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$N = 1$ SUPERSYMMETRY ALGEBRAS IN $d = 2, 3, 4 \bmod 8$

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ABSTRACT

We derive Poincaré, de Sitter and conformal supersymmetry algebras in all dimensions allowing Majorana spinors. We consider only minimal gradings (N=1), and show that these always exist. A brief discussion of fermionic central charges is given.

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1. - INTRODUCTION

In various discussions of auxiliary fields in 10 and 11 dimensions, it was noted that the structure of the theories considered seemed to reflect some underlying conformal supersymmetry[1]. However, the existence of superconformal symmetries in $d = 10, 11$ and other dimensions has been put into doubt. A closer examination reveals that these arguments for non-existence are implicitly based on on-shell properties of the theory. It is well known that the on-shell symmetries of relativistic field theories can only be generated by scalar operators with respect to the Lorentz group, with the exception of the generators of the Poincaré group itself or, for massless states, those of the conformal group and their spin 1/2 gradings. Generators of other on-shell symmetries in non-trivial representations of the Lorentz group are ruled out[2),3)]. This was in fact the starting point for the analysis of all possible supersymmetry algebras in four dimensions carried out in Ref. 3).

We will however disregard such on-shell theorems. Our ultimate interest is to gain information on the <u>off-shell</u> formulation of higher dimensional supergravity theories, and off-shell these theories may have many more unsuspected invariances. For these extra symmetries the non-existence of on-shell representations is irrelevant because they do not transform the physical states.

The usefulness of these extra symmetries has been stressed before[4)] and is well-illustrated by the construction of the highly non-linear $N = 1, 2$ supergravity theories and their matter couplings. In fact it was the conformal symmetry which clarified the structure of $N = 2$ supergravity to the extent that one could base a complete multiplet calculus on it. This led, for example, to the construction of an alternative off-shell formulation of the Poincaré theory.

In this paper we will show that without the on-shell restrictions it is possible to obtain $N = 1$ de Sitter and conformal superalgebras in all space-time dimensions which allow Majorana spinors. This last requirement is perhaps not necessary, but is useful in practice. Moreover, the interesting cases, when $d = 4, 10, 11,$ are included among them. The central element of our approach is the solution of Jacobi identities which, for graded algebras, read

$$[A,[B,C\}\} = [[A,B\},C\} - (-)^{bc}[[A,C\},B\}, \qquad (1.1)$$

with

$$(-)^{bc} = \begin{cases} -1 \ , \text{ if } B, C \text{ are fermionic,} \\ +1 \ , \text{ in all other cases.} \end{cases}$$

In order to do Dirac algebra manipulations in arbitrary numbers of dimensions, it was quite useful to develop a general procedure and notation for the reduction of products of Dirac-algebra elements. We use these notations, which are explained in the Appendix, throughout.

The main part of this paper is organized as follows. In Sections 2 and 3 we derive the de Sitter and conformal superalgebras respectively on the assumption that they have a minimal (N=1) grading. The techniques and procedures in both cases are quite similar, but we have tried to keep the sections as self-contained as possible so that they can be read independently. Section 4 contains a short discussion of fermionic central charges which can occur in Poincaré algebras and are claimed to be part of the underlying structure of $d = 11$ supergravity[5]. In Section 5 we collect our results and present our conclusions. It contains a review of the algebras we found. Finally, a connection between conformal algebras and de Sitter algebras in one more dimension is indicated.

2. - DE SITTER SUPERALGEBRAS

In this Section we will investigate the possibilities for a grading of the de Sitter algebra

$$\begin{aligned} [P_\mu, P_\nu] &= m M_{\mu\nu} , \\ [M_{\mu\nu}, P_\rho] &= 2 P_{[\mu} \delta_{\nu]\rho} , \\ [M_{\mu\nu}, M_{\rho\sigma}] &= 4 M_{[\nu}{}^{[\rho} \delta^{\sigma]}_{\mu]} . \end{aligned} \tag{2.1}$$

By a grading we mean that we allow an anticommuting spinor generator Q_a. Being a spinor, it has the following transformation rule under Lorentz rotations

$$[M_{\mu\nu}, Q] = -\tfrac{1}{2} \Gamma_{\mu\nu} Q . \tag{2.2}$$

We restrict ourselves to Majorana spinors, so (see Appendix) we can only treat the dimensions $d = 2, 3, 4 \bmod 8$. In this Section we allow only one spinor generator (a minimal grading). Generalizations will be discussed in Section 4.

To start our analysis we have to write down a general form for the $[P,Q]$ commutator. The only possibility is

$$[P_\mu , Q] = \frac{1}{2} x \Gamma_\mu Q , \tag{2.3}$$

where x is an arbitrary parameter. In this argument we used implicitly Lorentz covariance. This will always be done. As a result, Jacobi identities with an explicit $M_{\mu\nu}$ are automatically fulfilled. From (2.1) - (2.3) one could check for example the $[P,Q,M]$ Jacobi identity.

Having put forward the form of the $[P,Q]$ commutator, one can consider the $[P,P,Q]$ Jacobi identity. It results in

$$m = x^2 . \tag{2.4}$$

Using real generators and Γ matrices, x is real. Therefore we find only gradings of de Sitter algebras with a positive radius m^{-1}. In fact, (2.4) has two solutions for x. If we allow more Q's some could have a positive x and others a negative x. This will be discussed further in Section 4. Here we allow only one Q and therefore take one choice for x.

To investigate the consequences of the $[P,Q,Q]$ Jacobi identity, one needs an expression for the $\{Q_a,Q_b\}$ anticommutator. By definition, this must be symmetric in $a \leftrightarrow b$. In the Appendix it is indicated that for Majorana spinors the symmetric matrices are $(\Gamma^m C^{-1})_{ab}$ with $m = 1, 2 \bmod 4$. So a general form for $\{Q,Q\}$ is

$$\{Q_a , Q_b\} = \sum_k{}' \frac{1}{k!} (\Gamma^k C^{-1})_{ab} Z^k . \tag{2.5}$$

The sum over k runs over $1 \leq k \leq d$ with the restriction $k = 1, 2 \bmod 4$. The prime indicates that for odd dimensions the sum is only taken for $k \leq (d-1)/2$ (the independent ones). We can now use the $[P,Q,Q]$ Jacobi identity to find

$$[P, Z^i] = \sum_k x \left\{ {}^{1}{}_{k}{}^{i} \right\} Z^k . \tag{2.6}$$

In fact, the right-hand side contains only one term as $\{{}^1{}_k{}^i\}$ is only non-zero for $k = i \pm 1$. Since Z^k exist only for $k = 1, 2 \bmod 4$, only one of the two terms survives. Note however that no prime is written on the sum symbol in (2.6). Therefore in odd dimensions k can be higher than $(d-1)/2$. We define those Z^k by

$$\frac{1}{(d-k)!}\left(\Gamma^{d-k}C^{-1}\right)_{ab}Z^{d-k}=\frac{1}{k!}\left(\Gamma^{k}C^{-1}\right)_{ab}Z^{k},\ \text{for odd } d. \tag{2.7}$$

This relation implies

$$Z^{d-k}_{\mu_1\cdots\mu_{d-k}}=\frac{i}{k!}\,\epsilon_{\nu_k\cdots\nu_1\mu_1\cdots\mu_{d-k}}\,Z^{k}_{\nu_1\cdots\nu_k}\,. \tag{2.8}$$

So, for example in $d = 11$, from (2.6) and (2.8) we get

$$\left[P_\nu\,,\,Z^{5}_{\mu_1\cdots\mu_5}\right]=x\,Z^{6}_{\nu\mu_1\cdots\mu_5}=x\,\frac{i}{5!}\,\epsilon_{\rho_5\cdots\rho_1\nu\mu_1\cdots\mu_5}\,Z^{5}_{\rho_1\cdots\rho_5}\,. \tag{2.9}$$

In further Jacobi identities one needs an expression for $[Q,Z^i]$. We define arbitrary parameters y_i

$$[Q,Z^i]=(-)^i y_i\,\Gamma^i Q\,. \tag{2.10}$$

From the $[P,Q,Z]$ identity one finds

$$\text{if } x\neq 0\,,\ \text{then } y_{i+1}=y_i\,. \tag{2.11}$$

The $[Z,Q,Q]$ commutator provides

$$[Z^i,Z^j]=2y_i\sum_k\left\{{}^{i\,j}_{\ k}\right\}Z^k\,. \tag{2.12}$$

By definition, the left-hand side must be antisymmetric in $i \leftrightarrow j$. Using the methods described in the Appendix it is not difficult to prove the antisymmetry in $i \leftrightarrow j$ of $\{{}^{i\,j}_{\ k}\}$ if $i, j, k = 1, 2 \bmod 4$. Therefore we must have

$$y_i=y_j\,,\ \text{when } \sum_k\left\{{}^{i\,j}_{\ k}\right\}Z^k\neq 0\,. \tag{2.13}$$

