$N(k)$-QUASI EINSTEIN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

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Abstract. The object of the present paper is to study $N(k)$-quasi Einstein manifolds satisfying certain curvature conditions. Two examples have been constructed to prove the existence of such a manifold. Finally, a physical example of an $N(k)$-quasi Einstein manifold is given.

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g, \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g), respectively. According to ([1], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation :

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.2}$$

where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X, \xi) = \eta(X), \qquad g(\xi, \xi) = \eta(\xi) = 1, \tag{1.3}$$

for all vector fields X, Y.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [2] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (1.2). We shall call η the associated 1-form and the unit vector field ξ is called the generator of the manifold.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. So many studies about Einstein field equations are done. For example, in [11], Naschie turned the tables on the theory of elementary particles and showed that we could derive the expectation number of elementary particles

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of the standard model using Einstein's unified field equation or more precisely his somewhat forgotten strength criteria directly and without resorting to quantum field theory [12]. He also discussed possible connections between Gödel's classical solution of Einstein's field equations and E-infinity in [10]. Also quasi Einstein manifolds have some importance in the general theory of relativity. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds [9]. Further, quasi Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity [6].

The study of quasi Einstein manifolds was continued by Chaki [3], Guha [13], De and Ghosh [7], [8] and many others. The notion of quasi Einstein manifolds have been generalized in several ways by several authors. In recent papers, Özgür studied super quasi Einstein manifolds [19] and generalized quasi Einstein manifolds [20].

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by [23]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.4}$$

k being some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be a $N(k)$-quasi Einstein manifold [25]. In fact k is not arbitrary as the following:

In an n-dimensional $N(k)$-quasi Einstein manifold it follows that

$$k = \frac{a+b}{n-1}. \tag{1.5}$$

Now, it is immediate to note that in an n-dimensional $N(k)$-quasi Einstein manifold [17]

$$R(X,Y)\xi = \frac{a+b}{n-1}[\eta(Y)X - \eta(X)Y], \tag{1.6}$$

which is equivalent to

$$R(X,\xi)Y = \frac{a+b}{n-1}[\eta(Y)X - g(X,Y)\xi] = -R(\xi,X)Y. \tag{1.7}$$

From (1.4) we get

$$R(\xi,X)\xi = \frac{a+b}{n-1}[\eta(X)\xi - X]. \tag{1.8}$$

In [25] it was shown that an n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. Also in [18] Özgür, cited some physical examples of $N(k)$-quasi Einstein manifolds. In 2011, Taleshian and Hosseinzadeh [24] studied $N(k)$-quasi Einstein manifolds satisfying certain curvature conditions. Nagaraja [16] also studied $N(k)$-mixed quasi Einstein manifolds.

In 1968, Yano and Sawaki [22] defined and studied a tensor $\widetilde{C}$ on a Riemannian manifold of dimensional n which includes both conformal curvature tensor and concircular curvature tensor as particular cases. This tensor is known as quasi-conformal curvature tensor and is defined by

$$\begin{aligned} \widetilde{C}(X,Y)Z &= \lambda R(X,Y)Z \\ &\quad + \mu\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad - \frac{r}{n}\{\frac{\lambda}{(n-1)} + 2\mu\}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.9}$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$. Here λ and μ are arbitrary constants. If $\lambda = 1$ and $\mu = -\frac{1}{n-2}$, then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor. For an $n \geq 4$ dimensional Riemannian manifold, if $\widetilde{C} = 0$ then it is called quasi-conformally flat. Recently Mantica and Suh [15] studied quasi-conformally recurrent Riemannian manifolds.

