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$\mathcal{N} = 1$ Supersymmetric Renormalization Group Flows from IIB Supergravity

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We consider $\mathcal{N} = 1$ supersymmetric renormalization group flows of $\mathcal{N} = 4$ Yang-Mills theory from the perspective of ten-dimensional IIB supergravity. We explicitly construct the complete ten-dimensional lift of the flow in which exactly one chiral superfield becomes massive (the LS flow). We also examine the ten-dimensional metric and dilaton configurations for the "super-QCD" flow (the GPPZ flow) in which all chiral superfields become massive. We show that the latter flow generically gives rise to a dielectric 7-brane in the infra-red, but the solution contains a singularity that may be interpreted as a "duality averaged" ring distribution of 5-branes wrapped on S^2. At special values of the parameters the singularity simplifies to a pair of S-dual branes with (p, q) charge $(1, \pm 1)$.

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1. Introduction

Five-dimensional supergravities have proven to be a powerful tool in the study of holographic RG flows of field theories on $D3$-branes. This has been particularly well studied for the flows of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory under perturbations that involve either masses or vevs for bilinear operators [1-23]. The corresponding five-dimensional theory is thus gauged $\mathcal{N} = 8$ supergravity [24-26], but this is to be viewed as a consistent truncation of ten-dimensional IIB supergravity [27,28]. This paper will, once again, focus on such flows of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, but now from the ten-dimensional perspective, and as we will show, this approach will reveal some very interesting new features of these flows.

It has become increasingly evident that while the five-dimensional theories are a valuable tool, the five-dimensional perspective is somewhat limiting when it comes to physically interpreting the majority of these flows. To be more precise, almost all flows involve running to infinite values of the supergravity potential, that is, they are what one of our earlier collaborators dubbed *"Flows to Hades"*. In this limit the five-dimensional supergravity metric develops a singularity that appears superficially pathological. However when "lifted" to ten dimensions, the corresponding IIB supergravity solution is typically rather less singular, and may well admit a simple physical interpretation. This softening of the five-dimensional singularity arises partially because the "lift" to ten-dimensions involves multiplying the 5-metric by a warp-factor, and the asymptotic behaviour of the warp factor modifies the asymptotics of the five-metric. The simplest, but most illustrative example of this are the $\mathcal{N} = 4$ Coulomb branch flows of [29,30,8,9,11]: In five dimensions these all generate apparently peculiar metrics with singularities at $r = 0$, whereas the corresponding ten-dimensional metrics resolve the $r = 0$ singularity into a smooth distribution of $D3$-branes.

A second facet to the lift to ten dimensions is the IIB dilaton. The scalars of the five-dimensional theory are described by a coset model $\mathcal{S} \equiv E_{6(6)}/USp(8)$, which contains a submanifold: $\mathcal{S}_0 \equiv SL(6, \mathbb{R})/SO(6) \times SL(2, \mathbb{R})/SO(2)$. Perturbatively the scalars of $SL(6, \mathbb{R})/SO(6)$ correspond to metric perturbations on the S^5 of the $AdS_5 \times S^5$ compactification of the IIB theory. Similarly, the $SL(2, \mathbb{R})/SO(2)$ coset may be identified with the IIB dilaton and axion at the perturbative level. Moreover, it has been argued [8] that this identification remains true to all orders so long as the scalars of the gauged $\mathcal{N} = 8$ supergravity are restricted to $\mathcal{S}_0$. Indeed this is strongly substantiated by the five-dimensional description of the Coulomb branch flows.

On the other hand, it was first shown in [3] that when more general supergravity scalars are used (*i.e.* ones that correspond to fermion bilinears in the Yang-Mills theory, or corrrespond to $B_{\mu\nu}$ fields in the IIB theory) the deformation of the S^5 metric is rather more complicated. More recently, it was also shown that when the same supergravity scalars are non-trivial, the dilaton/axion coset, $SL(2,\mathbb{R})_{\rm IIB}/SO(2)$, is not the same as the $SL(2,\mathbb{R})_{5d}/SO(2)$ factor in five-dimensional coset model $E_{6(6)}/USp(8)$. In particular, even if the five-dimensional scalars of $SL(2,\mathbb{R})_{5d}/SO(2)$ are set to zero it was shown in [20,21] that the corresponding ten-dimensional dilaton and axion could be highly non-trivial.

To be more explicit, it was argued in [3] that the inverse metric, g^{pq}, on the deformed S^5 is given by

$$\Delta^{-\frac{2}{3}}\, g^{pq} \;=\; \frac{1}{a^2}\, K^{IJ\,p}\, K^{KL\,q}\, \widetilde{\mathcal{V}}_{IJab}\, \widetilde{\mathcal{V}}_{KLcd}\, \Omega^{ac}\, \Omega^{bd}\;, \tag{1.1}$$

where $\mathcal{V} = (\mathcal{V}^{IJab}, \mathcal{V}_{I\alpha}{}^{ab})$ is the scalar matrix of the $E_{6(6)}/USp(8)$ coset and $\widetilde{\mathcal{V}} = (\widetilde{\mathcal{V}}_{IJab}, \widetilde{\mathcal{V}}^{I\alpha}{}_{ab})$ is the inverse of $\mathcal{V}$ [26], $K^{IJ\,p}$ are Killing vectors on S^5, Ω^{ab} is the $USp(8)$ symplectic form, and $\Delta = \det^{1/2}(g_{mp}\, \overset{\circ}{g}{}^{pq})$, where $\overset{\circ}{g}{}^{pq}$ is the inverse of the "round" S^5 metric. The quatity Δ can be determined by taking the determinant of both sides of (1.1). For more details see [21].

The ten-dimensional solution is then reconstructed by taking:

$$ds_{10}^2 \;=\; \Omega^2\, ds_{1,4}^2 \;+\; ds_5^2\,. \tag{1.2}$$

where $ds_{1,4}^2$ is the metric of the $\mathcal{N}=8$ supergravity in five dimensions, $ds_5^2 = g_{mn}dy^m dy^n$ is the deformed S^5 metric given by (1.1), and $\Omega^2 = \Delta^{-\frac{2}{3}}$ is the warp factor.

In [21] it was further argued that if x^I are the cartesian coordinates that define the S^5 in $\mathbb{R}^6$ (with $\sum_I (x^I)^2 = 1$), and S is the IIB dilaton/axion matrix in $SL(2,\mathbb{R})_{\rm IIB}/SO(2)$, then one has

$$\Delta^{-\frac{4}{3}}\,(S\,S^T)^{\alpha\beta} \;=\; \text{const}\,\times\, \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\, \mathcal{V}_{I\gamma}{}^{ab}\, \mathcal{V}_{J\delta}{}^{cd}\, x^I x^J\, \Omega_{ac}\, \Omega_{bd}\,, \tag{1.3}$$

to all orders in the $\mathcal{N}=8$ supergravity fields. This is sufficient to determine the matrix S up to an $SO(2)$ gauge choice.

The argument that led to (1.1) showed that if consistent truncation were true then this was necessarily the exact form of the internal metric. This result has since been extensively tested [8,11,31-35,21,22]. The argument that led to (1.3) was similar to that for (1.1) but was based upon an additional (well motivated) assumption. It has also not been quite so well tested, but it will be implicitly tested further by some of the results in this paper.

The distinction between the five-dimensional and ten-dimensional $SL(2,\mathrm{I\!R})/SO(2)$'s is a fimiliar one in field theory. The $SL(2,\mathrm{I\!R})_{5d}/SO(2)$ should be viewed as the $\mathcal{N}=4$ coupling at the UV fixed point, whereas $SL(2,\mathrm{I\!R})_{\mathrm{IIB}}/SO(2)$ should be viewed as a running coupling of the theory on the branes. The importance of (1.3) is that it gives the running coupling as an explicit function of the UV coupling and the masses and vevs along the flow. The derivative of (1.3) with respect to r is thus a holographic beta function for the flow. One should also remember that the identification of the dilaton and axion as the running gauge coupling is based upon perturbation theory about the ten-dimensional IIB theory, and thus upon perturbations about the UV fixed point of the Yang-Mills theory. As in field theory, non-trivial operator mixings can and do occur along RG flows, and so this running coupling may become some other non-trivial coupling of the effective action as one flows toward the infra-red. Indeed, as we will see, the flow of [7] provides an example of this phenomenon.

The primary goal of this paper is to construct ten-dimensional "lifts" of two of the $\mathcal{N}=1$ supersymmetric RG flows and use then these lifts to study the near-brane asymptotics. The secondary purpose of the paper is to give further support for the formula (1.3) by showing that it correctly predicts the dilaton behaviour for these lifts.

We will begin in section 2 by reviewing some of the the essential details of the supergravity description of supersymmetric RG flows, and then go on to examine in detail the $\mathcal{N}=2$ supersymmetric subsectors of $\mathcal{N}=8$ supergravity that generate some of the possible flows of $\mathcal{N}=4$ Yang-Mills down to an $\mathcal{N}=1$ theory.

In section 3 we construct the *complete* ten-dimensional lift of the "Leigh-Strassler" (LS) flow [36,4,7]. This lift generalizes the recent compactification of the chiral IIB supergravity obtained in [34] (see, also [33]).

Sections 4, 5 and 6 contain a rather involved analysis of an $SO(3)$ invariant subsector of the $\mathcal{N}=8$ supergravity in five dimensions. This subsector represents the truncation of $\mathcal{N}=8$ supergravity down to $\mathcal{N}=2$ supergravity coupled to two hypermultiplets. In terms of the field theory on the brane, this sector involves breaking the $\mathcal{N}=4$ Yang-Mills to $\mathcal{N}=1$ with equal masses given to each of the chiral multiplets. A restricted version of this was studied in detail in [5,13,14,20,37,38] from the perspective of five-dimensional supergravity. In section 5 we examine this restriction of the the $SO(3)$ invariant sector, and show that it requires that the solution be S-dual, *i.e.*, self-dual with respect to $g\to 1/g$, where g is the $\mathcal{N}=4$ gauge coupling. In section 6 we will then go on to construct the ten-dimensional metric and dilaton configuration for the RG flow (GPPZ-flow), while in section 7 we examine the IR asymptotics, and show how dielectric 7-branes emerge at the near-brane (IR) end of the flow. In section 8 we take a closer look at the singularities in ten dimensions and discuss the relationship between our work and that of [39] and [40].

The reader may wish to skip from section 3 to sections 7 and 8 since the IR asymptotics can be readily understood without pushing through the details of consistent truncations. We have chosen to include some of the technical details in the intervening sections, partially to facilitate calculations by others working in this area, but also to highlight the special "self-dual" structure of the flows considered in [13]. The worst of the technical details have been relegated to an appendix.

In section 9 we construct the ten-dimensional lift of a restriction of the GPPZ-flow to one of the fields of the LS-flow. While this flow is "unphysical" from the perspective of the theory on the brane, it represents one of the very few $\mathcal{N}=1$ supersymmetric flows for which we have a complete, analytically known lift. A formal limit of this lift reproduces the $SU(3)$ compactification of the IIB supergravity discovered by Romans [41].

Finally, in section 10 we summarize our results, and try to draw some general threads out of what we have learnt in using supergravity to study RG flows holographically.

2. Some $\mathcal{N}=1$ supersymmetric flows

2.1. Supersymmetric flows in general

As is, by now, standard we generally take the five-dimensional metric to have the form:[1]

$$ds_{1,4}^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu - dr^2\,. \tag{2.1}$$

If the supergravity scalars are canonically normalized, with a scalar kinetic term $\frac{1}{2}\sum_j(\partial\varphi_j)^2$, then the supersymmetric flow equations take the form:

$$\frac{d\varphi_j}{dr} ~=~ \frac{1}{L}\,\frac{\partial W}{\partial\varphi_j}\,, \qquad \frac{dA}{dr} ~=~ -\frac{2}{3L}\,W\,. \tag{2.2}$$

The supergravity potential, $\mathcal{P}$, is obtained from the superpotential via

$$\mathcal{P} ~=~ \frac{g^2}{8}\sum_j\Big(\frac{\partial W}{\partial\varphi_j}\Big)^2 ~-~ \frac{g^2}{3}\,W^2\,. \tag{2.3}$$

In our conventions the length scale, L, is related to the coupling constant, g, by $g=2/L$.

[1] We use the mostly "$-$" convention.

We will only consider flows that in the UV start in the maximally supersymmetric vacuum with $\varphi_j = 0$. At this point both $\mathcal{P}$ and W have a critical point, and $W = -\frac{3}{2}$. We therefore take $A(r) \sim r/L$ as $r \to \infty$.

As r decreases there are two possibilities: either there is a "soft landing" in which the flow approaches another critical point of W, or W decreases without bound along the flow. If the other end of the flow is a critical point of W then one has $A(r) \sim r/\ell$ as $r \to -\infty$, for some value of $\ell < L$, and the metric once again approaches that of AdS_5.

If the flow goes to negatively infinite values of W, then $A'(r) \to +\infty$, and hence $A(r) \to -\infty$, and the five-dimensional space-time is singular. To be more precise, the superpotential is typically a sum of exponentials of φ_j, and one or more of the exponentials dominate the IR limit. One then easily solve for the asymptotics, and one typically finds $\varphi_j \sim a_j \log(r - c_j)$ for some constants a_j, c_j. It also turns out that $A(r) \sim \sum b_j \log(r - c_j)$ for some positive constants b_j. As a result, the cosmological term in (2.1) usually vanishes at finite r as some positive power: $(r-c)^{2b}$. The power depends upon the details, but rather little can be deduced from this behaviour alone: One really needs the ten-dimensional solution to understand the IR limit properly.

Our problem thus is to construct a solution to the field equations of the chiral IIB supergravity in ten dimensions [27,28], *i.e.* to find the metric, g_{MN}, the dilaton/axion field, B, and the antisymmetric tensor fields, A_{MN} and F_{MNPQRS}, expressed in terms of the fields A and φ_i in the flow, such that the ten dimensional equations of motion[2] become equivalent to the flow equations (2.2).