If there is at least one non-zero Z^m, we can look to $i = 2$, $j = m$. Then the sum in (2.13) reduces to

$$\left\{{}^{2\,m}_{\ m}\right\}Z^m\neq 0\quad,\ \text{for } m\neq d\,. \tag{2.14}$$

Therefore

$$Z^m\neq 0\quad\Rightarrow\quad y_m=y_2\equiv y\,,\ \text{for } m\neq d\,. \tag{2.15}$$

The case $m = d$ can only occur in $d = 2 \bmod 8$, but then if $x \neq 0$ or $y_1 \neq 0$ we must still have $y_d = y_{d-1}$ [by (2.10) or (2.13) with $i = 1$]. For the other cases, (2.15) shows that each non-zero y_i must be equal to y_2. The [Q,Z,Z] Jacobi identity restricts the y_k still further. If $\{{}^{i}{}_{k}{}^{j}\} \neq 0$ then

$$\begin{aligned} &\text{if } y_k = 0 \quad : \quad y_i = 0 \quad \text{or} \quad y_j = 0\,; \\ &\text{if } y_k \neq 0 \quad : \quad y_i = y_j = y_k \text{ or } y_i = y_j = 0. \end{aligned} \tag{2.16}$$

The solutions of these equations are different in each dimension. However, two solutions are general:

a) all $k : y_k = y$,

b) $y_{4k+1} = 0$, $y_{4k+2} = y$ except y_d which is arbitrary.

In odd dimensions (2.8) has as a consequence

$$y_{d-k} = y_k \quad , \; d \text{ odd}. \tag{2.17}$$

Therefore solution b) cannot exist in odd dimensions. However, (2.16) does not exclude some other solution. For example, in $d = 10$ there are still the following possibilities:

$$\begin{aligned} &\text{c)} \quad y_1 = y_2 = y_9 = y_{10} = y \quad &&; \quad y_5 = y_6 = 0\,; \\ &\text{d)} \quad y_1 = y_2 = y &&; \quad y_5 = y_6 = y_9 = y_{10} = 0\,; \\ &\text{e)} \quad y_2 = y_9 = y &&; \quad y_1 = y_5 = y_6 = y_{10} = 0; \\ &\text{f)} \quad y_2 = y \,,\, y_{10} \text{ arbitrary} &&; \quad y_1 = y_5 = y_6 = y_9 = 0. \end{aligned} \tag{2.18}$$

In higher dimensions, more constraints are imposed on the y_k. In $d = 11$, the condition (2.17) allows only solutions a), c) and e). In $d = 12$, apart from a) and b), d) is still possible. f) can also occur, but (2.15) then tells us that $y_{10} = y$ or $y_{10} = 0$.

The last Jacobi identity is the $[Q_a, Q_b, Q_c]$ identity. After Fierzing it reduces to the equations

$$2^{\nu} y_l \Gamma^l = 2 \sum_k{}' y_k \frac{(-)^k}{k!} \Gamma^k \Gamma^l \Gamma^k \,, \tag{2.19}$$

where k and ℓ are again $k,\ell = 1, 2 \bmod 4$. The prime on the sum symbol means that for odd dimensions, one only sums over $k = 1$ to $(d-1)/2$. Equation (A.20) shows that this is just a factor of two for odd dimensions. Using (A.19), (2.19) becomes

$$X \equiv -2\sum_k\sum_i (-)^{kl+i}\binom{l}{i}\binom{d-l}{k-i} y_k \overset{?}{=} 2^{d/2} y_l \begin{cases} 1 & \text{, for } d=2,4 \bmod 8. \\ \sqrt{2} & \text{, for } d=3 \bmod 8. \end{cases} \tag{2.20}$$

The summations run over

$$\begin{aligned} &i = \max(k+l-d, 0) \to \min(k,l) \\ &k = 1 \to d \quad , \text{ restricted to } \; k=1,2 \bmod 4. \end{aligned} \tag{2.21}$$

However, one can first sum over $(k-i)$ and then over i. In this way one can use Eqs. (A.10) if we now put

$$\forall i: \quad y_{4i+1} = y_1 \quad , \quad y_{4i+2} = y_2 . \tag{2.22}$$

This will allow us to prove the existence of the solutions a) and b) mentioned above. One obtains

$$\begin{aligned} X &= 2^{d/2}\left[y_2 \sin\tfrac{d\pi}{4} - y_1 \cos\tfrac{d\pi}{4}\right] && \text{, for } l = 1 \bmod 4, \\ &= 2^{d/2}\left[y_1 \sin\tfrac{d\pi}{4} - y_2 \cos\tfrac{d\pi}{4}\right] && \text{, for } l = 2 \bmod 4 (\neq d), \\ &= y_1\left(2^{d-1} + 2^{d/2}\right) - y_2 2^{d-1} && \text{, for } l = d \end{aligned} \tag{2.23}$$

Solution a) corresponds to $y_1 = y_2 = y$. For $d = 2, 3, 4 \bmod 8$, (2.23) then indeed satisfies (2.20). In solution b), $y_1 = 0$, $y_2 = y$. In $d = 4 \bmod 8$, this still satisfies (2.20). In $d = 2 \bmod 8$, (2.23) would give

$$\begin{aligned} X &= 2^{d/2} y && \text{, for } l = 1 \bmod 4, \\ &= 0 && \text{, for } l = 2 \bmod 4 (\neq d) \\ &= -2^{d-1} y && \text{, for } l = d . \end{aligned} \tag{2.24}$$

However, for this case we could choose the value of y_d arbitrary. For $k = d$, i can only be $i = \ell$ in (2.20). Therefore we get

$$\begin{aligned} X &= 2^{d/2} y + 2(y_d - y) \overset{?}{=} 0 && \text{, for } l = 1 \bmod 4, \\ &= 0 - 2(y_d - y) \overset{?}{=} y 2^{d/2} && \text{, for } l = 2 \bmod 4 (\neq d), \\ &= -2^{d-1} y - 2(y_d - y) \overset{?}{=} y_d 2^{d/2} && \text{, for } l = d . \end{aligned} \tag{2.25}$$

So, solution b) still exists if we choose

$$y_d = (1 - 2^{\frac{d}{2}-1})y . \tag{2.26}$$

Solutions a) and b) thus exist for all dimensions as indicated. Actually, all other solutions seem to violate the [Q,Q,Q] Jacobi identity, as we will show in the different dimensions.

For solution a) we can identify Z^1_μ with P_μ.

$$Z^1_\mu = \frac{2y}{x} P_\mu , \tag{2.27}$$

y is then just a normalization factor. One finds

$$Z^2_{\mu\nu} = 2y M_{\mu\nu} . \tag{2.28}$$

Poincaré algebras are obtained as the limit of the de Sitter algebras where $x = y = m = 0$. The Z^i then commute with all other generators except for the Lorentz rotations. $Z^{(2)}$, $Z^{(5)}$, etc. can be introduced, but are not necessary as in the de Sitter algebra.

Remark that by (2.11) solution b) is only possible for $x = 0$. We now look in more detail at several dimensions. Although Eq. (2.19) is a relation between the y_k for every value of y_ℓ, $\ell = 1, 2 \bmod 4$, it turns out that they all imply only one or two independent equations. These equations can be derived by Tables 1 - 4 [calculated with the aid of Eq. (A.19)], which give the values of $c(k,m)$ defined by

$$\frac{(-)^k}{k!} \Gamma^k \Gamma^m \Gamma^k = c(k,m)\Gamma^m . \tag{2.29}$$

The resulting independent equations are given for $d = 2, 3, 4, 10, 11$ and 12 in Table 5. One can always check the existence of solution a). Solution b) can also be found for the even dimensions. Note that in $d = 2$, Eq. (2.26) implies $y_2 = 0$ $(=y_1)$. Finally one can show the non-existence of other solutions c) to f).

