Tohoku Math. J.
60 (2008), 329–347

n-SASAKIAN MANIFOLDS

Dedicated to Professor Josef Dorfmeister and Jost-Hinrich Eschenburg

OWEN DEARRICOTT

(Received April 24, 2006, revised Novemer 26, 2007)

Abstract. We define a new class of manifolds called n-Sasakian manifolds that enjoy remarkable geometric properties. We furnish examples of such manifolds and make links to the study of isoparametric hypersurfaces. We demonstrate that these examples carry Einstein metrics.

Introduction. The study of 3-Sasakian geometry saw a strong resurgence during the 1990s [BG]. 3-Sasakian manifolds are remarkable Einstein manifolds; each comes along with a companion Einstein geometry on the leaf space of its 3-foliation, which is quaternionic Kähler and carries a second Einstein metric in its canonical variation. The first inhomogeneous examples of 3-Sasakian geometries were attained via the process of 3-Sasakian reduction of circle actions on spheres [BGM]. The reduction process gives a submanifold N of the sphere to which the 3-foliation remains tangent. N remains invariant under the circle action. The geometry on N in turn generates the desired 3-Sasakian geometry on the circle quotient of N.

In this paper we will discuss a generalization of 3-Sasakian manifolds which we call n-Sasakian manifolds. We demonstrate that examples of these geometries carry associated Einstein metrics and will see that their associated foliation enjoys properties in keeping with the 3-Sasakian picture described above. Like the reductions previously mentioned, the examples arise as quotients of submanifolds of the sphere, although these examples do not come from a reduction procedure but rather from the theory of isoparametric hypersurfaces.

1. n-Sasakian manifolds.

DEFINITION 1.1. Let $\pi : M \to B$ be a Riemannian orbifold submersion with totally geodesic leaves such that for any vector $V \in T_x F^n = \mathcal{V}_x$ (vertical vector) tangent to the leaf F and any pair of vectors X and $Y \in T_x M$ it holds that

$$R(X, Y)V = \langle Y, V\rangle X - \langle X, V\rangle Y$$

for each $x \in M$. Then M is said to be n-Sasakian, where $n = \dim F$.

2000 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C25.

Key words and phrases. CR submanifold, isoparametric hypersurface family, Einstein manifold.

During the course of this work the author was supported on an Australian Research Council grant at the Centre for Mathematics and its Applications at the Australian National University. The author thanks the CMA for their hospitality.

REMARK 1.2. For an n-Sasakian manifold O'Neill's structure equations give that the unnormalized sectional curvature on a mixed plane is given by $|X|^2|V|^2 = K(X, V) = |A_X V|^2$. Polarizing this identity twice implies that the O'Neill tensor induces an anti-symmetric Clifford representation of the vertical space on the horizontal space.

2. Homogeneous examples. In subsequent sections we will discuss examples of homogeneous n-Sasakian manifolds. These examples arise from considering contact CR structures that exist on the focal sets of particular isoparametric hypersurfaces with four principal curvatures.

The n-Sasakian geometric conditions are intertwined with the contact CR geometry of the sphere in an intricate manner. The Münzner equations for isoparametric hypersurface families also play a role in the relations they force between the second fundamental form of the focal set and various distributions they define. If the contact CR geometry 'agrees' with these relations, we get the n-Sasakian structure on the quotient of the focal set by the Hopf action. This quotient is a CR submanifold of a complex projective space.

3. On the geometry of contact CR submanifolds in odd-dimensional spheres. Rather than discuss CR manifolds directly, we instead think of them as quotients of contact CR submanifolds of Sasakian manifolds.

DEFINITION 3.1. A submanifold M of a Sasakian manifold is said to be *contact CR* if the structure field ξ is tangent to M and there is a smooth distribution $\mathcal{D}$ of M such that:

(i) $\mathcal{D}$ is invariant with respect to ϕ, i.e., $\phi(\mathcal{D}_x) \subset \mathcal{D}_x$ for each $x \in M$;

(ii) the complementary orthogonal distribution $\mathcal{D}^\perp$ is anti-invariant with respect to ϕ, i.e., $\phi(\mathcal{D}_x^\perp) \subset \mathrm{T}_x(M)^\perp = \nu_x(M)$ for all $x \in M$.

An important property of such manifolds is that the complementary orthogonal distribution is completely integrable and hence the submanifold is foliated.

Let S now denote the shape operator of M.

THEOREM 3.2. *A contact CR submanifold M satisfies $\mathcal{D}S_N\phi(X) = \mathcal{D}\phi(S_N X)$ for $X \in \mathcal{D}$ and $N \in \phi(\mathcal{D}^\perp)$ if and only if the leaves of the foliation are equidistant.*

PROOF. Let $X, Y \in \mathcal{D}$ and $V = a\xi + \phi(N)$. Then we have

$$\begin{aligned}
\langle \nabla_X V, Y\rangle &= a\langle \phi(X), Y\rangle + \langle \nabla_X \phi(N), Y\rangle = -a\langle X, \phi(Y)\rangle + \langle \phi(\nabla_X N), Y\rangle \\
&= -a\langle X, \phi(Y)\rangle - \langle \phi(S_N X), Y\rangle = -a\langle X, \phi(Y)\rangle - \langle S_N\phi(X), Y\rangle \\
&= -a\langle X, \phi(Y)\rangle - \langle \phi(X), S_N Y\rangle = -a\langle X, \phi(Y)\rangle + \langle X, \phi(S_N Y)\rangle \\
&= -a\langle X, \phi(Y)\rangle - \langle X, \phi(\nabla_Y N)\rangle = -a\langle X, \phi(Y)\rangle - \langle X, \nabla_Y \phi(N)\rangle \\
&= -\langle X, \nabla_Y V\rangle\,.
\end{aligned}$$ □

The aforementioned condition is the characterization of bundlelikeness of the foliation on M, namely, the condition we need for an orbifold submersion and for O'Neill's structure equations to hold.

DEFINITION 3.3. A contact CR submanifold M is said to be *mixed totally geodesic* if $S_N\mathcal{D} \subset \mathcal{D}$ for all vectors N normal to M.

THEOREM 3.4. *A contact CR bundlelike foliation on M satisfies $S_{\phi(\mathcal{D}^\perp)}\mathcal{D} \subset \mathcal{D}$ if and only if the leaves of the foliation are totally geodesic in M.*

PROOF. Let $U, V \in \mathcal{D}^\perp \ominus \boldsymbol{R}\xi$ and $X \in \mathcal{D}$. Then we have

$$\langle \nabla_U V, X\rangle = \langle \nabla_U \phi(N), X\rangle = \langle \phi(\nabla_U N), X\rangle = \langle \nabla_U N, \phi(X)\rangle = -\langle S_N U, \phi(X)\rangle\,.$$

The leaves are totally geodesic if and only if this quantity vanishes. Hence the equivalence follows. □

DEFINITION 3.5. A contact CR submanifold M has *contact totally geodesic leaves* in its ambient Sasakian manifold if $S_N V \in \boldsymbol{R}\xi$ for all normal N and $V \in \mathcal{D}^\perp \ominus \boldsymbol{R}\xi$.

THEOREM 3.6. *Let M be a contact CR submanifold of the sphere with a bundlelike foliation with contact totally geodesic leaves in the sphere. Then M/ξ is n-Sasakian, where $n = \dim \mathcal{D}^\perp - 1$.*

PROOF. Let B denote the second fundamental form of the submanifold M. Let X, Y, Z be tangent to M. Then we have

$$\bar{R}(X,Y)Z = R(X,Y)Z - S_{B_Y Z}X + S_{B_X Z}Y + (\nabla_X B)_Y Z - (\nabla_Y B)_X Z\,.$$

Suppose that X, Y are orthogonal to ξ. Then, since we are in a sphere, the last two terms come to 0, since they are normal to M. Also, $\langle B_X V, N\rangle = \langle X, S_N V\rangle = 0$ if $S_N V$ is a multiple of ξ. Hence we get

$$R_M(X,Y)V = \bar{R}(X,Y)V\,.$$

Now we consider the quotient M/ξ of the foliation of ξ. In this case, $\langle A_V Y, \xi\rangle = \langle \phi(V), Y\rangle$. V is carried by ϕ into the normal bundle of M. So it follows immediately from O'Neill's equation that

$$\langle R(X,Y)V, Z\rangle = \langle R_M(X,Y)V, Z\rangle\,,$$

from which it follows that the property

$$R(X,Y)V = \langle Y, V\rangle X - \langle X, V\rangle Y$$

holds on M/ξ. □

REMARK. Note that a contact CR submanifold with equidistant totally geodesic leaves satisfies $A_X\xi = \phi(X)$ and $A_X\phi(N) = -\phi(S_N X)$. It follows that

$$A_{A_X\phi(N)}\phi(K) = -\phi(S_K(-\phi(S_N X))) = \phi^2(S_K S_N X) = -S_K S_N X\,.$$

Hence the property of $\phi(\mathcal{D}^\perp)$ inducing a symmetric Clifford representation on $\mathcal{D}$ via the shape operator is equivalent to that of $\mathcal{D}^\perp \subset T(M/\xi)$ inducing an antisymmetric Clifford representation on $\mathcal{D} \subset T(M/\xi)$ via the O'Neill tensor of the foliation of the quotient, M/ξ.