The projective curvature tensor P and the concircular curvature tensor $\tilde{Z}$ in a Riemannian manifold (M^n, g) are defined by [26]

$$P(X,Y)W = R(X,Y)W - \frac{1}{n-1}[S(Y,W)X - S(X,W)Y], \tag{1.10}$$

$$\tilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y], \tag{1.11}$$

respectively. In [25], the authors have proved that conformally flat quasi Einstein manifolds are certain $N(k)$-quasi Einstein manifolds. The derivation conditions $R(\xi,X) \cdot R = 0$ and $R(\xi,X) \cdot S = 0$ have been studied in [23], where R and S denote the curvature and Ricci tensor respectively. Özgür and Tripathi [17] continued the study of the $N(k)$-quasi Einstein manifolds. In [17], the derivation conditions $\tilde{Z}(\xi,X) \cdot R = 0$ and $\tilde{Z}(\xi,X) \cdot \tilde{Z} = 0$ on $N(k)$-quasi Einstein manifolds were studied, where $\tilde{Z}$ is the concircular curvature tensor. Moreover in [17], for an $N(k)$-quasi Einstein manifold it was proved that $k = \frac{a+b}{n-1}$. Özgür in [18] studied the condition $R \cdot P = 0$, $P \cdot S = 0$ and $P \cdot P = 0$ for an $N(k)$-quasi Einstein manifold, where P denotes the projective curvature tensor and some physical examples of $N(k)$-quasi Einstein manifolds are given. Again, in 2008, Özgür and Sular [21] studied $N(k)$-quasi Einstein manifolds satisfying $R \cdot C = 0$ and $R \cdot \tilde{C} = 0$, where C and $\tilde{C}$ represent the conformal curvature tensor and the quasi-conformal curvature tensor, respectively. This paper is a continuation of previous studies.

The paper is organized as follows: After preliminaries in section 3, we study quasi-conformally recurrent $N(k)$-quasi Einstein manifolds. We prove that quasi-conformally recurrent manifold satisfies $R(\xi,X) \cdot \widetilde{C} = 0$. In section 4, we prove that for an $n \geq 4$ dimensional $N(k)$-quasi Einstein manifold, the conditions $\widetilde{C}(\xi,X) \cdot S = 0$, $\widetilde{C}(\xi,X) \cdot P = 0$, $\widetilde{C}(\xi,X) \cdot \widetilde{Z} = 0$ hold on the manifold if and only if $\lambda = \mu(2-n)$. Finally, we give two examples of an $N(k)$-quasi Einstein manifold and a physical example of an $N(k)$-quasi Einstein manifold.

2. **Preliminaries**

From (1.2) and (1.3) it follows that

$$S(X,\xi) = (a+b)\eta(X), \tag{2.1}$$

and

$$r = an + b, \tag{2.2}$$

where r is the scalar curvature of M^n.

From the definition of the quasi-conformal curvature tensor, we can write

$$\begin{aligned}\widetilde{C}(\xi,X)Y &= \lambda R(\xi,X)Y \\ &+\mu\{S(X,Y)\xi - S(\xi,Y)X + g(X,Y)Q\xi - g(\xi,Y)QX\} \\ &-\frac{r}{n}\{\frac{\lambda}{(n-1)} + 2\mu\}[g(X,Y)\xi - g(\xi,Y)X].\end{aligned}$$

Here using (1.7) and (2.1), we find

$$\widetilde{C}(\xi,X)Y = \{\lambda k - \frac{r}{n}\{\frac{\lambda}{(n-1)} + 2\mu) + \mu(2a+b)\}\{g(X,Y)\xi - \eta(Y)X\}.$$

Now using $r = an + b$, we find

$$\lambda k - \frac{r}{n}\{\frac{\lambda}{(n-1)} + 2\mu) + \mu(2a+b) = \lambda - \mu(2-n).$$

Then we obtain

$$\widetilde{C}(\xi,X)Y = \{\lambda - \mu(2-n)\}\{g(X,Y)\xi - \eta(Y)X\}. \tag{2.3}$$