In ten dimensions (as in all $d = 2 \bmod 8$) one can introduce Majorana-Weyl spinors. The 32 component Q_a can be decomposed in two chirality eigenstates

$$Q^{\pm} = \frac{1 \pm \Gamma^*}{2} Q . \tag{2.30}$$

In terms of these 16 component chiral generators (2.3) and (2.5) are

$$[P_\mu, Q^+] = \tfrac{1}{2} x \Gamma_\mu Q^-,$$
$$[P_\mu, Q^-] = \tfrac{1}{2} x \Gamma_\mu Q^+,$$
$$\{Q^+_a, Q^+_b\} = (\Gamma^+_\mu C^{-1})_{ab} Z^{+1}_\mu + \tfrac{1}{5!}(\Gamma^5_{\mu_1\cdots\mu_5} C^{-1})_{ab} Z^{+5}_{\mu_1\cdots\mu_5}, \quad (2.31)$$
$$\{Q^-_a, Q^-_b\} = (\Gamma^-_\mu C^{-1})_{ab} Z^{-1}_\mu + \tfrac{1}{5!}(\Gamma^5_{\mu_1\cdots\mu_5} C^{-1})_{ab} Z^{-5}_{\mu_1\cdots\mu_5},$$
$$\{Q^+_a, Q^-_b\} = (\mathbf{1}^+ C^{-1})_{ab} Z^0 + \tfrac{1}{2}(\Gamma^+_{\mu\nu} C^{-1})_{ab} Z^2_{\mu\nu} + \tfrac{1}{4!}(\Gamma^+_{\mu_1\cdots\mu_4} C^{-1}) Z^4_{\mu_1\cdots\mu_4}$$

The new quantities are expressed in terms of the old ones as follows

$$\mathbf{1}^\pm = \frac{1 \pm \Gamma^*}{2}, \quad \Gamma^\pm_\mu = \mathbf{1}^\pm \Gamma_\mu, \ \ldots$$
$$Z^{\pm 1}_\mu = Z^1_\mu \pm \frac{i}{9!} \epsilon_{\mu\nu_1\ldots\nu_9} Z^9_{\nu_1\ldots\nu_9},$$
$$Z^{\pm 5} = \tfrac{1}{2}(Z^5 \pm \tilde{Z}^5) \ ; \ \tilde{Z}^5_{\mu_1\cdots\mu_5} = \frac{i}{5!} \epsilon_{\mu_1\cdots\mu_5\nu_1\cdots\nu_5} Z^5_{\nu_1\cdots\nu_5}, \quad (2.32)$$
$$Z^0 = \frac{-i}{10!} \epsilon^{\mu_1\cdots\mu_{10}} Z^{10}_{\mu_1\cdots\mu_{10}},$$
$$Z^4_{\mu_1\cdots\mu_4} = \frac{-i}{6!} \epsilon_{\mu_1\cdots\mu_4\nu_1\cdots\nu_6} Z^6_{\nu_1\cdots\nu_6}.$$

If we consider Weyl generators (only one chirality, say Q^+), then only Z^{+1}_μ and a self dual Z^{+5} appear. From (2.31), P_μ now commutes with Q^+. So effectively $x = 0$ and we can only have a Poincaré algebra. This could already be expected from the fact that no two index tensor can appear in the commutator of chiral spinor generators. So $M_{\mu\nu}$ cannot appear in $\{Q^+, Q^+\}$. Therefore Majorana-Weyl spinors only allow Poincaré algebras. Z^{+1}_μ can still be identified with P_μ. Z^{+5} commutes with everything (except for Lorentz transformations) and is some sort of a central charge. It can be omitted from the algebra. However, it could play a role in obtaining off-shell field representations. Its reduction to four dimensions gives, among others, six scalar central charges.

We conclude that de Sitter algebras with Majorana spinors exist in all dimensions considered. In even dimensions one also has a smaller algebra with only $Z^{(4k+2)}$ appearing in the $\{Q,Q\}$ commutator. However, this is not a de Sitter algebra, in the sense that P_μ is not present, but a graded Lorentz algebra. It is in fact the generating algebra for the de Sitter algebra in the lower dimension. This will be discussed in Section 5.

3. - THE d-DIMENSIONAL CONFORMAL SUPERALGEBRA

In this Section we will extend the $N = 1$ conformal superalgebra in four space-time dimensions to any number of dimensions in which Majorana spinors can be defined, i.e., $d = 2, 3, 4 \bmod 8$. This superconformal algebra is defined by the following requirements:

1) its bosonic sector includes the standard d dimensional conformal Lie algebra;
2) it contains a <u>minimal</u> number of spinorial generators.

The first condition leads us to introduce the bosonic generators D, P_μ, K_μ and $M_{\mu\nu}$ for dilatations, translations, conformal boosts and Lorentz-rotations respectively. These generators satisfy the commutation relations:

$$
\begin{aligned}
&[D, P_\mu] = -P_\mu \quad , \quad [D, K_\mu] = K_\mu \ , \\
&[M_{\mu\nu}, P_\lambda] = \delta_{\nu\lambda} P_\mu - \delta_{\mu\lambda} P_\nu \, , \ [M_{\mu\nu}, K_\lambda] = \delta_{\nu\lambda} K_\mu - \delta_{\mu\lambda} K_\nu \, , \\
&[P_\mu, K_\nu] = 2(\delta_{\mu\nu} D - M_{\mu\nu}) \, , \\
&[M_{\mu\nu}, M_{\kappa\lambda}] = \delta_{\mu\lambda} M_{\nu\kappa} + \delta_{\nu\kappa} M_{\mu\lambda} - \delta_{\mu\kappa} M_{\nu\lambda} - \delta_{\nu\lambda} M_{\mu\kappa} \, ,
\end{aligned}
\tag{3.1}
$$

all other ones being zero. Equations (3.1) are valid for <u>any</u> number of dimensions.

In order to obtain a minimal grading of this algebra, it is necessary to introduce <u>two</u> spinorial generators Q_a, S_a, which are eigenvectors of the dilatation operator. One finds for their Weyl weight $-1/2$, $+1/2$ respectively. Moreover, the Jacobi identities with one fermionic and two bosonic operators require P and K to rotate S to Q and vice versa. Since Q and S are by definition Majorana spinors under the Lorentz rotations, we obtain the following commutation relations up to normalizations:

$$
[M_{\mu\nu}, Q_a] = -\tfrac{1}{2}(\Gamma_{\mu\nu} Q)_a \quad , \ [M_{\mu\nu}, S_a] = -\tfrac{1}{2}(\Gamma_{\mu\nu} S)_a \, ,
\tag{3.2}
$$

$$[D, Q_a] = -\tfrac{1}{2} Q_a \quad , [D, S_a] = \tfrac{1}{2} S_a \; ,$$
$$[P_\mu, Q_a] = 0 \quad , [P_\mu, S_a] = -(\Gamma_\mu Q)_a \, , \qquad \begin{matrix}(3.2)\\ \text{cont.}\end{matrix}$$
$$[K_\mu, Q_a] = (\Gamma_\mu S)_a \quad , [K_\mu, S_a] = 0 \; .$$

As remarked above, this grading exists, when Q, S are Majorana spinors, in all $d = 2, 3, 4 \bmod 8$; in other numbers of dimension one can only use Dirac spinors. The above algebra is, however, not complete. In particular we do not know the anticommutators of two spinorial generators. We will now construct a closure of the algebra defined by Eqs. (3.1) and (3.2), assuming that no further spinorial generators are introduced, as we required in condition 2).

For this purpose we define bosonic operators $Z^{(k)}_{\mu_1,\ldots,\mu_k}$, $Z'^{(k)}_{\mu_1,\ldots,\mu_k}$, $A^{(k)}_{\mu_1,\ldots,\mu_k}$, antisymmetric in all Lorentz indices, by

$$\{Q_a, Q_b\} = \sum_k{}' \frac{1}{k!} \left(\Gamma^k_{\mu_1\cdots\mu_k} C^{-1}\right)_{ab} Z^{(k)}_{\mu_1\cdots\mu_k} \, ,$$
$$\{S_a, S_b\} = \sum_k{}' \frac{1}{k!} \left(\Gamma^k_{\mu_1\cdots\mu_k} C^{-1}\right)_{ab} Z'^{(k)}_{\mu_1\cdots\mu_k} \, , \qquad (3.3)$$
$$\{Q_a, S_b\} = \sum_k{}' \frac{1}{k!} \left(\Gamma^k_{\mu_1\cdots\mu_k} C^{-1}\right)_{ab} A^{(k)}_{\mu_1\cdots\mu_k} \, .$$

In the following we will often simplify our notation by not writing out explicitly all (summations over) the Lorentz indices $\mu_1,\ldots,\mu_k$. As indicated by the prime on the summation signs, the sums on the right-hand side of Eqs. (3.3) range over $0 \le k \le d$ whenever d is even, but only over $0 < k < (d-1)/2$ for d odd, exactly as in Eq. (2.5). Restricting ourselves to the Majorana dimensions $d = 2, 3, 4 \bmod 8$, the first two anticommutators are symmetric in the spinor indices (ab), hence only the $Z^{(m)}$, $Z'^{(m)}$ with $m = 1, 2 \bmod 4$ occur.