2.2. Supersymmetric flows in particular: Truncations

A standard process by which one reduces the number of supergravity scalar fields to a more manageable subset is to impose invariance under a carefully chosen discrete or continuous symmetry of the action. The idea is that since the symmetry is an invariance of the action, any expansion of the action will be at least quadratic in non-singlet fields, and so it is consistent to set all non-singlet fields to zero, and solve the equations on the space of singlets alone. In this paper we will employ two such trunactions to arrive at distinct $\mathcal{N} = 2$ supergravities, coupled to matter and vector multiplets, and embedded in the $\mathcal{N} = 8$ theory. Such $\mathcal{N} = 2$ supergravities are certainly not new: there is a well established technology for constructing broad classes of such theories (see, e.g., [42,43,44]).

[2] We refer the reader to [27] and to our recent paper [21] for the explicit form of those equations.

The significance of the $\mathcal{N} = 2$ supergravities considered here is that they are dual to distinct $\mathcal{N} = 1$ Yang-Mills theories arising from massive flows of the $\mathcal{N} = 4$ theory.

The natural way to accomplish this is to use the $SO(6) \times SL(2, \mathrm{I\!R})$ symmetry, and under the $SO(6)$ the gravitini transform as $\mathbf{4}$ and $\bar{\mathbf{4}}$. To get an $\mathcal{N} = 2$ supergravity we use symmetries for which the $\mathbf{4}$ has only one singlet. An obvious candidate is to take $SU(3) \subset SO(6)$, under which $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$. Imposing $SU(3)$ invariance is, however, far too restrictive and leaves only one nontrivial scalar field. (This restricted case will, in fact, be discussed in detail in section 9.) Instead we pass to subgroups of this $SU(3)$. In the first instance we will impose invariance under $SU(2) \times U(1) \subset SU(3)$, and in the second we impose invariance under $SO(3) \subset SU(3) \subset SO(6)$, where the $SO(3)$ is the real subgroup of $SU(3)$.

Considering the entire spectrum of the gauged $\mathcal{N} = 8$ supergravity, the space of $SU(2) \times U(1)$ singlets consists of the following: a graviton, two gravitini, two vector fields, no tensor gauge fields, four spinors, and five scalars. These make up the $\mathcal{N} = 2$ supergravity multiplet coupled to one vector multiplet and one hypermultiplet. A more careful examination of the group theory shows that the scalar manifold is:

$$\mathcal{S}_{LS} \;=\; \frac{SU(2,1)}{SU(2) \times U(1)} \times SO(1,1)\,. \tag{2.4}$$

An $SU(1,1)$ subgroup of $SU(2,1)$ represents the dilaton/axion coset, while the other two non-compact generators of $SU(2,1)$ are the supergravity dual of a (complex) Yang-Mills fermion mass. The $SO(1,1)$ represents a diagonal element of the $SL(6, \mathrm{I\!R}) \subset E_{6(6)}$ and is dual to a Yang-Mills scalar mass. The dilaton/axion scalars will remain fixed in the flows considered here, and the corresponding reduction of the scalar manifold can be done by imposing invariance under another $U(1)$ so that the coset becomes $\frac{SU(1,1)}{U(1)} \times SO(1,1)$. As in [7], this can be parametrized by two real scalars, χ and α, along with the $U(1)$ symmetry of the denominator.

The $SO(3)$ invariant subsector of the $\mathcal{N} = 8$ theory consists of: a graviton, two gravitini, one vector field, no tensor gauge fields, four spinors, and eight scalars. The result is thus $\mathcal{N} = 2$ supergravity coupled to two hypermultiplets. The scalar manifold of this $\mathcal{N} = 2$ theory is now the quaternionic manifold:

$$\mathcal{S}_{QCD} \;=\; \frac{G_{2(2)}}{SU(2) \times SU(2)}\,. \tag{2.5}$$

We will discuss the parametrization of this extensively in sections 4 and 5.

3. The LS-flow

The $SU(2) \times U(1)$ invariant sector of the supergravity is dual to $\mathcal{N} = 4$ Yang-Mills perturbed by the bilinear operators with the same invariance. Specifically χ and α are, respectively, dual to a single fermion mass and the mass of the scalar[3] in the same chiral multiplet. The $SU(2)$ symmetry is a global symmetry of the two remaining massless chiral multiplets, while the $U(1)$ essentially gives rise to the $\mathcal{N} = 1$ R-symmetry. The $\mathcal{N} = 1$ flows in this sector thus include the flow considered by Leigh and Strassler in [36].

3.1. The five-dimensional flow

The field χ is canonically normalized, whereas α is not: The kinetic term is $-\frac{1}{2}(\partial\chi)^2 - 3(\partial\alpha)^2$. The superpotential is:

$$W = \frac{1}{4\rho^2}\Big[\cosh(2\chi)\,(\rho^6 - 2) - (3\rho^6 + 2)\Big]\,. \tag{3.1}$$

where $\rho = \exp(\alpha)$. The resulting field equations are:

$$\begin{aligned} \frac{d\chi}{dr} &= \frac{g}{2}\frac{\partial W}{\partial \chi} = \frac{g}{4}\,\frac{(\rho^6 - 2)\,\sinh(2\chi)}{\rho^2}\,, \\ \frac{d\rho}{dr} &= \frac{g}{12}\,\rho^2\,\frac{\partial W}{\partial \rho} = \frac{g}{12}\,\frac{\rho^6\,(\cosh(2\chi) - 3) + 2\cosh^2\chi}{\rho}\,. \end{aligned} \tag{3.2}$$

The superpotential (3.1), and the corresponding potential (2.3) have an $\mathcal{N} = 2$ supersymmetric critical point for $\chi = \frac{1}{2}\log(3)$ and $\alpha = \frac{1}{6}\log(2)$ [3]. As shown in [4,7], this critical point is the dual of the Leigh-Strassler conformal fixed point of [36]. The compactification of the chiral IIB supergravity corresponding to the critical point has recently been obtained in [34]. The flow itself can be obtained by solving (3.2) with the proper initial conditions in the UV [7].

One of the difficulties in studying this flow is that an explicit solution to (3.2) in a closed form is not known. Formally, one can derive a series solution for the trajectory, $\rho = \rho(\chi)$, which is of the form

$$\begin{aligned} \rho(\chi) &= \sum_{m=0}^{\infty}\sum_{n=2m}^{\infty} a_{mn}\,(\log\chi)^n\,\chi^{2m} \\ &= 1 + (\gamma - \tfrac{2}{3}\log\chi)\chi^2 \\ &\quad + \Big(\tfrac{16+42\gamma+171\gamma^2}{18} - \tfrac{2(7+57\gamma)}{9}\log\chi + \tfrac{38}{9}(\log\chi)^2\Big)\,\chi^4 + \ldots \end{aligned} \tag{3.3}$$

[3] The AdS_5-normalizable modes of the scalar α can additionally be interpreted in terms of vevs of scalars in the two massless chiral multiplets.

where γ is an integration constant parametrizing the trajectory. For futher numerical analysis we refer the reader to [7,19].

There exists also a closely related $\mathcal{N}=2$ flow which can be lifted to a solution of the chiral IIB supergravity in ten dimensions [21] for which the analogue of the series solution (3.3) *can* be summed in terms of elementary functions [21,22]. In the following we will use the general structure of those two solutions to obtain a ten dimensional lift of the present flow.

3.2. The lift to ten dimensions

As discussed in the introduction, both the ten-dimensional metric and the dilaton/axion field are given by the consistent truncation ansatz (1.2) and (1.3), respectively. In particular, the explicit form of the metric and the warp factor have already been obtained in [34,33]. Let us first recall this result.

In the cartesian coordinates on $\mathbb{R}^6$ with S^5 given as a unit sphere, $\sum(x^I)^2=1$, the internal metric is:

$$ds_5^2(\alpha,\chi) \;=\; \frac{a^2}{2}\frac{\operatorname{sech}\chi}{\xi}(dx^I Q^{-1}_{IJ}dx^J)+\frac{a^2}{2}\frac{\sinh\chi\tanh\chi}{\xi^3}(x^I J_{IJ}dx^J)^2\,. \tag{3.4}$$

Here Q is a diagonal matrix with $Q_{11}=\ldots=Q_{44}=e^{-2\alpha}$ and $Q_{55}=Q_{66}=e^{4\alpha}$, J is an antisymmetric matrix with $J_{14}=J_{23}=J_{65}=1$, and $\xi^2 \;=\; x^I Q_{IJ}x^J$. The warp factor is simply

$$\Omega^2 \;=\; \xi\;\cosh\chi\,. \tag{3.5}$$

The constant a, introduced to account for an arbitrary normalization of the Killing vectors, is fixed by requiring that at the $\mathcal{N}=8$ point the ten-dimensional metric becomes a product of $AdS_5\times S^5$ with equal radii, $L=2/g$ and $a/\sqrt{2}$, respectively, which gives $a\;=\;\sqrt{2}L$.

We need suitable spherical coordinates in which the $SU(2)\times U(1)^2$ symmetry of the metric becomes manifest. First define complex coordinates corresponding to J,

$$u^1 \;=\; x^1+ix^4\,,\qquad u^2 \;=\; x^2+ix^3\,,\qquad u^3 \;=\; x^5-ix^6\,, \tag{3.6}$$

and then reparametrize them using the group action[4]

$$\begin{pmatrix}u^1\\u^2\end{pmatrix} \;=\; \cos\theta\, g(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1\\0\end{pmatrix}\,,\qquad u^3 \;=\; e^{-i\phi}\sin\theta\,, \tag{3.7}$$

[4] Note that, unlike in [34], the $SU(2)$ doublet is inert under the ϕ rotation.

where $g(\alpha_1, \alpha_2, \alpha_3)$ is an $SU(2)$ matrix expressed in terms of Euler angles.

Define

$$\begin{aligned} X_1(r,\theta) &= \cos^2\theta + \rho(r)^6 \sin^2\theta\,, \\ X_2(r,\theta) &= \operatorname{sech}\chi(r)\, \cos^2\theta + \rho(r)^6 \cosh\chi(r)\, \sin^2\theta\,, \end{aligned} \tag{3.8}$$

By performing the change of variables (3.7), we find that

$$\xi = \frac{X_1^{1/2}}{\rho}\,, \tag{3.9}$$

and the ten-dimensional metric can be diagonalized in terms of the following frames:

$$\begin{aligned} e^{\mu+1} &= \frac{X_1^{1/4}(\cosh\chi)^{1/2}}{\rho^{1/2}}\, e^A\, dx^\mu\,, \qquad \mu = 0,\ldots,3\,, \\ e^5 &= \frac{X_1^{1/4}(\cosh\chi)^{1/2}}{\rho^{1/2}}\, dr\,, \\ e^6 &= \frac{2}{g}\, \frac{X_1^{1/4}}{\rho^{3/2}\,(\cosh\chi)^{1/2}}\, d\theta\,, \\ e^7 &= \frac{1}{g}\, \frac{\rho^{3/2}\, \cos\theta}{X_1^{1/4}\,(\cosh\chi)^{1/2}}\, \sigma_1\,, \\ e^8 &= \frac{1}{g}\, \frac{\rho^{3/2}\, \cos\theta}{X_1^{1/4}\,(\cosh\chi)^{1/2}}\, \sigma_2\,, \\ e^9 &= \frac{1}{g}\, \frac{\rho^{3/2}\, X_1^{1/4}\, \cos\theta}{X_2^{1/2}}\, \sigma_3\,, \\ e^{10} &= \frac{2}{g}\, \frac{X_2^{1/2}\, \sin\theta}{\rho^{3/2} X_1^{3/4}}\, d\phi + \frac{1}{g}\, \frac{\rho^{9/2}\, \sinh\chi\, \tanh\chi\, \cos^2\theta\, \sin\theta}{X_1^{3/4}\, X_2^{1/2}}\, \sigma_3\,, \end{aligned} \tag{3.10}$$

where σ_i, $i = 1, 2, 3$, are the $SU(2)$ left-invariant 1-forms normalized according to $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\, \sigma_j \wedge \sigma_k$.

An explicit evaluation of the dilaton/axion matrix (1.3) yields a somewhat surprising result: *the ten-dimensional dilaton/axion field remains constant along this flow.* This is surprising from the field theory perspective in that one might have expected a running coupling. What we find is that the dilaton and axion value provides a modulus for the theory all along the flow: at the UV point this is simply the gauge theory coupling constant, but in the infra-red this presumably defines the line of marginal perturbations identified in [36].

The constancy of the dilaton and axion is not so surprising from the supergravity perspective: The product structure of (2.4) means that the running fermion mass does not mix with the five-dimensional dilaton/axion $SL(2,\mathrm{I\!R})$ to produce a non-trivial ten-dimensional dilaton and axion. This fact simplifies considerably the ten-dimensional equations of motion.

Having exhausted the consistent truncation ansatz, our strategy is to use the field equations and the underlying symmetry to construct the remaining fields. As in [34,21], we start with the Einstein equations which should yield information about the field strengths of the antisymmetric tensor fields given that the left hand side, *i.e.* the Ricci tensor, is computable. The crucial observation here is that the Ricci tensor depends on the derivatives of $A(r)$ only, and thus by using repeatedly the flow equations (2.2) and (3.2), one can eliminate all derivatives with respect to the flow parameter, r, and be left with rational expressions in ρ and the hyperbolic functions of χ. It is also reasonable to expect that an explicit solution for trajectories (cf. (3.3)) will involve transcendental functions of $\cosh\chi$ and $\sinh\chi$ and thus we should attempt to solve the ten-dimensional equations by matching various ρ and χ terms independently. This is how the lift worked for the $\mathcal{N}=2$ flow in [21], where the explicit solution to the flow equations was not needed: the equations themselves were sufficient.