Analogous results hold in the CR case. In particular, we have the following.

THEOREM 3.7. *Let M be a CR submanifold of a complex projective space with a bundlelike foliation with totally geodesic leaves. Then M is n-Sasakian, where $n = \dim \mathcal{D}^{\perp}$.*

4. Contact CR structures on focal sets of isoparametric submanifolds of spheres with four principal curvatures. For basic notions we refer to [DN1–5] and [FKM]. In the interest of brevity we readily adopt the notation from these papers.

Consider an isoparametric system of hypersurfaces, each with four principal curvatures such that the isoparametric function is invariant under the action of $\mathrm{S}^1 \subset \boldsymbol{C}$.

It follows immediately that the associated triple product has the property

$$\{(za)(zb)(zc)\} = z\{abc\}.$$

With no loss of generality we consider the focal set corresponding to the minimum value of the isoparametric function, M_-. This corresponds to points c with $|c|^2 = 1$ and $\{ccc\} = 6c$. Differentiating these constraints reveals that the tangent space at c is given by

$$\mathrm{T}_c M_- = \{x \in V \mid \langle x, c\rangle = 0, \{ccx\} = 2x\},$$

or, in other words, $\mathrm{T}_c M_- = V_2(c)$ in the notation of [DN1]. The normal bundle to M_- in the sphere is given by $\nu_c M_- = V_0(c)$. Clearly, $V_\mu(zc) = zV_\mu(c)$ for $\mu = 0, 2$, and differentiation of the action of z reveals that $ic \in V_2(c)$.

Note that c and ic are not orthogonal tripotents [DN1, Section 4], but rather $c \in V_2(ic)$ and $ic \in V_2(c)$ (see [DN5] for insight into this case). Let $\mathcal{D}_c = V_2(c) \cap V_2(ic)$. Then we see that $i\mathcal{D}_c = \mathcal{D}_c$. Let $\mathcal{D}_c^{\perp}$ be its orthogonal complement in $V_2(c)$. Note that $(V_2(c) \cap V_0(ic)) \oplus \boldsymbol{R}(ic) \subset \mathcal{D}_c^{\perp}$. One has the following.

THEOREM 4.1. *$(V_2(c) \cap V_0(ic)) \oplus \boldsymbol{R}(ic) = \mathcal{D}_c^{\perp}$ for each $c \in M_-$ if and only if M_- is a contact CR submanifold of S^*.*

PROOF. If M_- is contact CR, then $\phi(\mathcal{D}^{\perp}) \subset V_0(c)$. Hence $i(\mathcal{D}^{\perp}) \subset V_0(c) \oplus \boldsymbol{R}c$ and thus $\mathcal{D}^{\perp} \subset V_0(ic) \oplus \boldsymbol{R}(ic)$. Conversely, the logic runs in reverse. □

In the next section we assume that we have a contact CR structure on M_-. Hence we can orthogonally decompose our vector space V as follows:

$$\begin{aligned} V = \boldsymbol{R}c &\oplus \{(V_2(c) \cap V_2(ic)) \oplus ((V_2(c) \cap V_0(ic)) \oplus \boldsymbol{R}(ic))\} \\ &\oplus \{(V_0(c) \cap V_2(ic)) \oplus (V_0(c) \cap V_0(ic))\}. \end{aligned}$$

5. On the contact CR geometry of the focal set.

THEOREM 5.1. *Let M_- be a contact CR submanifold of the sphere. Then its leaves are equidistant.*

PROOF. At each point c we may define a product via the triple product $x \circ y = \{xcy\}$. For $x \in V_2(c)$ and $u \in V_0(c)$ the shape operator is given by $S_u x = u \circ x$. We have the identity

$$i\{ucx\} = \{(iu)cx\} + \{u(ic)x\} + \{uc(ix)\},$$

where $u \in V_0(c)$ and $x \in V_2(c) \cap V_2(ic)$. We then see that the left-hand side is in $V_2(ic)$, the first two terms on the right-hand side are in $V_0(c)$ and $V_0(ic)$ respectively, and the final term is in $V_2(c)$. Under our assumption of the splitting, this shows that $\mathcal{D}i(u \circ x) = \mathcal{D}u \circ (ix)$ and $i(u \circ x) = u \circ (ix)$ if and only if S_u preserves $\mathcal{D}$ for each $u \in V_0(c) \cap V_2(ic)$. □

It should be noted that

$$S_u S_v x + S_v S_u x = \{uvx\} = 2\langle u, v\rangle x$$

for $u, v \in V_0(c) \cap V_2(ic)$ is equivalent to the condition that $\mathcal{D} \subset V_2(q)$ for every minimal tripotent $q \in V_0(c) \cap V_2(ic)$.

THEOREM 5.2. $(V_2(c) \cap V_0(ic)) \circ (V_2(ic) \cap V_0(c)) \subset \boldsymbol{R}(ic)$ *if and only if for* $q \in V_0(c) \cap V_2(ic)$ *and* $x \in V_2(c) \cap V_2(ic)$ *we have* $q \circ x \in \mathcal{D}$.

PROOF. Let $q \in V_2(ic) \cap V_0(c)$ and $v \in V_2(c) \cap V_0(ic)$. If $x \in V_2(c) \cap V_2(ic)$, then $\langle q \circ x, ic\rangle = \langle q, x \circ (ic)\rangle = 0$ and $\langle q \circ x, v\rangle = \langle x, q \circ v\rangle = 0$, assuming the hypothesis.

Conversely, since v is a scalar multiple of a tripotent and $v \in V_2(c)$, we have $c \in V_2(v/|v|)$ and hence $v \circ v = 2|v|^2 c$. Polarizing this we get $u \circ v = 2\langle u, v\rangle c$, where $u \in V_2(c) \cap V_0(ic)$. Thus $\langle q \circ v, u\rangle = \langle q, u \circ v\rangle = 0$ and $\langle q \circ v, x\rangle = \langle v, q \circ x\rangle = 0$. □

THEOREM 5.3. $(V_2(c) \cap V_0(ic)) \circ (V_2(ic) \cap V_0(c)) \subset \boldsymbol{R}(ic)$ *and* $V_0(c) \cap V_0(ic) = 0$ *if and only if* $V_2(c) \cap V_2(ic) \subset V_2(q)$ *for all minimal tripotents* $q \in V_2(ic) \cap V_0(c)$.

PROOF. Consider $x \in V_2(c) \cap V_2(ic)$ and the minimal tripotent $q \in V_0(c) \cap V_2(ic)$. Then

$$\{qqx\} = 2x - 2\{q(ic)\{q(ic)x\}_{V_0(ic)}\} - \{x(ic)\{qq(ic)\}_{V_0(ic)}\} + \{qqx\}_{V_0(ic)}\,.$$

The third term vanishes, since $\{qq(ic)\} = 2ic$. Since $n \circ x \in \mathcal{D}$, we have $\langle n \circ v, x\rangle = \langle n \circ x, v\rangle = 0$ and hence $\{q(ic)x\}_{V_0(ic)} = 0$. Hence we get

$$\{qqx\} = 2x + \{qqx\}_{V_0(ic)}\,.$$

Since $\langle \{qqx\}, v\rangle = \langle x, \{qqv\}\rangle = 0$, then $\{qqx\} = 2q \circ (q \circ x) \in \mathcal{D}$. Hence $\{qqx\} = 2x$.

Conversely, let iv be a unit element of $V_0(c) \cap V_0(ic)$. If $iu \in V_0(c) \cap V_2(ic)$ is a unit, then $\{(iu)(iu)(iv)\} = 2iv$ but v is also in there, so $\{(iu)(iu)v\} = 2v$. Then $v \in V_2(u) \cap V_2(iu)$. However, we have $V_2(c) \cap V_2(ic) = V_2(u) \cap V_2(iu)$ by assumption. Hence $V_0(c) \cap V_0(ic) = 0$.