The Lorentz transformation properties of Z, Z', A are manifest. Those under dilatations follow immediately from the definitions (3.3):

$$[D, Z^{(k)}] = -Z^{(k)} \, ,$$
$$[D, Z'^{(k)}] = Z'^{(k)} \, , \qquad (3.4)$$
$$[D, A^{(k)}] = 0 \; .$$

In particular, the original bosonic operators can be included among the Z, Z' and A as follows:

- P_μ transforms as $Z^{(1)}_\mu$;
- K_μ transforms as $Z'^{(1)}_\mu$;
- D, $M_{\mu\nu}$ transform as $A^{(0)}$, $A^{(2)}_{\mu\nu}$ respectively.

It turns out, that in actual calculations it is inconvenient to have defined $Z^{(k)}$, $Z'^{(k)}$ and $A^{(k)}$ in $d = 3 \bmod 8$ only for $k \leq (d-1)/2$. In order to be able to extend summations over all values of k, we define

$$A^{(d-k)}_{\mu_1\cdots\mu_{d-k}} = \frac{i}{k!}\,\epsilon_{\mu_d\cdots\mu_{d-k+1}\mu_1\cdots\mu_{d-k}}\,A^{(k)}_{\mu_{d-k+1}\cdots\mu_d}\,, \tag{3.5}$$

etc., as in Eq. (2.8). As an illustration of the usefulness of this definition, we determine the commutator $[P,Z'^{(k)}]$ by solving the Jacobi identity for P and two spinorial generators S:

$$[P_\mu,\{S_a,S_b\}] = \{[P_\mu,S_a],S_b\} + \{[P_\mu,S_b],S_a\}\,. \tag{3.6}$$

Straightforward substitution leads to

$$\sum_m{}' \frac{1}{m!}\,(\Gamma^m C^{-1})_{ab}[P,Z'^{(m)}] = -2\sum_k{}'\sum_m \frac{1}{k!}\begin{Bmatrix} 1 & k \\ & m \end{Bmatrix}(\Gamma^m C^{-1})_{ab}A^{(k)}. \tag{3.7}$$

There is no prime on the sum over m on the right-hand side. But this sum, as well as the primed sum over m on the left-hand side, is restricted to $m = 1, 2 \bmod 4$, for reasons explained above. For $d = 3 \bmod 8$ we now split the sum over m on the right-hand side of (3.7) into a primed sum plus a rest; we find:

$$-2\sum_k{}'\frac{1}{k!}\left(\sum_m{}'\begin{Bmatrix} 1 & k \\ & m \end{Bmatrix}(\Gamma^m C^{-1})_{ab} + \delta_{k,\frac{d-1}{2}}\begin{Bmatrix} 1 & \frac{d-1}{2} \\ & \frac{d+1}{2} \end{Bmatrix}(\Gamma^{\frac{d+1}{2}}C^{-1})_{ab}\right)A^{(k)}.$$

To rewrite this expression we use Eq. (A.17):

$$\frac{1}{k!}\begin{Bmatrix} \overleftarrow{i} & k \\ & m \end{Bmatrix} = \frac{1}{m!}\begin{Bmatrix} i & m \\ & k \end{Bmatrix}, \tag{3.8}$$

and

$$\Gamma^{d-k}_{\mu_1\cdots\mu_{d-k}} = -\frac{i}{k!}\,\epsilon_{\mu_1\cdots\mu_d}\,\Gamma^{d}_{\mu_d\cdots\mu_{d-k+1}}\,,\quad d = 3 \bmod 8\,. \tag{3.9}$$

Putting all this together using the definition (3.5) one obtains:

$$\sum_m{}' \frac{1}{m!} (\Gamma^m C^{-1})_{ab} [P, Z'^{(m)}] = -2 \sum_m{}' \sum_k \frac{1}{m!} \left\{ {1\ m \atop k} \right\} (\Gamma^m C^{-1})_{ab} A^{(k)}, \quad \text{for all } d. \tag{3.10}$$

where the sum over k now runs over <u>all</u> values of k. From this we learn the expression for $[P, Z']$

Similarly one can work out the other Jacobi identities with P or K and two spinorial generators. This leads to the further relations:

$$\begin{aligned}
&[P, Z^{(k)}] = 0 && , [K, Z^{(k)}] = -2 \sum_j \left\{ {k\ 1 \atop j} \right\} A^{(j)}, \\
&[P, Z'^{(k)}] = -2 \sum_j \left\{ {1\ k \atop j} \right\} A^{(j)} && , [K, Z'^{(k)}] = 0, \\
&[P, A^{(k)}] = \sum_j \left\{ {k\ 1 \atop j} \right\} Z^{(j)} && , [K, A^{(k)}] = \sum_j \left\{ {1\ k \atop j} \right\} Z'^{(j)}
\end{aligned} \tag{3.11}$$

The next step is to compute the commutation relations between Z, Z' or A and the fermionic generators Q, S. Because we do not allow the introduction of new fermionic generators, and because of the Weyl weight of the various operators, they must take the form

$$\begin{aligned}
&[Z^{(m)}, Q] = 0, && [Z^{(m)}, S] = b_m \overleftarrow{\Gamma}^m Q, \\
&[Z'^{(m)}, Q] = b'_m \overleftarrow{\Gamma}^m S, && [Z'^{(m)}, S] = 0, \\
&[A^{(k)}, Q] = a_k \overleftarrow{\Gamma}^k Q, && [A^{(k)}, S] = a'_k \Gamma^k S.
\end{aligned} \tag{3.12}$$

To determine the coefficents a_k, a'_k and b_m, b'_m, one solves the Jacobi identities with one spinorial generator Q, S, one bosonic operator P or K and one of the new bosonic generators Z, Z' or A. This leads to the following relations:

$$\begin{aligned}
&a_k = (-)^{k+1} a'_k \quad , \forall k; \\
&b_m = b'_m = 2a_{m+1} = 2a_{m-1}, \quad \forall m = 1, 2 \bmod 4,
\end{aligned} \tag{3.13}$$

and hence

$$a_k = a_{k+2}, \quad \forall k = 0, 1 \bmod 4, \text{ except: } k = d, \text{ for } d = 4 \bmod 8.$$

It now remains to calculate the commutators of Z, Z' and A among themselves. These follow from the Jacobi identities with two spinorial generators and Z, Z' or A. Arguments analogous to those leading to (3.12) now establish that

$$[A^{(k)},Z^{(m)}]=b_m\sum_n\left\{{km \atop n}\right\}Z^{(n)} \quad , [A^{(k)},Z'^{(m)}]=-b_m\sum_n\left\{{mk \atop n}\right\}Z'^{(n)} ,$$

$$[A^{(k)},A^{(l)}]=a_k\sum_j\left(\left\{{kl \atop j}\right\}-\left\{{lk \atop j}\right\}\right)A^{(j)} \quad , [Z^{(m)},Z'^{(n)}]=2b_m\sum_j\left\{{mn \atop j}\right\}A^{(j)} , \tag{3.14}$$

$$[Z^{(m)},Z^{(n)}]=0 \quad , [Z'^{(m)},Z'^{(n)}]=0.$$

As in (3.11) the sums on the right-hand side run over all values of the index, even in $d = 3 \bmod 8$, provided we use the definition (3.5) whenever $k \geq (d+1)/2$. As a result of this definition one finds $a_{d-k} = a_k$. Besides (3.14) we find more relations to be satisfied by the coefficients a_k, b_m, if we analyze Jacobi identities with one fermionic generator.

$$\left\{{kl \atop j}\right\}+\left\{{\tilde{k}\tilde{l} \atop \tilde{j}}\right\}\neq 0 \quad \Rightarrow \begin{cases} \text{if } a_j=0, \text{ then: } & a_k=0 \text{ or } a_l=0; \\ \text{if } a_j\neq 0, \text{ then: } & a_k=a_l=0 \text{ or } a_k=a_l=a_j; \end{cases}$$

$$\left\{{mn \atop j}\right\}\neq 0 \quad {}_{m,\,n\,=\,1,\,2 \bmod 4} \Rightarrow \begin{cases} \text{if } a_j=0, \text{ then: } & b_m=0 \text{ or } b_n=0; \\ \text{if } a_j\neq 0, \text{ then: } & b_m=b_n=0 \text{ or } b_m=b_n=2a_j. \end{cases} \tag{3.15}$$

