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N-Soliton Collision in the Manakov Model

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We investigate soliton collisions in the Manakov model, which is a system of coupled nonlinear Schrödinger equations that is integrable via the inverse scattering method. Computing the asymptotic forms of the general N-soliton solution in the limits $t \to \mp\infty$, we elucidate a mechanism that factorizes an N-soliton collision into a nonlinear superposition of $\binom{N}{2}$ pair collisions with arbitrary order. This removes the misunderstanding that multiparticle effects exist in the Manakov model and provides a new "set-theoretical" solution to the quantum Yang-Baxter equation. As a by-product, we also obtain a new nontrivial relation among determinants and extended determinants.

§1. Introduction

In recent years, there has been a surge of interest in some systems of coupled nonlinear Schrödinger (coupled NLS) equations because of their relevance in nonlinear optics.[1)–3)] Among such systems, this paper focuses on the following system of coupled NLS equations:

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\|\boldsymbol{q}\|^2\boldsymbol{q} = \boldsymbol{0}, \qquad \boldsymbol{q} = (q_1, q_2, \cdots, q_m). \tag{1·1}$$

Here $\|\boldsymbol{q}\|^2 \equiv \boldsymbol{q}\cdot\boldsymbol{q}^\dagger = \sum_{j=1}^{m} |q_j|^2$, where the superscript $\dagger$ denotes Hermitian conjugation. The subscripts t and x denote the partial differentiation with respect to these variables. It is well known that (1·1) is a completely integrable system.[4),5)] We call (1·1) the Manakov model, since the two-component ($m = 2$) case of (1·1) was solved for the first time by Manakov[4)] using the inverse scattering method (ISM). The extension of the ISM to the general m-component case is straightforward.[5)] Nevertheless, the value of m is extremely important when we consider soliton solutions. The $m = 2$ case is merely a special case of the general m case. The most interesting is the case in which the total number of solitons, say N, is equal to the number of components, m. Indeed, in the $N = m$ case, the coefficient vectors for the hyperbolic-secant-type envelope of solitons [see, e.g., $\boldsymbol{u}_1$ in (2·12)], which we refer to after normalization as *polarization vectors*, are just sufficient to span the vector space $\mathbb{C}^m$, in which $\boldsymbol{q}$ exists. Soliton solutions in the other cases ($N > m$ or $N < m$) are obtained from those in this case through a special choice of soliton parameters or the operation of a unitary transformation. In this paper, we consider the most general case, in which N and m are arbitrary positive integers. Then, the most interesting case, $N = m$, is automatically included.

Although the integrability of the Manakov model (1·1) has been established

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formally through application of the ISM, multi-soliton dynamics in the model remain to be clarified. There are two reasons for this. One reason is that the vector nature of (1·1), which supports the internal degrees of freedom of solitons, leads to complicated behavior. Indeed, even a two-soliton collision is highly nontrivial,[6)–9)] as not only does a displacement of the soliton centers depend on the initial polarization vectors but also the polarization vectors themselves are changed. Therefore, in this model, the effect of an N-soliton collision can never be written as the *algebraic* sum of the effects of pair collisions (at least, if we employ only the initial soliton parameters). This is quite different from the NLS case [i.e. (1·1) with $m = 1$], in which the effect of an N-soliton collision in fact can be written as the *algebraic* sum of the effects of pair collisions (in which the order of the pair collisions does not matter).[10)–12)] The other reason is that Manakov gave a rather misleading description of an N-soliton collision in Ref. 4). We quote the corresponding part, the first two sentences of the last paragraph of §2 in Ref. 4):

> Comparison of relations (17) and (18) indicates that an N-soliton collision does not, in general, reduce to a pair collision. This is clear, for example, from the fact that the expression for $\boldsymbol{S}_k^+$ contains $\boldsymbol{S}_j^+$ with $j > k$, which depend on the initial parameters of all the remaining solitons. (♣)