The resulting Ricci tensor is rather complicated to the extent that we will not attempt reproducing it here. Nevertheless we find two simple linear combinations that will become important in the following:

$$R_{77} \;=\; R_{88} \;=\; R_{11}\,, \tag{3.11}$$

and

$$R_{99}+R_{10\,10}-2R_{11} \;=\; 2g^2\,\frac{\rho^3\,\sinh\chi\,\tanh\chi}{X_1^{1/2}}\,. \tag{3.12}$$

We also find that the only nontrivial off-diagonal components are R_{56} and $R_{9\,10}$.

The 5-index antisymmetric tensor field, $F_{(5)}$, is taken to be of the similar form as for the $\mathcal{N}=2$ flow [21], namely

$$F_{(5)} \;=\; \mathcal{F}+*\mathcal{F}\,, \tag{3.13}$$

where

$$\mathcal{F} \;=\; dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge(w_r dr+w_\theta d\theta)\,, \tag{3.14}$$

with arbitrary functions $w_i(r,\theta)$. The self-duality equation is then satisfied by construction. The structure of the energy-momentum tensor, $T^{(5)}_{MN}$, is the same as in [21], namely

$$T^{(5)}_{11} \;=\; -T^{(5)}_{22} \;=\; \ldots \;=\; -T^{(5)}_{33} \;=\; T^{(5)}_{77} \;=\; \ldots \;=\; T^{(5)}_{10\,10} \;=\; \mathcal{A}^2+\mathcal{B}^2\,, \tag{3.15}$$

$$T^{(5)}_{55} \;=\; -T^{(5)}_{66} \;=\; \mathcal{A}^2-\mathcal{B}^2\,, \tag{3.16}$$

and

$$T^{(5)}_{56} \;=\; T^{(5)}_{65} \;=\; 2\mathcal{A}\mathcal{B}\,, \tag{3.17}$$

where

$$\begin{aligned} \mathcal{A} \;&=\; g\, e^{-4A}\, \frac{\rho^{7/2} (\operatorname{sech}\chi)^{3/2}}{X_1^{5/4}}\, w_\theta\;, \\ \mathcal{B} \;&=\; -2\, e^{-4A}\, \frac{\rho^{5/2} (\operatorname{sech}\chi)^{5/2}}{X_1^{5/4}}\, w_r\;. \end{aligned} \tag{3.18}$$

The most general $SU(2)\times U(1)$ symmetric Ansatz for the potential $A_{(2)}$ of the antisymmetric tensor field is

$$A_{(2)} \;=\; e^{-i\phi}\,(a_1\,d\theta - a_2\sigma_3 - a_3\,d\phi)\wedge(\sigma_1 - i\sigma_2)\,, \tag{3.19}$$

where $a_i(r,\theta)$ are some arbitrary functions. This generalizes the result in [34], except for the a_3 term which, unlike in [34], cannot be gauged away because of the r dependence. Also the the $U(1)$ charge -1 is different than in [34] because of the different ϕ-dependence of the spherical coordinates (3.7).

In the absence of the dilaton/axion field, the 3-index antisymmetric tensor field $G_{(3)}$ is simply $G_{(3)} = dA_{(2)}$. Since $d(\sigma_1 - i\sigma_2) = i(\sigma_1 - i\sigma_2)\wedge\sigma_3$, we find that $(\sigma_1 - i\sigma_2)$ is a factor in $G_{(3)}$ so that $G_{MNP}G^{MNP} = 0$, as required by the dilaton equation. It also implies that the energy-momentum tensor, $T^{(3)}_{MN}$, satisfies $T^{(3)}_{77} = T^{(3)}_{88} = T^{(3)}_{11}$. Then, given (3.15), the Einstein equations imply (3.11), which provides us with the first nontrivial test of the vanishing of the dilaton/axion.

Next we consider the solution to the linearized Einstein and Maxwell equations at the UV end of the flow. From the diagonal Einstein equations we recover, up to a sign, the usual Freund-Rubin Ansatz for the 5-index tensor,

$$F_{12345} \;=\; F_{678910} \;=\; -\frac{g}{2}\,. \tag{3.20}$$

Substituting this into the Maxwell equations together with

$$a_i(r,\theta) \;=\; e^{-r/L}\,\tilde a_i(\theta) + O(e^{-2(r/L)})\,, \tag{3.21}$$

we look for a regular solution that also satisfies the (9,10) Einstein equation. The latter does not involve the 5-index tensor and thus has the lowest order contribution from the

3-index tensor. It turns out that a required solution exists only for the choice of sign as in (3.20), and we we find

$$\tilde{a}_1(\theta) \;=\; \frac{2}{g^2}\,\cos\theta\,, \qquad \tilde{a}_2(\theta) \;=\; \frac{1}{g^2}\,\cos^2\theta\,\sin\theta\,, \qquad \tilde{a}_3(\theta) \;=\; -\frac{2}{g^2}\,\cos^2\theta\,\sin\theta\,. \quad (3.22)$$

Turning to the general case we examine the combination

$$T^{(3)}_{99}+T^{(3)}_{10\,10}-2\,T^{(3)}_{11} \;=\; -\frac{g^4}{4}\,\frac{\rho\,\cosh\chi}{X_1^{1/2}\,\cos^2\theta}\left(\frac{\partial a_1}{\partial r}\right)^2+\frac{g^6}{4}\,\frac{X_1^{3/2}\,\cosh\chi}{\rho^3\,\cos^4\theta\,\sin^2\theta}\,(a_2-a_3)^2\,, \quad (3.23)$$

corresponding to (3.12). The θ-dependence on the right hand side suggests that the functions $a_i(r,\theta)$ should be a simple modification of their linearized conterparts, $\tilde{a}_i(\theta)$. In particular,

$$\frac{\partial a_1}{\partial r} \;=\; \frac{1}{g}\,\frac{(\rho^6-2)\,\tanh\chi}{\rho^2}\,\cos\theta\,, \quad (3.24)$$

and

$$a_2-a_3 \;=\; \frac{1}{g^2}\,\frac{(\rho^6+2)\,\tanh\chi}{X_1}\,\cos^2\theta\,\sin\theta\,. \quad (3.25)$$

The first equation is simply integrated as

$$a_1(r,\theta) \;=\; \frac{2}{g^2}\,\tanh\chi\,\cos\theta\,. \quad (3.26)$$

Substituting (3.25) and (3.26) into the (9,10) Einstein equation we finally determine that

$$\begin{aligned} a_2(r,\theta) \;&=\; \frac{1}{g^2}\,\frac{\rho^6\,\tanh\chi}{X_1}\,\cos^2\theta\,\sin\theta\,,\\ a_3(r,\theta) \;&=\; -\frac{2}{g^2}\,\frac{\tanh\chi}{X_1}\,\cos^2\theta\,\sin\theta\,, \end{aligned} \quad (3.27)$$

and thus solve all the Einstein equations that do not involve the 5-index tensor.

The solution for the 5-index tensor is easily obtained from, for example, the (1,1), (5,5) and (5,6) Einstein equations, with any sign ambiguities resolved by comparing the result with the linearized limit. We find

$$\begin{aligned} w_r \;&=\; \frac{g}{8}\,e^{4A}\,\frac{\cosh^2\chi}{\rho^4}\,\big((\cosh(2\chi)-3)\,\cos^2\theta+\rho^6\,(2\rho^6\sinh^2\chi\,\sin^2\theta+\cos(2\theta)-3)\big)\,,\\ w_\theta \;&=\; \frac{e^{4A}}{8\rho^2}\,\big(2\cosh^2\chi+\rho^6\,(\cosh(2\chi)-3)\big)\,\sin(2\theta)\,, \end{aligned} \quad (3.28)$$

and verify that

$$\frac{\partial w_r}{\partial\theta} \;=\; \frac{\partial w_\theta}{\partial r}\,, \quad (3.29)$$

which shows that $w_r dr + w_\theta d\theta = dw$ for some function $w(r,\theta)$.

At this point all the fields have been determined and we verify that all the remaning Einstein equations, the Maxwell equations and the Bianchi identities are satisfied.

4. $\mathcal{N}=2$ supergravity with hypermultiplets

The truncations that we consider here are motivated by the flow considered in [13]. The idea was to consider an $\mathcal{N}=1$ supersymmetric flow in which all the chiral multiplets are given a mass, leaving only the massless vector multiplet. For simplicity, all the masses are set equal and so the flow has an $SO(3)$ symmetry rotating the three chiral multiplets into one another. As we will discuss below, the truncation of the supergravity to the $SO(3)$ invariant sector leaves eight scalar fields. However, in [13] only two of these scalars were considered, and while these were the scalars of physical interest, it was unclear as to whether they represented a consistent truncation of the full set of eight. It turns out that it is indeed consistent to truncate to these scalars, and one way to establish this is to show that they are the invariants under an additional discrete symmetry. We will also discuss this in some detail below since this discrete symmetry has some interesting consequences for the physics.

4.1. The $SO(3)$ invariant sector

The fermion mass matrix, and the corresponding supergravity scalars can be represented as a complex, symmetric matrix m_{ij}, $i,j=1,\ldots,4$. The flow described in [13] involves setting $m_{ij}=\mathrm{diag}(m,m,m,0)$. The $SO(3)$ invariance is thus the orthogonal rotations on $a,b=1,2,3$. In particular it is the real subgroup of $SU(3)\subset SU(4)=SO(6)$. The $\mathbf{4}$ of $SO(6)$ therefore decomposes as $\mathbf{4}\to\mathbf{3}+\mathbf{1}$ and $\mathbf{6}$ decomposes as $\mathbf{6}\to\mathbf{3}+\mathbf{3}$ of $SO(3)$. As mentioned earlier, the truncation to the space of $SO(3)$ singlets reduces the $\mathcal{N}=8$ supergravity theory to $\mathcal{N}=2$ supergravity coupled to two hypermultiplets, and the scalar manifold is given by (2.5).

In terms of the Yang-Mills theory on the branes, the eight scalars are dual the gauge coupling, the theta-angle, the scalar operators:

$$\mathcal{O}_1 \;=\; \sum_{j=1}^{3}\,\big(\mathrm{Tr}\big(X^jX^j\big)\;-\;\mathrm{Tr}\big(X^{j+3}X^{j+3}\big)\big)\,,\qquad \mathcal{O}_2\;=\;\sum_{j=1}^{3}\,\big(\mathrm{Tr}\big(X^jX^{j+3}\big)\,,\qquad(4.1)$$

and the two complex fermion bilinears:

$$\mathcal{O}_3\;=\;\sum_{a=1}^{3}\,\mathrm{Tr}\big(\lambda^a\lambda^a\big)\,,\qquad \mathcal{O}_4\;=\;\mathrm{Tr}\big(\lambda^4\lambda^4\big)\,.\qquad(4.2)$$

The coefficients of $\mathcal{O}_3$ and $\mathcal{O}_4$ are two complex, or four real parameters. One should also remember that the supergravity magically adjusts the proper amount of

$$\mathcal{O}_0 \equiv \sum_{j=1}^{6} \operatorname{Tr}\big(X^j X^j\big) \,.$$

4.2. Much ado about $G_{2(2)}$

Our first task is to find an effective way to parametrize the manifold (2.5). Recall that $E_{6(6)}$ has a maximal subgroup $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})_{5d}$. Here we have put a subscript $5d$ on this $SL(2,\mathbb{R})$ to distinguish it as that of the dilaton and axion of the ten-dimensional IIB theory. The $SO(3)$ is the compact subgroup of the diagonal $SL(3,\mathbb{R}) \subset SL(3,\mathbb{R}) \times SL(3,\mathbb{R}) \subset SL(6,\mathbb{R})$. The $G_{2(2)}$ subgroup of $E_{6(6)}$ that we seek in fact commutes with the diagonal $SL(3,\mathbb{R})$. Thus we have:

$$E_{6(6)} \supset SL(3,\mathbb{R}) \times G_{2(2)} \supset SO(3) \times G_{2(2)} \,, \tag{4.3}$$

for which

$$\mathbf{27} \to (\overline{\mathbf{6}},\mathbf{1}) + (\mathbf{3},\mathbf{7}) \to (\mathbf{1},\mathbf{1}) + (\mathbf{5},\mathbf{1}) + (\mathbf{3},\mathbf{7}) \,. \tag{4.4}$$

We now need to see how the invariances of the supergravity potential act on this manifold.

First note that the diagonal $SL(3,\mathbb{R})$ commutes with $SL(2,\mathbb{R})_X$ in $SL(6,\mathbb{R})$, where the subscript X is to distinguish from $SL(2,\mathbb{R})_{5d}$. Hence, the $G_{2(2)}$ contains $SL(2,\mathbb{R})_X \times SL(2,\mathbb{R})_{5d}$. The non-compact generators of $SL(2,\mathbb{R})_X$ are dual to the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of (4.1).

Of the original $SO(6)$ invariance of the scalar potential, only the $SO(2)$ subgroup of $SL(2,\mathbb{R})_X$ survives. In addition, the potential is invariant under $SL(2,\mathbb{R})_{5d}$. This four parameter family of invariances reduces the eight-manifold to four independent parameters.

