b}$$

Here the weight adjustment factor $\boldsymbol{C}$ is multiplied so that $\mathcal{D}_\alpha{}^{\mathrm{P}}$ carries the desired weight $(w, n)=(0, 0)$ in versions (I) and (III) ((0, −3/2) in the version (II)).*) With the help of Eq. (4·31), all the commutation relations of covariant derivatives $\mathcal{D}_A{}^{\mathrm{P}}$ in various Poincaré supergravities are systematically derivable from the $\mathcal{D}_A{}^{(u)}$ algebra in our conformal framework, e.g., compare Eq. (4·24) with the corresponding Poincaré expressions in Refs. 24), 31), 32) and 15).

Equation (4·31) implies that the chiral projection operator Π^{P} of Poincaré theory defined by $\mathcal{D}_{\dot{\alpha}}{}^{\mathrm{P}}\, \Pi^{\mathrm{P}}=0$ also coincides with $\Pi^{(u)}$ aside from the $\boldsymbol{C}$ factors in (4·31b):

$$\Pi^{\mathrm{P}}=\Pi^{(u)}\bar{\boldsymbol{C}}^2 .$$

[For instance, (4·29b) reproduces Π^{P} of Wess and Zumino[24] for $\boldsymbol{u}=\boldsymbol{\Sigma}_0$.]

Therefore the chiral multiplets $\boldsymbol{\Sigma}_A{}^{\mathrm{P}}$ and linear multiplets $\underset{\sim}{\mathcal{L}}_A{}^{\mathrm{P}}$ in Poincaré supergravities (I), (III), defined by $\mathcal{D}_{\dot{\alpha}}{}^{\mathrm{P}}\, \boldsymbol{\Sigma}_A{}^{\mathrm{P}}=0$ and $\Pi^{\mathrm{P}}\underset{\sim}{\mathcal{L}}_A{}^{\mathrm{P}}=0$, respectively, are nothing but the $\boldsymbol{u}$-chiral and $\boldsymbol{u}$-linear multiplets when $\boldsymbol{u}$ is taken to be the corresponding compensating multiplet (I) $\boldsymbol{\Sigma}_0$, (II) $\boldsymbol{L}_0$ or (III) $\underset{\sim}{\mathcal{L}}_0$. Therefore from the results of previous subsections we see that chiral and linear multiplets exist only for purely undotted spinor index case in the old minimal Poincaré supergravity (I), while they exist for arbitrary external Lorentz indices in the new minimal (II) and non-minimal (III) versions. [Further they exist for arbitrary chiral weights in the case of (II).] This result is well-known for the old minimal[14] and non-minimal[15] versions, but is less known for the new minimal case.

More importantly, our finding here in this section implies further that such $\boldsymbol{u}$-chiral and $\boldsymbol{u}$-linear multiplets for general $\boldsymbol{u}$ (not necessarily set equal to the compensating multiplet) can be defined and exist equally in any Poincaré versions. This is clear because the correspondent of the $\boldsymbol{u}$-associated derivative is definable also by using only Poincaré spinor derivative $\mathcal{D}_\alpha{}^{\mathrm{P}}$ since the usual conformal spinor derivative $\mathcal{D}_\alpha$ in Eq. (4·6) can be expressed in terms of $\mathcal{D}_\alpha{}^{\mathrm{P}}$ with the help of the compensating multiplet. This fact, in particular, means that *there exist "chiral multiplets"* (as well as "linear multiplets") *with arbitrary Lorentz index A even in the old minimal Poincaré theory* provided that we prepare an associate multiplet $\boldsymbol{L}$ or $\underset{\sim}{\mathcal{L}}$ (which has to be a matter multiplet).

*) This is because $\mathcal{D}_\alpha{}^{\mathrm{P}}$ is a mapping of Poincaré multiplets $\underset{\sim}{\mathcal{V}}_A{}^{\mathrm{P}}$ to Poincaré multiplets and $\underset{\sim}{\mathcal{V}}_A{}^{\mathrm{P}}$ are identified with superconformal ones $\underset{\sim}{\mathcal{V}}_A{}^{(0,0)}$ carrying zero Weyl and chiral weights in the versions (I) and (III). Whereas in the new minimal version (II), in which the notion of chiral weight is still present, the Poincaré multiplets are denoted by $\underset{\sim}{\mathcal{V}}_A{}^{\mathrm{P}(n)}$ with their chiral weight n and identified with superconformal ones $\underset{\sim}{\mathcal{V}}_A{}^{(0,n)}$.