The relations (17) referred to here are given by

$$\boldsymbol{S}_N^+ = \left\{ \prod_{n<N} \alpha_{11}^{-1}(\zeta_N, \zeta_n) \right\} \hat{\alpha}^T(\zeta_N, \zeta_1, \boldsymbol{S}_1^-) \cdots \hat{\alpha}^T(\zeta_N, \zeta_{N-1}, \boldsymbol{S}_{N-1}^-) \boldsymbol{S}_N^-,$$

$$\begin{aligned} \boldsymbol{S}_i^+ = &\left\{ \prod_{k>i} \alpha_{11}(\zeta_i, \zeta_k) \right\} \left\{ \prod_{n<i} \alpha_{11}^{-1}(\zeta_i, \zeta_n) \right\} \hat{\alpha}^*(\zeta_i^*, \zeta_{i+1}, \boldsymbol{S}_{i+1}^+) \cdots \hat{\alpha}^*(\zeta_i^*, \zeta_N, \boldsymbol{S}_N^+) \\ &\cdot \hat{\alpha}^T(\zeta_i, \zeta_1, \boldsymbol{S}_1^-) \cdots \hat{\alpha}^T(\zeta_i, \zeta_{i-1}, \boldsymbol{S}_{i-1}^-) \boldsymbol{S}_i^-, \qquad i = 1, 2, \cdots, N-1, \end{aligned}$$

while the relations (18) are given by

$$\begin{aligned} \boldsymbol{S}_2^+ &= \alpha_{11}^{-1}(\zeta_2, \zeta_1)\, \hat{\alpha}^T(\zeta_2, \zeta_1, \boldsymbol{S}_1^-) \boldsymbol{S}_2^-, \\ \boldsymbol{S}_1^+ &= \alpha_{11}(\zeta_1, \zeta_2)\, \hat{\alpha}^*(\zeta_1^*, \zeta_2, \boldsymbol{S}_2^+) \boldsymbol{S}_1^-. \end{aligned}$$

We now briefly explain the situation considered and the notation used in Ref. 4). The equations (17) of Ref. 4) represent the solution for the collision of N solitons (to which we refer as solitons-1, 2, $\cdots$, N), while the equations (18) represent that for two solitons. We note that the equations (18) are obtained from (17) by setting $N = 2$. The equations (17) were derived through analysis based on and intuition gained from the ISM. It is assumed that a soliton designated by a larger number moves faster along the x-axis. That is, soliton-i *overtakes*[*)] solitons-1, 2, $\cdots$, $i-1$ and is overtaken by solitons-$i+1$, $i+2$, $\cdots$, N as time t passes from $-\infty$ to $+\infty$. The velocity of soliton-i and its amplitude are determined by the complex parameter ζ_i, which is time independent. The quantities $\boldsymbol{S}_i$ are column vectors with

*) Throughout this paper, we use the term "overtake" if only the relative velocity is positive. Thus it can be used for head-on collisions, etc.

two complex components, corresponding to the choice of $m = 2$ in Ref. 4). The vector norm $\|\boldsymbol{S}_i\| \left[= (\boldsymbol{S}_i^\dagger \cdot \boldsymbol{S}_i)^{\frac{1}{2}}\right]$ determines the center position of soliton-i, while the normalized vector $\boldsymbol{S}_i/\|\boldsymbol{S}_i\|$ gives its polarization vector after the operation of Hermitian conjugation. The superscripts $+$ and $-$ denote the final state ($t \to +\infty$) and the initial state ($t \to -\infty$), respectively. α_{11} is a scalar function and $\hat{\alpha}$ is a 2×2 matrix function. We do not give their explicit forms, which are not important in the following discussion. The superscripts T and $*$ denote the operations of transposition and complex conjugation, respectively.

Although the assertion (♣) is somewhat ambiguous, the most natural and reasonable interpretation seems to be the following:

> Let us try to explain (17) by assuming that the N solitons collide pairwise in accordance with (18). Then, the first equation in (17) can be understood as follows. Soliton-N first overtakes soliton-$N-1$ in its initial state, i.e. soliton-$N-1$, which has not collided with other solitons. Next, soliton-N overtakes soliton-$N-2$, which has not collided with other solitons. ... Finally, it overtakes soliton-1, which has not collided with other solitons.
>
> Similarly, the second equation in (17) can be understood as follows. Soliton-i overtakes solitons-$i-1$, $i-2$, $\cdots$, 1 in this order, none of which has collided with other solitons. Next, soliton-i is overtaken by solitons-N, $N-1$, $\cdots$, $i+1$ in their final states, i.e. those which will not collide with other solitons.
>
> If we attempt to diagram these events, we immediately encounter a contradiction. This *indicates* that an N-soliton collision cannot be described as a sequence of pair collisions, since the matrices $\hat{\alpha}$ for different sets of arguments do not commute in general.

The logic of the above interpretation is not mathematically rigorous, but it seems to be correct if the complex structure of (17) is taken into account. It might also be possible to interpret the assertion (♣) in a different way. In any case, it appears that an N-soliton collision in the Manakov model (1·1) does not reduce to a pair collision, and thus that some multi-particle effects exist in the Manakov model. However, in fact this is not true.