There are, of course, many ways to parametrize the manifold, but the simplest form that we have found is discovered by using the $SU(2)$ that is diagonal[5] in the denominator $SU(2)$'s of (2.5). Under this $SU(2)$ the eight non-compact generators decompose as a $\mathbf{5}+\mathbf{3}$. It turns out that the $O(2)$ subgroup of $SL(2,\mathbb{R})_X$ and the non-compact generators of $SL(2,\mathbb{R})_{5d}$ can be used to set the non-compact generators of the $\mathbf{3}$ to zero, leaving the $\mathbf{5}$. Remarkably enough, this $\mathbf{5}$ extends the $SU(2) = SO(3)$ to another $SL(3,\mathbb{R})$. Thus we

[5] Care is needed here since there is also an anti-diagonal embedding, but this does not have the invariance structure that we need.

will parametrize the scalar potential using this $SL(3,\mathrm{I\!R})/SO(3)$. There is still the residual invariance of the compact generator of $SL(2,\mathrm{I\!R})_{5d}$. This acts on $SL(3,\mathrm{I\!R})$ as a rotation in the first and second entries. Explicit details of how this particular gauge choice is made in $G_{2(2)}$ may be found in Appendix A. (For another explicit parametrization of the coset, see [45].)

Finally, to parametrize $S \in SL(3,\mathrm{I\!R})/SO(3)$, we will write it as

$$S \;=\; \mathcal{O}(\theta_1,\theta_2,\theta_3)^{-1}\; D\; \mathcal{O}(\theta_1,\theta_2,\theta_3)\,, \quad \text{where} \quad D \;=\; \mathrm{diag}(\rho_1\,,\rho_2\,,(\rho_1\rho_2)^{-1})\,, \tag{4.5}$$

and where $\mathcal{O}(\theta_1,\theta_2,\theta_3)$ is a general $SO(3)$ rotation matrix. It is usually convenient to parametrize such a rotation matrix using Euler angles, *i.e.* by fundamental rotations, $R_{ij}(\theta)$, through an angle θ in the i-j-plane:

$$\mathcal{O}(\theta_1,\theta_2,\theta_3) \;=\; R_{12}(\theta_1)\; R_{23}(\theta_2)\; R_{12}(\theta_3)\,. \tag{4.6}$$

If one uses this form of $\mathcal{O}$ then the residual invariance will mean that the potential is independent of the angle θ_3.

4.3. The scalar sector of the $\mathcal{N}=2$ theory

Using the parametrization described above, we find the following expression for the scalar potential of the $SO(3)$-invariant subsector:

$$\begin{aligned} \mathcal{P} \;=\; &-\frac{3g^2}{8} \;-\; \frac{3}{128}\left(\rho_1^2-\rho_1^{-2}\right)\left(\rho_2^2-\rho_2^{-2}\right) \;-\; \frac{3}{32}\left(\rho_1^2+\rho_1^{-2}+\rho_2^2+\rho_2^{-2}\right) \\ &+\; \frac{3}{128}\left(\rho_1-\rho_1^{-1}\right)^3\left(\rho_1\,\rho_2^2-\rho_1^{-1}\,\rho_2^{-2}\right)\sin^2(\theta_2) \\ &-\; \frac{3}{128}\left(\rho_1\,\rho_2^{-1}-\rho_1^{-1}\,\rho_2\right)\left(\rho_1\,\rho_2-\rho_1^{-1}\,\rho_2^{-1}\right)^3\sin^2(\theta_1)\sin^2(\theta_2)\,. \end{aligned} \tag{4.7}$$

One can easily check that this potential yields no other critical points other than the ones discovered in [3].

One of the key elements of five-dimensional supergravity is the matrix, W_{ab}, that appears in the supersymmetry transformation of the gravitino [26]. It is the eigenvalues of this matrix that generically provides a superpotential in $\mathcal{N}=1$ supersymmetric sub-sectors [7]. On the $SO(3)$ invariant $G_{2(2)}$ sector, we find that W_{ab} consists of four two-by-two blocks, three of which are identical. (This structure is required by $SO(3)$ invariance.) The multiplicity-one block corresponds to the indices $(3,7)$ and will be denoted $\mathcal{M}_1$, while the multiplicity-three block corresponds to the index pairs: $(1,5)$, $(2,6)$, $(4,8)$, and will be denoted $\mathcal{M}_2$. Writing

$$\mathcal{M}_j \;=\; \begin{pmatrix} \alpha_j + i\beta_j & -i\gamma_j \\ -i\gamma_j & \alpha_j - i\beta_j \end{pmatrix}\,, \quad j = 1,2\,, \tag{4.8}$$

one has

$$\begin{aligned}
\alpha_1 \;&=\; -\frac{3}{8\,\rho_1^2\rho_2^2}\,\Big[\rho_1\rho_2\big(1+\rho_1^2\big)\,\big(1+\rho_2^2\big) \;+\; \rho_2\big(1-\rho_1\big)\,\big(1+\rho_1^2\big)\,\big(1-\rho_1\rho_2^2\big)\,\sin^2(\theta_2) \\
&\qquad + \big(\rho_1-\rho_2\big)\,\big(1-\rho_1\rho_2\big)\,\big(1+\rho_1^2\rho_2^2\big)\,\sin^2(\theta_1)\sin^2(\theta_2)\Big]\,, \\
\alpha_2 \;&=\; -\frac{1}{8\,\rho_1^2\rho_2^2}\,\Big[\rho_1\big(1+5\rho_2^2+5\rho_1^2\rho_2^2+\rho_1^2\rho_2^4\big)\,\sin^2(\theta_1)\sin^2(\theta_2) \\
&\qquad + \;\rho_2\big(1+5\rho_1^2+5\rho_1^2\rho_2^2+\rho_1^4\rho_2^2\big)\,\cos^2(\theta_1)\sin^2(\theta_2) \\
&\qquad + \;\rho_1\rho_2\big(5+\rho_1^2+\rho_2^2+5\rho_1^2\rho_2^2\big)\,\cos^2(\theta_2)\Big]\,, \\
\beta_1 \;&=\; \frac{3}{8\,\rho_1^2\rho_2^2}\,\big(\rho_1-\rho_2\big)\,\big(1-\rho_1\rho_2\big)\,\big(1+\rho_1^2\rho_2^2\big)\,\sin(\theta_1)\cos(\theta_1)\sin(\theta_2)\,, \\
\beta_2 \;&=\; -\frac{1}{8\,\rho_1^2\rho_2^2}\,\big(\rho_1-\rho_2\big)\,\big(1-\rho_1\rho_2\big)\,\big(1-4\rho_1\rho_2+\rho_1^2\rho_2^2\big)\,\sin(\theta_1)\cos(\theta_1)\sin(\theta_2)\,, \\
\gamma_1 \;&=\; \frac{3}{8\,\rho_1^2\rho_2^2}\,\Big[\rho_2\big(1-\rho_1\big)\,\big(1+\rho_1^2\big)\,\big(1-\rho_1\rho_2^2\big) \;+ \\
&\qquad \big(\rho_1-\rho_2\big)\,\big(1-\rho_1\rho_2\big)\,\big(1+\rho_1^2\rho_2^2\big)\,\sin^2(\theta_1)\Big]\sin(\theta_2)\cos(\theta_2)\,, \\
\gamma_2 \;&=\; \frac{1}{8\,\rho_1^2\rho_2^2}\,\Big[\rho_2\big(1-\rho_1\big)\,\big(1-4\rho_1+\rho_1^2\big)\,\big(1-\rho_1\rho_2^2\big) \;+ \\
&\qquad \big(\rho_1-\rho_2\big)\,\big(1-\rho_1\rho_2\big)\,\big(1-4\rho_1\rho_2+\rho_1^2\rho_2^2\big)\,\sin^2(\theta_1)\Big]\sin(\theta_2)\cos(\theta_2)\,.
\end{aligned} \tag{4.9}$$

The eigenvalues of W_{ab} thus come in complex conjugate pairs with degeneracies 3 and 1, and are given by:

$$\lambda_j \;=\; \alpha_j \;\pm\; i\,\sqrt{\beta_j^2+\gamma_j^2}\,, \quad j=1,2\,. \tag{4.10}$$

From previous experience, it is these eigenvalues that can give rise to superpotentials, and in particular it is λ_1 that could potentially be the superpotential for an $\mathcal{N}=1$ theory.

The kinetic term of the subsector parametrized by ρ_1, ρ_2 and $\theta_j, j = 1,2,3$ is given by:

$$
\begin{aligned}
&-\tfrac{1}{2}\big((\partial\varphi_1)^2 \;+\; (\partial\varphi_2)^2 \;+\; (\partial\varphi_1)(\partial\varphi_2)\big) \;-\; 2\,\sinh^2(\varphi_1-\varphi_2)\,(\partial\theta_1)^2 \\
&\quad -\; 4\,\sinh^2(\varphi_1-\varphi_2)\,\cos\theta_2\,(\partial\theta_1)\,(\partial\theta_3) \\
&\quad -\; 2\,\big(\sinh^2(\varphi_1+2\varphi_2)+\sinh(\varphi_1-\varphi_2)\sinh 3(\varphi_1+\varphi_2)\,\sin^2\theta_1\big)\,(\partial\theta_2)^2 \\
&\quad +\; 4\,\sinh(\varphi_1-\varphi_2)\sinh 3(\varphi_1+\varphi_2)\,\sin\theta_1\,\cos\theta_1\,\sin\theta_2\,(\partial\theta_2)\,(\partial\theta_3) \\
&\quad -\; 2\,\big(\sinh^2(\varphi_1-\varphi_2)+\sinh(3\varphi_1)\,\sinh(\varphi_1+2\varphi_2)\,\sin^2\theta_2 \\
&\qquad\qquad -\sinh(\varphi_1-\varphi_2)\sinh 3(\varphi_1+\varphi_2)\,\sin^2\theta_1\,\sin^2\theta_2\big)\,(\partial\theta_3)^2\,,
\end{aligned}
\tag{4.11}
$$

where $\rho_1=e^{\varphi_1}$ and $\rho_2=e^{\varphi_2}$.

The complexity, both literal and figurative, of the eigenvalues (4.10) makes the isolation of $\mathcal{N}=1$ superpotentials very difficult. To facilitate this process, it is instructive to consider the field theory duals of the supergravity scalars, and see how to reduce the problem further.

5. Further truncations of the $\mathcal{N}=2$ theory

Finding flows in the full set of scalars of the $\mathcal{N}=2$ theory is still rather difficult, and so we simplify the problem further and reduce the number of scalars by imposing discrete symmetries.

5.1. The self-dual truncation

The route taken in [13] was to keep only $\mathcal{O}_3$ and $\mathcal{O}_4$ (and implicitly $\mathcal{O}_0$). The corresponding supergravity scalars were denoted by m and σ respectively, and the residual $U(1)$ invariances can be used to take m and σ to be real. While the results of [13] are certainly correct, there were a few omissions of detail, and as we will see at least one of these details reveals some significant physics.

Setting $\theta_j=0$ and $\rho_1=e^{\frac{m}{\sqrt3}+\sigma}$, $\rho_2=e^{\frac{m}{\sqrt3}-\sigma}$ in the paramtrization above yields a diagonal W_{ab} with one (multiplicity two) eigenvalue:

$$
W \;=\; -\tfrac{3}{4}\Big(\cosh(2\,\sigma)+\cosh\big(\tfrac{2\,m}{\sqrt3}\big)\Big)\,. \tag{5.1}
$$

The other eigenvalue is $-\frac{1}{4}\big(\cosh(2\sigma)+5\cosh\big(\frac{2m}{\sqrt3}\big)\big)$. The potential (4.7) reduces to:

$$
\mathcal{P} \;=\; -\tfrac{3g^2}{16}\left[2-\tfrac14\cosh(4\sigma)+\tfrac14\cosh\big(\tfrac{4}{\sqrt3}m\big)+\cosh\big(2\sigma+\tfrac{2}{\sqrt3}m\big)+\cosh\big(2\sigma-\tfrac{2}{\sqrt3}m\big)\right]\,, \tag{5.2}
$$

and the kinetic term takes the standard form:

$$-\tfrac{1}{2}\,(\partial m)^2 - \tfrac{1}{2}\,(\partial\sigma)^2\,. \tag{5.3}$$

One can easily check that:

$$\mathcal{P} \;=\; \frac{g^2}{8}\Big(\frac{\partial W}{\partial\sigma}\Big)^2 + \frac{g^2}{8}\Big(\frac{\partial W}{\partial m}\Big)^2 - \frac{g^2}{3}W^2\,,$$

and that there is no such equality for the other eigenvalue of W.

From this it is tempting to postulate [13] that, as in [7], $\mathcal{N}=1$ supersymmetric flows are given by taking:

$$\frac{d\varphi_j}{dr} \;=\; \frac{g}{2}\,\frac{\partial W}{\partial\varphi_j}\,, \qquad A' \;=\; -\frac{g}{3}\,W\,, \tag{5.4}$$

with $\varphi_1 = m$ and $\varphi_2 = \sigma$. However, to verify this one really needs to check the vanishing of the supersymmetry variations of the spin-$\frac{1}{2}$ fields: this we have confirmed in detail.

One other detail that is not immediately apparent in [13] is the consistency of truncating to the m and σ fields. In supergravity one might be concerned that the other fields of the $G_{2(2)}$ coset do not decouple, while in field theory one might be concerned that turning on m and σ may cause other fields to flow. Fortunately, explicit computation reveals that [13] is correct, and that this is a consistent truncation, however it would be more satisfying if this fact were understood as a result of a symmetry condition. This is indeed possible.

Consider the following matrices:

$$\begin{pmatrix} 0 & \mathbf{I}_{3\times3} \\ -\mathbf{I}_{3\times3} & 0 \end{pmatrix}\,, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{5.5}$$

where these are to be viewed as elements of $SL(6,\mathbb{R})$ and $SL(2,\mathbb{R})_{5d}$ in $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})_{5d} \subset E_{6(6)}$. These are invariances of the supergravity potential: indeed they are elements of the invariance group $SO(6)\times SL(2,\mathbb{R})_{5d}$. The simultaneous action of these two matrices negates the non-compact generators of $SL(2,\mathbb{R})_X$ and $SL(2,\mathbb{R})_{5d}$, and leaves invariant precisely the (complex) parameters m and σ. Thus this discrete symmetry effects the desired consistent truncation, and shows exactly why the other fields do not run in these models.