The curvature conditions $\widetilde{C}\cdot S$, $\widetilde{C}\cdot P$ and $\widetilde{C}\cdot\widetilde{Z}$ are defined by

$$(\widetilde{C}(U,X)\cdot S)(Y,Z) = -S(\widetilde{C}(U,X)Y,Z) - S(Y,\widetilde{C}(U,X)Z), \tag{2.4}$$

$$\begin{aligned}(\widetilde{C}(U,X)\cdot P)(Y,Z,W) &= \widetilde{C}(U,X)P(Y,Z)W - P(\widetilde{C}(U,X)Y,Z)W \\ &-P(Y,\widetilde{C}(U,X)Z)W - P(Y,Z)\widetilde{C}(U,X)W,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}(\widetilde{C}(U,X)\cdot \widetilde{Z})(Y,Z,W) &= \widetilde{C}(U,X)\widetilde{Z}(Y,Z)W - \widetilde{Z}(\widetilde{C}(U,X)Y,Z)W \\ &-\widetilde{Z}(Y,\widetilde{C}(U,X)Z)W - \widetilde{Z}(Y,Z)\widetilde{C}(U,X)W,\end{aligned} \tag{2.6}$$

respectively.

3. Quasi-conformally recurrent $N(k)$ -quasi Einstein manifold

In [21], Özgür and Sular proved that in an $N(k)$-quasi Einstein manifold the condition $R(\xi,X)\cdot\widetilde{C} = 0$ holds on M^n if and if only if either $a = -b$ or, M^n is conformally flat with $\lambda = \mu(2-n)$. In this section we study quasi-conformally recurrent $N(k)$-quasi Einstein manifolds.

A non-flat Riemannian manifold M is said to be quasi-conformally recurrent [15] if the quasi conformal curvature tensor $\widetilde{C}$ satisfies the condition $\nabla\widetilde{C} = A\otimes\widetilde{C}$, where A is an everywhere non-zero 1-form. We now define a function f on M by $f^2 = g(\widetilde{C},\widetilde{C})$, where the metric g is extended to the inner product between the tensor fields in the standard fashion. Then we know that $f(Yf) = f^2A(Y)$. So from this we have $Yf = fA(Y)$, because $f \neq 0$. This implies that

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

Hence

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f.$$

Therefore we get

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})f = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M by our assumption, we obtain

$$dA(X,Y) = 0, \text{ that is, the 1-form } A \text{ closed.} \tag{3.1}$$

Now from

$$(\nabla_X \widetilde{C})(U,V)Z = A(X)\widetilde{C}(U,V)Z,$$

we get

$$(\nabla_U \nabla_V \widetilde{C})(X,Y)Z = \{UA(V) + A(U)A(V)\}\widetilde{C}(X,Y)Z.$$

Hence using (3.1) we get

$$(R(X,Y)\widetilde{C})(U,V)Z = [2dA(X,Y)]\widetilde{C}(U,V)Z = 0.$$

Therefore, for a quasi-conformally recurrent manifold, we have

$$R(X,Y)\widetilde{C} = 0 \quad \text{for all} \quad X, Y. \tag{3.2}$$

An equivalent proof can be given as follows: From the conditon $\nabla_i \widetilde{C}^m_{jkl} = A_i \widetilde{C}^m_{jkl}$ one gets easily $\nabla_i(\widetilde{C}^m_{jkl}\widetilde{C}^{jkl}_m) = 2A_i(\widetilde{C}^m_{jkl}\widetilde{C}^{jkl}_m)$ and thus putting $f = \widetilde{C}^m_{jkl}\widetilde{C}^{jkl}_m$, we recover locally the closedness of the 1-form A.

Hence by Theorem 4.3 of Özgür and Sular [21], we can state the following:

Theorem 1. *An $N(k)$-quasi Einstein manifold is quasi-conformally recurrent if and only if either $a = -b$ or, M^n is conformally flat with $\lambda = \mu(2-n)$.*

4. **Main results**

In this section we give the main results of the paper. At first we give the following :

Theorem 2. *Let M^n be an n-dimensional, $n \geq 4, N(k)$-quasi Einstein manifold. Then M^n satisfies the condition $\tilde{C}(\xi,X)\cdot S = 0$ if and only if $\lambda = \mu(2-n)$.*

