LE MATEMATICHE
Vol. LXII (2007) - Fasc. I, pp. 121-127

n-BLOCKS COLLECTIONS ON FANO MANIFOLDS AND SHEAVES WITH REGULARITY $-\infty$

E. BALLICO - F. MALASPINA

Let X be a smooth Fano manifold equipped with a " nice " n-blocks collection in the sense of [3] and $\mathcal{F}$ a coherent sheaf on X. Assume that X is Fano and that all blocks are coherent sheaves. Here we prove that $\mathcal{F}$ has regularity $-\infty$ in the sense of [3] if Supp$(\mathcal{F})$ is finite, the converse being true under mild assumptions. The corresponding result is also true when X has a geometric collection in the sense of [2].

1. Introduction.

Let X be an n-dimensional smooth projective variety over $\mathbb{C}$. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{O}_X - mod)$ denote the bounded category of $\mathcal{O}_X$-sheaves. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that X has a geometric collection in the sense of [2] or an n-blocks collection in the sense of [3]. L. Costa and R. M. Miró-Roig defined the notion of regularity for $\mathcal{F}$ and asked a characterization of all $\mathcal{F}$ whose regularity is $-\infty$ ([2], Remark 3.3). In section 2 we will recall the definitions contained in [2] and [3] and

Entrato in redazione il Entrato in redazione il 13 Giugno 2007.
1991 *Mathematics Subject Classification*: 14J60.
Keywords and phrases. m-block collection; Fano manifold; coherent sheaves on a Fano manifold; order of regularity.
This author was partially supported by MIUR and GNSAGA of INdAM (Italy).

used in our statements below. After the statements we will discuss our motivations and give a very short list of interesting varieties to which these results may be applied.

We prove the following results.

Theorem 1. *Assume that X is Fano and that it has an n-blocks collection $\mathcal{B}$ whose members are coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on X. If $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$, then* $\mathrm{Supp}(\mathcal{F})$ *is finite. If all right mutations of all elements of $\mathcal{B}$ are locally free and* $\mathrm{Supp}(\mathcal{F})$ *is finite, then $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$.*

Corollary 1. *Assume that X is Fano and that it has a geometric collection $\mathcal{G}$ whose members are coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on X. If $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{G}$, then* $\mathrm{Supp}(\mathcal{F})$ *is finite. If all right mutations of all elements of $\mathcal{G}$ are locally free and* $\mathrm{Supp}(\mathcal{F})$ *is finite, then $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{G}$.*

We recall that any projective manifold with a geometric collection is Fano ([2], part (2) of Remark 2.16). Any n-dimensional smooth quadric $Q_n \subset \mathbf{P}^{n+1}$ has an n-block collection whose members are locally free ([3], Example 3.2 (2)). It has a geometric collection if and only if n is odd. Any Grassmannian G has an n-block collection (with $n := \dim(G)$) whose members are locally free sheaves ([3], Example 3.7 (4)). For the Fano 3-folds V_5 and V_{22} D. Faenzi found a geometric collection whose members are locally free ([4], [5]).

Castelnuovo-Mumford regularity was introduced by Mumford in [8], Lecture 14, for a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$. He ascribed the idea to Castelnuovo for the following reason. Let $C \subset \mathbf{P}^n$ be a closed subvariety and $H \subset \mathbf{P}^n$ be a general hyperplane. Then we have an exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C\cap H}(t) \to 0 \tag{1}$$

Castelnuovo used the corresponding classical (pre-sheaves) concepts of linear systems to get informations on C from informations on $C \cap H$ plus other geometrical or numerical assumptions on C. The key properties of Castelnuovo-Mumford regularity is that if $\mathcal{F}$ is m-regular, then it is $(m+1)$-regular and $\mathcal{F}(m)$ (or $\mathcal{I}_C(m)$) is spanned. Since [8] several hundred papers studied this notion, which is now also a key property in computational algebra. Let X be a projective scheme, H an ample line bundle on X and $\mathcal{F}$ a coherent sheaf on X. The definition in [8], Lecture 14, apply verbatim, just writing $\mathcal{F} \otimes H^{\otimes t}$ instead of $\mathcal{F}(t)$. This is also called Castelnuovo-Mumford regularity with respect to the polarized pair (X, H). X may have several non-proportional polarizations. It is better to collect all informations for all polarizations in a single integer (the regularity) not in a string of integers, one for each proportional class of polarizations on X. This is the reason for the definitions given by Hoffman-Wang for products of projective varieties ([6]) and by Maclagan and Smith for toric varieties ([7]). Even when X has only one polarization the search for generalizations of Beilinson's spectral sequence from $\mathbf{P}^n$ to X gave a strong motivation to introduce the notions of regularities for geometric collections ([2], Th. 2.21) and n-block collections ([3], Th. 3.10). The reader will notice that to prove Theorem 1 and Corollary 1 we will use neither the main definitions of [2] and [3] nor the machinery of derived categories. We will only use the formal properties (like " spannedness " or " m-regularity implies $(m+1)$-regularity ") proved in [2] and [3] (see eq. (2) in section 2 for an explanation of the word " spannedness "). We hope that our results will be extended and used if other notions of regularity will appear in the literature.