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Appendix A

—— *Notations and Conventions for Two Component Spinors and Second Rank Anti-Symmetric Tensors* ——

The relations between 4-component spinor $\psi_{\tilde{\alpha}}$ and 2-component ones ψ_α, $\psi_{\dot{\alpha}}$ are

$$\psi_{\tilde{\alpha}}=\begin{pmatrix}\psi_\alpha \\ \psi^{\dot{\alpha}}\equiv\varepsilon^{\dot{\alpha}\dot{\beta}}(\psi_\beta)^*\end{pmatrix}\equiv\begin{pmatrix}\psi_{\rm R}\\ \psi_{\rm L}\end{pmatrix},\quad \bar{\psi}\equiv\psi^T C=(\psi^\alpha,\ \psi_{\dot{\alpha}})\equiv(\bar{\psi}_{\rm R},\ \bar{\psi}_{\rm L}),$$

$$C^{\tilde{\alpha}\tilde{\beta}}=\begin{pmatrix}-\varepsilon^{\alpha\beta} & \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}}\end{pmatrix},\qquad C^{-1}=\begin{pmatrix}\varepsilon_{\alpha\beta} & 0\\ 0 & -\varepsilon^{\dot{\alpha}\dot{\beta}}\end{pmatrix}. \qquad (\mathrm{A\cdot1})$$

Here $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ are anti-symmetric 2×2 matrix with $\varepsilon^{12}=\varepsilon_{12}=+1$ and the raising and lowering of spinor index are defined by

$$\psi^\alpha=\varepsilon^{\alpha\beta}\psi_\beta\,,\qquad \psi_\alpha=\psi^\beta\varepsilon_{\beta\alpha}\,.\quad(\varepsilon^{\alpha\beta}\varepsilon_{\gamma\beta}=\delta_\gamma{}^\alpha) \qquad (\mathrm{A\cdot2})$$

(The same convention is adopted for dotted spinors.) Then γ-matrices become

$$\gamma_m=\begin{pmatrix}0 & (\sigma_m)_{\alpha\dot{\beta}}\\ (\bar{\sigma}_m)^{\dot{\alpha}\beta} & 0\end{pmatrix},\qquad (\sigma_m)_{\alpha\dot{\beta}}=(-i\boldsymbol{\sigma},1),$$

$$\gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \sigma_{mn}=\frac{1}{4}[\gamma_m,\gamma_n]=\begin{pmatrix}(\sigma_{mn})_\alpha{}^\beta & 0\\ 0 & (\bar{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}}\end{pmatrix}. \qquad (\mathrm{A\cdot3})$$

Notice that $C\gamma_m C^{-1}=-\gamma_m{}^T$ says

$$(\bar{\sigma}_m)^{\dot{\alpha}\beta}=\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}(\sigma_m)_{\delta\dot{\gamma}}=(\sigma_m)^{\beta\dot{\alpha}}\,. \qquad (\mathrm{A\cdot4})$$

The completeness relations of σ_m and σ_{mn} matrices are

$$(\sigma^m)_{\alpha\dot{\beta}}(\sigma^m)_{\gamma\dot{\delta}}=2\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\beta}\dot{\delta}},\quad (\sigma_m)_{\alpha\dot{\beta}}(\sigma_n)^{\alpha\dot{\beta}}=2\delta_{mn}\,, \qquad (\mathrm{A\cdot5})$$

$$(\sigma^{mn})_\alpha{}^\gamma(\sigma_{mn})_\beta{}^\delta-\delta_\alpha{}^\gamma\delta_\beta{}^\delta=-2\delta_\alpha{}^\delta\delta_\beta{}^\gamma\,. \qquad (\mathrm{A\cdot6})$$

For any second rank anti-symmetric tensors F_{mn}, we define

$$\tilde{F}^{mn}\equiv\frac{1}{2}\varepsilon^{mnkl}F_{kl}\,,\qquad F^{\pm}_{mn}\equiv\frac{1}{2}(F_{mn}\pm\tilde{F}_{mn}). \qquad (\mathrm{A\cdot7})$$

(Notice that 1/2 was not introduced for the dual $\tilde{F}^{mn}$ in Refs. 9) and 13)). Then σ_{mn} matrix on the chiral spinors $\psi_{\rm R}=\mathcal{P}_{\rm R}\psi[\equiv(1/2)(1-\gamma_5)\psi]$ and $\psi_{\rm L}=\mathcal{P}_{\rm L}\psi$ has a definite self-dual property:

$$\sigma_{mn}\psi_{\rm R}=\sigma^{-}_{mn}\psi_{\rm R}\,,\qquad \sigma_{mn}\psi_{\rm L}=\sigma^{+}_{mn}\psi_{\rm L}\,. \qquad (\mathrm{A\cdot8})$$