Again, in $d = 3 \bmod 8$ the only independent conditions are those for $k, \ell \leq (d-1)/2$. Equation (3.13) implied already that the independent a_k are those with $k = 1, 2 \bmod 4$, and a_d if $d = 4 \bmod 8$. If k, ℓ and j take those values appearing in the first equation of (3.15), then this is exactly the same as in the de Sitter case Eq. (2.16). Therefore the solutions are immediately restricted to those mentioned there. The extra conditions from (3.15) eliminate some possibilities. The general solutions a) and b) also exist in this case.

a) all $A^{(k)}$, $Z^{(m)}$, $Z'^{(m)}$ exist and $b_m = 2a_k \equiv 2a$ for all k and all $m = 1, 2 \bmod 4$; a is a free parameter.

b) All $A^{(k)}$, $Z^{(m)}$, $Z'^{(m)}$ with k odd and m even vanish, and accordingly $a_{2p+1} = b_{2n} = 0$. The non-zero coefficients satisfy $b_{2n+1} = 2a_{2p} \equiv 2a$, except a_d in $d = 4 \bmod 8$, which remains a free parameter. This solution does however not exist for $d = 3 \bmod 8$ because of the relation $a_k = a_{d-k}$, connecting odd and even a_k. Other solutions are:

c) $b_1 = 2a_2 = 2a_0$, but all other coefficients vanish.

In odd dimensions, this exists also with $b_{d-1} = 2a_d = 2a_{d-2} = b_1$. Finally, this last case, for odd dimensions (e.g., in $d = 11$: $b_1 = b_{10} \neq 0$) also exists in one dimension less (e.g., in $d = 10$)

d) in $d = 2 \bmod 8$: $b_1 = 2a_2 = 2a_0 = b_d = 2a_{d-2}$, other coefficients are zero.

Finally, there is one more Jacobi identity to be solved, namely one with three spinorial generators of mixed kind. These identities are all equivalent, while the ones with three identical fermionic generators are trivially satisfied due to the Weyl weight of the various operators. For any solution (a_k, b_m) the mixed Jacobi identity reduces to a type of Fierz relation to be checked:

$$b_m \Gamma^m = \sum_k{}' \frac{(-)^k}{k!\, 2^{\nu-1}}\, a_k \Gamma^k \Gamma^m \Gamma^k \,, \tag{3.16}$$

where $\nu = [d/2]$. We show that this identity holds for solutions a) and b) mentioned above. First we note that the prime on the sum in (3.16) can be removed if we multiply the right-hand side by a factor of 2 in odd dimensions. This follows from the relation (A.18). Using (A.17), we can reduce (3.16) to a check of the equation

$$X \equiv \sum_k \sum_i (-)^{i+km+\tilde{k}+1} \binom{m}{i}\binom{d-m}{k-i} a_k = \left(\tfrac{1}{2} b_m\right) 2^{d/2} \begin{cases} 1, & d = 2,4 \bmod 8; \\ \sqrt{2}, & d = 3 \bmod 8. \end{cases} \tag{3.17}$$

Here the sums run over:

$$k:\ 0 \to d\,,$$
$$i:\ \max(0, k+m-d) \to \min(k,m)\,.$$

Define $\ell = k - i$ and interchange the order of summation. Then the ranges of the sums become:

$$\ell:\ 0 \to d-m\,,$$
$$i:\ 0 \to m\,.$$

Clearly, the minus signs in (3.17) are the same for k and $(k+4)$, and consequently for ℓ and $(\ell+4)$. Therefore we can split the sum and use (A.10). The result depends on i, but again modulo 4. So we can use (A.10) once more for the sum over i. Putting

$$a_{2k+1} = a_1\,, \quad a_{2k} = a_0\,, \quad \forall k\,,$$

we can simplify the expression for X to:

$$X = 2^{d/2}\left[a_0 \cos\left(\frac{d-2m}{4}\pi\right) - (-)^m a_1 \sin\left(\frac{d-2m}{4}\pi\right)\right].$$

To prove the first solution, one puts $a_0 = a_1 = 1/2\, b_m$ and uses $m = 1, 2 \bmod 4$, $d = 2, 3, 4 \bmod 8$. Equation (3.17) then follows straightforwardly. The second solution is obtained by taking $a_1 = b_{2n} = 0$, $a_0 = 1/2\, b_{1+4n} = a$ and this yields (3.17) immediately for $d = 2 \bmod 8$. For $d = 4 \bmod 8$ there is the additional complication that a_d is a free parameter. In (3.17) $k = d$ restricts i to $i = m$; therefore one finds

$$X = -a_d + a \qquad , m = 1 \bmod 4,$$

$$X = a\, 2^{d/2} + (a_d - a) \qquad , m = 2 \bmod 4,$$

and (3.17) is satisfied if

$$a_d = \left(1 - 2^{d/2}\right) a . \tag{3.18}$$

For example, in $d = 4$ this gives $a_4 = -3a$. In N extended supersymmetry in $d = 4$ we get here $a_4 = (N-4)a$, which agrees with this result.

We have proved the existence of at least one solution for $d = 3 \bmod 8$, and two solutions for $d = 2, 4 \bmod 8$. In the second one, only half the number of bosonic generators Z, Z', A is present. For $d = 2 \bmod 8$ this second solution corresponds to taking Majorana Weyl spinors for Q, S. In $d = 4$ it is the usual $N = 1$ superconformal algebra.

To rule out solutions c) and d) we analyze Eq. (3.16) in detail for all possible numbers of dimensions. Using (3.13) and Tables 1 - 4 we establish the conditions resulting from (3.16) in $d = 4, 10, 11, 12$ respectively. They are collected in Table 6. It is evident that all solutions considered above satisfy these equations. Other solutions to Eq. (3.15) are ruled out.

4. - EXTRA SPINORGENERATORS

In Section 2 we derived the most general de Sitter or Poincaré algebra under the assumption that there exist only one spinorgenerator. Our motivation was that going to higher dimensions replaces the need to look for higher N algebras. However, recently it was shown that in 11 dimensions spinorial central charges can play a role[5]. These are extra generators which enter in $[Z,Q]$ (or $[P,Q]$) commutators, but commute themselves with all other generators. They are not equivalent to the first introduced Q generator and therefore the theory is not a higher N theory. We will now investigate the possibilities for extra fermionic generators.

The first place where we used the fact that only one spinorgenerator is present was in Eq. (2.3) where we wrote that $[P,Q]$ is proportional to Q. If we relax this condition, we should write

$$[P_\mu, Q] = \Gamma_\mu Q' ,$$
$$[P_\mu, Q'] = \Gamma_\mu Q'' . \tag{4.1}$$

Then the $[P,P,Q]$ Jacobi identity would imply

$$Q'' = \frac{x^2}{4} Q .$$

If $x \neq 0$, then we can define generators Q_1 and Q_2 such that Eq. (4.1) is diagonalized

$$Q_1 = Q' + \frac{x}{2} Q \quad , \quad Q_2 = Q' - \frac{x}{2} Q . \tag{4.2}$$

Then Eq. (3.1) is replaced by

$$[P_\mu, Q_1] = \frac{x}{2} \Gamma_\mu Q_1 ,$$
$$[P_\mu, Q_2] = -\frac{x}{2} \Gamma_\mu Q_2 . \tag{4.3}$$

So, for the de Sitter case $(x \neq 0)$, Eq. (2.3) is still the most general case if we allow no higher representation spinors. We can continue as in Section 2 with Q_1 only. The generator Q_2 provides a duplication of the algebra with the other choice of $x = \pm\sqrt{m}$. This has therefore to be considered in a treatment of higher N de Sitter algebras or better, (N,M) de Sitter algebras where N generators have $x = +\sqrt{m}$, and M have $x = -\sqrt{m}$.

For the Poincaré case $(x=0)$, (4.1) reduces to

$$[P_\mu, Q] = \Gamma_\mu Q' ,$$
$$[P_\mu, Q'] = 0 . \tag{4.4}$$

Now Q' satisfies the same commutation law with P_μ as Q does in the Poincaré case of Section 2. So, for a general treatment we could continue with Q' and a rule such as

$$\{Q'_a, Q'_b\} = \sum_k{}' \frac{1}{k!} (\Gamma^k C^{-1})_{ab} Z'^{(k)} . \tag{4.5}$$

However, we will only treat here the case when Q' are some sort of fermionic central charges (apart from their commutation rule with $M_{\mu\nu}$ which shows that they are spinors). This is the situation found in the geometric theory of $d = 11$ on-shell supergravity[5)]. We discuss the following case (with $P_\mu = Z^1_\mu$)

$$\{Q_a, Q_b\} = \sum_k{}' \frac{1}{k!} (\Gamma^k C^{-1})_{ab} Z^{(k)} ,$$
$$[Q , Z^{(i)}] = (-)^i z_i \Gamma^i Q' . \qquad (4.6)$$

(All other commutators are zero.)