More significant is the fact that this symmetry uses the modular inversion of $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})_{5d}$, combined with an $SO(6)$ rotation. This should therefore be a symmetry of the underlying string theory as well, and the invariance under (5.5) forces the UV string coupling, and hence the Yang-Mills coupling on the brane to its self-dual value.

It is thus hard to see, from the field theory why the super-QCD flow of [13] should provide a model for electric and magnetic confinement.

To understand [13] more completely, one should note that the modular inversion is combined with a spatial inversion of S^5 in which the first three cartesian coordinates, x^1, x^2, x^3, are exchanged with the second three, x^4, x^5, x^6. This means that if one sees a characteristic "electric behavior" by approaching on the $(1,2,3)$-axes then one must be able to see the dual "magnetic behavior" by approaching on the $(4,5,6)$-axes. As a result one sees that the confining behavior observed in [13] must be a pathology induced in Wilson and 't Hooft loops by approaching the S^5 from a very special direction. In reality an apparently confining loop can lower its energy by slightly modifying its direction of approach, and thereby become screened. Thus the confining behavior of [13] is no more physical than that of [46]: it is simply an artefact of an unstable symmetry axis. This interpretation is consistent with the analysis of [13] in which the string tensions were read off as eigenvalues of the B-field kinetic terms. The selection of an eigenvalue is tantamount to selecting a direction on S^5, and so the confining eigenvalues should be wiped out by screening effects unless the S^5 is approached from very special directions [6].

5.2. The parity-invariant sub-sector

Before we leave the subject of consistent truncations, it is worthwhile cataloging another potentially interesting subsector. As we have seen, the flows of [13] are self-dual, and it would be nice to have a "tame" sector in which the five-dimensional dilaton could possibly flow. One way to get such a sector is to require the parity symmetry:

$$\begin{pmatrix} \mathbf{I}_{3\times 3} & 0 \\ 0 & -\mathbf{I}_{3\times 3} \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.6}$$

The former matrix lies in $O(6)$, and the latter is in $GL(2,\mathbb{R})$. While the usual stated symmetry of the supergravity theory is $SO(6)\times SL(2,\mathbb{R})_{5d}$, it is actually symmetric under $(O(6)\times SL^{\pm}(2,\mathbb{R})_{5d})/\mathbb{Z}_2$, where $SL^{\pm}(2,\mathbb{R})$ denotes the subset of $GL(2,\mathbb{R})$ with determinant ± 1, and the division by $\mathbb{Z}_2$ requires that the determinants are equal in each factor. The parity symmetry (5.6) projects out the operator $\mathcal{O}_2$, the five-dimensional axion, and enforces a reality condition on m and σ. This symmetry commutes with the $SO(3)$ symmetry, but it removes the $U(1)\times U(1)$ symmetries of $SL(2,\mathbb{R})_X \times SL(2,\mathbb{R})_{5d}$. We are thus left with four supergravity scalars, one of which is the five-dimensional dilaton.

[6] This last observation was made in discussions with Joe Minahan.

These four scalars turn out to be the non-compact directions of yet another $SL(2,\mathrm{I\!R})\times SL(2,\mathrm{I\!R})$ in $G_{2(2)}$. In terms of the generators of Appendix A, these $SL(2,\mathrm{I\!R})$'s are given by $L_1^{(1)} = -\frac{1}{2}(X_4+X_8)$, $L_2^{(1)} = \frac{1}{2}(X_1+X_5)$, $L_3^{(1)} = \frac{1}{2}(J_3+K_3)$, and $L_1^{(2)} = \frac{1}{2}(X_4-X_8)$, $L_2^{(2)} = \frac{1}{2}(X_1-X_5)$, $L_3^{(2)} = -\frac{1}{2}(3J_3-K_3)$, with $L_1^{(j)}$ being compact. We could proceed as above to get at the scalar structure, but the previous parametrization did not handle the dilaton cleanly: while it does not appear in the potential, the dilaton kinetic term will mix in a complicated manner with the other scalars. Here we use a different gauge where the kinetic term is simple, but the potential appears to depend upon all four scalars. Each $SL(2,\mathrm{I\!R})$ is parametrized using:

$$\exp\big(-\phi_j L_1^{(j)}\big)\ \exp\big(\alpha_j L_3^{(j)}\big)\ \exp\big(\phi_j L_1^{(j)}\big)\,,\qquad j=1,2\,. \tag{5.7}$$

Define

$$\begin{aligned} W \;=\; & -\frac{3}{2}\cosh(\alpha_1)\cosh(\alpha_2)\big(\cosh^2(\alpha_2)-e^{4i\,\phi_2}\sinh^2(\alpha_2)\big) \\ & -\frac{3}{8}\sinh(\alpha_1)\sinh(\alpha_2)\big(e^{2i\,(\phi_1+\phi_2)}+3\,e^{2i\,(\phi_1+3\phi_2)}\big) \\ & +\frac{3}{4}\,i\,\sinh(\alpha_1)\sinh(3\,\alpha_2)\sin(2\,\phi_2)\,e^{2i\,(\phi_1+2\phi_2)}\,, \end{aligned} \tag{5.8}$$

then one has

$$\mathcal{P} \;=\; \frac{1}{8}\left|\frac{\partial W}{\partial\alpha_1}\right|^2 + \frac{1}{24}\left|\frac{\partial W}{\partial\alpha_2}\right|^2 - \frac{1}{3}\,\big|W\big|^2\,. \tag{5.9}$$

Note that there are no derivatives of W with respect to ϕ_j on the right-hand side of this equation. There is also an asymmetry between α_1 and α_2 in (5.8) because the two $SL(2,\mathrm{I\!R})$'s are different: the non-compact generators of $G_{2(2)}$ form a $(\mathbf{2},\mathbf{4})$ of these two groups.

In this parametrization the kinetic term takes the form:

$$-\tfrac{1}{2}\,(\partial\alpha_1)^2-\tfrac{3}{2}\,(\partial\alpha_2)^2-2\sinh^2(2\,\alpha_1)\,(\partial\phi_1)^2-6\sinh^2(2\,\alpha_2)\,(\partial\phi_2)^2\,. \tag{5.10}$$

6. The metric and dilaton background

6.1. The metric

As desribed in the introduction, the ten-dimensional background metric is given by the warped product (1.2), with an internal metric on the deformed S^5 given by (1.1). It is natural to use the $SO(3)$ in describing the internal geometry, and indeed we will describe the deformed S^5 in terms of a degenerate fibration of $SO(3)$ over the space of its orbits.

We will also only consider the metric corresponding to the two scalar subspace of section 5.1 and [13].

We start by thinking of the "round" S^5 as the unit sphere in $\mathbb{R}^6$, but with the cartesian coordinates split into two groups of three: (u^i, v^j), $i, j = 1,2,3$. The $SO(3)$ symmetry acts upon these simultaneously in the vector representation. The internal 5-manifold is still the unit sphere:

$$u^2 + v^2 \;=\; 1\,, \tag{6.1}$$

but the general metric on this deformed S^5 is given by:

$$ds_5^2 \;=\; \xi^{-\frac{3}{2}}\, d\hat{s}_5^2\,,$$

where

$$\begin{aligned} d\hat{s}_5^2 \;&= a_1\, du^i du^i + 2\, a_2\, du^i dv^i + a_3\, dv^i dv^i \\ &\quad + a_4\, \big(d(u\cdot v)\big)^2 + 2\, a_5\, (v^i du^i)(v^j du^j) + 2\, a_6\, (u^i du^i)(v^j dv^j)\,. \end{aligned} \tag{6.2}$$

The coefficient functions are then given by:

$$\begin{aligned} a_1 \;&=\; \frac{1}{4\,\mu^2\,\nu^4}\;(1+\mu^2\,\nu^2)\;\big((1+\mu^2\,\nu^2)\,\nu^2\,u^2 \;+\; (\mu^2+\nu^6)\,v^2\big)\,,\\ a_2 \;&=\; -\,\frac{1}{4\,\mu^2\,\nu^4}\;(1-\nu^4)\;(1-\mu^2\,\nu^2)\;(\mu^2+\nu^2)\;u\cdot v\,,\\ a_3 \;&=\; \frac{1}{4\,\mu^2\,\nu^4}\;(1+\mu^2\,\nu^2)\;\big((\mu^2+\nu^6)\,u^2 \;+\; (1+\mu^2\,\nu^2)\,\nu^2\,v^2\big)\,,\\ a_4 \;&=\; \frac{1}{16\,\mu^4\,\nu^6}\;(1-\mu^2\,\nu^2)^2\;(1+\mu^2\nu^2)\;(\mu^2+\nu^6)\,,\\ a_5 \;&=\; \frac{1}{8\,\mu^4\,\nu^4}\;(1-\mu^4\,\nu^4)\;(\mu^4-\nu^4)\,,\\ a_6 \;&=\; -\,\frac{1}{8\,\mu^2\,\nu^6}\;(1-\nu^8)\;(\mu^4-\nu^4)\,, \end{aligned} \tag{6.3}$$

where

$$\mu \equiv e^{\sigma}\;, \qquad \nu \equiv e^{\frac{m}{\sqrt{3}}}\;. \tag{6.4}$$

The warp-factor, ξ, is given by:

$$\begin{aligned} \xi^2 \;=\; \frac{1}{16\,\mu^4\,\nu^8}\;\Big[&\nu^2\,(1+\mu^2\,\nu^2)^3\;(\mu^2+\nu^6) \;+\; (1-\nu^4)^2\;(\mu^2-\nu^2)^2\;(1+\mu^2\,\nu^2)^2\;u^2\,v^2\\ &-(1-\mu^2\,\nu^2)^2\;(1-\nu^4)^2\;(\mu^2+\nu^2)^2\;(u\cdot v)^2\Big]\,. \end{aligned} \tag{6.5}$$

Note that at $\mu = \nu = 1$ the internal metric given by (6.2) and (6.3), on the surface (6.1), collapses to that of the round sphere of unit radius. Moreover, at $\mu = \nu = 1$ one has $\xi = 1$. As usual, we define Δ by $\Delta^2 \equiv \det(g_{mp} \overset{\circ}{g}{}^{pq})$, where g_{mp} is the internal metric on S^5 given by (6.2) and $\overset{\circ}{g}{}^{pq}$ is the inverse of the "round" internal metric at $\mu = \nu = 1$. We then have

$$\xi \;\equiv\; \Delta^{-\frac{4}{3}}\,, \tag{6.6}$$

and the complete ten-dimensional metric is:

$$ds_{10}^2 \;=\; \xi^{\frac{1}{2}}\; ds_{1,4}^2 \;+\; \xi^{-\frac{3}{2}}\; d\hat{s}_5^2\,. \tag{6.7}$$

The foregoing metric on S^5 is far from elementary, but a natural way to think of it is as an $\mathbb{RP}^3$ fibered over $\mathbb{P}^1/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The $\mathbb{RP}^3$ fiber is, of course, $SO(3) \equiv S^3/\mathbb{Z}_2$, and the base is the the orbit space of this $SO(3)$ on S^5. This base has the topology of a disk.

To see this explicitly one can use the $SO(3)$ action to reduce u and v to:

$$u \;=\; \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ \cos\theta \end{pmatrix}\,, \qquad v \;=\; \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ \sin\theta\;\sin\phi \\ \sin\theta\;\cos\phi \end{pmatrix}\,. \tag{6.8}$$

The remaining non-zero elements satisfy $(u_3)^2 + (v_2)^2 + (v_3)^2 = 1$, and so naively describe an S^2. However, any two coordinates can be negated by an $SO(3)$ rotation and so we divide by the inversions: $v_2 \to -v_2$ and $u_3 \to -u_3, v_3 \to -v_3$. Thus we can make the restrictions $v_2 \geq 0$, $u_3 \geq 0$. In terms of the polar coordinates, one has: $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi$. It should also be noted that for $\theta = \pi/2$ the coordinate ϕ becomes redundant. Given the $\mathbb{Z}_2$ identifications for general θ and ϕ, the base may be thought of as a quarter sphere, which has the topology of a disk. We can parametrize it in terms of the coordinates:

$$w_1 = 2\,u\cdot u - 1 = \cos(2\theta)\,, \quad w_2 = 2\,u\cdot v = \sin(2\theta)\;\cos\phi\,, \quad 0 \leq w_1^2 + w_2^2 \leq 1\,. \tag{6.9}$$

The fiber is regular except at the edges of the disk, *i.e.* at points where

$$w_1^2 \;+\; w_2^2 \;=\; 1 \qquad \Leftrightarrow \qquad \sin\theta\,\cos\theta\,\sin\phi = 0\,. \tag{6.10}$$

At these points either u or v vanishes, or u and v are colinear. At such points the fiber degenerates to a $\mathbb{P}^1$. We should stress that even though this description as an $\mathbb{RP}^3$ fibration is singular, the overall manifold at generic values of μ and ν is still a perfectly smooth, but deformed, S^5.