The main goal of this paper is to clear up this misunderstanding. We explicitly demonstrate a mechanism that factorizes an N-soliton collision in the Manakov model (1·1) into a *nonlinear* superposition of pair collisions. Here, we have used the term "nonlinear" to express the fact that the considered superposition is no longer additive. For the sake of definiteness, we explain in advance what we prove in terms of the Manakov notation, which also gives the definition of *factorization* in this paper. We first interpret the equations (18) as forming a nonlinear mapping with two complex parameters, $f(\zeta_2, \zeta_1)$, which maps the initial state $\{\boldsymbol{S}_2^-, \boldsymbol{S}_1^-\}$ into the final state $\{\boldsymbol{S}_2^+, \boldsymbol{S}_1^+\}$. Then, we can use the mapping $f(\zeta_j, \zeta_k)$ to evaluate in an N-soliton collision the effect of the two-soliton collision whereby soliton-j in a state $\boldsymbol{S}_j$ overtakes soliton-k in a state $\boldsymbol{S}_k$. For a given order of $\binom{N}{2}$ pair collisions, we consider the corresponding composition of the $\binom{N}{2}$ mappings: $f(\zeta_j, \zeta_k)$, $N \geq j > k \geq 1$.

Then, regardless of the order of the pair collisions,*) the composed mapping maps the initial state $\{\boldsymbol{S}_N^-, \boldsymbol{S}_{N-1}^-, \cdots, \boldsymbol{S}_1^-\}$ exactly to the final state $\{\boldsymbol{S}_N^+, \boldsymbol{S}_{N-1}^+, \cdots, \boldsymbol{S}_1^+\}$ given by the equations (17).

To prove this factorization, we do not employ the Manakov results, the equations (17) and (18), for the following two reasons. One reason is that, although it is ingenious and seems to be correct, the derivation of (17) in Ref. 4) is neither very rigorous nor understandable to the reader not familiar with the ISM. The other reason is that the equations (17) are not tractable for our purpose. In this paper, we employ a more straightforward approach to obtain another formula for the asymptotic behavior of N solitons. We start from an explicit formula for the N-soliton solution of the matrix NLS equation derived using the ISM in Ref. 13). Through a simple reduction, we obtain an explicit formula for the general N-soliton solution of the Manakov model (1·1).[14] To make the paper self-contained, we first set $N = 2$ and compute the asymptotic forms of the two-soliton solution in the $t \to \mp\infty$ limits in our notation. These solutions define the collision laws of two solitons in the Manakov model, which are essentially the same as those given by the equations (18). Next, we consider the general N case and compute the asymptotic forms of the N-soliton solution in the $t \to \mp\infty$ limits. To express the polarization vectors appearing in the asymptotic forms concisely, we extend the definition of a determinant in such a way that the last column of an extended determinant consists of vectors. This determinant represents a vector defined in terms of the Laplace expansion with respect to the last column. We find a beautiful relation which casts the Hermitian product between such extended determinants into the form of a product of conventional determinants. Using this relation and the Jacobi formula for determinants, we prove that an N-soliton collision in the Manakov model (1·1) can be factorized into a nonlinear superposition of $\binom{N}{2}$ pair collisions in arbitrary order. This reveals**) the following two properties of the Manakov model:

(a) An N-soliton collision is composed of a nonlinear superposition of $\binom{N}{2}$ pair collisions of certain order.

(b) The composition of $\binom{N}{2}$ mappings corresponding to pair collisions in an N-soliton collision yields the same mapping for every possible order of composition.

We remark that solitons are not mass points, and do not have compact support. Rather, they are structures with infinitely long tails. Even in a two-soliton collision, it takes an infinite time for the solitons to completely recover their own shapes. Taking this into consideration, it is more meaningful to understand this factorization conceptually than phenomenologically. It is unlikely that properties (a) and (b) are related directly. We note that proving (a) is equally difficult for every order of pair collisions and that proving (b) for arbitrary N reduces to proving it for $N = 3$, that is, the following:

*) Here, we are referring to the order in which *neighboring* solitons collide pairwise. That is, unlike the $m = 1$ case (scalar NLS), we do not consider virtual collisions between non-neighboring solitons.

**) To deduce property (b) from the factorization rigorously in the present setting, some discussion is needed. This is given in §5.

(b') The composition of three mappings corresponding to pair collisions in a three-soliton collision does not depend on the (conceptional) order of pair collisions.