Convenient formulae for (anti-)self-dual tensors are

$$F^-_{ma}G^-_{mb}+(a\leftrightarrow b)=\frac{1}{2}\delta_{ab}(F^-\cdot G^-), \tag{A·9}$$

$$F^-_{ma}G^+_{mb}-(a\leftrightarrow b)=0\,,\qquad F^-\cdot G^+=0 \tag{A·10}$$

with $F\cdot G\equiv F_{mn}G^{mn}$. Further for arbitrary anti-symmetric tensor G_{mn},

$$F^+_{ma}G_{mb}-(a\leftrightarrow b)\equiv H^+_{ab}\quad \text{is}\quad \text{selfdual}\,,$$

$$F^-_{ma}G_{mb}-(a\leftrightarrow b)\equiv H^-_{ab}\quad \text{is anti-selfdual}\,. \tag{A·11}$$

The formula following from (A·8) and (A·10) is also useful:

$$\gamma_n\gamma_k\psi_L F_-{}^{mn}=\gamma_m\gamma_n\psi_L F_-{}^{nk}\,. \tag{A·12}$$

Appendix B

—— *Properties of Superconformal Curvatures* $R_{\mu\nu}(X^A)$ ——

We collect here various identities of the curvatures which are used to check the superconformal algebra on the general multiplets $\underset{\sim}{\mathcal{V}}_A$ in §2. Most of them result from the constraints[5])

$$R_{\mu\nu}(P^m)=0\,,\qquad R_{\mu\nu}(Q)\gamma^\nu=0\,,$$

$$R_{\nu\lambda}(M^{mn})e^{m\lambda}e^n{}_\mu-\frac{1}{2}R_{\lambda\mu}(Q)\gamma_\nu\psi^\lambda+\frac{1}{2}i\tilde{R}_{\mu\nu}(A)=0\,, \tag{B·1}$$

and/or the Bianchi identities (cf., Refs. 5), 12), 13)):

$$R^R_{\mu\nu}(Q)\equiv R_{\mu\nu}(Q)\mathcal{P}_R=-\tilde{R}^R_{\mu\nu}(Q),\qquad R^L_{\mu\nu}(Q)=+\tilde{R}^L_{\mu\nu}(Q),$$

$$R_{\mu\nu}(Q)\sigma^{\mu\nu}=0\,,\qquad \varepsilon^{\mu\nu\rho\sigma}R_{\nu\rho}(Q)\gamma_\sigma=0\,,$$

$$R_{ab}(Q)\gamma.\sigma..\sigma^{ab}=0\,,\qquad R_{ab}(Q)\gamma_\mu\gamma_\nu\sigma^{ab}=4R_{\nu\mu}(Q),$$

$$\gamma^a\sigma^{mn}\eta(R_{mn}(Q)\gamma_a\varepsilon)=0\,,\qquad (\varepsilon,\,\eta\text{: arbitrary spinor})$$

$$\varepsilon^{mabc}R^{\rm cov}_{na}(M^{bc})=\varepsilon^{mabc}R^{\rm cov}_{ab}(M^{cn})=-2\tilde{R}_{mn}(D)=iR_{mn}(A),$$

$$R^{\rm cov}_{ma}(M^{an})+(m\leftrightarrow n)=0\,,\qquad R^{\rm cov-}_{mn}(M_+{}^{ab})=R^{\rm cov+}_{mn}(M_-{}^{ab})=0\,,$$

$$R^{\rm cov}_{mn}(M^{ab})-R^{\rm cov}_{ab}(M^{mn})=\delta_{ma}R_{nb}(D)+3\text{-terms}\,,$$

$$R_{ab}(A)=i\varepsilon^{ab\mu\nu}(\bar{\psi}_\mu\varphi_\nu-4f_\mu{}^m e_{m\nu}-2\partial_\mu b_\nu),\qquad \varepsilon^{klmn}D_lR_{mn}(A)=0\,,$$

$$R^{\rm cov}_{mn}(S)\sigma^{mn}=0\,,\qquad \varepsilon^{klmn}(D_lR_{mn}(Q)+R^{\rm cov}_{mn}(S)\gamma_l)=0\,,$$

$$R^{\rm cov}_{mn}(S)-2[R^{\rm cov}_{km}(S)\sigma_{kn}-(m\leftrightarrow n)]-\tilde{R}^{\rm cov}_{mn}(S)\gamma_5=0\,,$$

$$R^{\rm cov}_{mn}(S)\sigma_{kl}+R^{\rm cov}_{kl}(S)\sigma_{mn}=R^{\rm cov}_{mk}(S)\sigma_{nl}+3\text{-terms}\,,$$