6.2. The dilaton

Using the scalar fields of section 5.1, we computed the right-hand side of (1.3) with $x^I = (u^i, v^j)$. There is an important consistency check in that taking the determinant on both sides of (1.3) must give the same expression for Δ as that given by (6.6) and (6.5). This does indeed work. Furthermore, we obtain the following components for $\mathcal{M} = S\,S^T$:

$$\begin{aligned} \mathcal{M}_{11} \ &= \frac{1}{4\,\xi\,\mu^2\,\nu^4}\ (1+\mu^2\,\nu^2)\,\big((\mu^2+\nu^6)\,\cos^2\theta + \nu^2(1+\mu^2\nu^2)\,\sin^2\theta\big)\,, \\ \mathcal{M}_{12} \ = \mathcal{M}_{21} \ &= \ \frac{1}{4\,\xi\,\mu^2\,\nu^4}\ (1-\nu^4)\,(1-\mu^2\,\nu^2)\,(\mu^2+\nu^2)\sin\theta\,\cos\theta\,\cos\phi \\ \mathcal{M}_{22} \ &= \frac{1}{4\,\xi\,\mu^2\,\nu^4}\ (1+\mu^2\,\nu^2)\,\big(\nu^2(1+\mu^2\nu^2)\,\cos^2\theta + (\mu^2+\nu^6)\,\sin^2\theta\big)\,. \end{aligned} \tag{6.11}$$

Thus we have an extremely non-trivial dilaton/axion background. Intriguingly enough, the matrix elements of $\mathcal{M}$ are, up to a factor of ξ, exactly the same as the metric coefficients a_1, a_2 and a_3. Thus the dilaton is controlling the relative sizes of the u-sphere and v-sphere, while the axion controls the fibering of one over the other.

7. The flows and their infra-red asymptotics

The mathematics of the flows were thoroughly described in [13], and we first summarize these results in our conventions. The solution to (5.4) for $\mu = e^\sigma$ and $\nu = e^{\frac{2}{\sqrt{3}}m}$ is:

$$\mu \ = \ \sqrt{\frac{1+\lambda t^3}{1-\lambda t^3}}\,, \qquad \nu \ = \ \sqrt{\frac{1+t}{1-t}}\,, \tag{7.1}$$

and

$$A(r) \ = \ \tfrac{1}{6}\log\big(t^{-3} - \lambda^2\,t^3\big) \ + \ \tfrac{1}{2}\log\big(t^{-1} - t\big) \ + \ C_1\,, \tag{7.2}$$

where

$$t \ = \ \exp\big[-\big(\frac{r}{L} - C_1\big)\big]\,, \qquad \lambda \ = \ \exp\big[3\big(C_2 - C_1\big)\big]\,, \tag{7.3}$$

and where the C_j are constants of integration for the flows of m and σ. Indeed, near the UV limit one has:

$$\begin{aligned} &m \ \sim m_0 e^{-\frac{r}{L}}\,, \quad \sigma \ \sim \ \sigma_0\,e^{-3\frac{r}{L}}\,, \\ &m_0 \ \equiv \ \tfrac{\sqrt{3}}{2}\,e^{C_1}\,, \quad \sigma_0 \ \equiv \ \tfrac{1}{3}\,e^{3C_2}\,, \quad \lambda \ = \ \tfrac{9\sqrt{3}}{8}\,\frac{\sigma_0}{m_0^3}\,. \end{aligned} \tag{7.4}$$

Thus the constants of integration therefore represent the values of the mass and gaugino consensate introduced in the UV theory. The constant of integration in $A(r)$ has been chosen so that $A(r) \sim \frac{r}{L} + O(e^{-r/L})$ as $r \to \infty$. It was argued in [13] that the physical flows have $\lambda \le 1$, and thus have the fermion mass scale greater that the gaugino condensate scale.

7.1. Asymptotics for $\lambda < 1$

For $\lambda < 1$ the five-dimensional metric becomes singular at $r = C_1 L$, or at $t = 1$ [13]. The ten-dimensional metric is, however, much less singular, and indeed resolves into a ring distribution of what appear to be 7-branes. To see this we start by parametrizing the vectors u, v by:

$$u \;=\; \mathcal{R}\begin{pmatrix} 0 \\ 0 \\ \cos\theta \end{pmatrix}, \qquad v \;=\; \mathcal{R}\begin{pmatrix} 0 \\ \sin\theta\,\sin\phi \\ \sin\theta\,\cos\phi \end{pmatrix}, \qquad 0 \le \theta \le \pi/2\,,\; 0 \le \phi \le \pi\,, \tag{7.5}$$

where $\mathcal{R}$ is a generic $SO(3)$ rotation matrix. One then decomposes $R^{-1}dR$ into the left invariant 1-forms, σ_i, $i = 1, 2, 3$, normalized according to $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\,\sigma_j \wedge \sigma_k$.

In the limit $t \to 1$ the warp factor, ξ, diverges according to:

$$\xi \;\sim\; \frac{1}{(1-t)^2}\,(\sin\theta\,\cos\theta\,\cos\phi)^{\frac{1}{2}}\,. \tag{7.6}$$

The factor of $\xi^{1/2}$ in (6.7) makes two important modifications to the five-dimensional metric. First, it exactly cancels the vanishing of e^{2A} as $t \to 1$, and leaves a finite coefficient. Secondly, it suggests the change of variable $\chi \equiv 2(1-t)^{1/2}$ to regularize the radial behaviour.

The net result of this is that the ten-dimensional metric has the following leading behaviour in χ as $\chi \to 0$.

$$\begin{aligned} ds^2 =& \tfrac{L^2}{\sqrt{2}}(1 - w_1^2 - w_2^2)^{\frac{1}{4}}\,\Big[\tfrac{2}{L^2}\,(1-\lambda^2)^{1/3}\,e^{2C_1}\,\big(\eta_{\mu\nu}dx^\mu dx^\nu\big) - d\chi^2 - \tfrac{1}{4}\,\chi^2\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\Big] \\ &-\; \tfrac{L^2}{\sqrt{2}}\,(1 - w_1^2 - w_2^2)^{-\frac{3}{4}}\,\left[\frac{2(1-\lambda)}{(1+\lambda)}\,dw_1^2 \;+\; \frac{(1+\lambda)}{2(1-\lambda)}\,dw_2^2\right]. \end{aligned} \tag{7.7}$$

Observe that the metric in the first square bracket is *locally* that of a flat Lorentzian 7-brane, while the metric in the second square bracket is that of a flat Euclidean metric on the disk. The warp factor:

$$\zeta \;\equiv\; 1 - w_1^2 - w_2^2 \;=\; 4\sin^2\theta\,\cos^2\theta\,\sin^2\phi\,, \tag{7.8}$$

is only singular on the ring at the edge of the disk. Moreover the powers of ζ that appear in (7.7) are precisely those appropriate to a dimensional reduction of ten dimensional physics to (7+1)-dimensional physics on the brane. Thus we see that in the IR limit the D3-brane physics appears to be oxidizing to 7-brane physics.

To be more explicit, far from the brane one sees the usual $D3$-brane throat, but as one approaches $t = 1$, or $r = C_1 L$, the throat rounds out into a seven-brane world. Now recall that $e^{2C_1} = \frac{4}{3} m_0^2$ (see (7.4)), where m_0 is the mass of the chiral multiplets. Thus the distance that one descends down the throat before encountering the 7-brane is set by the UV mass, m_0. Also note that the scale in front of the $D3$-brane metric is $(1-\lambda^2)^{1/3}\, e^{2C_1}$, and so the supergravity description of this flow terminates at a $D3$-brane scale determined by the chiral multiplet mass and by the gaugino condensate. The larger the chiral multiplet mass, the closer to the UV it terminates, but the nearer λ is to 1, the nearer the IR the flow goes.

The seven-brane form of the metric is precisely consistent with the infra-red limit of the dilaton. As $t \to 1$ the matrix $\mathcal{M}$ in (6.11) limits to:

$$\mathcal{M} \ = \ \frac{1}{\sin\phi} \begin{pmatrix} \cot\theta & \cos\phi \\ \cos\phi & \tan\theta \end{pmatrix} . \tag{7.9}$$

This dilaton/axion configuration is regular everywhere except exactly where (7.8) vanishes.

There is also an interesting topological issue: while the first metric factor in (7.7) is locally flat, it is actually $\mathbb{R}^{3,1} \times \mathbb{R}^4/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ negates four of the spatial coordinates. It thus has an A_1 singularity. The reason for this is that the apparently spherical section of the metric (7.7) represented by the left invariant one-forms, σ_j, is the metric on $SO(3) = S^3/\mathbb{Z}_2$ and not the metric on $SU(2) = S^3$. This is the origin of the modding by $\mathbb{Z}_2$.

This suggests that the string theory will see new massless states associated with branes wrapping this vanishing 2-cycle.

7.2. Asymptotics for $\lambda = 1$

If one looks at (7.7) one sees that various coefficients either vanish or diverge as $\lambda \to 1$. In a more careful treatment of the asymptotics these coefficients are, respectively, replaced by positive or negative powers of the radial coordinate χ. To be more explicit, first note that the five-dimensional metric (2.1) now behaves according to:

$$ds^2_{1,4} = (1-t)^{4/3}\ e^{2C_1}\ 2^{\frac{4}{3}}\, 3^{\frac{1}{3}}\, \eta_{\mu\nu} dx^\mu dx^\nu - dr^2 \,. \tag{7.10}$$

The warp factor is now asymptotic to:

$$\xi \ \sim \ \frac{1}{(1-t)}\, \widehat{\Omega}\,, \quad \text{where} \quad \widehat{\Omega} \equiv \tfrac{1}{3} \big(3\, \cos^2(2\theta) + 4\, \sin^2(2\theta)\, \sin^2(\phi)\big)^{1/2} . \tag{7.11}$$

Once again one introduces the change of variables: $\chi \equiv 2(1-t)^{1/2}$, and one then finds that the ten-dimensional metric takes the form:

$$ds^2 \ \sim \ \widehat{\Omega}^{\frac{1}{2}} \left[2^{\frac{2}{3}}\, 3^{\frac{1}{3}}\, e^{2C_1}\, \chi^{\frac{2}{3}} \left(\eta_{\mu\nu} dx^\mu dx^\nu\right) - L^2\, d\chi^2\right] \ - \ L^2\, \widehat{\Omega}^{-\frac{3}{2}} \left[\frac{1}{3\chi^2}\, dw_2^2 \ + \ d\tilde{s}_4^2\right], \tag{7.12}$$

where $d\tilde{s}_4^2$ is a complicated, but regular metric on $\mathbb{RP}^3$ and in the θ direction.

The dilaton matrix, $\mathcal{M}$, takes the form

$$\mathcal{M} \ \sim \ \tfrac{1}{3}\, \widehat{\Omega}^{-1} \begin{pmatrix} 1+2\cos^2(\theta) & 2\,\sin(2\theta)\,\cos\phi \\ 2\,\sin(2\theta)\,\cos\phi & 1+2\sin^2(\theta) \end{pmatrix}. \tag{7.13}$$

The metric and the dilaton no longer have a ring singularity, but only have a singularity at the points $\theta = \pm\frac{\pi}{4}, \phi = 0$. On the other hand, the metric now has a singularity at $\chi = 0$. It is not so simple to give this metric a geometric interpretation, particularly since one of the internal directions is blowing up as $\chi \to 0$. On the other hand, in contradistinction to the $\lambda < 1$ flows, the $D3$-brane coefficient vanishes as $\chi \to 0$, which, in principle, suggests that the flow might be able to probe further into the infra-red.

Interestingly enough, the metric and dilaton becomes a little more regular near the apparently singular region $\theta = \pm\pi/4, \phi = 0$. Setting $\theta = \frac{\pi}{4} + \psi$, $t = 1 - \frac{1}{2}\chi^2$ and expanding in small χ, ψ and ϕ we find:

$$\begin{aligned} ds^2 \ &\sim 2\,\widetilde{\Omega}^{\frac{1}{2}} \left[3^{\frac{1}{3}}\, e^{2C_1}\, \chi^{\frac{2}{3}} \left(\eta_{\mu\nu} dx^\mu dx^\nu\right) - L^2\, d\chi^2\right] \\ &\quad - \ \tfrac{1}{2\sqrt{3}}\, L^2\, \widetilde{\Omega}^{-\frac{3}{2}}\, \chi^2 \left[\tfrac{16}{3}\, d\psi^2 \ + \ d\phi^2 + \phi^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \right], \end{aligned} \tag{7.14}$$

where

$$\widetilde{\Omega} \equiv \tfrac{2}{3}\left(3\,\psi^2 + \phi^2 + \chi^4\right)^{1/2}. \tag{7.15}$$

Note that the metric (7.14) has round $\mathbb{RP}^3$ fibers, but there is a conical singularity at $\phi = 0$ [7]. The dilaton matrix becomes:

$$\mathcal{M} \ \sim \ \tfrac{2}{3}\, \widetilde{\Omega}^{-1}\, \mathcal{Q} \begin{pmatrix} 2 & -\psi \\ -\psi & 2\psi^2 + \frac{1}{2}\,\phi^2 + \frac{1}{2}\,\chi^4 \end{pmatrix} \mathcal{Q}^T, \tag{7.16}$$

where $\mathcal{Q}$ is a rotation by $\theta = \pi/4$.

[7] In our conventions the non-conical metric would be: $d\phi^2 + \frac{1}{4}\phi^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$

8. The ring singularity: Looking for 5-branes

One of the motivations of [40] was to relate the supergravity flows to the non-commutative geometry suggested by the Yang-Mills superpotential. In particular, it was shown in [47,48] that the chiral superfields of the supersymmetric vacuum must obey:

$$\big[\Phi_i\,,\Phi_j\big] \;=\; -\frac{m}{\sqrt{2}}\epsilon_{ijk}\;\Phi_k\,. \tag{8.1}$$

Since these are the commutation relations of $SU(2)$, the possible vacua are classified by the maps of $SU(2)$ into the gauge group $SU(N)$. Indeed, if the mass parameter, m, is real then the only solution to (8.1) is to take Φ_j to be some real combination of the anti-hermitian generators of $SU(N)$. It was thus argued in [39,40] that to find a ground state of the $\mathcal{N}=1$ theory, only the real part of Φ_i can develop a vev, and then given that $\sum_j Tr(|\Phi_j|^2) \sim m^2$, it follows that the vacuum state of the $\mathcal{N}=1$ theory should correspond to the $D3$-branes becoming dielectric 5-branes that wrap a non-commutative S^2.