The property (b') is called the Yang-Baxter property. The validity of this property means that the collision laws of two solitons in the Manakov model (1·1) give a "set-theoretical" solution to the quantum Yang-Baxter equation.[15] To the best of the author's knowledge, this solution is new. We mention that another "set-theoretical" solution to the quantum Yang-Baxter equation is studied by Veselov[16] through investigation of the matrix KdV equation.

A few interesting ideas[17)–19)] have been proposed to explain in a general manner the pairwise nature of soliton collisions in integrable systems. However, those ideas seem to be too intuitive in their present forms to complete a mathematically rigorous proof. For instance, it is not obvious that a pair collision is unaffected by the other solitons given only that they are far away, or that the final result is invariant when the order of multiple limits is changed. In addition, unlike this work, those works do not elucidate a nice mechanism of factorization.

Finally, we would like to comment on the literature concerning the Manakov model (1·1). Multi-soliton solutions of the Manakov model have already been obtained using the Hirota method[20)–22)] (see also Refs. 23) and 24) for results obtained with another method). In this sense, although it is very useful, the explicit formula for the N-soliton solution obtained in this paper may not be essentially new.[*] The main contribution of this work is the elucidation of the pairwise nature of soliton collisions in the Manakov model. The results of this paper were obtained by the author in the summer of 2000 and presented at the autumn meeting of the Physical Society of Japan in that year. Very recently, he encountered some papers[25)–27),**] which pose the same problem (but do not solve it completely). In particular, in Ref. 27), which actually appeared after the first submission of this paper, Kanna and Lakshmanan gave a "proof" of the pairwise collision nature of a three-soliton collision. Unfortunately, the "proof" of Kanna-Lakshmanan[27] is *absolutely incorrect*. Indeed, in addition to a few fatal mistakes, their "proof" as a whole is a typical example of *circular reasoning*.

The remainder of this paper is organized as follows. In §2, we derive an explicit formula for the N-soliton solution of the Manakov model.[14] In §3, we obtain the collision laws of two solitons. In §4, we compute the asymptotic forms of the N-soliton solution in the limits $t \to \mp\infty$. In §5, we elucidate a mechanism that factorizes an N-soliton collision into a nonlinear superposition of pair collisions and discuss the Yang-Baxter property. Section 6 is devoted to concluding remarks.

§2. Explicit formula for the general N-soliton solution

In this section, considering a reduction of a formula given in Ref. 13), we derive an explicit formula for the general N-soliton solution of the Manakov model (1·1).

In Ref. 13), under vanishing boundary conditions, we applied the ISM to nonlin-

[*] In any case, our formula has the advantage of compactness in its own right.

[**] Actually, the work of Park and Shin[25] is very similar to that of Steudel.[28]

ear evolution equations associated with the generalized Zakharov-Shabat eigenvalue problem:

$$\left[\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right]_x = \left[\begin{array}{cc} -\mathrm{i}\zeta I & Q \\ -Q^\dagger & \mathrm{i}\zeta I \end{array} \right] \left[\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right]. \tag{2·1}$$

Here, ζ is the spectral parameter, I is the $m \times m$ unit matrix, and Q is an $m \times m$ matrix-valued potential function. The first two of the nonlinear evolution equations associated with (2·1) are the matrix NLS equation,

$$\mathrm{i}Q_t + Q_{xx} + 2QQ^\dagger Q = O, \tag{2·2}$$

and the matrix complex mKdV equation,

$$Q_t + Q_{xxx} + 3(Q_x Q^\dagger Q + QQ^\dagger Q_x) = O. \tag{2·3}$$

We mention that integrable space-discretizations of (2·2) and (2·3) were found recently[29)] (see also the relevant work in Refs. 22) and 30)–32)). The general N-soliton solution of (2·2) or (2·3) is expressed as[13)]

$$Q(x,t) = -2\mathrm{i}(\underbrace{I\ I\ \cdots\ I}_{N})\ S^{-1} \left(\begin{array}{c} C_1(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_1^* x} \\ C_2(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_2^* x} \\ \vdots \\ C_N(t)^\dagger \mathrm{e}^{-2\mathrm{i}\zeta_N^* x} \end{array} \right), \tag{2·4}$$

where the $mN \times mN$ matrix S is given by

$$S_{jk} = \delta_{jk} I - \sum_{l=1}^{N} \frac{\mathrm{e}^{2\mathrm{i}(\zeta_l - \zeta_j^*)x}}{(\zeta_l - \zeta_k^*)(\zeta_l - \zeta_j^*)} C_j(t)^\dagger C_l(t), \qquad 1 \le j,k \le N. \tag{2·5}$$

Here, ζ_j are discrete eigenvalues in the upper-half plane of ζ ($\mathrm{Im}\,\zeta_j > 0$), each of which determines a bound state in the potential Q. The quantities $C_j(t)$ are $m \times m$ nonzero matrices whose time dependences are given by

$$C_j(t) = C_j(0)\mathrm{e}^{4\mathrm{i}\zeta_j^2 t}, \qquad j = 1, 2, \cdots, N, \tag{2·6}$$

for the matrix NLS equation (2·2), and

$$C_j(t) = C_j(0)\mathrm{e}^{8\mathrm{i}\zeta_j^3 t}, \qquad j = 1, 2, \cdots, N,$$

for the matrix complex mKdV equation (2·3), respectively.