Suppose now that $\{qqx\} = 2x$. Then $q \circ (q \circ x) = x$ on $V_2(c) \cap V_2(c)$. Moreover, $2q \circ x = \{(ic)(ic)q \circ x\} = q \circ \{(ic)(ic)x\}$ and $2q \circ x = \{ccq \circ x\} = q \circ \{ccx\}$. Thus the ± 1-eigenspaces are preserved and $q \circ x = x_+ - x_- \in V_2(c) \cap V_2(c)$. □

COROLLARY 5.4. *Let* M_- *be contact CR such that the leaves of the contact CR structure are totally geodesic. Then* M_- *is a generic contact CR submanifold.*

PROOF. $\phi(\mathcal{D}^\perp) = V_2(ic) \cap V_0(c) = V_0(c)$, since $V_0(c) \cap V_0(ic) = 0$ from the previous result. □

COROLLARY 5.5. *Let M_- be contact CR in a sphere. Then the leaves are contact totally geodesic in the sphere if and only if the leaves of the contact CR structure are totally geodesic.*

PROOF. This is a direct consequence of the results above. □

COROLLARY 5.6. *Let M_- be contact CR in the sphere. Then M_- is mixed totally geodesic if and only if the leaves of the contact CR structure are totally geodesic.*

PROOF. This is a direct consequence of the results above. □

THEOREM 5.7. *If M_- is a contact CR submanifold of a sphere with totally geodesic leaves, then M_-/ξ is (m_2+1)-Sasakian.*

PROOF. Let M_- be contact CR. Then the CR structure is bundlelike and the totally geodesic condition on the leaves implies that the leaves are contact totally geodesic in the sphere. It then follows that M_-/ξ is n-Sasakian with $n = \dim \mathcal{D}^\perp - 1 = m_2 + 1$. □

THEOREM 5.8. $(V_2(c) \cap V_0(ic)) \circ (V_2(ic) \cap V_0(c)) \subset \boldsymbol{R}(ic)$ *if and only if* $u \circ (iv) = \langle u, v\rangle ic$ *for* $u, v \in V_2(c) \cap V_0(ic)$.

PROOF. If $v \in V_2(c) \cap V_0(ic)$, then $iv \in V_2(ic) \cap V_0(c)$. Note that

$$i((iv) \circ (iv)) = -2((iv) \circ v) + i\{vvc\}\,.$$

Hence we have

$$0 = -2((iv) \circ v) + i(2|v|^2 c)\,,$$

which implies that

$$(iv) \circ v = |v|^2 (ic)\,.$$

Polarizing this, we have $u \circ (iv) + v \circ (iu) = 2\langle u, v\rangle ic$. Hence

$$\langle u \circ iv, ic\rangle = \langle iv, u \circ ic\rangle = \langle v, iu \circ ic\rangle = \langle iu \circ v, ic\rangle\,.$$

The equivalence is now clear. □

6. Some identities for certain isoparametric triples. In this section we summarize some consequences of the geometric conditions of the previous section for the products $\circ$ and $\{\cdots\}$.

We assume that $V_0(c) \cap V_0(ic) = 0$ and $u \circ (iv) = \langle u, v\rangle ic$ for $u, v \in V_2(c) \cap V_0(ic)$, and

$$V_2(c) = (V_2(c) \cap V_2(ic)) \oplus ((V_2(c) \cap V_0(ic)) \oplus \boldsymbol{R}(ic))\,.$$

Let $x \in V_2(c) \cap V_2(ic)$. From this it also follows that $u \circ (ic) = iu$, $(iv) \circ (ic) = v$ and $x \circ u = 0$. We also obtain from known basic relations that $x \circ (ic) = 0$, $x \circ c = 2x$, $u \circ c = 2u$, $(iv) \circ c = 0$ and that $\circ$ is determined on $\mathcal{D}$, since the triple restricts to a dual FKM-subtriple on $\mathcal{D}$.

We write out some consequences of the above assumptions for the triple product $\{\cdots\}$ by way of making preparation for the theorem of the next section. Let $u, v \in V_2(c) \cap V_0(ic)$

and $x, y \in V_2(c) \cap V_2(ic) \equiv W$. Then we find

$$\begin{aligned}
\{(iu)(iv)(iw)\} &= 2(\langle u, v\rangle(iw) + \langle v, w\rangle(iu) + \langle w, u\rangle(iv))\,,\\
\{(iu)(iv)x\} &= 2\langle u, v\rangle x\,,\\
\{(iu)(iv)w\} &= (iu)\circ((iv)\circ w) + (iv)\circ((iu)\circ w)\\
&= (iu)\circ(\langle v, w\rangle(ic)) + (iv)\circ(\langle u, w\rangle(ic))\\
&= \langle v, w\rangle u + \langle u, w\rangle v\,,\\
\{(iu)(iv)(ic)\} &= (iu)\circ((iv)\circ(ic)) + (iv)\circ((iu)\circ(ic))\\
&= (iu)\circ v + (iv)\circ u = 2\langle u, v\rangle(ic)\,,\\
\{xyv\}_2 &= 2\langle x, y\rangle v - v\circ(x\circ y)_0 = 2\langle x, y\rangle v + \langle v, i(x\circ y)_0\rangle(ic)\\
&= 2\langle x, y\rangle v - \langle iv\circ x, y\rangle(ic)\,,\\
\{uvw\} &= 2(\langle u, v\rangle w + \langle v, w\rangle u + \langle w, u\rangle v)\,,\\
\{uvx\} &= 2\langle u, v\rangle x\,,\\
\{yv(iw)\} &= \langle iw, y\circ v\rangle c + [v\circ(iw\circ y) + y\circ(v\circ iw)]_0 + \{yv(iw)\}_2\\
&= \{yv(iw)\}_2\,.
\end{aligned}$$

Consider $U = u + \alpha c \in (V_2(c) \cap V_0(ic)) \oplus \boldsymbol{R}c \equiv Y$. Then we find

$$\begin{aligned}
\{UUU\} &= \{uuu\} + 3\alpha\{uuc\} + 3\alpha^2\{ccu\} + \alpha^3\{ccc\}\\
&= 6|u|^2u + 6|u|^2\alpha c + 6\alpha^2 u + 6\alpha^3 c = 6|U|^2 U\,,\\
\{UUx\} &= \{uux\} + 2\alpha\{ucx\} + \alpha^2\{ccx\}\\
&= 2|u|^2 x + 2\alpha^2 x = 2|U|^2 x\,,\\
\{UU(iV)\} &= \{uu(iV)\} + 2\alpha\{uc(iV)\} + \alpha^2\{cc(iV)\}\\
&= \{uu(iv)\} + \beta\{uu(ic)\} + 2\alpha\{uc(iv)\} + 2\alpha\beta\{uc(ic)\}\\
&\quad + \alpha^2\{cc(iv)\} + \alpha^2\beta\{cc(ic)\}\\
&= 2\langle u, v\rangle iu + 2\alpha\langle u, v\rangle(ic) + 2\alpha\beta iu + \alpha^2\beta ic\\
&= 2(\langle u, v\rangle + \alpha\beta)(iu + \alpha(ic)) = 2\langle U, V\rangle iU\,.
\end{aligned}$$

Let U, V be unit. If $U = V$, then $\{UU(iU)\} = 2iU$. If U, V are orthogonal, then $\{UU(iV)\} = 0$. Also it holds that

$$\begin{aligned}
i\{UUx\} &= 2\{U(iU)x\} + \{UU(ix)\}\,,\\
2ix &= 2\{U(iU)x\} + 2ix\,,\\
\{U(iU)x\} &= 0\,.
\end{aligned}$$

By polarization we obtain

$$\{U(iV)x\} = -\{V(iU)x\}\,.$$

From above we see that $V_2(c) \cap V_2(ic) \subset V_2(U)$. Moreover, $V_2(c) \cap V_2(ic) = V_2(U) \cap V_2(iU) \subset V_2(U) \cap V_2(iV)$, and U and iV are orthogonal tripotents. Hence we have

$$\{U(iV)\{U(iV)x\}\} = x\,.$$

7. Isoparametricity of such triples.

THEOREM 7.1. *Let $V = Y \oplus iY \oplus W$ be a vector space and suppose that W carries an almost complex structure. Define a symmetric triple $\{\cdots\}$ on V satisfying the following relations for $u, v \in Y$ and $x \in W$.*

(i) $\{uuu\} = 6|u|^2 u$.

(ii) $\{(iu)(iu)(iu)\} = 6|u|^2 iu$.

(iii) $\{uux\} = 2|u|^2 x$.

(iv) $\{(iu)(iu)x\} = 2|u|^2 x$.

(v) $\{uu(iv)\} = 2\langle u, v\rangle iu$.

(vi) $\{u(iv)(iv)\} = 2\langle u, v\rangle v$.

(vii) $\{u(iu)x\} = 0$.

(viii) *For $|u| = |v| = 1$, $v \in u^\perp$ we have $\{u(iv)\{u(iv)x\}\} = x$, that is, $u^\perp$ induces a Ferus-Karcher-Münzner (FKM) system on W defined by $P_k x = \{u(iv_k)x\}$.*

(ix) *$\{\cdots\}$ on W is a dual FKM triple defined via any of the equivalent systems induced from $u^\perp$.*

Then V is an isoparametric triple system splitting orthogonally as described previously with respect to tripotents u and iu.

PROOF. Suppose we have such a triple. We will show that it is isoparametric. We begin with unit $c \in Y$ and $x \in W$ and aim to show that c is a tripotent. We first work under the assumption that $\{xxx\} = 3|x|^2 x$ but will later remove this assumption.