Proof. Assume that an $N(k)$-quasi Einstein manifold satisfies

$$\widetilde{C}(\xi,X)\cdot S = 0.$$

Then we get from (2.4)

$$S(\widetilde{C}(\xi,X)Y,Z) + S(Y,\widetilde{C}(\xi,X)Z) = 0. \tag{4.1}$$

Using (2.3) in (4.1) we get

$$\{\lambda - \mu(2-n)\}[g(X,Y)S(\xi,Z) - \eta(Y)S(X,Z) + g(X,Z)S(Y,\xi) - \eta(Z)S(Y,X)] = 0.$$

Then either

$$\lambda - \mu(2-n) = 0,$$

or,

$$g(X,Y)S(\xi,Z) - \eta(Y)S(X,Z) + g(X,Z)S(Y,\xi) - \eta(Z)S(Y,X) = 0. \tag{4.2}$$

Putting $Y = \xi$ in (4.2) we find

$$S(X,Z) = (a+b)g(X,Z), \tag{4.3}$$

which implies that the manifold is an Einstein manifold which contradicts the definition of $N(k)$-quasi Einstein manifold. Then only $\lambda - \mu(2-n) = 0$ holds.

Conversely, let $\lambda = \mu(2-n)$, then from (2.3) we have $\widetilde{C}(\xi,X)Y = 0$. Hence we get $\widetilde{C}(\xi,X)\cdot S = 0$. This completes the proof. $\square$

If $\lambda = \mu(2-n)$, then from the definition of quasi-conformal curvature tensor it follows that $\widetilde{C} = \lambda C$. Thus we can state the following:

Corollary 1. *In an $N(k)$-quasi Einstein manifold satisfying the condition $\widetilde{C}(\xi, X)\cdot S = 0$, conformally flatness and quasi-conformally flatness are equivalent.*

Now we give the following:

Theorem 3. *Let M^n be an n-dimensional, $n \geq 4, N(k)$- quasi Einstein manifold. Then M^n satisfies the condition $\widetilde{C}(\xi, X)\cdot P = 0$ if and only if $\lambda = \mu(2-n)$.*

Proof. Suppose that the $N(k)$-quasi Einstein manifold satisfies

$$\widetilde{C}(\xi, X)\cdot P = 0.$$

Then from (2.5), we get

$$\widetilde{C}(\xi, X)P(Y, Z)W - P(\widetilde{C}(\xi, X)Y, Z)W - P(Y, \widetilde{C}(\xi, X)Z)W - P(Y, Z)\widetilde{C}(\xi, X)W = 0.$$

Using (1.10) and (2.3) we obtain

$$\begin{aligned}&\{\lambda - \mu(2-n)\}\{g(X, P(Y, Z)W)\xi - \eta(P(Y, Z)W))X - g(X, Y)P(\xi, Z)W\\&+\eta(Y)P(X, Z)W - g(X, Z)P(Y, \xi)W + \eta(Z)P(Y, X)W - g(X, W)P(Y, Z)\xi\\&+\eta(W)P(Y, Z)X\} = 0,\end{aligned}$$

which implies either $\lambda - \mu(2-n) = 0$ or,

$$\begin{aligned}&g(X, P(Y, Z)W)\xi - \eta(P(Y, Z)W))X - g(X, Y)P(\xi, Z)W\\&+\eta(Y)P(X, Z)W - g(X, Z)P(Y, \xi)W + \eta(Z)P(Y, X)W - g(X, W)P(Y, Z)\xi\\&+\eta(W)P(Y, Z)X = 0.\end{aligned} \tag{4.4}$$

Taking the inner product of both sides of (4.4) with ξ, we have

$$\begin{aligned}&g(X, P(Y, Z)W) - \eta(P(Y, Z)W))\eta(X) - g(X, Y)\eta(P(\xi, Z)W)\\&+\eta(Y)\eta(P(X, Z)W) - g(X, Z)\eta(P(Y, \xi)W) + \eta(Z)\eta(P(Y, X)W)\\&+\eta(W)\eta(P(Y, Z)X) = 0.\end{aligned} \tag{4.5}$$

Hence with the help of (1.10) the equation (4.5) is reduced to

$$0 = P(Y, Z, W, X) + \frac{b}{n-1}\{g(X, Z)g(Y, W) - g(X, Y)g(Z, W)\}, \tag{4.6}$$

where $P(Y, Z, W, X) = g(X, P(Y, Z)W)$.