2. The main definitions and the proofs.

Let X be an n-dimensional smooth projective variety over $\mathbb{C}$. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{O}_X - mod)$ denote the bounded category of $\mathcal{O}_X$-sheaves. For all objects $A, B \in \mathcal{D}$ set $\mathrm{Hom}^\bullet(A, B) := \oplus_{k\in\mathbb{Z}} \mathrm{Ext}^k_{\mathcal{D}}(A, B)$. An object $A \in \mathcal{D}$ is said to be *exceptional* if $\mathrm{Hom}^\bullet(A, A)$ is an 1-dimensional algebra generated by the identity. An ordered collection $(A_0, \dots, A_m)$ of objects of $\mathcal{D}$ will be called an *exceptional collection* if each A_i is exceptional and $\mathrm{Ext}^\bullet_{\mathcal{D}}(A_k, A_j) = 0$ for all $0 \le j < k \le m$. A collection $(A_0, \dots, A_m)$ is said to be *strongly exceptional* if it is exceptional and $\mathrm{Ext}^i_{\mathcal{D}}(A_j, A_k) = 0$ for all (i, j, k) such that $i \neq 0$ and $j \le k$. A collection $(A_0, \dots, A_m)$ is said to be full if it generates $\mathcal{D}$. This implies $\mathcal{D} \cong \mathbb{Z}^{\oplus(m+1)}$. Now asssume that X admits a fully exceptional collection $\sigma = (A_0, \dots, A_n)$. For any $A, B \in \mathcal{D}$ the right mutation $R_B A$ of A and the left mutation $L_A B$ of B are defined by the following distinguished triangles

$$R_B A[-1] \to A \to \mathrm{Hom}^{\times\bullet}(A, B) \otimes B \to R_B A$$

$$L_A B \to \mathrm{Hom}^\bullet(A, B) \otimes A \to L_A B[1]$$

([2], Definition 2.4). For every integer i such that $1 \leq i \leq n$, define the i-th right mutation $R_i\sigma$ and the i-th left mutation $L_i\sigma$ of σ by the formulas

$$R_i\sigma := (A_0, \dots A_{i-2}, A_i, R_{A_i}A_{i-1}, A_{i+1}, \dots A_n)$$

$$L_i\sigma := (A_0, \dots, A_{i-2}, L_{A_{i-1}}A_i, A_{i-1}, A_{i+1}, \dots, A_n)$$

(a switch of two elements of σ and the application to one of them of a right or left mutation) ([2], Definition 2.6). For any $j \geq 2$, set $R^{(j)}A_i := R_{A_{i+j}} \circ \cdots \circ R_{A_{i+1}}A_i \in \mathcal{D}$ and define in a similar way the iterated left mutations $L^{(i)}$ ([2], Notation 2.7). Set $A_{n+i} := R^{(n)}A_{i-1}$ for all $0 \leq i \leq n$ and $A_{-i} := L^{(n)}A_{n-i+1}$ for all $1 \leq i \leq n$. Iterating the use of $R^{(n)}$ and $L^{(n)}$ we get the elix $\{A_i\}_{i\in\mathbb{Z}}$ with $A_i \in \mathcal{D}$ for all i ([2], Definition 2.12). For instance, if $X = \mathbf{P}^n$, then $(A_0, \dots, A_n) := (\mathcal{O}_{\mathbf{P}^n}, \mathcal{O}_{\mathbf{P}^n}(1), \dots, \mathcal{O}_{\mathbf{P}^n}(n))$ is a geometric collection and $\{\mathcal{O}_{\mathbf{P}^n}(t)\}_{t\in\mathbb{Z}}$ is the corresponding elix. Let $\mathcal{F}$ be a coherent sheaf on X. $\mathcal{F}$ is said to be m-regular with respect to the geometric collection $\sigma = (A_0, \dots, A_n)$ if $\mathrm{Ext}^q(R^{(-p)}A_{-m+p}, \mathcal{F}) = 0$ for all integers q, p such that $q > 0$ and $-n \leq p \leq 0$. The regularity of $\mathcal{F}$ is the minimal integer m such that $\mathcal{F}$ is m-regular (or $-\infty$ if it is m-regular for all $m \in \mathbb{Z}$). An exceptional collection $(A_0, \dots, A)$ is called a *block* if $\mathrm{Ext}^i_{\mathcal{D}}(A_j, A_k) = 0$ for all i, j, k such that $k \neq j$. An m-block collection of elements of $\mathcal{D}$ is an exceptional collection which may be partitioned into $m+1$ consecutive blocks. Assume that X has an n-block collection whose elements generate $\mathcal{D}$. Let $\mathcal{F}$ be a coherent sheaf on X. In [3], Definition 4.5, there is a definition of regularity of $\mathcal{F}$; it requires only technical modifications with respect to the simpler case of a geometric collection: they gave similar definitions of left and right mutations and elices. Then the definition of m-regularity is again given by certain Ext-vanishings. If a coherent sheaf $\mathcal{F}$ is m-regular with respect to a geometric collection σ or an n-block collection σ, then it gives a resolution