First $\{ccc\} = 6c$. If $\langle c, u\rangle = 0$, then $\{ccu\} = 2|c|^2 u = 2u$, $\{cc(iu)\} = 2\langle u, c\rangle ic = 0$ and $\{cc(ic)\} = 2\langle c, c\rangle ic = 2ic$. Hence the vector space $V = W \oplus Y \oplus iY$ splits up as $V = \boldsymbol{R}c \oplus V_2(c) \oplus V_0(c)$, where

$$V_2(c) = W \oplus c^\perp \oplus \boldsymbol{R}(ic)\,, \quad V_0(c) = ic^\perp\,.$$

Hence $\mathcal{M}(c, a) = 0$ for all $a \in V$ by [DN1, Theorem 2.2].

Let $iu, iv \in V_0(c)$. Then $\{(iu)c(iv)\} = \langle u, c\rangle v + \langle v, c\rangle u = 0$. Hence we obtain [DN1, Equation (2.3)]. Let $x \in W$. Then

$$\begin{aligned}\{(iv)c(x + u + \alpha(ic))\} &= \{(iv)cx\} + \{(iv)cu\} + \alpha\{(iu)c(ic)\}\\ &= \{(iv)cx\} + \langle u, v\rangle ic + \langle v, c\rangle iu + \alpha\langle v, c\rangle c + \alpha\langle c, c\rangle v\\ &= \{(iv)cx\} + \langle u, v\rangle ic + \alpha v\\ &\in W \oplus c^\perp \oplus \boldsymbol{R}(ic)\,,\end{aligned}$$

which implies [DN1, Equation (2.4)]. On the other hand, it holds that

$$\begin{aligned}\{(x+u+\alpha(ic))c(y+v+\beta(ic))\} &= \{xyc\}+\{uvc\}+\alpha\beta\{(ic)(ic)c\}\\ &\quad +\{ucy\}+\{vcx\}+\alpha\{vc(ic)\}+\beta\{uc(ic)\}\\ &= \{xyc\}+2\langle u,v\rangle c+2\alpha\beta c+\alpha iv+\beta iu\,.\end{aligned}$$

By definition, the Y component of $\{xyc\}$ is $2\langle x,y\rangle c$, and moreover $\langle\{xyc\},ic\rangle = \langle\{c(ic)x\},y\rangle = 0$. We conclude that $\{(x+u+\alpha(ic))c(y+v+\beta(ic))\}$ has the $\boldsymbol{R}c\oplus V_2(c)$ component $2\langle x,y\rangle c+2\langle u,v\rangle c+2\alpha\beta c$, namely, $2\langle x+u+\alpha ic,\, y+v+\beta ic\rangle c$. Hence [DN1, Equation (2.5)] follows. We conclude via [DN1, Theorem 2.3(a)] that $\mathcal{M}(c,a,b)=0$ for all $a,b\in V$.

Now, let $iu, iv, iw \in V_0(c)$. It follows immediately that

$$\{(iu)(iv)(iw)\} = 2(\langle u,v\rangle iw+\langle v,w\rangle iu+\langle w,u\rangle iv)\,,$$

from which follows [DN1, Equation (2.6)].

For $x+u+\alpha ic\in V_2(c)$ and $iv\in V_0(c)$ we have

$$\begin{aligned}\{(iv)(iv)(x+u+\alpha ic)\} &= \{(iv)(iv)x\}+\{(iv)(iv)u\}+\alpha\{(iv)(iv)(ic)\}\\ &= 2|v|^2x+2\langle u,v\rangle v+2\alpha|v|^2ic\,,\\ \{(iv)c\{(iv)c(x+u+\alpha ic)\}\} &= \{(iv)c(\{(iv)cx\}+\langle u,v\rangle\{(iv)c(ic)\}+\alpha v)\}\\ &= \{(iv)c\{(iv)cx\}\}+\langle u,v\rangle v+\alpha|v|^2ic\\ &= |v|^2x+\langle u,v\rangle v+\alpha|v|^2ic\,,\end{aligned}$$

hence $\{(iv)(iv)(x+u+\alpha ic)\} = 2\{(iv)c\{(iv)c(x+u+\alpha ic)\}\}$. [DN1, Equation (2.7)] subsequently follows by polarization.

In the presence of the other relations, [DN1, Equation (2.8)] and [DN1, Equation (2.9)] are logically equivalent. For this reason we confirm only [DN1, Equation (2.9)] here. Since $V_2(c) = W\oplus c^{\perp}\oplus\boldsymbol{R}(ic)$ we only need confirm [DN1, Equation (2.9)] for left hand sides $\{vvx\}$, $\{vvv\}$, $\{v(ic)x\}$, $\{(ic)(ic)x\}$, $\{(ic)(ic)v\}$, $\{vv(ic)\}$, $\{(ic)(ic)(ic)\}$, $\{xxx\}$, $\{xx(ic)\}$ and $\{xxv\}$, where $v\in c^{\perp}$ and $x\in W$. [DN1, Equation (2.9)] then follows in general from the multilinearity of $\{\cdots\}$ and from polarization.

$\{vvx\} = 2|v|^2x$. $\{vvc\} = 2|v|^2c$, and hence $\{vvc\}_0 = 0$, $\{vcx\} = 2\langle v,c\rangle x = 0$. Hence the 2 component of the right-hand side of [DN1, Equation (2.9)] is $2|v|^2c$.

$\{(ic)(ic)x\} = 2x$. $\{(ic)cx\} = 0$ and $\{(ic)(ic)c\} = 2c$. Hence $\{(ic)(ic)c\}_0 = 0$, so that the 2 component of the right-hand side of [DN1, Equation (2.9)] is $2x$.

$\{vvv\} = 6|v|^2v$, $\{vvc\} = 2|v|^2c$, and hence $\{vvc\}_0 = 0$ so that the 2 component of the right-hand side of [DN1, Equation (2.9)] is $6|v|^2v$.

$\{(ic)(ic)(ic)\} = 6ic$. $\{(ic)(ic)c\} = 2c$, and hence the 2 component of the right-hand side of [DN1, Equation (2.9)] is $6c$.

$\{xxx\} = 3|x|^2x$ and the right-hand side is $6|x|^2x-3x\circ(x\circ x)_0$, which is consistent with being a dual FKM triple by the relation of Faulkner.

$\{v(ic)x\} = -\{(iv)cx\}$ by assumption. $\{vc(ic)\} = iv$, $\{vcx\} = 0$ and $\{(ic)cx\} = 0$.

$x \circ (v \circ (ic))_0 = \{(iv)cx\}$. Hence [DN1, Equation (2.9)] holds.
$\{xxv\} = 2|x|^2 v + \{xxv\}_{iY}$.

$$\langle \{xxv\}, ic\rangle = \langle x, \{v(ic)x\}\rangle = -\langle x, (iv)\circ x\rangle = -\langle x\circ x, iv\rangle = -\langle (x\circ x)_0, iv\rangle\,.$$

$\langle (x\circ x)_0, iv\rangle ic = v \circ (x\circ x)_0$. Hence [DN1, Equation (2.9)] holds.
$\{xx(ic)\} = 2|x|^2 ic + \{xx(ic)\}_Y$.

$$\langle \{xx(ic)\}, v\rangle = \langle x, \{v(ic)x\}\rangle = -\langle x, (iv)\circ x\rangle = -\langle x\circ x, iv\rangle = \langle i(x\circ x)_0, v\rangle\,.$$

$v\circ (x\circ x)_0 = -i(x\circ x)_0$. Hence [DN1, Equation (2.9)] holds.

Via [DN1, Theorem 2.3(c)] this shows that $c \in Y$, with $|c| = 1$, is a minimal tripotent. It follows essentially verbatim that every $iv \in V_0(c)$ is also a minimal tripotent. Now, set $v_2 = x + u + \alpha ic$, which gives in [DN1, Theorem 3.11(a)]. We already know that

$$\begin{aligned}
v_2\circ v_2 &= 2|u|^2c + 2\alpha^2 c + 2\alpha iu + \{xxc\} = 2(|x|^2+|u|^2+\alpha^2)c + \{xxc\}_{iY},\\
\{v_2v_2v_2\} &= \{xxx\} + \{uuu\} + \alpha^3\{(ic)(ic)(ic)\}\\
&\quad + 3\{xxu\} + 3\alpha\{xx(ic)\} + 3\alpha^2\{(ic)(ic)x\} + 3\alpha^2\{(ic)(ic)u\}\\
&\quad + 3\{uux\} + 3\alpha\{uu(ic)\} + 6\alpha\{u(ic)x\}\\
&= 3|x|^2x + 6|u|^2u + 6\alpha^3 ic + 6|u|^2x + 6|x|^2u + 3\{xxu\}_{iY} + 6\alpha|x|^2 ic\\
&\quad + 3\{xx(ic)\}_Y + 6\alpha^2 x + 6\alpha\{u(ic)x\}\\
&= (3|x|^2 + 6|u|^2 + 6\alpha^2)x + 6(|u|^2+|x|^2)u + 6(\alpha^2+|x|^2)\alpha(ic)\\
&\quad + 3\{xxu\}_{iY} + 3\alpha\{xx(ic)\}_Y + 6\alpha\{u(ic)x\}\,.
\end{aligned}$$

Hence $\langle \{v_2v_2v_2\}, v_2\circ v_2\rangle = 3\langle \{xxu\}_{iY}, \{xxc\}_{iY}\rangle$. We have $|\{xxv\}_{iY}|^2 = |x|^4|v|^2$ by Riesz' representation theorem so that $\langle \{xxu\}_{iY}, \{xxc\}_{iY}\rangle = |x|^4\langle u, c\rangle = 0$. Hence we have [DN1, Theorem 3.11(b)].