In the conformal algebra, this insertion is not allowed without also introducing other bosonic operators in $\{Q',Q\}$. For example, the $[S,Z^i,Z^j]$ Jacobi identity cannot be satisfied with the insertion (4.6). For the Poincaré case we have only the $\{Q,Q,Q\}$ Jacobi identity which is non-trivial. This identity is exactly the same equation for z_i as we found in Section 2 for the y_i [Eqs. (2.19) and (2.20) or Table 5]. The difference with Section 2 is that while the y_i were still restricted by Eq. (2.16), the z_i have only to satisfy the relations from the [Q,Q,Q] Jacobi identity, which are tabulated in Table 5. We recall that there are certainly the solutions

a) $z_k = z$ for all k

b) $z_{4k+1} = 0$ $z_{4k+2} = z$ (in even dimensions), but $z_d = (1-2^{(d/2)-1})z$.

However, more general solutions are of course possible. In Ref. 5) two other solutions occur. The equation of Table 5 for $d = 11$ corresponds to their equation (6.4). (Note that they do not define the factor $1/k!$ in the $\{Q,Q\}$ anticommutator.)

5. - CONCLUSIONS

Using only Jacobi identities we showed the existence of de Sitter and Poincaré superalgebras in dimensions which allow Majorana spinors. The de Sitter algebra is

$$\{Q_a, Q_b\} = \sum_k{}' \frac{1}{k!} (\Gamma^k C^{-1})_{ab} Z^k ,$$
$$[Q , Z^{(k)}] = (-)^k y_k \Gamma^k Q . \qquad (5.1)$$
$$[Z^{(i)}, Z^{(j)}] = 2 y_i \sum_k \left\{ {i \atop k} {j \atop } \right\} Z^{(k)} .$$

We repeat that here $i, j, k = 1, 2 \bmod 4$. In odd dimensions the prime on the first summation indicates that $k \leq (d-1)/2$. This prime is not on the last summation symbol. In odd dimensions we define

$$Z^{(d-k)}_{\mu_1\cdots\mu_{d-k}} = \frac{i}{k!} \epsilon_{\nu_k\cdots\nu_1\mu_1\cdots\mu_{d-k}} Z^{(k)}_{\nu_1\cdots\nu_k} . \qquad (5.2)$$

The y_i must still satisfy equations which have only the following solutions:

a) for all k

$$y_k = y . \tag{5.3}$$

We can define

$$Z^{(1)}_{\mu} = 2\frac{y}{x} P_{\mu} , \qquad Z^{(2)}_{\mu\nu} = 2y M_{\mu\nu} ; \tag{5.4}$$

(y is then just a normalization factor).

The Poincaré theory can be obtained in the limit $x = y = 0$. For the Poincaré theory Z^2, Z^5, etc. can be introduced but are not necessary, while they are unavoidable in the de Sitter algebra. In 10 dimensions ($d = 2 \bmod 8$) introduction of Majorana-Weyl spinors is possible only for the Poincaré case. Apart from the Lorentz transformations the only non-zero commutator is

$$\{Q_a, Q_b\} = (\Gamma_\mu C^{-1})_{ab} P_\mu + \frac{1}{5!} (\Gamma^5_{\mu_1 \cdots \mu_5} C^{-1})_{ab} Z^+_{\mu_1 \cdots \mu_5} , \tag{5.5}$$

where Q is chiral and Z^+ is self-dual.

b) In even dimensions

$$\begin{aligned} &y_k = 0 , && \text{for } k \text{ odd;} \\ &y_k = y , && \text{for } k \text{ even,} \\ &\quad \text{except: } && y_d = (1 - 2^{\frac{d}{2}-1}) y , \; d = 2 \bmod 8 . \end{aligned} \tag{5.6}$$

In this case P_μ is not present in the algebra, hence we have in fact a graded Lorentz algebra.

The conformal algebra in d dimensions is

$$\begin{aligned} &\{Q_a, Q_b\} = \sum_m{}' \frac{1}{m!} (\Gamma^m C^{-1})_{ab} Z^{(m)} , \\ &\{S_a, S_b\} = \sum_m{}' \frac{1}{m!} (\Gamma^m C^{-1})_{ab} Z'^{(m)} , \\ &\{Q_a, S_b\} = \sum_k{}' \frac{1}{k!} (\Gamma^k C^{-1})_{ab} A^{(k)} , \\ &[A^{(k)}, Q_a] = a_k (\overleftarrow{\Gamma}^k Q)_a \quad , \quad [A^{(k)}, S_a] = (-)^{k+1} a_k (\Gamma^k S)_a , \end{aligned} \tag{5.7}$$

$$[Z^{(m)}, S] = 2a_{m-1}\overleftarrow{\Gamma}^{m}Q \quad , [Z'^{(m)}, Q] = 2a_{m-1}\overleftarrow{\Gamma}^{m}S ,$$

$$[A^{(k)}, Z^{(m)}] = 2a_k \sum_n \left\{ {km \atop n} \right\} Z^{(n)} \quad , [A^{(k)}, Z'^{(m)}] = -2a_k \sum_n \left\{ {mk \atop n} \right\} Z'^{(n)} , \tag{5.7 cont.}$$

$$[Z^{(m)}, Z'^{(n)}] = 4a_{m-1} \sum_k \left\{ {mn \atop k} \right\} A^k ,$$

all other commutators are zero.

In Eq. (5.7) k runs over $0 \to d$ (or $(d-1)/2$ for Σ' in odd dimensions), while m and n must be 1, 2 mod 4. In odd dimensions Eq. (5.2) is defined for A, Z and Z'. Again there are two possible solutions

a) for all k: $\quad a_k = a$;

b) (only for d even): $\quad a_k = 0, \; k$ odd;

$$a_k = a, \; k \text{ even}, \qquad \text{except: } a_d = (1-2^{d/2})a, \; d = 4 \bmod 8. \tag{5.8}$$

In solution b) only $Z^{(m)}$, $Z'^{(m)}$ with odd m and $A^{(k)}$ with even k exist. In all cases one can make the identification

$$Z^{(1)}_\mu = -2aP_\mu \quad , Z'^{(1)}_\mu = 2aK_\mu ,$$
$$A^{(0)} = -2aD \quad , A^{(2)}_{\mu\nu} = 2aM_{\mu\nu} , \tag{5.9}$$

and a is again a normalization factor. In $d = 4$ solution b) is the standard conformal algebra, which has P, K and M as the only non-scalar bosonic generators. $A^{(4)}$ is then the $U(1)$ generator. In 10 dimensions solution b) allows Majorana-Weyl generators Q and S; if Q is chiral, S is antichiral and vice versa.

We have indicated the numbers of generators of the various $d = 10$, 11 and 12 algebras in Table 7. Note that we always choose to include the smallest conformal algebra, e.g., the one with Majorana-Weyl spinors and duality conditions in $d = 10$. One then finds some interesting relations between the algebras. The superconformal algebra has the same number of generators as the super de Sitter algebra in one more dimension. Moreover, the generators of the de Sitter algebra in d dimensions can be decomposed with respect to the Lorentz subgroup of the d - 1 dimensional conformal group. This is always true for the ordinary Lie algebras. Denoting the d dimensional indices by $\hat{\mu},\ldots$ and the (d-1) dimensional by $\mu,\ldots$ one can write

$$\begin{aligned} P_{\hat{\mu}} &= \left(\tfrac{1}{2}(P_\mu - mK_\mu),\ xD\right), \\ M_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} M_{\mu\nu} & \frac{1}{2}\left(\frac{1}{x}P_\mu + xK_\mu\right) \\ -\frac{1}{2}\left(\frac{1}{x}P_\nu + xK_\nu\right) & 0 \end{pmatrix}, \end{aligned} \tag{5.10}$$

where again $x^2 = m$. Similarly, the ordinary de Sitter Lie-algebras can be obtained from the higher dimensional Lorentz algebra by writing

$$M_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} M_{\mu\nu} & \frac{i}{x}P_\mu \\ -\frac{i}{x}P_\nu & 0 \end{pmatrix}, \tag{5.11}$$

satisfying

$$[M_{\hat{\mu}\hat{\nu}}, M^{\hat{\rho}\hat{\sigma}}] = 4\, M_{[\hat{\nu}}{}^{[\hat{\rho}}\,\delta^{\hat{\sigma}]}_{\hat{\mu}]} \ . \tag{5.12}$$