Let us consider a reduction of the N-soliton solution of the matrix NLS equation to that of the Manakov model. We restrict the matrix Q to the form

$$Q = \left[\begin{array}{cccc} q_1 & q_2 & \cdots & q_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right] \equiv \left[\begin{array}{c} \boldsymbol{q} \\ O \end{array} \right], \tag{2·7}$$

so that the matrix NLS equation (2·2) is reduced to the Manakov model (1·1). In this case, the matrices $C_j(t)^\dagger$ must have the same form as Q, from their definition,[13),14)]

$$C_j(t)^\dagger = \begin{bmatrix} c_j^{(1)} & c_j^{(2)} & \cdots & c_j^{(m)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \equiv \mathrm{i} \begin{bmatrix} \boldsymbol{c}_j(t) \\ O \end{bmatrix}, \quad j = 1, 2, \cdots, N. \tag{2·8}$$

Conversely, if $C_j(t)^\dagger$ take the form (2·8), $Q(x,t)$ given by the formula (2·4) with (2·5) fits the form (2·7). Then, the formula can be compressed into a compact form,[14)]

$$\boldsymbol{q}(x,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N}(T^{-1})_{jk}\mathrm{e}^{-2\mathrm{i}\zeta_k^* x}\boldsymbol{c}_k(t),$$

where the $N \times N$ matrix T is given by

$$T_{jk} = \delta_{jk} - \sum_{l=1}^{N} \frac{\mathrm{e}^{2\mathrm{i}(\zeta_l - \zeta_j^*)x}}{(\zeta_l - \zeta_k^*)(\zeta_l - \zeta_j^*)} \boldsymbol{c}_j(t) \cdot \boldsymbol{c}_l(t)^\dagger, \quad 1 \le j,k \le N.$$

Thanks to (2·6) and (2·8), the time dependence of $\boldsymbol{c}_j(t)$ is given by

$$\boldsymbol{c}_j(t) = \mathrm{e}^{-4\mathrm{i}\zeta_j^{*2}t}\boldsymbol{c}_j(0), \quad j = 1, 2, \cdots, N.$$

The above set of equations gives a formula for the general N-soliton solution of the Manakov model (1·1) under vanishing boundary conditions. Let us rewrite this into a form convenient to investigate the asymptotic behavior. We first rewrite it as

$$\boldsymbol{q}(x,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N}(W^{-1})_{jk}\mathrm{e}^{-\mathrm{i}[(\zeta_k + \zeta_k^*)x + 2(\zeta_k^2 + \zeta_k^{*2})t]}\boldsymbol{c}_k(0),$$

where the $N \times N$ matrix W is given by

$$\begin{aligned} W_{jk} = \delta_{jk}\mathrm{e}^{-\mathrm{i}[(\zeta_j - \zeta_j^*)x + 2(\zeta_j^2 - \zeta_j^{*2})t]} - \sum_{l=1}^{N} \frac{\boldsymbol{c}_j(0) \cdot \boldsymbol{c}_l(0)^\dagger}{(\zeta_l - \zeta_k^*)(\zeta_l - \zeta_j^*)} \mathrm{e}^{\mathrm{i}[(\zeta_l - \zeta_l^*)x + 2(\zeta_l^2 - \zeta_l^{*2})t]} \\ \times \mathrm{e}^{\mathrm{i}[(\zeta_l + \zeta_l^*)x + 2(\zeta_l^2 + \zeta_l^{*2})t]}\mathrm{e}^{-\mathrm{i}[(\zeta_j + \zeta_j^*)x + 2(\zeta_j^2 + \zeta_j^{*2})t]}, \quad 1 \le j,k \le N. \end{aligned}$$