Then by using (1.10) and putting $X = Y = e_i$ in (4.6), where $\{e_i\}$ is ortonormal basis at each point of the manifold and taking summation over i, $1 \leqslant i \leqslant n$, we obtain

$$bg(Z, W) = 0.$$

This means that $b = 0$ which implies that the manifold is an Einstein manifold which contradicts the definition of an $N(k)$-quasi Einstein manifold. Then only the relation $\lambda - \mu(2-n) = 0$ holds. Conversely, let $\lambda = \mu(2-n)$, then from (2.3), we have $\widetilde{C}(\xi, X)Y = 0$. Hence $\widetilde{C}(\xi, X)\cdot P = 0$. This completes the proof. □

Remark 1. *The Corollary 1 also holds in this case.*

Theorem 4. *Let M^n be an n-dimensional, $n \geq 4, N(k)$-quasi Einstein manifold. Then M^n satisfies the condition $\widetilde{C}(\xi, X)\cdot \widetilde{Z} = 0$ if and only if $\lambda = \mu(2-n)$.*

Proof. We suppose that

$$\widetilde{C}(\xi, X) \cdot \widetilde{Z} = 0.$$

Then we get from (2.6)

$$\widetilde{C}(\xi, X)\widetilde{Z}(Y, V)W - \widetilde{Z}(\widetilde{C}(\xi, X)Y, V)W - \widetilde{Z}(Y, \widetilde{C}(\xi, X)V)W - \widetilde{Z}(Y, V)\widetilde{C}(\xi, X)W = 0.$$

So from(1.11) and (2.3), we obtain

$$\begin{aligned}&\{\lambda - \mu(2-n)\}\{g(X, \widetilde{Z}(Y, V)W)\xi - \eta(\widetilde{Z}(Y, V)W))X - g(X, Y)\widetilde{Z}(\xi, V)W \\ &+\eta(Y)\widetilde{Z}(X, V)W - g(X, V)\widetilde{Z}(Y, \xi)W + \eta(V)\widetilde{Z}(Y, X)W - g(X, W)\widetilde{Z}(Y, V)\xi \\ &+\eta(W)\widetilde{Z}(Y, V)X\} = 0.\end{aligned}$$

Then either $\lambda - \mu(2-n) = 0$ or,

$$\begin{aligned}g(X, \widetilde{Z}(Y, V)W)\xi &- \eta(\widetilde{Z}(Y, V)W))X - g(X, Y)\widetilde{Z}(\xi, V)W \\ &+\eta(Y)\widetilde{Z}(X, V)W - g(X, V)\widetilde{Z}(Y, \xi)W + \eta(V)\widetilde{Z}(Y, X)W - g(X, W)\widetilde{Z}(Y, V)\xi \\ &+\eta(W)\widetilde{Z}(Y, V)X = 0.\end{aligned} \tag{4.7}$$

Taking inner product with ξ the equation (4.7), we get

$$\begin{aligned}&g(X, \widetilde{Z}(Y, V)W) - \eta(\widetilde{Z}(Y, V)W))\eta(X) - g(X, Y)\eta(\widetilde{Z}(\xi, V)W) \\ &+\eta(Y)\eta(\widetilde{Z}(X, V)W) - g(X, V)\eta(\widetilde{Z}(Y, \xi)W) + \eta(V)\eta(\widetilde{Z}(Y, X)W) - g(X, W)\eta(\widetilde{Z}(Y, V)\xi) \\ &+\eta(W)\eta(\widetilde{Z}(Y, V)X) = 0.\end{aligned}$$

Using (2.6) we obtain

$$g(X, R(Y, V)W) - k\{g(X, Y)g(V, W) - g(X, V)g(Y, W)\} = 0. \tag{4.8}$$

Taking $X = Y = e_i$ in (4.8), we obtain

$$S(Y, W) = (a + b)g(Y, W),$$

which implies that the manifold is an Einstein manifold which contradicts the definition of an $N(k)$-quasi Einstein manifold. Then we have $\lambda = \mu(2-n)$.

