Ricerche di Matematica
https://doi.org/10.1007/s11587-022-00719-z

n^2 of dissipative couplings are sufficient to guarantee the exponential decay in elasticity

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Received: 5 April 2022 / Revised: 20 May 2022 / Accepted: 13 June 2022

Abstract
In this paper, we prove that the solutions to the problem determined by an elastic material with n^2 coupling dissipative mechanisms decay in an exponential way for every (bounded) geometry of the body, where n is the dimension of the domain, and whenever the coupling coefficients satisfy a suitable condition. We also give several examples where the solutions do not decay when the rank of the matrix of the coupling mechanisms is less than n^2 (2 in dimension 2 and 6 in dimension 3).

Keywords Thermoelasticity · Dissipation mechanism · Energy decay

Mathematics Subject Classification 74F05 · 74H20 · 74H40 · 35B40 · 35L35

1 Introduction

The decay of system solutions of thermoelasticity with respect to time has received much attention over the years. This system proposes the coupling of a conservative mechanism (elasticity) with a dissipative mechanism (temperature) and some questions as the dissipative mechanism controls the conservative one causing the decay of solutions. We may recall that Dafermos [2] proved the decay of solutions in the one-dimensional case, as well as the impossibility of decay for certain two-dimensional

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Published online: 06 July 2022

geometries, providing examples of undamped and isothermal mechanical oscillations. Considerable progress has been made in this direction in recent years. Currently, it is known the exponential decay of the solutions in the one-dimensional case and that, for dimension greater than one, we can obtain temporary decay for many geometries (see [5, 6, 9, 12]). In fact, many results for alternative theories of heat have also been obtained [10, 11].

Since we cannot expect the exponential decay of the solutions for the classical theories of thermoelasticity, we can ask ourselves about the adequate number of coupled dissipation mechanisms that we must impose to guarantee the exponential decay of the solutions. To our knowledge, this question has not still been considered in the scientific literature and therefore, we do not know the answer to this question. In this work, we will show that, if we introduce a number of dissipation mechanisms equal to the square of the dimension of the solid geometry, and the rank of the coupling matrix is maximum, then we can guarantee the exponential decay of the solutions. This imposition will imply the anisotropy of the coupling coefficients since, in the isotropic case, the condition of the range of the coupling coefficients will not be satisfied. Extensions of this result can be obtained immediately for other dimensions, but we prefer to work with the simplest case to highlight the method. It is worth noting that our method follows the usual arguments for proving the exponential decay in the one-dimensional case. We will also provide examples of problems with undamped solutions in the case where the rank of the coupling matrix is less than n^2. To be precise, we will give examples in the case that the matrix of the coupling coefficients has rank 2 (in dimension 2) or rank 6 (in dimension 3). In conclusion, it will be enough to impose dissipative mechanisms by determining a matrix of rank n^2 to guarantee the exponential decay for any domain, but the question remains whether this same conclusion can be applied to the rank 3 (of the coupling matrix) in dimension 2: or for range 7 or 8 in dimension 3.

In the next section, we propose a two-dimensional problem where four coupled dissipative mechanisms are imposed. The existence and uniqueness of solution is also proved. Then, in the third section, we assume that the rank of the matrix of coupling coefficients is maximum and we will obtain that the decay is exponential for every geometry. Finally, we consider a two-dimensional case where the rank of the coupling coefficient matrix is less than four, and we prove the existence of undamped mechanical solutions. The generalization for dimension three is also pointed out.

2 Preliminaries

In this section, we define the thermomechanical problem and we recall some basic results.

Let us consider a two-dimensional bounded domain B with a sufficiently smooth boundary which is made of an elastic material coupled with four dissipative mechanisms. From a physical point of view, we can think that the dissipative mechanisms are determined by two temperatures and two mass diffusion processes. In order to propose a simpler system, we assume that the material is homogeneous and isotropic in the mechanical and thermal parts, but we do not impose this requirement on the coupling coefficients. Our system is written as follows (see [3]):

$$\begin{aligned} \rho \ddot{u}_i &= \mu \Delta u_i + (\lambda + \mu) u_{j,ji} + A^l_{ik} \theta_{l,k}, \\ m_{lp} \dot{\theta}_p &= k_{lp} \Delta \theta_p + A^l_{ik} \dot{u}_{i,k} + \xi^l_{pq} (\theta_p - \theta_q), \end{aligned} \tag{1}$$

where $i, j, k = 1, 2$ and $l, q, p = 1, \ldots, 4$.