Next, we introduce the parametrization

$$\begin{aligned} &\zeta_j = \xi_j + \mathrm{i}\eta_j \ \ (\xi_j \in \mathbb{R},\ \eta_j > 0), \\ &\boldsymbol{c}_j(0) = 2\eta_j \mathrm{e}^{-\alpha_j}\boldsymbol{u}_j \ \ (\alpha_j \in \mathbb{R},\ \|\boldsymbol{u}_j\| = 1), \end{aligned}$$

and employ the following abbreviations:

$$\begin{aligned} &\tau_j \equiv -\mathrm{i}[(\zeta_j - \zeta_j^*)x + 2(\zeta_j^2 - \zeta_j^{*2})t] = 2\eta_j(x + 4\xi_j t), \\ &\Theta_j \equiv (\zeta_j + \zeta_j^*)x + 2(\zeta_j^2 + \zeta_j^{*2})t = 2\xi_j x + 4(\xi_j^2 - \eta_j^2)t. \end{aligned}$$

In this way, we obtain the simplest formula for the general N-soliton solution of the Manakov model (1·1),

$$\boldsymbol{q}(x,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N}(U^{-1})_{jk}\mathrm{e}^{-\mathrm{i}\Theta_k}\boldsymbol{u}_k, \tag{2·9}$$

where the $N\times N$ matrix U is given by

$$U_{jk} = \frac{\mathrm{e}^{\tau_j+\alpha_j}}{2\eta_j}\delta_{jk} + \sum_{l=1}^{N}\lambda_{jkl}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_j)}, \quad 1\le j,k\le N, \tag{2·10}$$

with

$$\lambda_{jkl} = -\frac{2\eta_l(\boldsymbol{u}_j\cdot\boldsymbol{u}_l^\dagger)}{(\zeta_l-\zeta_k^*)(\zeta_l-\zeta_j^*)}. \tag{2·11}$$

If we set $N=1$ in the above formula, we obtain the one-soliton solution,

$$\boldsymbol{q}(x,t) = 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1. \tag{2·12}$$

Therefore, we understand the significance of each parameter/coordinate as follows:

$2\eta_j$: amplitude of soliton-j,
$-4\xi_j$: velocity of soliton-j's envelope,
τ_j : coordinate for observing soliton-j's envelope,
Θ_j : coordinate for observing soliton-j's carrier waves,
$\boldsymbol{u}_j$: polarization vector of soliton-j ($\|\boldsymbol{u}_j\|=1$).

To be precise, in the case of two or more solitons, the *real* polarization vectors are not invariant and change under soliton collision. The vector $\boldsymbol{u}_j$ defines the *bare* polarization of soliton-j, which is realized when it becomes the rightmost soliton. This point is demonstrated below. In the following, we assume that all the soliton velocities are distinct, so that every soliton collides with all others.

§3. Two-soliton collision

In this section, we compute the asymptotic forms of the two-soliton solution in the limits $t\to\mp\infty$, which define the collision laws of two solitons in the Manakov model (1·1).

We first write out the two-soliton solution given by (2·9) with $N=2$. According to (2·10), the matrix U in this case takes the form

$$U = \begin{bmatrix} \dfrac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1} + \displaystyle\sum_{l=1}^{2}\lambda_{11l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_1)} & \displaystyle\sum_{l=1}^{2}\lambda_{12l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_1)} \\ \displaystyle\sum_{l=1}^{2}\lambda_{21l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_2)} & \dfrac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \displaystyle\sum_{l=1}^{2}\lambda_{22l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_2)} \end{bmatrix}.$$

Then, the two-soliton solution is given by

$$\boldsymbol{q}(x,t) = \frac{2}{\det U}\left\{\left[\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \sum_{l=1}^{2}(\lambda_{22l}-\lambda_{21l})\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_2)}\right]\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1 \right.$$
$$\left. + \left[\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1} + \sum_{l=1}^{2}(\lambda_{11l}-\lambda_{12l})\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_1)}\right]\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2\right\}, \quad (3{\cdot}1)$$

with

$$\det U = \frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\sum_{l=1}^{2}\lambda_{22l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_2)}$$
$$+ \frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2}\sum_{l=1}^{2}\lambda_{11l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_1)}$$
$$+ \mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-(\tau_2+\alpha_2)}\sum_{\{l_1,l_2\}=\{1,2\}}\begin{vmatrix}\lambda_{11l_1} & \lambda_{12l_1}\\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}. \quad (3{\cdot}2)$$

Here, we have simplified the expression of $\det U$ using the relations

$$\begin{vmatrix}\lambda_{11l} & \lambda_{12l}\\ \lambda_{21l} & \lambda_{22l}\end{vmatrix} = 0, \qquad l = 1, 2, \quad (3{\cdot}3)$$

which can be proved straightforwardly.