Conversely, let $\lambda = \mu(2-n)$, then from (2.3), we have $\widetilde{C}(\xi, X)Y = 0$. Hence $\widetilde{C}(\xi, X) \cdot \widetilde{Z} = 0$. □

Remark 2. *The Corollary* 1 *also holds in this case.*

Corollary 2. *From Theorems* 1-4 *the following statements are equivalent:*
i) $\widetilde{C}(\xi, X) \cdot S = 0$,
ii) $\widetilde{C}(\xi X) \cdot P = 0$,
iii) $\widetilde{C}(\xi X) \cdot \widetilde{Z} = 0$,
iv)$\lambda = \mu(2-n)$.

5. Examples of an $N(k)$-quasi Einstein manifold

Example 1. *Let us consider a semi-Riemannian metric g on $\mathbb{R}^4$ by*

$$ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2. \tag{5.1}$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma^2_{11} = \Gamma^2_{33} = -\frac{1}{2x^2}, \qquad \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^3_{23} = \frac{1}{2x^2},$$

$$R_{1221} = R_{2332} = -\frac{1}{2x^2}, \quad R_{1331} = \frac{1}{4x^2}, \quad R_{1232} = 0,$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor R_{ij} *are*

$$R_{11} = R_{33} = -\frac{1}{4(x^2)^2}, \quad R_{22} = -\frac{1}{(x^2)^2}, \quad R_{44} = 0.$$

It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4,g)$ *is*

$$-\frac{3}{2(x^2)^3} \neq 0.$$

We choose the 1-form A as follows

$$\begin{aligned} A_i(x) &= \sqrt{\frac{\{4(x^2)^2+1\}x^2}{6\{(x^2)^2+1\}}}, \text{ for } i = 1,3 \\ &= \sqrt{\frac{2x^2}{3}}, \text{ for } i = 1,2 \\ &= 0, \text{ otherwise} \end{aligned}$$

at any point $x \in \mathbb{R}^4$. *We take the associated scalars as follows:*

$$a = \frac{1}{x^2} \quad \text{and} \quad b = -\frac{3}{2}\frac{1+(x^2)^2}{(x^2)^3}.$$

Here we have

$$R_{11} = ag_{11} + bA_1A_1, \tag{5.2}$$

$$R_{22} = ag_{22} + bA_2A_2, \tag{5.3}$$

$$R_{33} = ag_{33} + bA_3A_3. \tag{5.4}$$

R.H.S. of (5.2) is $ag_{11} + bA_1A_1 = -\frac{1}{4(x^2)^2} = R_{11} =$ *L.H.S of (5.2). Similarly, we can verify (5.3) and (5.4). Now,*

$$\frac{a+b}{n-1} = \frac{\frac{1}{x^2} - \frac{3}{2}\frac{1+(x^2)^2}{(x^2)^3}}{3} = -\frac{3+(x^2)^2}{6(x^2)^3}.$$

In an n-dimensional $N(k)$*-quasi Einstein manifold, the relation*

$$r = na + b,$$

holds. Here we find that $r = 4a + b$ *holds for this example. Therefore,* (M^4, g) *is an* $N(-\frac{3+(x^2)^2}{6(x^2)^3})$*-quasi Einstein manifold.*

Example 2. *We consider the Riemannian metric g on* $\mathbb{R}^4$

$$ds^2 = g_{ij}dx^idx^j = x^1(x^3)^4(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2, \tag{5.5}$$

where $i, j = 1, 2, 3, 4$. *Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are following :*

$$\begin{aligned} \Gamma^3_{11} &= -2x^1(x^3)^3, \quad \Gamma^2_{11} = \frac{1}{2}(x^3)^4, \quad \Gamma^2_{13} = 2x^1(x^3)^3, \\ \Gamma^1_{12} &= \Gamma^3_{23} = \frac{1}{2x^2}, \quad R_{1331} = 6x^1(x^3)^2, \quad R_{11} = 6x^1(x^3)^2. \end{aligned}$$