Here, u_i denotes the displacement, θ_1, θ_2 are the two temperatures, θ_3, θ_4 are the other two dissipative variables, ρ is the mass density, λ, μ are the Lamé constants, k_{ij} are related with the thermal and diffusive conductivity, m_{ij} are related with the thermal and diffusive capacity, A^l_{ij} are the coupling terms for which we do not impose any condition yet, and ξ^l_{kj} corresponds to the coefficients associated to the relative temperatures (or concentrations). As we commented it before, we consider the case of two temperatures and two concentrations. Therefore, it is natural to assume that $\xi^2_{12} = -\xi^1_{12} = l_1$, $\xi^2_{34} = -\xi^1_{34} = l_2$, $\xi^4_{12} = -\xi^3_{12} = l_2$, $\xi^4_{34} = -\xi^3_{34} = l_3$ and the other combinations of $\xi^q_{lp} = 0$. In this work, we also follow this choice.

From now on, we will assume that

(i) $\rho > 0$, $\mu > 0$, $\lambda + \mu > 0$.

(ii) The matrices (m_{ij}), (k_{ij}) are symmetric[1], that is, $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$ for $i, j = 1, \ldots, 4$.

(iii) The matrices (m_{ij}), (k_{ij}) and (l_i) are positive definite, that is, there exist three positive constants C_1, C_2 and C_3 such that

$$\begin{aligned} &m_{ij}\xi_i\xi_j \geq C_1\xi_i\xi_i \quad \text{for} \quad i, j = 1, \ldots 4, \\ &k_{ij}\xi_i\xi_j \geq C_2\xi_i\xi_i \quad \text{for} \quad i, j = 1, \ldots 4, \\ &l_1\xi_1^2 + 2l_2\xi_1\xi_2 + l_3\xi_2^2 \geq C_3(\xi_1^2 + \xi_2^2). \end{aligned}$$

Remark 1 The interpretation of condition (i) is clear. We mean that the natural assumption is that the mass density is positive. The conditions on the Lamé constants are proposed to guarantee that the elasticities are positive definite (see [4, page 19]) because we are working in dimension two. The assumption on the term (m_{ij}) is natural if we take into account the study proposed in [5]. As we have pointed out in a previous footnote, the symmetry of the tensor k_{ij} is proposed to simplify the analysis, but it is not needed “a priori”. Moreover, it is natural to assume that m_{ij} is positive definite to guarantee that the energy defined by the thermal and concentrate components is positive. The conditions on (k_{ij}) and (l_i) are related with the well-known property of a heat (mass diffusion) conductor. These conditions on the Lamé constants do not come from the axioms of thermomechanics but it is usual to consider them when we deal with the elastic stability.

To define a well posed problem, we need to prescribe the initial conditions, for a.e. $\boldsymbol{x} \in B$,

$$u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}), \quad \theta_j(\boldsymbol{x}, 0) = \theta_j^0(\boldsymbol{x}), \tag{2}$$

[1] We assume that the matrix k_{ij} is symmetric to simplify the problem, but we must recall that this is not a consequence of the basic axioms of thermomechanics.

where $i = 1, 2$ and $j = 1, \ldots, 4$, and the boundary conditions, for a.e. $\boldsymbol{x} \in \partial B$ and $t > 0$,

$$u_i(\boldsymbol{x}, t) = 0, \quad \theta_{k,j}(\boldsymbol{x}, t) n_j(\boldsymbol{x}) = 0, \quad i, j = 1, 2, \; k = 1, \ldots, 4, \tag{3}$$

where n_j $(j = 1, 2)$ is the normal vector to the boundary of the domain B.