$$0 \to \mathcal{L}_{-n} \to \cdots \to \mathcal{L}_{-1} \to \mathcal{L}_0 \to \mathcal{F} \to 0 \tag{2}$$

in which each $\mathcal{L}_i \in \mathcal{D}$ is constructed from $\mathcal{F}$ and the elements of σ taking tensor products ([2], between 3.1 and 3.2 for geometric collections, [3], eq. (4.2), for n-blocks). If the elements of σ are coherent sheaves (resp. localy free coherent sheaves), then each $\mathcal{L}_i$ is a coherent sheaf (resp. a locally free coherent sheaf). In the case of Castelnuovo-Mumford regularity the corresponding result is true. It shows how the Castelnuovo-Mumford regularity bounds the degrees of the syzygies. This is the key reason for its use in computational algebra.

The following well-known result answers the corresponding problem for Castelnuovo-Mumford regularity.

Lemma 1. *Let X be a projective scheme, L an ample line bundle on X and $\mathcal{F}$ a coherent sheaf on X. The following conditions are equivalent:*

(a) *$\mathcal{F}$ is supported by finitely many points of X;*

(b) *$\mathcal{F} \otimes L^{\otimes t}$ is spanned for all $t \ll 0$;*

(c) *$h^i(X, \mathcal{F} \otimes L^{\otimes t}) = 0$ for all $i > 0$ and all $t \in \mathbb{Z}$.*

Proof. Obviously, (a) implies (b) and (c). Now assume that (b) holds, but that $\dim(\mathrm{Supp}(\mathcal{F})) > 0$. Take an integral projective curve $C \subseteq \mathrm{Supp}(\mathcal{F})$. Since the restriction of a spanned sheaf is spanned, $\mathcal{F}|C$ satisfies (c) with respect to the ample line bundle $R := L|C$. Let $f : D \to C$ be the normalization. Set $M := f^*(R)$. M is an ample line bundle on D. Since D is a smooth curve, the coherent sheaf $f^*(\mathcal{F})$ is either a torsion sheaf or the direct sum of a torsion sheaf T and a vector bundle E with positive rank. To prove (a) we must check that $f^*(\mathcal{F})$ is torsion. Assume $E \neq 0$. Since the pull-back of a spanned sheaf is spanned, $E \otimes M^{\otimes t}$ is spanned for all $t \in \mathbb{Z}$. Since $\deg(E \otimes R^{\otimes t}) = \deg\,(E) + t\cdot\,\mathrm{rank}(E)\cdot\,\deg\,(M) < 0$ for $t \ll 0$, $E \otimes R^{\otimes t}$ is not spanned for $t \ll 0$, contradiction. Let $x \geq 1$ be an integer such that $L^{\otimes x}$ is very ample. If $\mathcal{F}$ satisfies (c) for the line bundle L, then it satisfies the same condition for the line bundle $L' := L^{\otimes x}$. Hence to check that (c) implies (a) we may assume that L is very ample. Fix an integer t. Since $h^i(X, \mathcal{F} \otimes L^{\otimes(t-i-1)}) = 0$ for all $i > 0$, $\mathcal{F} \otimes L^{\otimes t}$ is spanned ([8], p. 100). Thus (b) holds and hence (a) holds. □