Also the scalar curvature $r = 0$. *We take the scalars* a *and* b *as follows :*

$$a = x^1x^3 \text{ and } b = -4x^1x^3.$$

We choose the 1*-form* A *as follows :*

$$\begin{aligned} A_i(x) &= \frac{1}{2}\sqrt{x^1(x^3)^4 - 6x^3} \text{ for } i = 1 \\ &= 0, \text{ otherwise.} \end{aligned}$$

From the definition we get

$$R_{11} = ag_{11} + bA_1A_1. \tag{5.6}$$

R.H.S. of (5.6) is $ag_{11} + bA_1A_{11} = 6x^1(x^3)^2 = R_{11} =$ *L.H.S of (5.6). Now,*

$$a + b = x^1x^3 - 4x^1x^3 = -3x^1x^3.$$

So, this is an example of $N(-x^1x^3)$*-quasi Einstein manifold. In this example, we take the scalars* a *and* b*, such that the condition* $r = an + b$ *is satisfied i.e., the condition* $4a + b = 0$ *is satisfied.*

6. Physical Example of an $N(k)$-quasi-Einstein Manifold

This example is concerned with an $N(k)$-quasi-Einstein manifold in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of the Lorentzian manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. Here we consider a perfect fluid $(PRS)_4$ spacetime of non-zero scalar curvature and having the basic vector field U as the timelike vector field of the fluid, that is, $g(U, U) = -1$. An n-dimensional semi-Riemannian manifold is said to be pseudo Ricci-symmetric [4] if the Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X). \tag{6.1}$$

Such a manifold is denoted by $(PRS)_n$.
For the perfect fluid spacetime, we have the Einstein equation without cosmological constant as

$$S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa T(X, Y), \tag{6.2}$$

where κ is the gravitational constant, T is the energy-momentum tensor of type $(0, 2)$ given by

$$T(X, Y) = (\sigma + p)B(X)B(Y) + pg(X, Y), \tag{6.3}$$

with σ and p as the energy density and isotropic pressure of the fluid respectively. Using (6.3) in (6.2)we get

$$S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa[(\sigma + p)B(X)B(Y) + pg(X, Y)]. \tag{6.4}$$

Taking a frame field and contracting (6.4) over X and Y we have

$$r = \kappa(\sigma - 3p). \tag{6.5}$$

Using (6.4) in (6.5), we see that

$$S(X,Y)=\kappa[(\sigma+p)B(X)B(Y)+\frac{(\sigma-p)}{2}g(X,Y)]. \tag{6.6}$$

Putting $Y=U$ in (6.6) and since $g(U,U)=-1$, we get

$$S(X,U)=-\frac{\kappa}{2}[\sigma+3p]B(x). \tag{6.7}$$

Again for $(PRS)_4$ spacetime [4], $S(X,U)=0$. This condition will be satisfied by the equation (6.7) if

$$\sigma+3p=0 \quad \text{as} \quad \kappa\neq 0 \ \text{ and } \ A(X)\neq 0. \tag{6.8}$$

Using (6.5) and (6.8) in(6.6), we see that

$$S(X,Y)=\frac{r}{3}[B(X)B(Y)+g(X,Y)]. \tag{6.9}$$

Thus we can state the followings:

Theorem 5. *A perfect fluid pseudo Ricci-symmetric spacetime is an $N(\frac{2r}{9})$-quasi-Einstein manifold.*

Remark 3. *Equation (6.9) recovers a result of Guha* [14] *which says that a perfect fluid pseudo Ricci-symmetric spacetime is a quasi Einstein manifold with each of its associates scalars equal to $\frac{r}{3}$, r being the scalar curvature. Also, this result has been mentioned by De and Gazi* [5].

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