We note that the solutions to problem (1)-(3) satisfy the equality:

$$E(t) + 2\int_0^t D(s)\, ds = E(0),$$

where

$$E(t) = \int_B \left(\rho v_i v_i + 2\mu e_{ij} e_{ij} + \lambda e_{ii} e_{jj} + m_{lk}\theta_l\theta_k\right) da,$$

and

$$D(t) = \int_B \left(k_{lk}\nabla\theta_l\nabla\theta_k + l_1(\theta_1 - \theta_2)^2 + 2l_2(\theta_1 - \theta_2)(\theta_3 - \theta_4) + l_3(\theta_3 - \theta_4)^2\right) da,$$

where $i, j = 1, 2, l, k = 1, \ldots, 4$, and e_{ij} is the strain tensor given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{4}$$

In view of the assumptions (i)-(iii) we can guarantee the stability of solutions. We will see that, under suitable conditions, we will obtain the exponential decay.

The existence and uniqueness of the solutions to problem (1)-(3) can be obtained by means of the semigroup of linear operators theory. So, we define the Hilbert space

$$\mathcal{H} = [W_0^{1,2}(B)]^2 \times [L^2(B)]^2 \times \left[L_*^2(B)\right]^4,$$

where $W_0^{1,2}(B)$ and $L^2(B)$ are the usual Sobolev spaces and

$$L_*^2(B) = \{f \in L^2(B), \int_B f\, da = 0\}, \quad W_*^{1,2}(B) = W^{1,2}(B) \cap L_*^2(B).$$

We can consider the elements $U = (u_i, v_i, \theta_l)$ and $U^* = (u_i^*, v_i^*, \theta_l^*)$ in this space and, inspired in the energy equation, we define the inner product:

$$< U, U^* >= \int_B \left(\rho v_i \bar{v}_i^* + m_{kl}\theta_k\bar{\theta}_l^* + 2\mu e_{ij}\bar{e}_{ij}^* + \lambda e_{ii}\bar{e}_{jj}^*\right) da,$$

where a superposed bar means the conjugated complex and we continue using the expression of e_{ij} as in (4).

In view of the previous assumptions and taking into account the comments of the book by Marsden and Hughes [8, p. 345], we can conclude that this is an inner product with a norm which is equivalent to the usual one in the Hilbert space $\mathcal{H}$.

Now, our problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = (\boldsymbol{u}^0, \boldsymbol{v}^0, \boldsymbol{\theta}^0), \tag{5}$$

where we have the operator

$$\mathcal{A}\begin{pmatrix} u_i \\ v_i \\ \theta_d \end{pmatrix} = \begin{pmatrix} v_i \\ \rho^{-1}(\mu\Delta u_i + (\lambda+\mu)u_{j,ji} + A_{ik}^p\theta_{p,k}) \\ n_{dl}(k_{lp}\Delta\theta_p + A_{ik}^l v_{i,k} + \xi_{pq}^l(\theta_p - \theta_q)) \end{pmatrix}.$$

Here, $n_{dl}m_{lp} = \delta_{dp}$, with δ_{dp} being the delta of Kronecker.

We note that the domain of the operator $\mathcal{A}$, denoted by dom($\mathcal{A}$), is given by the elements of the Hilbert space $\mathcal{H}$ such that $v_i \in W_0^{1,2}(B)$, $u_i \in W^{2,2}(B)$ and $\theta_i \in W^{2,2}(B) \cap L_*^2(B)$ and so, it is a dense subspace of the Hilbert space.

We now give a couple of lemmata which will be used later.

Lemma 1 *For every $U \in dom(\mathcal{A})$, we have that*

$$Re < \mathcal{A}U, U > \leq 0.$$

Proof In view of the field equations, the boundary conditions and the use of the divergence theorem, we obtain

$$\begin{aligned}\text{Re} < \mathcal{A}U, U > = -\int_B \Big(& k_{ij}\theta_{i,m}\overline{\theta_{j,m}} + l_1(\theta_1-\theta_2)\overline{(\theta_1-\theta_2)} + l_3(\theta_3-\theta_4)\overline{(\theta_3-\theta_4)} \\ & + l_2[(\theta_1-\theta_2)\overline{(\theta_3-\theta_4)} + (\theta_3-\theta_4)\overline{(\theta_1-\theta_2)}]\Big)da.\end{aligned}$$

Keeping in mind the assumptions on the coefficients k_{ij} and l_i we see that the lemma is proved. □

Lemma 2 *Zero belongs to the resolvent of the operator $\mathcal{A}$.*

Proof Let us consider $F = (f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4)$ an element of the Hilbert space $\mathcal{H}$. We should prove the existence of an element $U \in$ dom($\mathcal{A}$) such that

$$\mathcal{A}U = F. \tag{6}$$

We can write:

$$\begin{aligned}& v_i = f_i, \\ & \mu\Delta u_i + (\lambda+\mu)u_{j,ji} + A_{ik}^l\theta_{l,k} = \rho f_{2+i}, \\ & k_{lp}\Delta\theta_p + A_{ik}^l v_{i,k} + \xi_{pq}^l(\theta_p - \theta_q) = m_{lp}g_p.\end{aligned}$$

Here, $i, j, k = 1, 2$ and $p, q, l = 1, \dots, 4$.