Next, we assume that

$$\xi_1(=\mathrm{Re}\,\zeta_1) > \xi_2(=\mathrm{Re}\,\zeta_2)$$

and investigate the asymptotic behavior of $\boldsymbol{q}(x,t)$ as $t \to \mp\infty$. This is accomplished by identifying the dominant terms in the numerator of (3·1) and its denominator (3·2). We here note the relation $\tau_1/\eta_1 = \tau_2/\eta_2 + 8(\xi_1-\xi_2)t$.

In the limit $t \to -\infty$, we have

$$\frac{\tau_1}{\eta_1} \ll \frac{\tau_2}{\eta_2}.$$

In this case, we have to consider separately the two regions (1^-) and (2^-) defined below. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in all other regions.

(1^-) finite τ_1, $\quad \tau_2 \to +\infty$

In this case, the dominant terms are those which contain the factor e^{τ_2}. Then, using the relation $\lambda_{111} = 1/(2\eta_1)$, we obtain

$$\boldsymbol{q} \simeq \frac{2\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1}{\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2}\lambda_{111}\mathrm{e}^{-(\tau_1+\alpha_1)}}$$
$$= 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1. \quad (3{\cdot}4)$$

(2^-) $\tau_1 \to -\infty$, finite τ_2

Here, the dominant terms are those which contain the factor $\mathrm{e}^{-\tau_1}$. Then, we obtain

$$\boldsymbol{q} \simeq \frac{2\left[(\lambda_{221}-\lambda_{211})\mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_1 + (\lambda_{111}-\lambda_{121})\mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2\right]}{\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2}\lambda_{111}\mathrm{e}^{-(\tau_1+\alpha_1)} + \mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-(\tau_2+\alpha_2)}\sum_{\substack{\{l_1,l_2\}\\=\{1,2\}}}\begin{vmatrix}\lambda_{11l_1} & \lambda_{12l_1}\\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}}. \quad (3·5)$$

In terms of ϕ_{12} defined by

$$\mathrm{e}^{-2\phi_{12}} \equiv \frac{1}{\lambda_{111}\lambda_{222}}\sum_{\{l_1,l_2\}=\{1,2\}}\begin{vmatrix}\lambda_{11l_1} & \lambda_{12l_1}\\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}, \quad (3·6)$$

we can rewrite the asymptotic form (3·5) as

$$\boldsymbol{q} \simeq 2\eta_2\,\mathrm{sech}(\tau_2+\alpha_2+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_2} \times \mathrm{e}^{\phi_{12}}\left[\left(\frac{\lambda_{221}-\lambda_{211}}{\lambda_{111}}\right)\boldsymbol{u}_1 + \left(1-\frac{\lambda_{121}}{\lambda_{111}}\right)\boldsymbol{u}_2\right]. \quad (3·7)$$

Here, ϕ_{12} is always taken as real, since (3·6) can be rewritten as [cf. (3·3) and (2·11)]

$$\begin{aligned}\mathrm{e}^{-2\phi_{12}} &= \frac{1}{\lambda_{111}\lambda_{222}}\begin{vmatrix}\lambda_{111}+\lambda_{112} & \lambda_{121}+\lambda_{122}\\ \lambda_{211}+\lambda_{212} & \lambda_{221}+\lambda_{222}\end{vmatrix}\\ &= (2\eta_1)(2\eta_2)\begin{vmatrix}\mathrm{i}\frac{\boldsymbol{u}_1\cdot\boldsymbol{u}_1^\dagger}{\zeta_1-\zeta_1^*} & \mathrm{i}\frac{\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger}{\zeta_2-\zeta_1^*}\\ \mathrm{i}\frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_1^\dagger}{\zeta_1-\zeta_2^*} & \mathrm{i}\frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2^\dagger}{\zeta_2-\zeta_2^*}\end{vmatrix}\Bigg/\begin{vmatrix}\mathrm{i}\frac{2\eta_1}{\zeta_1-\zeta_1^*} & \mathrm{i}\frac{2\eta_1}{\zeta_1-\zeta_2^*}\\ \mathrm{i}\frac{2\eta_2}{\zeta_2-\zeta_1^*} & \mathrm{i}\frac{2\eta_2}{\zeta_2-\zeta_2^*}\end{vmatrix}\\ &= \left|\frac{\zeta_1-\zeta_2}{\zeta_1-\zeta_2^*}\right|^2\left\{1+\frac{(\zeta_1-\zeta_1^*)(\zeta_2-\zeta_2^*)}{\left|\zeta_1-\zeta_2^*\right|^2}\left|\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right|^2\right\}\ (>0).\end{aligned}$$

In the limit $t \to +\infty$, we have

$$\frac{\tau_1}{\eta_1} \gg \frac{\tau_2}{\eta_2}.$$

In this case, we have to consider separately the two regions (2^+) and (1^+) defined below. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in all other regions.