Writing

$$\{v_2v_2v_2\} = Ax + Bu + C(\alpha ic) + 3\{xxu\}_{iY} + 3\{xx(ic)\}_Y + 6\alpha\{u(ic)x\}\,,$$

we find

$$\begin{aligned}
|\{v_2v_2v_2\}|^2 &= A^2|x|^2 + B^2|u|^2 + C\alpha^2 + 2(6A+3B+3C)\langle\{u(ic)x\}, x\rangle\\
&\qquad + 9|x|^4|u|^2 + 9|x|^4\alpha^2 + 36|u|^2\alpha^2|x|^2\,,\\
\langle\{v_2v_2v_2\}, v_2\rangle &= A|x|^2 + B|u|^2 + C\alpha^2 + 12\alpha\langle\{u(ic)x\}, x\rangle\,,\\
6A+3B+3C &= 6(3|x|^2 + 6|u|^2 + 6\alpha^2) + 3(6(|u|^2+|x|^2)) + 3(6(\alpha^2+|x|^2))\\
&= 54(|x|^2+|u|^2+\alpha^2) = 54|v_2|^2\,,
\end{aligned}$$

$$
\begin{aligned}
&|\{v_2v_2v_2\}|^2 - 9|v_2|^2\langle\{v_2v_2v_2\}, v_2\rangle \\
&\quad = (A^2 - 9A|v_2|^2)|x|^2 + (B^2 - 9B|v_2|^2)|u|^2 \\
&\qquad + (C^2 - 9C|v_2|^2)\alpha^2 + 108|v_2|^2\langle\{u(ic)x\}, x\rangle \\
&\qquad - 9(12\langle\{u(ic)x\}, x\rangle) + 9|x|^4|u|^2 + 9|x|^4\alpha^2 + 36|u|^2\alpha^2|x|^2 \\
&\quad = A(A - 9|v_2|^2)|x|^2 + B(B - 9|v_2|^2)|u|^2 \\
&\qquad + C(C - 9|v_2|^2)\alpha^2 + 9|x|^4|u|^2 + 9|x|^4\alpha^2 + 36|u|^2\alpha^2|x|^2\,,
\end{aligned}
$$

$$
\begin{aligned}
A - 9|v_2|^2 &= 3|x|^2 + 6|u|^2 + 6\alpha^2 - 9(|x|^2 + |u|^2 + \alpha^2) \\
&= -6|x|^2 - 3|u|^2 - 3\alpha^2 = -3(2|x|^2 + |u|^2 + \alpha^2)\,.
\end{aligned}
$$

Since $A = 3(|x|^2 + 2|u|^2 + 2\alpha^2)$, we have

$$
\begin{aligned}
A(A - 9|v_2|^2) &= -9(2|x|^4 + 5|x|^2|u|^2 + 5|x|^2\alpha^2 + 4|u|^4 + 4\alpha^4 + 8|u|^2\alpha^2)\,, \\
B - 9|v_2|^2 &= 6|x|^2 + 6|u|^2 - 9(|x|^2 + |u|^2 + \alpha^2) \\
&= -3|x|^2 - 3|u|^2 - 9\alpha^2 = -3(|x|^2 + |u|^2 + 3\alpha^2)\,.
\end{aligned}
$$

Since $B = 6(|x|^2 + |u|^2)$, we then have

$$
\begin{aligned}
B(B - 9|v_2|^2) &= -18(|x|^4 + 2|x|^2|u|^2 + |u|^4 + 3\alpha^2|u|^2 + 3\alpha^2|x|^2)\,, \\
C - 9|v_2|^2 &= 6|x|^2 + 6\alpha^2 - 9(|x|^2 + |u|^2 + \alpha^2) = -3|x|^2 - 9|u|^2 - 3\alpha^2 \\
&= -3(|x|^2 + 3|u|^2 + \alpha^2)\,.
\end{aligned}
$$

Since $C = 6(|x|^2 + \alpha^2)$, we also have

$$
C(C - 9|v_2|^2) = -18(|x|^4 + 2|x|^2\alpha^2 + \alpha^4 + 3\alpha^2|u|^2 + 3|u|^2|x|^2)\,.
$$

Therefore, it follows that

$$
\begin{aligned}
&A(A - 9|v_2|^2)|x|^2 + B(B - 9|v_2|^2)|u|^2 + C(C - 9|v_2|^2)\alpha^2 \\
&\quad = -9(2|x|^4 + 5|x|^2|u|^2 + 5|x|^2\alpha^2 + 2|u|^4 + 2\alpha^4 + 4|u|^2\alpha^2)|x|^2 \\
&\qquad - 18(|x|^4 + 2|x|^2|u|^2 + |u|^4 + 3\alpha^2|u|^2 + 3\alpha^2|x|^2)|u|^2 \\
&\qquad - 18(|x|^4 + 2|x|^2\alpha^2 + \alpha^4 + 3\alpha^2|u|^2 + 3|u|^2|x|^2)\alpha^2 \\
&\quad = -9(2|x|^6 + 7|x|^4|u|^2 + 7|x|^4\alpha^2 + 2|u|^6 + 16|u|^2\alpha^2|x|^2 \\
&\qquad\quad + 2\alpha^6 + 6|u|^2\alpha^4 + 6|x|^2\alpha^4 + 6\alpha^2|u|^4)\,, \\
&|\{v_2v_2v_2\}|^2 - 9|v_2|^2\langle\{v_2v_2v_2\}, v_2\rangle \\
&\quad = 9(2|x|^6 + 6|x|^4|u|^2 + 6|x|^4\alpha^2 + 2|u|^6 + 12|u|^2\alpha^2|x|^2 \\
&\qquad + 2\alpha^6 + 6|u|^2\alpha^4 + 6|x|^2\alpha^4 + 6\alpha^2|u|^4) \\
&\quad = -18(|x|^6 + 3|x|^4|u|^2 + 3|x|^4\alpha^2 + 2|u|^6 + 6|u|^2\alpha^2|x|^2 \\
&\qquad\quad + \alpha^6 + 3|u|^2\alpha^4 + 3|x|^2\alpha^4 + 3\alpha^2|u|^4) = -18|v_2|^3\,.
\end{aligned}
$$

Hence we have [DN1, Theorem 3.11(c)]. On the other hand, $\dim V_0(c) = m_2 + 1$ and $\dim V_2(c) = 2m_1 + m_2$, since W is even-dimensional and $c^\perp \oplus \mathrm{R}(ic)$ has dimension $m_2 + 2$. Therefore, $\dim V_2(c) - m_2$ is even and is at least 2. Hence we have [DN1, Theorem 3.11(d), Part (1)].

$$\begin{aligned}
\{(iv)c(x+u+\alpha(ic))\} &= \{(iv)cx\} + \langle u, v\rangle ic + \alpha v\,,\\
\{(iv)c\{(iv)c(x+u+\alpha(ic))\}\} &= \{(iv)c\{(iv)cx\}\} + \langle u, v\rangle v + \alpha\{(iv)cv\}\\
&= x + \langle u, v\rangle v + \alpha ic\,,\\
\langle\{(iv)c\{(iv)cv_2\}\}, v_2\rangle &= |x|^2 + \langle u, v\rangle^2 + \alpha^2\,.
\end{aligned}$$

Fixing v splits the linear operator into the 1-eigenspace, $W \oplus \boldsymbol{R}v \oplus \boldsymbol{R}(ic)$ and the 0-eigenspace, $v^\perp \cap c^\perp$. The trace of the above operator is just the dimension of the 1-eigenspace, which is evidently $2m_1$. Hence we have [DN1, Theorem 3.11(d), Part (2)]. We conclude that v is isoparametric via [DN1, Theorem 3.11].