The first two equations imply that

$$k_{lp}\Delta\theta_p + \xi^l_{pq}(\theta_p - \theta_q) = m_{lp}g_p - A^l_{ik}f_{i,k}. \tag{7}$$

If we denote $G_l = m_{lp}g_p - A^l_{ik}f_{i,k}$ we see that $G : [W^{1,2}_*(B)]^4 \to \mathbb{C}$ defined by $G(\theta_l) =< G_l, \theta_l >_{L^2(B)}$ is a linear bounded operator.

At the same time, the form

$$B(\theta_p, \vartheta_k) = \int_B k_{lp}\nabla\theta_p\nabla\bar{\vartheta}_l\, da - \int_B \xi^l_{pq}(\theta_p - \theta_q)\bar{\vartheta}_l\, da$$

defines a coercive and bounded bilinear form in $[W^{1,2}_*(B)]^4$. Thus, Lax-Milgram lemma implies that system (7) admits a solution belonging to $[W^{2,2}(B) \cap W^{1,2}_*(B)]^4$. If we now substitute θ_i in the remaining equations, it follows that

$$\mu\Delta u_i + (\lambda + \mu)uj, ji = \rho f_{2+i} - A^l_{ik}\theta_{l,k}.$$

Again, we can check that we can apply the Lax-Milgram lemma to obtain a solution (u_1, u_2). Furthermore, we can see that

$$\|U\| \leq K\|F\|,$$

where K is a constant independent of F and so, the lemma is proved. □

In view of the previous lemmata, we can apply the Lumer-Phillips corollary to the Hille-Yosida theorem (see [7, p.3]) to obtain the following result.

Theorem 1 *The operator $\mathcal{A}$ generates a C^0-semigroup of contractions.*

3 Exponential decay

In this section, we show the exponential decay of the solutions generated by the semigroup obtained in the previous section. However, in order to prove this result we need to harden the assumptions on the coupling coefficients. In fact, in this section we assume that the matrix:

$$\begin{pmatrix} A^1_{11} & A^1_{12} & A^2_{21} & A^2_{22} \\ A^2_{11} & A^2_{12} & A^2_{21} & A^2_{22} \\ A^3_{11} & A^3_{12} & A^3_{21} & A^3_{22} \\ A^4_{11} & A^4_{12} & A^4_{21} & A^4_{22} \end{pmatrix}$$

has rank 4.

To prove the exponential decay result, we recall that it is enough to show that the imaginary axis is contained in the resolvent of the operator and that the asymptotic condition

$$\limsup_{|\beta|\to\infty} ||(i\beta I - \mathcal{A})^{-1}|| < \infty$$

holds (see [7, p. 4]).

Theorem 2 *The solutions to problem (5) decay in an exponential way. That is, there exist two positive constants M and c such that for every $U(0)$ in the domain of the operator $\mathcal{A}$:*

$$||U(t)|| \le M||U(0)|| \exp(-ct) \quad \textit{for } t \ge 0. \tag{8}$$

Proof We can prove the conditions stated above following a similar argument. It can be obtained by assuming that it does not hold and arriving to a contradiction. Let us assume that there exists an element of the imaginary axis contained in the spectrum. By using a standard argument (see [7, p.25]), there will exist a sequence of real numbers $\omega_n \to \omega \neq 0$ and a sequence of unit norm vectors U_n in the domain of the operator such that

$$||(i\omega_n I - \mathcal{A})U_n|| \to 0. \tag{9}$$

This is equivalent to assume that

$$\begin{aligned}
&i\omega_n u_{jn} - v_{jn} \to 0 \text{ in } W^{1,2}(B),\\
&i\omega_n \rho v_{jn} - (\mu\Delta u_{jn} + (\lambda+\mu)\tfrac{\partial^2 u_{kn}}{\partial x_k \partial x_j} + A^l_{jk}\theta_{ln,k}) \to 0 \text{ in } L^2(B),\\
&i\omega_n m_{pq}\theta_{qn} - (k_{pq}\Delta\theta_{qn} + \xi^p_{rs}(\theta_r - \theta_s) + A^p_{ml} v_{mn,l}) \to 0 \text{ in } L^2(B).
\end{aligned}$$