Proof of Theorem 1. Fix a coherent sheaf $\mathcal{F}$. Let $\mathcal{E}$ be the helix of blocks generated by $\mathcal{B}$ ([3], Definition 4.1). All elements of $\mathcal{E}$ are coherent sheaves, not just complexes ([3], Corollary 4.4) and their elements satisfies a periodicity modulo $n+1$: $\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes \omega_X$ ([3], lines between 4.3 and 4.4). First assume that $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$, i.e. that it is m-regular with respect to $\mathcal{B}$ for all $m \ll 0$. Fix $m \in \mathbb{Z}$. The

m-regularity of $\mathcal{F}$ implies that it is a quotient of a finite sum $\mathcal{L}_0$ of sheaves of the form E_s^{-m} appearing in the blocks of $\mathcal{B}$ ([3], Definition 4.5). Since $\mathcal{F}$ is t-regular for all $t \ll 0$, the periodicity property of $\mathcal{E}$ shows that for all integers $t \leq 0$, $\mathcal{F}$ is a quotient of a finite direct sum of sheaves of the form $\mathcal{L}_0 \otimes \omega_X^{\otimes(-t)}$. Since X is Fano, ω_X^* is ample. Take $L := \omega_X^*$ and copy the proof that (b) implies (a) in Lemma 1. We get that $\mathrm{Supp}(\mathcal{F})$ is finite.

Now assume that $\mathrm{Supp}(\mathcal{F})$ is finite and that all right mutations of elements of $\mathcal{B}$ are locally free. Let A be any of these mutations. Since A is locally free, the local Ext-functors $Ext^i(A, \mathcal{F})$ vanish for all $i > 0$. Hence the local-to-global spectral sequence for the Ext-functors gives $\mathrm{Ext}^i(A, \mathcal{F}) \cong H^i(X, Hom(A, \mathcal{F}))$ for all $i \geq 0$. Since $\mathrm{Supp}(\mathcal{F})$ is finite, we get $\mathrm{Ext}^q(A, \mathcal{F}) = 0$ for all $q > 0$. Hence for every integer m the sheaf $\mathcal{F}$ satisfies the definition of m-regularity given in [3], Definition 4.5. Since $\mathcal{F}$ is m-regular with respect to $\mathcal{B}$ for all m, its regularity is $-\infty$. □

Proof of Corollary 1. This result is a particular case of Theorem 1, because the definition of regularity for geometric collections given in [2] agrees with the definition of regularity for n-blocks collections given in [3] (see [3], Remark 4.7). It may be proved directly, just quoting [2], Remark 2.14, to get the periodicity property $\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes \omega_X$ and [2], Proposition 3.8, to get the surjection $\mathcal{L}_0 \to \mathcal{F}$. □

Remark 1. In [1] J. V. Chipalkatti defined a notion of regularity for a coherent sheaf $\mathcal{F}$ on a Grassmannian. He remarked that $\mathcal{F}$ have regularity $-\infty$ (according to his definition) if and only if its support is finite ([1], part 4) of Remark 1.2).

Remark 2. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n \times \mathbf{P}^m$. Hoffman and Wang introduced a bigraded definition of regularity ([6]). The definition of ampleness and [6], Prop. 2.8, imply that if $\mathcal{F}$ is (a, b)-regular in the sense of Hoffman-Wang for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, then $\mathrm{Supp}(\mathcal{F})$ is finite. The converse is obvious. As remarked in [3], Remark 5.2, Hoffman-Wang definition and its main properties may be extended verbatim to arbitrary multiprojective spaces $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$.

REFERENCES

[1] J. V. Chipalkatti, A generalization of Castelnuovo regularity to Grassman varieties, Manuscripta Math. 102 (2000), n. 4, pp. 447-464.

[2] L. Costa - R. M. Miró-Roig, Geometric collections and Castelnuovo-Mumford regularity, arXiv:math/0609561, Math. Proc. Cambridge Phil. Soc., to appear.

[3] L. Costa - R. M. Miró-Roig, m-blocks collections and Castelnuovo-Mumford regularity in multiprojective spaces, Nagoya Math. J. 186 (2007), pp. 119-155.

[4] D. Faenzi, Bundles over the Fano threefold V_5, Comm. Algebra 33 (2005), n. 9, pp. 3061-3080.

[5] D. Faenzi, Bundles over Fano threefolds of type V_{22}, Ann. Mat. Pura Appl. (4) 186 (2007), n. 1, pp. 1-25.

[6] J. W. Hoffman - H. H. Wang, Castelnuovo-Mumford regularity in biprojective spaces, Adv. Geom. 4 (2004), n. 4, pp. 513-536.

[7] D. Maclagan - G. G. Smith, Multigraded Castelnuovo-Mumford regularity, J. Reine Angew. Math. 571 (2004), pp. 179-212.

[8] D. Mumford, Lectures on curves on an algebraic surface, Princeton University Press, Princeton, N.J., 1966.

E. Ballico
Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
email: ballico@science.unitn.it

F. Malaspina
Università di Torino
via C. Alberto 10
10123 Torino, Italy
email:francesco.malaspina@unito.it