We backtrack a moment and assume that we have a more general FKM system on W. We can show that indeed the resulting triple again is isoparametric without much alteration of the working above. The dual triple $\{xxx\}' = 9|x|^2x - \{xxx\}$ has $\{xxx\}' = 3|x|^2x + 3x\circ(x\circ x)_0$, consistent with the relation of Faulkner. Since $-\{xxx\}+6|x|^2x = 3x\circ(x\circ x)_0$ has no mention of c on the left, it is independent of the choice of minimal tripotent c. Hence $|(x\circ x)_0|^2 = \langle x\circ(x\circ x)_0, x\rangle$ is also independent of c. Now $|\{xxv\}_{iY}|^2 = |v|^2|\{x(v/|v|)x\}_{iY}|^2 = |v|^2|(x\circ x)_0|^2$. Hence by polarization we get $\langle\{xxu\}_{iY}, \{xxc\}_{iY}\rangle = |(x\circ x)_0|^2\langle u, c\rangle = 0$. In consequence, we have [DN1, Theorem 3.11(b)]. We write $\{xxx\} = 3(|x|^2 - x\circ(x\circ x)_0) + 3|x|^2$ and try to understand the polynomial relation by comparison with the special case.

$$\langle\{v_2v_2v_2\}, v_2\rangle = 3(|x|^4 - |(x\circ x)_0|^2) + A|x|^2 + B|u|^2 + C\alpha^2 + 12\langle\{u(ic)x\}, x\rangle\,,$$

$$\begin{aligned}
||x|^2x - x\circ(x\circ x)_0|^2 &= |x|^6 - 2|x|^2\langle x, x\circ(x\circ x)_0\rangle + |x\circ(x\circ x)_0|^2\\
&= |x|^6 - 2|x|^2|(x\circ x)_0|^2 + |x|^2|(x\circ x)_0|^2\\
&= |x|^2(|x|^4 - |(x\circ x)_0|^2)\,,
\end{aligned}$$

$$\langle|x|^2x - x\circ(x\circ x)_0, |x|^2x\rangle = |x|^2(|x|^4 - |(x\circ x)_0|^2)\,.$$

We consider the expression $\langle\{u(ic)x\}, |x|^2x - x\circ(x\circ x)_0\rangle$, and the product $\circ$ relative to ic. We then get

$$\begin{aligned}
\langle\{u(ic)x\}, |x|^2x - x\circ(x\circ x)_0\rangle &= \langle u\circ x, |x|^2x\rangle - \langle u\circ x, x\circ(x\circ x)_0\rangle\\
&= |x|^2\langle u, (x\circ x)_0\rangle - |x|^2\langle u, (x\circ x)_0\rangle = 0\,.
\end{aligned}$$

Now we have

$$\begin{aligned}
&|\{v_2v_2v_2\}|^2\\
&\quad = 9||x|^2x - x\circ(x\circ x)_0|^2 + 6A\langle|x|^2x - x\circ(x\circ x)_0, |x|^2x\rangle\\
&\qquad + A^2|x|^2 + B^2|u|^2 + C\alpha^2 + 2(6A + 3B + 3C)\langle\{u(ic)x\}, x\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ 9|x|^4|u|^2 + 9|x|^4\alpha^2 + 36|u|^2\alpha^2|x|^2 - 9(|u|^2+\alpha^2)(|x|^4 - |(x\circ x)_0|^2)\\
&= (9|x|^2 + 6(3|x|^2 + 6|u|^2 + 6\alpha^2))(|x|^4 - |(x\circ x)_0|^2)\\
&\quad - 9(|u|^2+\alpha^2)(|x|^4 - |(x\circ x)_0|^2)\\
&\quad + A^2|x|^2 + B^2|u|^2 + C\alpha^2 + 2(6A+3B+3C)\langle\{u(ic)x\}, x\rangle\\
&\quad + 9|x|^4|u|^2 + 9|x|^4\alpha^2 + 36|u|^2\alpha^2|x|^2\\
&= 27(|x|^2 + |u|^2 + \alpha^2)(|x|^4 - |(x\circ x)_0|^2)\\
&\quad + A^2|x|^2 + B^2|u|^2 + C\alpha^2 + 2(6A+3B+3C)\langle\{u(ic)x\}, x\rangle\\
&\quad + 9|x|^4|u|^2 + 9|x|^4\alpha^2 + 36|u|^2\alpha^2|x|^2\,,
\end{aligned}
$$

$$
\begin{aligned}
&9|v_2|^2\langle\{v_2v_2v_2\}, v_2\rangle\\
&= 9(|x|^2 + |u|^2 + \alpha^2)(3(|x|^4 - |(x\circ x)_0|^2)\\
&\quad + A|x|^2 + B|u|^2 + C\alpha^2 + 12\langle\{u(ic)x\}, x\rangle)\\
&= 27(|x|^2 + |u|^2 + \alpha^2)(|x|^4 - |(x\circ x)_0|^2)\\
&\quad + 9(|x|^2 + |u|^2 + \alpha^2)(3A|x|^2 + B|u|^2 + C\alpha^2 + 12\langle\{u(ic)x\}, x\rangle)\,.
\end{aligned}
$$

We see that the only difference is in the first and second terms and, since $27|v_2|^2 - 27|v_2|^2 = 0$, the polynomial is identically zero. Hence we have [DN1, Theorem 3.11(c)]. The remainder follows as previously.

As to the orthogonal splitting we have

$$V_2(c)\cap V_2(ic) = W,\quad V_2(c)\cap V_0(ic) = c^\perp\,,\quad V_2(ic)\cap V_0(c) = ic^\perp\,,\quad V_0(ic)\cap V_0(c) = 0\,.$$

By hypotheses the triple obeys the conditions for the first three theorems of the previous section. □

8. Classification of such triples. 8.1. The dual FKM condition.

THEOREM 8.1. *Let V and $\{\cdots\}$ be as in the previous section. Suppose W contains a minimal tripotent. Then $V, \{\cdots\}$ is a dual FKM triple with $m_2(V) = m_2(W)$.*

PROOF. Consider a triple of the kind specified in the previous section. Suppose $c, u \in Y$ such that $\langle c, u\rangle = 0$. Then $iu \in ic^\perp$ and c are orthogonal tripotents, see [DN1, Section 4]. We then have

$$
\begin{gathered}
V_2(c) = W \oplus c^\perp \oplus \boldsymbol{R}ic\,,\quad V_0(c) = ic^\perp\,,\\
V_2(iu) = W \oplus \boldsymbol{R}u \oplus iu^\perp\,,\quad V_0(iu) = u^\perp\,,\\
V_{12} = V_2(c)\cap V_2(iu) = W \oplus \boldsymbol{R}u \oplus \boldsymbol{R}ic\,,\quad V_{11}^- = V_{22}^- = 0\,,\\
V_{10} = c^\perp \cap u^\perp\,,\quad V_{20} = i(c^\perp\cap u^\perp)\,.
\end{gathered}
$$

If $v \in V_{10}$, then $V_2(v) = W \oplus v^\perp \oplus \boldsymbol{R}iv$. If $iw \in V_{20}$, then $V_2(iw) = W \oplus \boldsymbol{R}w \oplus iw^\perp$. If $a \in V_{10}$, then $iw \circ a = \langle a, w\rangle ic$ and hence $iw \circ V_{10} = \boldsymbol{R}ic$. Similarly, $\{v(iu)V_{20}\} = \boldsymbol{R}u$.

Note that

$$U(iw) = V_{12} \cap V_2(iw) \ominus (iw \circ V_{10}) = W\,,$$
$$U(v) = V_{12} \cap V_2(v) \ominus (v \circ V_{20}) = W\,.$$

Hence $W = Q$ via [DN5, Lemma 2.2]. Moreover, W is dual to a formal FKM triple with $m_2(Q) = m_2(V)$.

Following [DN5, Section 2.5], let $v \in V_{10}$, $iw \in V_{20}$ and $x \in W$. Then $\{v(iw)x\} \in W$. Hence $\{V_{10}V_{20}Q\} \subset Q$. So $Y_{12} = V_{12} \ominus Q = \boldsymbol{R}u \oplus \boldsymbol{R}ic$. Now, $\{u(ic)x\}$, $\{uux\}$ and $\{(ic)(ic)x\}$ are all in $W = Q$ and $T(Y_{12})Q \subset Q$. Note that

$$v \circ iw = \langle v, w\rangle ic\,.$$

Hence $V_{10} \circ V_{20} = \boldsymbol{R}ic$ and $iu \circ ic = u$. Therefore, $C(V_{10} \circ V_{20}) = \boldsymbol{R}u \oplus \boldsymbol{R}ic$.