From the dissipation inequality, we can obtain that $\nabla\theta_{in} \to 0$ as n tends to infinite for $i = 1, \ldots, 4$. Therefore, if we consider the last four convergences and we divide by ω_n, we obtain that

$$\omega_n^{-1} k_{jl}\Delta\theta_{ln} + iA^j_{pq}u_{pn,q} \to 0 \quad \text{in} \quad L^2(B).$$

We now multiply by $A^j_{pq}u_{pn,q}$ the jth-convergence, and we take into account that

$$< \omega_n^{-1}\Delta\theta_{in}, A^j_{pq}u_{pn,q} >= - < \nabla\theta_{in}, \omega_n^{-1}\nabla(A^j_{pq}u_{pn,q}) > .$$

We note that, in view of the assumptions on the Lamé constants, the operator defining the elastic part is positive definite. Then, we have that $\omega_n^{-1}u_{in}$ is bounded in $W^{2,2}(B)$.

It then follows that

$$A^i_{pq}u_{pn,q} \to 0 \text{ in } L^2(B) \quad \text{for } i = 1, \ldots, 4 \text{ and } p, q = 1, 2.$$

In view of the assumptions on the coefficients A^i_{pq}, we obtain that u_{in} tends to zero in $W^{1,2}(B)$ for $i = 1, 2$. Then, we also find that v_{in} $(i = 1, 2)$ tends to zero in $L^2(B)$ and we arrive to a contradiction.

Now, let us to assume that the asymptotic condition does not hold. Therefore, there exist a sequence of $\omega_n \to \infty$ and a sequence of U_n in the domain of the operator, with unit norm, such that condition (9) holds. In this case, we can repeat the previous argument to arrive to a contradiction and the asymptotic condition is proved. □

It is worth noting that the assumption on the coefficients A^i_{pq} does not hold in the case that we assume that the coupling terms are also isotropic. In this case, we can recall the example proposed by Dafermos [2] to obtain undamped mechanical solutions. In fact, if we assume that the domain satisfies the following condition:

Condition D. There exists a nonzero field $\phi_i \in [W_0^{1,2}(B)]^n$ such that $\phi_{i,jj} + \gamma^2\phi = 0$ and $\phi_{i,i} = 0$, where $\gamma \neq 0$,

then, there exist undamped isothermal mechanical solutions.

We note that, in dimension one, there are not intervals satisfying Condition D. However, in dimension greater than one, this condition is satisfied for several symmetric domains. For instance, if the domain is a ball: there is an infinite number of eigenvalues γ satisfying this condition (see [1]).

The arguments proposed here can be extended without difficulty to the three-dimensional case. It is clear that, in this situation, we would need nine coupling dissipative mechanisms. Furthermore, the analysis can be adapted straightforwardly to the case where the elastic and dissipative parts are anisotropic, whenever the elasticity tensor defines a positive definite operator and the dissipative part is positive definite.

4 Some counterexamples

In the previous section, we have proved that, when the matrix of the coupling coefficients has the maximum rank, we have that the solutions decay in an exponential way for all the systems and for every geometry. Now, the natural question is if we can relax this condition on the rank of the matrix of the coupling coefficients and to continue having this property for every system and every geometry[2]. It does not seem to be easy to answer this question, but it is possible to find examples where there are undamped isothermal solutions for several systems and for every geometry when the rank of the matrix is lower than n^2. As we have pointed out in the previous section, the isotropic case for the coupling terms leads to a matrix of rank one and the undamped solution proposed by Dafermos [2] only applies for certain particular geometries.