If one wants to extend this procedure to the graded Lie algebras, one finds from Table 7 that the d = 12 Lorentz algebra must be split in two. Duality should relate $Z^{(10)}$ to $Z^{(2)}$ and reduce $Z^{(6)}$ to a self-dual tensor. The resulting d = 11 de Sitter superalgebra generates, as mentioned, the d = 10 conformal superalgebra. Besides this (unique) d = 11 de Sitter superalgebra there is also the graded Lorentz algebra in that dimension, provided one interprets the vectorial generator as the dual of a $Z^{(10)}$ operator. This means that it generates the d = 10 de Sitter superalgebra as well, which has exactly the same number of components as the conformal superalgebra in 10 dimensions. Thus there is a one-to-one correspondence between the generators of the de Sitter and conformal algebras in d = 10. Through this correspondence the chiral part of Q (de Sitter) becomes Q (conformal), while the antichiral part becomes S (conformal). Similar correspondences exist between the superalgebras in other dimensions, e.g., in d = 2, 3, 4. As a result, the classification of these supersymmetry algebras in terms of the standard Osp and SU graded Lie algebras is the same, as indicated in Table 7.

Finally we remark that we have investigated here only the usual kind of superalgebras which are finite and have structure constants as opposed to infinite dimensional ones with structure functions. That is, we have considered rigid supersymmetries, not local ones. Local algebras might still allow extra possibilities such as transformations involving derivatives (e.g., gauge transformations). This is known to happen, for example, in $N = 2$ supergravity in $d = 4$, where central charges are not allowed in the rigid superconformal algebra, but do occur in the local algebra[6].

APPENDIX

This appendix describes our conventions for the Dirac algebra in arbitrary dimensional spaces and a number of useful formulae and calculational procedures. We use the symbol $\Gamma^{(i)}$ to denote a completely antisymmetric product of i gamma matrices

$$\left(\Gamma^{i}_{\mu_1\cdots\mu_i}\right)_{ab} = \left(\Gamma_{[\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_i]}\right)_{ab} . \tag{A.1}$$

The antisymmetrization, denoted by the square brackets, is such that the over-all weight is 1. In (A.1) we write for the various indices:

a, b,... for spinor indices

μ, ν,... for Lorentz indices

i, j,... for the number of Lorentz indices.

As usual, spinor indices are often omitted. We also frequently omit Lorentz indices. Then i in $X^{(i)}$ indicates the index structure $[\mu_1\cdots\mu_i]$. Summation over repeated Lorentz indices is always understood, even if they are implicit. However, a repeated $i, j,\ldots$ does not imply summation over $i, j,\ldots$ if this is not explicitly mentioned. For example, in $X^{(i)}\,Y^{(i)}$ summation over $[\mu_1\cdots\mu_i]$ is to be understood, but only for the value of i indicated; there is thus no summation over all possible values of i. In odd dimensions summations are sometimes restricted to $0 \leq k < (d-1)/2$. This is indicated by a prime on the summation symbol. If the order of Lorentz indices on $X^{(i)}$ is reversed we denote this by the symbol $\overleftarrow{X}^{(i)}$. It follows immediately that

$$\begin{aligned} \overleftarrow{X}^{(i)} &= X^{(i)} , \quad \text{for } i = 0,1 \bmod 4, \\ \overleftarrow{X}^{(i)} &= -X^{(i)} , \quad \text{for } i = 2,3 \bmod 4. \end{aligned} \tag{A.2}$$

Spinors in d dimensions have 2^{ν} components, where

$$\nu = \mathrm{Int}\left[\frac{d}{2}\right] . \tag{A.3}$$

In Ref. 7) it was analyzed in which dimensions Majorana spinors exist. In these dimensions there must exist a charge conjugation matrix satisfying

$$C = -C^{T} , \quad \Gamma_{\mu}^{T} = -C\Gamma_{\mu}C^{-1} . \tag{A.4}$$

From this it follows that

$$\left(\Gamma^{i}C^{-1}\right)^{T} = (-)^{i+1}\,\overleftarrow{\Gamma}^{i}\,C^{-1} \equiv \Gamma^{\tilde{i}}\,C^{-1} \tag{A.5}$$

The notation $\tilde{i}$ is convenient in calculations and can be defined by

$$X^{(\tilde{i})} = (-)^{i+1} \overleftarrow{X}^{(i)} = \begin{cases} X^{(i)} & , \; i = 1,2 \bmod 4; \\ -X^{(i)} & , \; i = 0,3 \bmod 4. \end{cases} \tag{A.6}$$

Clearly, from (A.5), in Majorana dimensions $\Gamma^{(i)}C^{-1}$ is symmetric in its spinor indices whenever $i = 1, 2 \bmod 4$. Instead of repeating the complete proof given in Ref. 7), we will present here a counting argument. In d dimensions there are $\binom{d}{i}$ independent matrices $\Gamma^{(i)}$. Because

$$\sum_{i=0}^{d} \binom{d}{i} = 2^d , \tag{A.7}$$

we learn that in <u>even</u> dimensions the complete set of $\Gamma^{(i)}$ spans the $2^\nu \times 2^\nu$ dimensional algebra of matrices in spinor space. In odd dimensions it suffices to take $i = 0,\ldots,(d-1)/2$ because (A.7) is a factor 2 too high and because

$$\binom{d}{d-i} = \binom{d}{i} . \tag{A.8}$$

Alternatively, one may restrict oneself to the odd or even $\Gamma^{(i)}$, which are connected through a duality transformation. Their number is also 2^{d-1}, as follows from (A.8).

Of the $2^\nu \times 2^\nu$ independent matrices $\Gamma^{(i)}$ there are N_s symmetric and N_a antisymmetric ones with

$$\begin{aligned} N_s &= 2^{\nu-1}(2^\nu+1) , \\ N_a &= 2^{\nu-1}(2^\nu-1) . \end{aligned} \tag{A.9}$$

Now, for $d \geq 1$, the following summation formulae are to be noted[8)]

$$\begin{aligned}
&\text{a} \quad \binom{d}{0} + \binom{d}{4} + \ldots = \left(2^{d-2} + 2^{\frac{d}{2}-1} \cos \frac{d\pi}{4}\right) , \\
&\text{b} \quad \binom{d}{1} + \binom{d}{5} + \ldots = \left(2^{d-2} + 2^{\frac{d}{2}-1} \sin \frac{d\pi}{4}\right) , \\
&\text{c} \quad \binom{d}{2} + \binom{d}{6} + \ldots = \left(2^{d-2} - 2^{\frac{d}{2}-1} \cos \frac{d\pi}{4}\right) , \\
&\text{d} \quad \binom{d}{3} + \binom{d}{7} + \ldots = \left(2^{d-2} - 2^{\frac{d}{2}-1} \sin \frac{d\pi}{4}\right) .
\end{aligned} \tag{A.10}$$

We see that for even dimensions b and c sum up to N_s if and only if $d = 2, 4 \bmod 8$. For odd dimensions c gives N_s provided $d = 3, 5 \bmod 8$. But only for $d = 3 \bmod 8$ does (A.8) connect $\Gamma^{(1 \bmod 4)}$ with $\Gamma^{(2 \bmod 4)}$, hence only for $d = 2, 3, 4 \bmod 8$ can (A.5) be satisfied.

Similarly one can show that Majorana-Weyl spinors exist only in $d = 2 \bmod 8$. In these dimensions one can introduce

$$\Gamma^* \equiv \frac{i}{d!} \epsilon^{\mu_1 \cdots \mu_d} \Gamma^d_{\mu_1 \cdots \mu_d} , \tag{A.11}$$

satisfying

$$\Gamma^k_{\mu_k \cdots \mu_1} \Gamma^* = \frac{i}{(d-k)!} \epsilon_{\mu_1 \cdots \mu_d} \Gamma^{d-k}_{\mu_{k+1} \cdots \mu_d} , \tag{A.12}$$

$$\Gamma^{*2} = 1 \quad , \quad C \Gamma^* C^{-1} = - \Gamma^{*T} .$$

Then the projection operators $1/2(1 \pm \Gamma^*)$ commute with the Majorana condition $\psi = C\bar{\psi}^T$.