To connect this with the results here, recall that for finite N and for commuting vevs, the Φ_j may be thought of as the cartesian coordinates transverse to the $D3$-branes. More generally, the solutions here have an $SO(3)$-invariance: in (8.1) the (real) $SO(3)$ acts the indices i,j,k, with the real and imaginary parts of Φ_i transforming separately, each as a triplet of $SO(3)$. Thus the real and imaginary parts of Φ_i correspond to the coordinates u_i and v_i on S^5. If we were to obtain precisely the solution of [40] then the 5-branes should emerge in the limit in which $v_j \equiv 0$: Instead we find a ring singularity when $\vec{u}$ and $\vec{v}$ are parallel.

The key to understanding this apparent discrepancy comes from looking at the flows with $\lambda = 0$. In supergravity these flows have an additional $U(1)$ symmetry that is generated by the simultaneous action of the matrices (5.5) considered as $SO(2)$ generators in $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$. This symmetry rotates $\vec{u}$ into $\vec{v}$ while performing an "S-duality rotation" in the $SL(2,\mathbb{R})$. Because this symmetry is embedded partially in the $SL(2,\mathbb{R})$, this $U(1)$ will not be a symmetry of the field theory at finite N: at best it will reduce to some discrete subgroup of the $SL(2,\mathbb{Z})$, S-duality symmetry of the finite N, $\mathcal{N}=4$ Yang-Mills theory.

Returning to our ring singularity, one sees that it is essentially given by $\phi = 0$, and $-\pi/2 \le \theta \le \pi/2$. Note that we have doubled the range of θ used in (7.5): This is enables us to set $\phi = 0$ and still cover the region with u and v anti-parallel ($\phi = \pi$). This range of θ thus covers the whole ring singularity. On this locus, the $SO(2)$ action is represented by a rotation in θ, and hence the symmetry sweeps out the ring. It follows that as we go around the ring, the singularity must be undergoing a continuous "S-duality rotation" in $SL(2,\mathbb{R})$.

This picture is confirmed by a more careful analysis of the behaviour of the dilaton near the singularity. In the previous section we took the limit in which $(1-w_1^2-w_2^2)$ was finite, and $(1-t)$ was becoming vanishingly small, that is, we considered a generic *interior* point of the disk defined by w_1 and w_2. The asymptotic behaviour of the metric and dilaton depend upon the order of these limits, and we now consider them in the opposite order. We will also restore λ, but keep $\lambda < 1$. The dilaton matrix now has the asymptotic form:

$$\mathcal{M} \;=\; Q\cdot\begin{pmatrix} \mathcal{U}\,\frac{1}{\sqrt{1-t}} & 0 \\ 0 & \mathcal{U}^{-1}\,\sqrt{1-t}\end{pmatrix}\cdot Q^{-1}\,, \tag{8.2}$$

where

$$\mathcal{U} \;\equiv\; \Big(\frac{2(1-\lambda^2)}{1+2\,\lambda\,\cos(4\,\theta)+\lambda^2}\Big)^{\frac{1}{2}}\,, \qquad Q \;\equiv\; \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\,. \tag{8.3}$$

Note that for $\theta = 0$ ($v=0$) and $\theta = \pm\pi/2$ ($u=0$) this dilaton configuration is precisely that which is appropriate for NS 5-branes and $D5$-branes, and that these limits are exchanged by the $\pi/2$ rotation corresponding to $\tau\to -1/\tau$. In between we have a continuous rotation $Q\in SO(2)\subset SL(2,\mathbb{R})$, and this is precisely the same as the rotation between $\vec{u}$ and $\vec{v}$ that takes us around the ring.

It is also instructive to parametrize the $SL(2,\mathbb{R})$ matrix in terms of the coupling, τ. One then finds:

$$\tau \;=\; \frac{i\mathcal{U}\,\cos\theta \;-\; \sqrt{1-t}\,\sin\theta}{\sqrt{1-t}\cos\theta \;+\; i\mathcal{U}\,\sin\theta} \;\sim\; \cot\theta \quad \text{as } t\to 1\,. \tag{8.4}$$

As one goes around the ring one finds that the coupling runs from infinity down to zero along the positive real axis. At finite N, a singularity at $Im(\tau)=0$ can be interpreted in terms (p,q)-branes provided that τ approaches a rational point on the real axis. It is only in the limit $N\to\infty$ that we can get a smooth distribution (p,q)-branes.

One can also analyse the metric in the limit in which $(1-w_1^2-w_2^2)$ vanishes faster than $(1-t)$, and one sees qualitatively different behaviour from (7.7). There is also a hint of the 5-branes wrapping an S^2 [39,40]. Indeed, it should be recalled that in our description of the S^5 geometry, the $\mathbb{RP}^3$ fiber degenerates to an S^2 on the ring singularity, and this is the S^2 upon which the 5-branes must wrap. Again we focus upon the flows with $\lambda < 1$.

We consider the metric near $t = 1$ but with $\phi = 0$ in (7.5). The residual directions are thus the radial, or t direction, the $D3$-branes, the S^2 fiber, and the angle, θ. We find:

$$
\begin{aligned}
ds_8^2 \ \sim \ \Big(\frac{1 + 2\,\lambda\, \cos(4\,\theta) + \lambda^2}{2(1-\lambda^2)}\Big)^{\frac{1}{4}} \Big(& - (1-t)^{-\frac{3}{4}}\,(dr^2 + d\theta^2) \ - \ (1-t)^{+\frac{5}{4}}\, d\Omega_2^2 \\
& + \ 2\,(1-\lambda^2)^{1/3}\,(1-t)^{+\frac{1}{4}}\, e^{2C_1}\,\big(\eta_{\mu\nu} dx^\mu dx^\nu\big)\Big)\,.
\end{aligned}
\tag{8.5}
$$

One can regularize the radial metric by setting $t \sim 1 + \chi^{8/5}$, in which case the radial and S^2 part of the metric become $d\chi^2 + \chi^2 d\Omega_2^2$. Thus the 2-spheres are collapsing in a natural manner. The metric in the θ direction is blowing up, which is reminiscent of approaching the ring singularity of a rotating Kerr-Newman black-hole. The scale factor in front of the $D3$-branes is now vanishing, which suggests that one can now access the far infra-red limit.

The foregoing limits of the dilaton and metric only depend in a rather mild way upon λ for $\lambda < 1$, and indeed are structurally identical to configurations with $\lambda = 0$. This means that the physical interpretation of the ring singularity should be the same for all $\lambda < 1$, and suggests that if $\lambda < 1$ then the gaugino vev is becoming irrelevant to the infra-red structure, which is then dominated by the flow in the mass, m. In this limit, the additional $U(1)$ symmetry is restored, and the ring singularity is a duality averaged family of 5-branes.

It is natural to wonder if the 5-brane identification becomes clearer for $\lambda = 1$ since the "ring of 5-branes" collapses into two singularities at $\theta = \pm\frac{\pi}{4}$. Indeed, at these points one has $\tau = \pm 1$, which are not only rational, but are consistent with $(1, \pm 1)$ 5-branes. As we have already noted, the five-dimensional flows considered here are self S-dual, and so finding such a pair of branes is the simplest possible solution we could have found. This solution no longer has the "unphysical" $U(1)$ symmetry, and therefore makes sense at finite N, and is also a good candidate for a string background. The asymptotic analysis of the metric in (7.14) does not, however, appear to be consistent with the 5-brane interpretation. In particular, the $\mathbb{RP}^3$ remains round rather than collapsing to an S^2.

Thus, if one approaches the core of the supergravity solution from a generic direction one sees a 7-brane, and the scale of the $D3$-brane world goes to a finite value. If one approaches the core from a direction that is consistent with having an infra-red vacuum in the field theory, then one encounters some lower dimensional "branes," and the scale in front of the $D3$-branes can now run to the far infra-red. If gaugino vev is too small then one finds a duality averaged ring of 5-branes in the core. If the gaugino vev is tuned to its maximum possible, and indeed critical, value then the "duality" symmetry is not restored in the infra-red, and core contains two discrete singularities with dilaton/axion (p, q) charges of $(1, \pm 1)$. The structure of the metric in this limit appears rather different from that of 5-branes wrapping an S^2.

9. The $SU(3)$ invariant flow

The complexity of the metric (6.2) makes it extremly difficult to study the full lift of the GPPZ-flow to ten dimensions. We can, however, consider a further truncation of this flow to a $SU(3)$ invariant subspace of the scalar manifold (2.5), obtained by considering the flow of the σ field only, *i.e.* with m set to zero. This yields an $\mathcal{N}=1$ flow with the superpotential, cf. (5.1),

$$W(\sigma) \;=\; -\frac{3}{2}\cosh^2\sigma\,, \tag{9.1}$$

and an explicit solution given by [13] (cf. (7.1))

$$\sigma(r) \;=\; \frac{1}{2}\log\left(\frac{1+\lambda t^3}{1-\lambda t^3}\right), \qquad A(r) \;=\; \frac{1}{6}\log\left(t^{-6}-\lambda^2\right)+\log(\frac{\ell}{L})\,, \tag{9.2}$$

where

$$t \;=\; e^{-r/L}\,, \qquad \lambda \;=\; e^{C}\,, \tag{9.3}$$

and where C and ℓ are integration constants.

One should note that the $SU(3)$ invariant scalar submanifold of this flow is the same as the submanifold parametrized by the χ field of the non-supersymmetric $SU(3)$ flow in [1,2] and also the χ field of the LS-flow in section 3. However, the latter involves a different superpotential and the fields, χ and ρ, form a coupled system in which a truncation to the χ field alone is inconsistent. Indeed, if we set $\rho=1$ in (3.2), the only solution is $\chi=0$. Nevertheless, we may still use those results from the lift of the LS-flow that do not depend explicitly on the flow equations.

The potential (2.3) is, of course the same for the superpotentials (3.1), with $\rho=1$, and (5.1), with $m=0$, when we identify $\chi=\sigma$. There is a critical point of the potential at $\sigma\equiv\chi=\frac{1}{2}\log(2-\sqrt{3})$ [26], which corresponds to the compactification of the chiral IIB supergravity for which the internal manifold is a $U(1)$ bundle over $\mathbb{CP}_2$ [41].[8] The present flow turns out to be a simple deformation of that solution.

This is rather easy to see if we work with the metric (3.4). Consider the complex coordinates (3.6) and set

$$u^i \;=\; u^3\zeta^i\,, \quad i=1,2\,, \qquad u^3=(1+\zeta^1\bar\zeta_1+\zeta^2\bar\zeta_2)^{-1/2}e^{i\phi}\,, \tag{9.4}$$

[8] See, [49] for a recent discussion of $U(1)$ bundles over $\mathbb{CP}_n$'s.

where ζ^i, $i = 1, 2$ are the standard complex coordinates on $\mathbb{CP}_2$ and ϕ is the coordinate along the $U(1)$ fiber of the projection $S^5 \to \mathbb{CP}_2$. Convenient real coordinates are, see e.g. [50],

$$\begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = \tan\theta \, g(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{9.5}$$

where, as usual, α_i are the $SU(2)$ Euler angles. The ten-dimensional metric (6.7) can now be recast into the following form:

$$ds_{10}^2 = \cosh\sigma \left(e^{2A} dx_\mu dx^\mu - dr^2 \right) - L^2 \left(\cosh\sigma (d\phi - A_{\rm FS})^2 + \frac{1}{\cosh\sigma} ds_{\rm FS}^2 \right) , \tag{9.6}$$

where $ds^2_{\rm FS}$ is the Fubini-Study metric on $\mathbb{CP}_2$,

$$ds^2_{\rm FS} = d\theta^2 + \frac{1}{4} \sin^2\theta \left((\sigma_1)^2 + (\sigma_2)^2 + \cos^2\theta \, (\sigma_3)^2 \right) , \tag{9.7}$$

and

$$A_{\rm FS} = \frac{1}{2} \sin^2\theta \, \sigma_3 , \tag{9.8}$$

is the $U(1)$ potential. We choose the 10-beins e^M, $M = 1, \ldots, 10$, as follows

$$e^{\mu+1} \propto dx^\mu , \qquad e^5 \propto dr , \qquad e^6 \propto d\theta , \qquad e^{6+i} \propto \sigma_i , \qquad e^{10} \propto d\phi + \ldots . \tag{9.9}$$

Recall that for the compactification in [41], the antisymmetric tensor field is simply given by $G_{(3)} \propto du^1 \wedge du^2 \wedge du^3$ with the potential

$$A_{\rm R} = \frac{1}{12} e^{3i\phi} \sin\theta \left(2i \, d\theta \wedge (\sigma_1 + \sigma_2) + \frac{1}{2} \sin(2\theta) (\sigma_1 + i\sigma_2) \wedge \sigma_3 \right) . \tag{9.10}$$

It has also been argued in [41] (see, also [50]) that the $SU(3)$ symmetry essentially determines this potential up to an overall scale. Thus, rather than starting with the result of section 3, which would require passing to the other spherical coordinates, we simply consider the following Ansatz:

$$G_{(3)} = dA_{(2)} , \qquad A_{(2)} = f_{(3)} \, A_R . \tag{9.11}$$

Similarly, we take

$$F_{(5)} = \mathcal{F} + *\mathcal{F} , \qquad \mathcal{F} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge df_{(5)} . \tag{9.12}$$

Finally, the same calculation as in section 3 implies that the dilaton/axion field vanishes and its field equation is satisfied because of the chiral factor $\sigma_1 + i\sigma_2$ in $G_{(3)}$.