We set

$$V^{\infty} = V_{11} + V_{10} + V_{22} + V_{20} + C(V_{10} \circ V_{20})\,,$$
$$V^{\infty} = \boldsymbol{R}c + c^{\perp} \cap u^{\perp} + \boldsymbol{R}iu + iu^{\perp} \cap ic^{\perp} + \boldsymbol{R}u \oplus \boldsymbol{R}ic\,,$$
$$V^{\infty} = V \ominus W, \quad Q^{\infty} = V \ominus V^{\infty} = W\,.$$

Now suppose that W contains a minimal tripotent. Then V^{∞} is a subtriple of V that is dual to a formal FKM triple and $m_2(V) = m_2(V^{\infty})$. Since we have assumed that W contains a minimal tripotent, $m_1(W) \geq 0$. W and V^{∞} are dual to FKM triples, and $m_2(V^{\infty}) = m_2(W) = m_2(V)$ and $V = W \oplus V^{\infty}$. Thus V is the dual of an FKM triple via [DN5, Theorem 2.7]. □

8.2. Possible multiplicities. Let us now consider what multiplicities are possible. On the one hand, $2m_1 - 2 = \dim W$ and, since $m = m_2$ is the size of the Clifford system on W, we must have that $m_1 - 1 = k\delta(m_2)$. On the other hand, $m_1 = l\delta(m_2) - m_2 - 1$, since V is the dual of a FKM system. But now this means that $m_2 + 2 = (l - k)\delta(m_2)$. Namely, $\delta(m_2)$ divides $m_2 + 2$. This is only possible for extremely low values of m_2, namely, $m_2 = 0, 1, 2, 6$.

8.3. 3-Sasakian manifolds. Consider the FKM system [FKM] with $m = 2$ defined by the action $P_k : \boldsymbol{H}^n \to \boldsymbol{H}^n$ given by $P_k(q) = -iqe_k$, where $k = 0, 1, 2$. The maximal set for the polynomial F of Ferus, Karcher and Münzner is then given by $|q|^2 = 1$ with $\langle P_k(q), q\rangle = 0$ for $k = 0, 1, 2$. In this context this can be written as $(iq, q) = 0$ and $(q, q) = 1$, where the standard quaternionic hermitian inner product is understood. A vector tangent to the maximal set can thus be thought of as $(x, q) + (q, x) = 0$ and $(ix, q) + (iq, x) = 0$. Let $N = P_k(q)$ be normal to the maximal set. Then $S_N x$ is the component of $-P_k(x)$ tangent to the maximal set.

Normal vectors to $\mathrm{T}_q M_+$ are of the form $iq\bar{p}$, where $p \in \operatorname{Im} \boldsymbol{H}$, and likewise $V_3(iq)$ consists of vectors of the form $q\bar{p}'$, where $p' \in \operatorname{Im} \boldsymbol{H}$. On M_+ we have $\langle q\bar{p}', iq\bar{p}\rangle = 0$ therefore $V_3(iq) \subset \mathrm{T}_q M_+$.

We characterize the set $\mathcal{D}_q$ as also having $(x, iq) + (iq, x) = 0$ and $(ix, iq) + (i(iq), x) = 0$. Hence $-(ix, q) + (iq, x) = 0$ and $(x, q) - (q, x) = 0$. This may be simply written as $(iq, x) = 0$ and $(q, x) = 0$. For $x \in \mathcal{D}_q$ we see that P_k preserves $\mathcal{D}_q$, by simply multiplying these previous two equations on the right by e_k. This means that S_N preserves $\mathcal{D}_q$ for all q

maximal. These maximal sets are homogeneous 3-Sasakian manifolds. For a discussion of these in the broader context of 3-Sasakian geometry, see [BG] and [BGM].

8.4. Sasakian, 3-Sasakian and 7-Sasakian manifolds. We now consider various examples with $m_2 = 0, 2, 6$. Consider $J : \boldsymbol{F}^n \oplus \boldsymbol{F}^n \to \boldsymbol{F}^n \oplus \boldsymbol{F}^n$ defined by sending (a, b) to $(-b, a)$. Define $P_k : V \to V$ by $P_k(o) = -Joe_k$, where o is a $2n$-tuple with ordinates in $\boldsymbol{F}$. Here $\boldsymbol{F} = \boldsymbol{C}, \boldsymbol{H}, \boldsymbol{O}$. As before, one can quickly argue the set of maximal points is o with $|o|^2 = 1$ and $(Jo, o) = 0$. The set $\mathcal{D}_o$ can be characterized by $(Jo, x) = 0$ and $(o, x) = 0$, where the standard hermitian inner product is understood. We cannot use the same argument regarding the shape operator as above for P_k, since $\boldsymbol{F}$ is not necessarily associative. Instead we use a more geometric argument. The normal bundle consists of vectors of the form Jof with f purely imaginary and the antiholomorphic distribution of oe with e purely imaginary.

To see this, consider the inner products $\langle Joe, of\rangle$. We would like to see that these are all zero. This is equivalent to showing $(Jo, oe) = 0$. The key consideration here is that $\boldsymbol{F}$ is a normed division algebra. If $\boldsymbol{F} = \boldsymbol{C}, \boldsymbol{H}$, then the associativity makes matters simple, since

$$\langle Joe, of\rangle = -\langle Jo, (of)e\rangle = -\langle Jo, o(fe)\rangle = 0\,.$$

In the case of $\boldsymbol{F} = \boldsymbol{O}$ this argument does not work in general. However, one can make sense of the case when $n = 1$ and $\boldsymbol{F} = \boldsymbol{O}$. Writing $o = (a, b)$, we have that $\bar{b}a$ is real and, thus, by the identity $x(\bar{x}y) = |x|^2 y$, it follows that $(\bar{b}a)f = \bar{b}(af)$, from which we conclude that $\langle Joe, of\rangle = 0$. In this case, $\mathcal{D}_o^{\perp}$ constitutes the entire tangent space. In general, for $n > 1$, writing $o = ((a_1, \ldots, a_n), (b_1, \ldots, b_n))$, one would need to confirm that if $\sum_k \bar{b}_k a_k$ is real, then $(\sum_k \bar{b}_k a_k) f = \sum_k \bar{b}_k(a_k f)$ for each purely imaginary f to have the above desired property. In these cases the non-associativity of $\boldsymbol{O}$ proves problematic. To see this, we consider the case $n = 2$. We put $a_1 = j$, $b_1 = i$ and $a_2 = jl$, $b_2 = il$. Then $\bar{b}_1 a_1 + \bar{b}_2 a_2 = -ij + (li)(jl) = -k + k = 0$. However, $a_1 i = -k$ and $a_2 i = (jl)i = -i(jl) = (ij)l = kl$, so $\bar{b}_1(a_1 i) + \bar{b}_2(a_2 i) = -i(-k) + (li)(kl) = -2j$. Hence the condition fails for $n > 1$.

We return to the previous cases. On the one hand, $\langle oe, of\rangle = \langle e, f\rangle$, so differentiation of this in the direction $x \in \mathcal{D}_o$ gives

$$\langle xe, of\rangle + \langle oe, xf\rangle = 0\,.$$

On the other hand, $\langle oe, Jof\rangle = 0$, so we differentiate in the direction Jx and see that

$$\langle Jxe, Jof\rangle + \langle oe, J(Jx)f\rangle = 0\,,$$

that is,

$$\langle xe, of\rangle - \langle oe, xf\rangle = 0\,.$$

Adding these two equations together, we get $\langle xe, of\rangle = 0$.

Similarly, differentiating $\langle oe, Jof\rangle = 0$ in the direction x, we get

$$\langle xe, Jof\rangle + \langle oe, Jxf\rangle = 0\,.$$

Also, differentiating $\langle oe, of\rangle = \langle e, f\rangle$ in the direction Jx, we get

$$\langle Jxe, of\rangle + \langle oe, Jxf\rangle = 0\,,$$

which may be rewritten as

$$-\langle xe, Jof\rangle + \langle oe, Jxf\rangle = 0\,,$$

and hence we conclude that $\langle xe, Jof\rangle = 0$. But now $x \in \mathcal{D}_o$, so we have $\langle xe, o\rangle = 0$ and $\langle xe, Jo\rangle = 0$. Collecting all of this information together, we see that $xe \in \mathcal{D}_o$. Hence $S_{-Joe}x = -Jxe \in \mathcal{D}_o$. This construction generates 1- and 3-Sasakian manifolds and a single 7-Sasakian manifold.

8.5. A 5-Sasakian manifold. To finish our discussion we look at the case with multiplicities (5, 4). This is the only remaining permissible case which is not dual FKM. We consider the homogeneous triple defined in [DN1] on $A_5(\boldsymbol{C})$. Consider a fixed maximal tripotent

$$\omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} J & 0\\ 0 & 0\end{pmatrix}.$$

One quickly calculates the spaces

$$V_3(\omega) = \left\{\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix} \,\middle|\, A = JA^*J\right\}$$

and

$$V_3(i\omega) = \left\{\begin{pmatrix} C & 0\\ 0 & 0\end{pmatrix} \,\middle|\, C = -JC^*J\right\},$$

whereby it follows that

$$\langle A, C\rangle = -\langle JA^*J, JC^*J\rangle = -\langle A, C\rangle\,,$$

and hence $V_3(i\omega) \subset V_1(\omega)$. Thus $\mathcal{D}_\omega$ consists of matrices $\left(\begin{smallmatrix} 0 & b\\ -b^T & 0\end{smallmatrix}\right)$ in agreement with the horizontal space of the submersion mentioned in [DE]. Moreover, $\mathcal{D}_\omega^\perp$ is the vertical distribution of this submersion whose leaves we know to be totally geodesic, since they occur as the fixed point set of involutions.