$$\varepsilon^{klmn}(D_lR_{mn}(D)-2R^{\rm cov}_{mn}(K_l))=0\,,\qquad R^{\rm cov}_{mn}(K^n)=0\,,$$

$$D_nR^{\rm cov}_{mn}(M^{ab})=2R^{\rm cov}_{ab}(K^m). \tag{B·2}$$

Here $R^{\rm cov}_{\mu\nu}(X^A)$'s are the additionally Q-covariantized curvatures corresponding to the modification (1·3) of Q-transformation law:[5)]

$$R_{\mu\nu}^{\rm cov}(M^{ab})=R_{\mu\nu}(M^{ab})+\frac{1}{2}R_{ab}(Q)(\gamma_\nu\psi_\mu-\gamma_\mu\psi_\nu),$$

$$R_{\mu\nu}^{\rm cov}(S)=R_{\mu\nu}(S)+\frac{1}{4}i\{\bar{\psi}_\mu\gamma^\rho(\gamma_5R_{\rho\nu}(A)-\tilde{R}_{\rho\nu}(A))-(\mu\leftrightarrow\nu)\},$$

$$R_{\mu\nu}^{\rm cov}(K^m)=R_{\mu\nu}(K^m)+\frac{1}{4}\{(2R_{\rho\mu}^{\rm cov}(S)\sigma^{m\rho}+\tilde{R}_{\rho\mu}^{\rm cov}(S)\gamma_5e^{m\rho})\psi_\nu-(\mu\leftrightarrow\nu)\}. \qquad (\mathrm{B}\cdot 3)$$

We list also some necessary transformation laws:

$$\delta_Q(\varepsilon)R_{mn}(Q)=\frac{1}{2}\bar{\varepsilon}\sigma_{ab}R_{mn}^{\rm cov}(M^{\rm ab})+\frac{1}{4}i\bar{\varepsilon}(\tilde{R}_{mn}(A)-\gamma_5R_{mn}(A)),$$

$$\delta_Q(\varepsilon)R_{mn}^{\rm cov}(M^{ab})=\Big[-\frac{1}{2}D_mR_{ab}(Q)\gamma_n\varepsilon+\frac{1}{2}R_{mn}^{\rm cov}(S)\sigma_{ab}\varepsilon$$
$$+\Big\{\delta_{bn}\Big(R_{cm}^{\rm cov}(S)\sigma_{ac}+\frac{1}{2}\tilde{R}_{am}^{\rm cov}(S)\gamma_5\Big)\varepsilon-(a\leftrightarrow b)\Big\}\Big]-(m\leftrightarrow n),$$

$$\delta_Q(\varepsilon)R_{mn}(A)=-R_{mn}^{\rm cov}(S)i\gamma_5\varepsilon\,,\quad \delta_Q(\varepsilon)R_{mn}(D)=-\frac{1}{2}\tilde{R}_{mn}^{\rm cov}(S)\gamma_5\varepsilon\,,$$

$$\delta_Q(\varepsilon)R_{mn}^{\rm cov}(S)=-R_{mn}^{\rm cov}(K^a)\bar{\varepsilon}\gamma_a+\frac{1}{4}i\bar{\varepsilon}\{(\gamma_5D_mR_{an}(A)+D_m\tilde{R}_{an}(A))-(m\leftrightarrow n)\}\gamma_a\,,$$

$$\delta_S(\zeta)R_{ab}(Q)=\delta_K(\xi_K{}^m)R_{ab}(Q)=0\,,$$

$$\delta_S(\zeta)R_{ab}(A)=R_{ab}(Q)i\gamma_5\zeta\,,\qquad \delta_K(\xi_K{}^m)R_{ab}^{\rm cov}(S)=R_{ab}(Q)\gamma_m\xi_K{}^m\,,$$

$$\delta_S(\zeta)R_{mn}^{\rm cov}(M^{ab})=(R_{mn}(Q)\sigma^{ab}+2R_{ab}(Q)\sigma_{mn})\zeta. \qquad (\mathrm{B}\cdot 4)$$

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Note added: Although the present paper dealt with the local symmetry case (i.e., supergravity), the rigid case of superconformal algebra has long been discussed from the very start of the supersymmetry. In the rigid context, the particular weight condition $w=n$ for chiral multiplets (of Lorentz scalar case) was already known in the first Wess-Zumino paper[33)] and was explicitly stated by Dondi and Sohnius (for $N=2$).[34)] The superconformal multiplets with arbitrary external Lorentz indices were studied also in rigid case by the authors of Refs.35). We thank Molotkov and Mikhov for bringing their work to our attention.