To determine the two unknown functions $f_{(3)}(r)$ and $f_{(5)}(r)$, we start with Einstein equations. The Ricci tensor is diagonal

$$\begin{aligned} R_{MN} \;=\; & f_1 \,\mathrm{diag}\,(\;\; 1, \;\; -1, \;\; -1, \;\; -1, \;\; -1, \;\;\;\; 1, \;\;\;\; 1, \;\;\;\; 1, \;\;\;\; 1, \;\;\;\; 1\,) \\ & +f_2 \,\mathrm{diag}\,(\;\; 0, \;\;\;\; 0, \;\;\;\; 0, \;\;\;\; 0, \;\; -1, \;\;\;\; 0, \;\;\;\; 0, \;\;\;\; 0, \;\;\;\; 0, \;\;\;\; 0\,)\,, \end{aligned} \tag{9.13}$$

where

$$f_1 \;=\; \frac{1}{2L^2}\cosh(\sigma)(7+\cosh(2\sigma))\,, \qquad f_2 \;=\; -\frac{18}{L^2}\sinh\sigma\,\tanh\sigma\,, \tag{9.14}$$

as are the energy momentum tensors,

$$\begin{aligned} T^{(3)}_{11} \;=\; -T^{(3)}_{22} \;=\; \ldots \;=\; -T^{(3)}_{44} \;=\; T^{(3)}_{66} \;=\; \ldots \;=\; T^{(3)}_{99} & \\ =\; \frac{\cosh\sigma}{18L^6}(9f_{(3)}^2 + L^2(f'_{(3)})^2)\,, & \end{aligned} \tag{9.15}$$

$$\begin{aligned} T^{(3)}_{55} \;&=\; \frac{\cosh\sigma}{6L^6}(-3f_{(3)}^2 + L^2(f'_{(3)})^2)\,, \\ T^{(3)}_{10\,10} \;&=\; \frac{\cosh\sigma}{18L^6}(27f_{(3)}^2 - L^2(f'_{(3)})^2)\,, \end{aligned} \tag{9.16}$$

and

$$T^{(5)}_{11} \;=\; -T^{(5)}_{22} \;=\; \ldots - T^{(5)}_{55} \;=\; T^{(5)}_{66} \;=\; \ldots \;=\; T^{(5)}_{10\,10} \;=\; \frac{4e^{-4A}\,(f'_{(5)})^2}{\cosh^5\sigma}\,. \tag{9.17}$$

where the $'$ denotes the derivative with respect to the flow coordinate r.

Clearly, we should have $T^{(3)}_{66} = \ldots = T^{(3)}_{10\,10}$, which yields

$$f'_{(3)} \;=\; \pm\frac{3}{L}\,f_{(3)}\,. \tag{9.18}$$

with the boundary conditions, $f_{(3)}(\infty) = f'_{(3)}(\infty) = 0$. The solution is $f_3(r) \;=\; C_3\,e^{-3r/L}$. Substituting this back into the Einstein equations we get, *a priori* three equations for $f'_{(5)}$ and the integration constant C_3, but it turns out that they are solved by

$$(f'_{(5)})^2 \;=\; \frac{\ell^8}{L^{10}}\,\frac{e^{12r/L}\left(2e^{6r/L}-3e^{2C}\right)^2}{4\left(e^{6r/L}-e^{2C}\right)^{8/3}}\,, \tag{9.19}$$

and $C_3 \;=\; 3\,L^2\,e^C$.

Next we use the Maxwell equations which determine the sign of $f'_{(5)}$. We also verify that the required Bianchi identities are satisfied. Finally, integrating (9.19) and re-expressing the result as a function of σ we obtain the following solution for the antisymmetric tensor fields:

$$f_{(3)} \;=\; 3L^2\tanh\sigma\,, \qquad f_{(5)} \;=\; \frac{\ell^4}{4L^4}\lambda^{4/3}\cosh^{2/3}\sigma\,\coth^{4/3}\sigma\,. \tag{9.20}$$

We conclude with some comments about the formal properties of this solution, in particular of the metric (9.6). We can recast it in the form

$$ds_{10}^2 \;=\; \frac{1}{\sqrt{F}}\,(dx_\mu dx^\mu) - \sqrt{F}\,ds_6^2\,, \tag{9.21}$$

where the function F is the analogue of the harmonic functions in the "brane-type" solutions, and consider the metric ds_6^2 on the six-dimensional manifold comprised of the flow coordinate, r, and the internal manifold. It is easy to check that

$$ds_6^2 \;=\; a(\rho)^2(dy)^2 + b(\rho)^2(yJdy)^2\,, \tag{9.22}$$

where y^i are unrestricted cartesian coordinates in R^6 and $\rho^2 = y\cdot y$ is the radial variable related to the original flow by

$$\rho \;=\; \rho_0\left(\coth\frac{\sigma}{2}\right)^{1/3}\,. \tag{9.23}$$

The relation to the previous coordinates on the sphere is $y^I = \rho x^I$. Setting $\rho_0 = \lambda = 1$, the functions $a(\rho)$ and $b(\rho)$ are given by

$$a(\rho) \;=\; \frac{\ell}{2^{1/3}}\left(1-\frac{1}{\rho^6}\right)^{1/3}\,, \qquad b(\rho) \;=\; 2^{2/3}\ell\,\frac{1}{(\rho^6-1)^{2/3}}\,, \tag{9.24}$$

and

$$F \;=\; 2^{4/3}\frac{L^4}{\ell^4}\,\frac{\rho^4(\rho^6-1)^{2/3}}{(\rho^6+1)^2}\,. \tag{9.25}$$

As expected, we find that F is not harmonic with respect to the six-dimensional metric (9.22) nor is the latter a flat metric. However, ds_6^2 turns out to be Ricci flat – a fact that certainly should have some significance.

10. Conclusions

We now have several non-trivial lifts of five-dimensional supergravity solutions to their ten-dimensional counterparts. As was evident in [8], and in the "super-QCD" flow presented here, it is essential to work with the ten-dimensional solutions if one is to understand properly the infra-red asymptotics of the supergravity descriptions of these flows. The five-dimensional solutions simply do not suffice.

An integral ingredient in understanding how to construct the lifts to ten dimensions is the relationship between the ten-dimensional dilaton and its five-dimensional counterpart.

As was remarked in [21], the expression (1.3), in principle, provides us with an analytic relation between the running gauge coupling, the $\mathcal{N} = 4$ coupling, the scale of the theory, and the running of the fermion and boson masses. In practice, the detailed interpretation of (1.3), and its connection with an NSVZ beta function, is more vexatious. The problem is the precise relationship of supergravity and field theory quantities, for example, the field theory scale and the supergravity radius, or the invariants of the Higgs vevs, and the angular behaviour of the supergravity solution. There is also the possibility of operator mixing, as we saw in the LS-flow. In addition to this, it should also be remembered that the supergravity solution is a strong coupling result, and so it may not actually be possible to track the details all the way to weak coupling results like the NSVZ beta-function. Thus the supergravity description exhibits all the proper structure, and general behaviour, but detailed connections with the weak coupling results remain elusive.

This raises the further question as to the extent that one should expect to be able to probe the infra-red limit using the supergravity solution. The answer to this question seems to depend upon the example. For the LS flow the solution runs all the way to the new critical point, and approaches a conformal theory. Thus the supergravity solution can "integrate out" the massive chiral multiplet and access the region of the field theory at mass scales far below the mass of the chiral multiplet. For "Flows to Hades" the supergravity approximation will break down near the singularity and so from the naive, five-dimensional perspective one would expect that the supergravity approximation will fail at some scale short of the infra-red. As was seen in [21], and in most of the solutions here, the ten-dimensional solution can resolve structure in the singularity and sometimes allow us to interpret the phase.

In this paper we saw how the ten-dimensional solution can also throw up a new infra-red obstacle: the oxidation of the $D3$-branes into 5-branes and 7-branes. We saw in section 7 that for the "super-QCD" flow in which all the chiral multiplets are given the same mass, m_0, the $D3$-brane throat generically "rounds out" into a 7-brane at a radial coordinate value of $r \sim m_0 L$. However, for special directions on the S^5, the flow approaches a singularity that may be interpreted as a ring distribution of 5-branes. This meshes well with expectations from field theory in that there is only a ground state in the infra-red if the vevs of the complex scalar fields, Φ_j, are real. If this condition is not met, then the flow runs into a "brick wall," and the scale in front of the $D3$-brane part of the 7-brane metric goes to a finite limit: The infra-red limit in which the chiral multiplets decouple is inaccessible. On the other hand, if the vevs of the the Φ_j are indeed real, then the flow runs to the ring of 5-branes, and as has been argued in [39,40], the field theory superpotential naturally leads to such dielectric 5-branes. Our results show some new elements of this 5-brane story: First, if the gaugino condensate is too small, we find the 5-branes smeared out into a ring. This is because of a restoration of a $U(1)$ duality symmetry in the infra-red, and the ring is a "duality" smeared family of 5-branes. If the gaugino condensate runs with its critical initial value (*i.e.* maximum possible physical value) the flow does not "round out" into the 7-brane solution, but limits to some form of $(1,1)$ and $(1,-1)$ "branes."

We are thus brought back to the issue of how to get the "correct" flow in that it properly describes the phases of $\mathcal{N}=1$ QCD. Perhaps the most compelling features of the the dielectric 5-brane story of [39,40] is that it very naturally distinguishes between electric and magnetic confinement in terms of the kinds of strings that can end upon different species of 5-brane. In the five-dimensional supergravity theory, this behaviour is still only visible as the result of some fine tuning. This was apparent in [13] where it was argued that Wilson loops exhibited confinement and 't Hooft loops exhibited screening. In the light of (5.5) we see that if there is indeed such behaviour for some Wilson loops, then Wilson loops that approach the core of the solution from duality flipped ($u \leftrightarrow v$) direction will exhibit the dual behaviour. As a result, a real physical Wilson loop will always be screened as it is lowered into the core of the solution: if it approaches from the "confining" direction, it will always be energetically favorable to change its orientation slightly, and thereby screen the quarks. In short, confining behaviour in the solution of [13] must be an artefact of fine tuning the direction of approach, much like the confining behaviour found in [46]. This very general argument, based on (5.5), shows that the "super-QCD" flow of [13] cannot result in purely an $NS5$ or $D5$ brane in the core. Indeed the best we could do is find a $(1,1)$ brane paired with a $(1,-1)$ brane.

From the detailed analysis of the vev of the gaugino condensate we have learnt that the structure of the infra-red limit is a discontinuous function of the initial conditions of vevs. In the ten-dimensional solution this means that the infra-red physics will depend upon precisely what, and how, normalizable modes are running. Physically, given an infra-red vacuum, one expects an exactly fixed relationships between the mass of the chiral multiplet and the vevs of various operators. For the flows considered here, the natural choice is to take $\lambda = 1$: Our computations suggest that such flows appear to run to a solution that makes sense for finite N. Therefore, in seeking out the IR limit of the field theory it is tempting to take the Holmesian approach of eliminating the impossible, and concluding that whatever remains, however improbable, must be the truth: Namely that the physical flow to the $\mathcal{N}=1$ theory in the far infra-red must be the one with $\lambda = 1$. Indeed, a similar conclusion was reached in [19], but for rather different reasons. What we are missing is the possibility of running other non-trivial vevs that, in ten-dimensions, correspond to higher supergravity modes. It is presumably the running of these normalizable modes that makes the difference between a ring of duality averaged 5-branes, a pair of $(1, \pm 1)$ branes, and a pure $D5$-brane, or pure $NS5$-brane.

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Appendix A. Explicit $G_{2(2)}$ Matrices

The $G_{2(2)}$ Matrix

The following matrices generate $G_{2(2)}$ in its seven-dimensional representation.

$$J_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The subgroups $SL(2,\mathbb{R})_{5d}$ and $SL(2,\mathbb{R})_X$ are generated by J_i and K_i respectively. The compact generators are J_1, K_1, X_3, X_4, X_7 and X_8, while $J_1, J_2, J_3,\ K_1$ generate the invariances of the potential. The full scalar manifold can be parametrized by matrices of the form:

$$\begin{aligned} M \ =\ & \exp(a_1 X_1 + a_2 X_2 + a_5 (X_5 - \tfrac{1}{2} X_1) + a_6 (X_6 - \tfrac{1}{2} X_2) \\ & + a_7 J_2 + a_8 J_3 - a_3 K_2 + a_4 K_3)\ . \end{aligned} \tag{A.1}$$

Now observe that:

$$\begin{aligned} \big[K_1,\, (X_5 - \tfrac{1}{2} X_1) \big] \ &=\ -\,(X_6 - \tfrac{1}{2} X_2)\,, \\ \big[K_1,\, (X_6 - \tfrac{1}{2} X_2) \big] \ &=\ (X_5 - \tfrac{1}{2} X_1)\,. \end{aligned}$$

We can therefore use the K_1 invariance to take $a_6 = 0$. We can then use the $SL(2,\mathbb{R})_{5d}$ to set $a_7 = -a_3$ and $a_8 = a_4$. Doing this we get a five-parameter family of matrices, with an unused J_1 invariance. Introduce the following change of basis matrix:

$$B \ =\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\ ,$$

then

$$B\ M\ B^{-1} \ =\ \begin{pmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \quad \text{where} \quad P \ =\ \begin{pmatrix} a_1 + \frac{1}{2} a_5 & a_2 & a_3 \\ a_2 & -a_1 + \frac{1}{2} a_5 & a_4 \\ a_3 & a_4 & -a_5 \end{pmatrix}\ .$$

This explicitly defines the embedding of the non-compact part of $SL(3,\mathbb{R})$ into $G_{2(2)}$.

Finally, let $H_1 = -(J_1 - K_1)$, $H_2 = (X_3 - X_7)$ and $H_3 = (X_4 + X_8)$. Then these matrices define the $SO(3)$ subgroup of the foregoing $SL(3,\mathbb{R})$ into $G_{2(2)}$, and indeed,

$$B\ \exp\left(\sum_{j=1}^{3}\ c_j\ H_j \right)\ B^{-1} \ =\ \begin{pmatrix} A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \quad \text{where} \quad A \ =\ \begin{pmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{pmatrix}\ .$$

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