9. The Einstein condition. When $n = 3$ the definition of n-Sasakian manifolds presented here is a weaker notion than the traditional one which also requires that the V-bundle be principal and leads to the conclusion that such a 3-Sasakian manifold is Einstein. One might think that in general an n-Sasakian manifold should be Einstein, but this fails. For example, a Sasakian geometry need not be Einstein. However, the particular examples of n-Sasakian geometries discussed in the previous section do have associated Einstein metrics. In this section we explain this. We draw heavily from [B].

LEMMA 9.1. *The V-bundle connection associated with an n-Sasakian manifold is Yang-Mills.*

PROOF. By O'Neill's fundamental equations of a submersion we have

$$0 = \langle R(X, Y)X, U\rangle = \langle(\nabla_X A)_X Y, U\rangle\,.$$

Hence, letting X run through a basis for $\mathcal{H}$ and summing, we get $\check{\delta} A = 0$. □

THEOREM 9.2. *Let M be a generic CR submanifold of complex projective space that has bundlelike leaves which are totally geodesic in complex projective space. Then $F \subset M \to N$ carries two Einstein metrics in its canonical variation*, *provided* $\dim F > 1$.

PROOF. Again the proof is via structure equations. First, using the Gauss equation, we look at the normalized sectional curvature of the submanifold, that is,

$$K(E, F) = |E|^2|F|^2 - \langle E, F\rangle^2 + 3\langle JE, F\rangle^2 + \langle B_E E, B_F F\rangle - |B_E F|^2 .$$

Now let $F = F_i$ in the above expression, where $\{F_i\}$ form an orthonormal basis for the tangent space of the submanifold, and sum over all i. We then have

$$\mathrm{Ric}(E) = (\dim M - 1)|E|^2 + 3|X|^2 + \langle B_E E, H\rangle - \sum_i |B_E F_i|^2 ,$$

where H is the mean curvature vector of the submanifold.

The first important observation is that $H = 0$. First of all, the totally geodesic condition on the leaves yields $B_F F = 0$ for vertical F. So we take $H = \sum_i B_{F_i} F_i$ with $\{F_i\}$ a basis for $\mathcal{H} = \mathcal{D}$. M is n-Sasakian, where $n = \dim \mathcal{V} = \dim \mathcal{D}^{\perp}$. Moreover, the generic condition gives that the shape operator induces a symmetric Clifford representation of the normal bundle on the horizontal space. To see $H = 0$, fix a normal vector N and choose a basis $\{F_i\}$ for $\mathcal{H}$ which is split between the ± 1-eigenspaces of S_N. Here we use the condition $\dim F > 1$. Then we obtain

$$\begin{aligned}\langle H, N\rangle &= \sum_i \langle B_{F_i} F_i, N\rangle = \sum_i \langle S_N F_i, F_i\rangle \\ &= r(1) + r(-1) = 0 ,\end{aligned}$$

where $2r = \dim \mathcal{H}$.

We now turn our attention to the last term $B_E F_i = B_X F_i$, which is zero if F_i is vertical, so that we may take our sum over a basis for $\mathcal{H}$. By X here we mean the horizontal component of E. Then we have

$$\begin{aligned}\sum_i |B_X F_i|^2 &= \sum_{i,j} \langle B_X F_i, N_j\rangle^2 = \sum_{i,j} \langle S_{N_j} X, F_i\rangle^2 \\ &= \sum_j |S_{N_j} X|^2 = \dim \mathcal{V} |X|^2 ,\end{aligned}$$

$$\begin{aligned}\mathrm{Ric}(E) &= (\dim M - 1)|E|^2 + (3 - \dim \mathcal{V})|X|^2 , \\ \mathrm{Ric}^N(X) &= \mathrm{Ric}(X) + 2(A_X, A_X) , \\ \mathrm{Ric}^N(X) &= \mathrm{Ric}(X) + 2\dim \mathcal{V}|X|^2 , \\ \mathrm{Ric}^N(X) &= (\dim M + \dim \mathcal{V} + 2)|X|^2 .\end{aligned}$$

Hence N is Einstein with Einstein constant $\dim M + \dim \mathcal{V} + 2$. On the other hand,

$$\mathrm{Ric}^F(U) = \mathrm{Ric}(U) - (AU, AU)\,,$$
$$\mathrm{Ric}^F(U) = (\dim M - 1)|U|^2 - \dim \mathcal{H}|U|^2\,,$$
$$\mathrm{Ric}^F(U) = (\dim \mathcal{V} - 1)|U|^2\,.$$

Hence F is Einstein with Einstein constant $\dim \mathcal{V} - 1$. For the canonical variation to have two Einstein metrics contained in it, we require that

$$(\dim M + \dim \mathcal{V} + 2)^2 - 3(\dim \mathcal{V} - 1)(\dim \mathcal{H} + 2\dim \mathcal{V}) > 0\,,$$

or, equivalently,

$$(\dim M + \dim \mathcal{V} + 2)^2 > 3(\dim \mathcal{V} - 1)(\dim M + \dim \mathcal{V})\,.$$

We have already that $\dim \mathcal{H} \geq 2\dim \mathcal{V} - 2$, since the normal bundle induces a Clifford representation on the horizontal space. Hence, $\dim M + 2 \geq 3\dim \mathcal{V}$. The required inequality follows immediately. □

COROLLARY 9.3. *Let M, N, F be as above. Then the leaves F are real projective plane and N is a Kähler-Einstein manifold.*

PROOF. Since the leaves F are anti-invariant totally geodesic submanifolds of complex projective space, they are real projective spaces, see [A]. Let $X, Y, Z \in \mathcal{D}$. Then we have

$$0 = \langle \bar{\nabla}_X(JY) - J(\bar{\nabla}_X Y), Z\rangle = \langle \nabla^*_X(JY) - J(\nabla^*_X Y), Z\rangle\,.$$

Hence N is Kähler. We have already seen that N is Einstein. □

It follows that all examples in the previous section contain two distinct Einstein metrics in their canonical variation over a Kähler-Einstein manifold. The condition that $\dim F > 1$ may be dropped, since the equality of dimensions of the ± 1-eigenspaces of S_N follows from noting that the map sending (a, b) to $(-bi, ai)$ has (a, ai) and $(a, -ai)$ as its ± 1-eigenspaces respectively, together with the fact that the examples are homogeneous. This gives that M is minimal and the remainder of the proof follows as before.

REFERENCES

[A] K. ABE, Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions, Tôhoku Math. J. 25 (1973), 425–444.

[B] A. L. BESSE, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete, 10. Springer-Verlag, Berlin, 1987.

[BG] C. BOYER AND K. GALICKI, 3-Sasakian manifolds in Surveys in differential geometry: Essays on Einstein manifolds, 123–184, Surv. Differ. Geom., VI, Int. Press, Boston, Mass., 1999.

[BGM] C. BOYER, K. GALICKI AND B. MANN, The geometry and topology of 3-Sasakian manifolds, J. Reine Angew. Math. 455 (1994), 183–220.

[DE] O. DEARRICOTT AND J.-H. ESCHENBURG, Totally geodesic embeddings of 7-manifolds in positively curved 13-manifolds, Manuscripta Math. 114 (2004), 447–456.

[DN1] J. DORFMEISTER AND E. NEHER, An algebraic approach to isoparametric hypersurfaces in spheres. I, Tôhoku Math. J. 35 (1983), 187–224.

[DN2] J. DORFMEISTER AND E. NEHER, An algebraic approach to isoparametric hypersurfaces in spheres. II, Tôhoku Math. J. 35 (1983), 225–247.

[DN3] J. DORFMEISTER AND E. NEHER, Isoparametric triple systems of FKM-type I, Abh. Math. Sem. Univ. Hamburg 53 (1983), 191–216.

[DN4] J. DORFMEISTER AND E. NEHER, Isoparametric triple systems of FKM-type II, Manuscripta Math. 43 (1983), 13–44.

[DN5] J. DORFMEISTER AND E. NEHER, Isoparametric triple systems with special Z-structure, Algebras Groups Geom. 7 (1990), 21–94.

[FKM] D. FERUS, H. KARCHER AND H.-F. MÜNZNER, Cliffordalgebren und neue isoparametrische Hyperflächen, Math. Zeitschrift 177 (1981), 479–502.

DARTMOUTH COLLEGE
DEPARTMENT OF MATHEMATICS
6188 KEMENY HALL
HANOVER, NH 03755–3551
U.S.A.

E-mail address: owen.dearricott@dartmouth.edu