We will give now examples where the rank of the matrix of coefficients is greater than one. They correspond to the case that $\lambda + \mu = 0$ and so, we consider the system:

$$
\begin{aligned}
\rho \ddot{u}_1 &= \mu \Delta u_1 + A_{11}\theta_{1,1} + A_{22}\theta_{2,2},\\
\rho \ddot{u}_2 &= \mu \Delta u_2 - A_{11}\theta_{1,1} - A_{22}\theta_{2,2},\\
m_1\dot{\theta}_1 &= k_1\Delta\theta_1 - l_1(\theta_1 - \theta_2) - l_2(\theta_3 - \theta_4) + A_{11}\dot{u}_{1,1} - A_{11}\dot{u}_{2,1},\\
m_2\dot{\theta}_2 &= k_2\Delta\theta_2 + l_1(\theta_1 - \theta_2) + l_2(\theta_3 - \theta_4) + A_{22}\dot{u}_{1,2} - A_{22}\dot{u}_{2,2},\\
m_3\dot{\theta}_3 &= k_3\Delta\theta_3 - l_2(\theta_1 - \theta_2) - l_3(\theta_3 - \theta_4),\\
m_4\dot{\theta}_4 &= k_4\Delta\theta_4 + l_2(\theta_1 - \theta_2) + l_3(\theta_3 - \theta_4).
\end{aligned}
$$

[2] We note that this condition can be seen equivalent to determine the number of dissipative mechanisms.

In the case that we assume that coefficients A_{11} and A_{22} are different from zero, our matrix has rank two. Therefore, if we consider the problem determined by the initial conditions:

$$\begin{aligned}&u_1(x,0)=u_2(x,0)=u^0(x),\quad \dot{u}_1(x,0)=\dot{u}_2(x,0)=v^0(x),\\&\theta_i(x,0)=0\quad\text{for}\quad i=1,\ldots,4,\end{aligned}$$

we obtain isothermal undamped mechanical solutions. We note that this kind of solutions is also found for every two-dimensional domain independently of the geometry.

These examples can be generalized to the three-dimensional case, where we note that the rank of the matrix of coefficients should be nine to guarantee the exponential decay for every system and every geometry using the argument of the previous section. Moreover, if the matrix of the coupling coefficients takes the form:

$$\begin{pmatrix}1&0&0&-1&0&0&0&0&0\\0&1&0&0&-1&0&0&0&0\\0&0&1&0&0&-1&0&0&0\\0&0&0&1&0&0&-1&0&0\\0&0&0&0&1&0&0&-1&0\\0&0&0&0&0&1&0&0&-1\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\end{pmatrix},$$

we can also obtain isothermal undamped solutions whenever we assume initial conditions of the type

$$\begin{aligned}&u_i(\boldsymbol{x},0)=u_i^0(\boldsymbol{x}),\quad \dot{u}_i(\boldsymbol{x},0)=v_i^0(\boldsymbol{x})\quad\text{for}\quad i=1,2,3,\\&\theta_i(\boldsymbol{x},0)=0\quad\text{for}\quad i=1,\ldots,9.\end{aligned}$$

In short, we have seen that, in dimension two, there are problems where there is no decay of some solutions when the rank of the matrix is two, and the same conclusion has been obtained for the dimension three when the rank of the matrix is six.

5 Conclusion

In this paper, we have considered the case of thermoelastic homogeneous materials. Though we have considered the isotropic case for both the elastic and thermal parts, we allow the anisotropy for the coupling terms.

We recall now the main results obtained in this work:

1. We have proved that, with n^2 dissipation mechanisms and when the rank of the matrix defining the coupling coefficients is maximum, the solutions decay in an exponential way for every (bounded) geometry of the solid.

2. We have provided an example when the matrix of coupling terms has rank 2 (in dimension 2) or rank 6 (in dimension 3) such that there are undamped isothermal mechanical oscillations.
3. The cases when the matrix of coupling terms has rank 3 in dimension 2 (7 or 8 in dimension 3) are still open and we do not know if the decay is exponential for every domain and every system or there are examples with slower decay.

Another issue to be considered in the near future is to see if the introduction of new dissipative mechanisms can produce a faster rate of decay of the solutions for some particular cases.

Acknowledgements The authors thank the anonymous reviewer whose comments have improved the final quality of the article.
This paper is part of the projects PGC2018-096696-B-I00 and PID2019-105118GB-I00, funded by the Spanish Ministry of Science, Innovation and Universities and FEDER "A way to make Europe".

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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