The completeness relation for the independent Γ matrices gives rise to the Fierz identity

$$M_{ab} = \frac{1}{2^\nu} \sum_k{}' \frac{1}{k!} \Gamma^k_{ab} \, \mathrm{Tr}\left(M \overleftarrow{\Gamma}^k\right) , \tag{A.13}$$

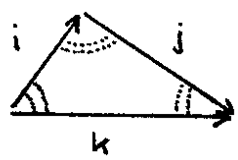
where the trace is over the spinor indices.

With the aid of this formula one can learn how to decompose products of $\Gamma^{(i)}$s into irreducible components. In particular one finds

$$\Gamma^i \Gamma^j = \sum_k \left\{ {i \, j \atop k} \right\} \Gamma^k , \tag{A.14}$$

where the Clebsch-Gordan coefficients, denoted by the curled brackets, are defined as follows

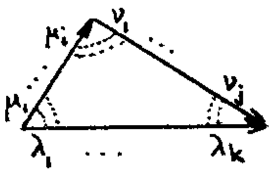
$$\left\{ {i \, j \atop k} \right\} = \frac{i! \; j!}{s! \, t! \, u!} \tag{A.15}$$



with

$$s = \tfrac{1}{2}(i+j-k)$$
$$t = \tfrac{1}{2}(i-j+k)$$
$$u = \tfrac{1}{2}(-i+j+k)$$

The triangular figure denotes a product of Kronecker δ symbols, antisymmetrized with weight one as in the corresponding Γ's. The precise rule is as follows: write the indices $\mu_1,\ldots,\mu_i$ on the i line in the direction of the arrow; similarly for (j) and (k). The connection lines then indicate Kronecker δ's with the indices it connects as arguments. If no lines cross, the over-all sign is positive; for each crossing of connection lines, a minus sign is to be included. Explicitly



$$= \delta^{[\mu_1}_{[\lambda_1}\cdots\delta^{\mu_t}_{\lambda_t}\,\delta^{\mu_{t+1}}_{[\nu_s}\cdots\delta^{\mu_i]}_{\nu_1}\,\delta^{\lambda_{t+1}}_{\nu_{s+1}}\cdots\delta^{\lambda_k]}_{\nu_j]}\,. \tag{A.16}$$

Apart from a few numerical factors, which we redefined, this notation was introduced in Ref. 9). The main advantage is that no indices have to be written out explicitly and tedious index manipulations can be avoided. One may turn and reflect diagrams at will, without introducing minus signs. Reversing the arrows in the Figure means reversing the order of all antisymmetrized sets of indices. This results in a sign change following the rule (A.2). As an example of diagram manipulations, we prove Eq. (3.8):

$$\frac{1}{k!}\left\{\begin{matrix}\overleftarrow{i} & k \\ & m\end{matrix}\right\} = \frac{1}{m!}\left\{\begin{matrix} i & m \\ & k\end{matrix}\right\} \tag{A.17}$$

Using (A.15) we can immediately check the numerical factors. Therefore we have to prove

$$= \tag{A.18}$$

The arrow over i in the left-hand side of (A.17) is taken into account by the reversal of the i arrow in the left-hand side of (A.18). Equation (A.18) is clearly valid because the figures are related by a reflection with respect to the bisectrix of the angle formed by the (k) and (m) lines.

Finally we present another useful formula for the reduction of products of Γ matrices

$$\overleftarrow{\Gamma}^k\,\Gamma^l\,\Gamma^k = k!\,(-)^{kl}\sum_i \binom{l}{i}\binom{d-l}{k-i}(-)^i\,\Gamma^l\,. \tag{A.19}$$

We remind the reader that the repeated k index on the left-hand side of (A.19) implies a sum over the Lorentz indices, but not over k itself. From (A.19) it follows by resummation $(i'=\ell-i)$ that

$$\frac{1}{k!}\overleftarrow{\Gamma}^{k}\,\Gamma^{\ell}\,\Gamma^{k} = \frac{1}{(d-k)!}\overleftarrow{\Gamma}^{d-k}\,\Gamma^{\ell}\,\Gamma^{d-k}\,(-)^{\ell(d+1)} \qquad \text{(A.20)}$$

k \ m	1	2
0	1	1
1	2	0
2	0	2
3	2	0
4	-1	1

Table 1: c(k,m) for d = 4

k \ m	1	2	5	6	9	10
0	1	1	1	1	1	1
1	8	-6	0	2	-8	10
2	-27	-13	5	3	-27	-45
3	-48	2	0	8	48	-120
4	42	-14	10	2	42	210
5	0	28	0	12	0	252
6	42	14	10	-2	42	-210
7	48	8	0	8	-48	-120
8	-27	13	5	-3	-27	45
9	-8	-6	0	2	8	10
10	1	-1	1	-1	1	-1

Table 2: c(k,m) for d = 10

k \ m	1	2	5
0	1	1	1
1	9	-7	1
2	-35	-19	5
3	-75	21	5
4	90	-6	10
5	42	42	10

Table 3: $c(k,m)$ for $d = 11$; $(k,m) \leq (d-1)/2$

k \ m	1	2	5	6	9	10
0	1	1	1	1	1	1
1	10	-8	2	0	-6	8
2	-44	-26	4	6	-12	-26
3	-110	40	10	0	2	-40
4	165	15	5	15	-27	15
5	132	48	20	0	36	-48
6	0	84	0	20	0	84
7	132	-48	20	0	36	48
8	-165	15	-5	15	27	15
9	-110	-40	10	0	2	40
10	44	-26	-4	6	12	-26
11	10	8	2	0	-6	-8
12	-1	1	-1	1	-1	1

Table 4: $c(k,m)$ for $d = 12$

d	relations
2	$y_1=y_2$
3	no relation
4	no relation
10	$-8y_1-27y_2+42y_6-8y_9+y_{10}=0$ $5y_2-16y_5+10y_6+y_{10}=0$
11	$y_1+5y_2-6y_5=0$
12	$y_1+2y_2-6y_5+5y_9-2y_{10}=0$ $y_2-2y_6+y_{10}=0$

Table 5: Independent relations contained in the $[Q,Q,Q]$ Jacobi identity for graded de Sitter algebras

d	relations
4	$-3a_0+4a_1-a_4=0$
10	$a_0-2a_4+a_8=0$ $2a_0+5a_1-6a_5-2a_8+a_9=0$
11	$a_0+a_1-2a_4=0$
12	$-25a_0-32a_1+99a_4-11a_8-32a_9+a_{12}=0$ $7a_0+35a_4-64a_5+21a_8+a_{12}=0$

Table 6: Independent relations contained in the $[Q,Q,S]$ Jacobi identity for the superconformal algebras

d	Lorentz algebra B F	de Sitter algebra B F	Conformal algebra B F
10	$Z^{(2)}$: 45 Q:32 $Z^{(6)}$:210 $Z^{(10)}$: 1 256 32 SU(16|1)	P : 10 Q:32 M : 45 $Z^{(5)}$:252 $Z^{(6)}$:210 $Z^{(9)}$: 10 $Z^{(10)}$: 1 528 32 Osp(32|1)	P,K : 20 Q:16 $Z^{(5)}, Z'^{(5)}$:252 S:16 D : 1 M : 45 $A^{(4)}$:210 528 32 Osp(32|1)
11		P : 11 Q:32 M : 55 $Z^{(5)}$:462 528 32 Osp(32|1)	P,K : 22 Q:32 $Z^{(2)} Z'^{(2)}$:110 S:32 $Z^{(5)}, Z'^{(5)}$:924 D : 1 $A^{(1)}$: 11 M : 55 $A^{(3)}$:165 $A^{(4)}$:330 $A^{(5)}$:462 2080 64 Osp(64|1)
12	M : 66 Q:64 $Z^{(6)}$:924 $Z^{(10)}$: 66 1056 64 Osp(32|1)+Osp(32|1)	P : 12 Q:64 M : 66 $Z^{(5)}$:792 $Z^{(6)}$:924 $Z^{(9)}$:220 $Z^{(10)}$: 66 2080 64 Osp(64|1)	P,K : 24 Q:64 $Z^{(5)} Z'^{(5)}$:1584 S:64 $Z^{(9)} Z'^{(9)}$: 440 D, $A^{(12)}$: 2 M, $A^{(10)}$: 132 $A^{(4)}, A^{(8)}$: 990 $A^{(6)}$: 924 4096 128 SU(64|1)

Table 7: Number of bosonic (B) and fermionic (F) generators of the N = 1 supersymmetry algebras in d = 10, 11, 12 and their classification in terms of standard graded Lie algebras.

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