(2^+) $\tau_1 \to +\infty$, finite τ_2

In this case, the dominant terms are those which contain the factor e^{τ_1}. Then, using the relation $\lambda_{222} = 1/(2\eta_2)$, we obtain

$$\begin{aligned}\boldsymbol{q} &\simeq \frac{2\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2}{\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\lambda_{222}\mathrm{e}^{-(\tau_2+\alpha_2)}}\\ &= 2\eta_2\,\mathrm{sech}(\tau_2+\alpha_2)\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2. \quad (3·8)\end{aligned}$$

(1^+) finite τ_1, $\tau_2 \to -\infty$

In this case, the dominant terms are those which contain the factor $\mathrm{e}^{-\tau_2}$. Then, with the help of (3·6), we obtain

$$\boldsymbol{q} \simeq \frac{2\left[(\lambda_{222}-\lambda_{212})\mathrm{e}^{-(\tau_2+\alpha_2)}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1+(\lambda_{112}-\lambda_{122})\mathrm{e}^{-(\tau_2+\alpha_2)}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_2\right]}{\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\lambda_{222}\mathrm{e}^{-(\tau_2+\alpha_2)}+\mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-(\tau_2+\alpha_2)}\sum_{\substack{\{l_1,l_2\}\\=\{1,2\}}}\begin{vmatrix}\lambda_{11l_1} & \lambda_{12l_1}\\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}}$$
$$= 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_1}\times\mathrm{e}^{\phi_{12}}\left[\left(1-\frac{\lambda_{212}}{\lambda_{222}}\right)\boldsymbol{u}_1+\left(\frac{\lambda_{112}-\lambda_{122}}{\lambda_{222}}\right)\boldsymbol{u}_2\right]. \tag{3·9}$$

Taking the sum of (3·4) and (3·7), or (3·8) and (3·9), with a slight simplification, we arrive at the following theorem.

Theorem 3.1. *The asymptotic forms of the two-soliton solution of the Manakov model* (1·1) *are as follows* (*see also Fig.* 1)*:*
as $t\to-\infty$*,*

$$\boldsymbol{q}\simeq 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1+2\eta_2\,\mathrm{sech}(\tau_2+\alpha_2+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_{\{1\},2};$$

as $t\to+\infty$*,*

$$\boldsymbol{q}\simeq 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_{\{2\},1}+2\eta_2\,\mathrm{sech}(\tau_2+\alpha_2)\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2.$$

Here ϕ_{12} *and* $\boldsymbol{u}_{\{1\},2}$, $\boldsymbol{u}_{\{2\},1}$ *are given by*

$$\mathrm{e}^{-2\phi_{12}}=\left|\frac{\zeta_1-\zeta_2}{\zeta_1-\zeta_2^*}\right|^2\left\{1+\frac{(\zeta_1-\zeta_1^*)(\zeta_2-\zeta_2^*)}{\left|\zeta_1-\zeta_2^*\right|^2}\left|\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right|^2\right\},$$

and

$$\boldsymbol{u}_{\{1\},2}=\mathrm{e}^{\phi_{12}}\frac{\zeta_1^*-\zeta_2^*}{\zeta_1-\zeta_2^*}\left\{\boldsymbol{u}_2-\frac{\zeta_1-\zeta_1^*}{\zeta_1-\zeta_2^*}\left(\boldsymbol{u}_2\cdot\boldsymbol{u}_1^\dagger\right)\boldsymbol{u}_1\right\},$$
$$\boldsymbol{u}_{\{2\},1}=\mathrm{e}^{\phi_{12}}\frac{\zeta_2^*-\zeta_1^*}{\zeta_2-\zeta_1^*}\left\{\boldsymbol{u}_1-\frac{\zeta_2-\zeta_2^*}{\zeta_2-\zeta_1^*}\left(\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right)\boldsymbol{u}_2\right\}.$$

Theorem 3.1 defines the collision laws of two solitons in the Manakov model, which we use in §5 to factorize an N-soliton collision into pair collisions. Here, we mention some important properties of the collision laws:

- A two-soliton collision changes neither the amplitudes of the solitons nor the modulus of the Hermitian product of the polarization vectors. In fact, recalling that $\|\boldsymbol{u}_1\|=\|\boldsymbol{u}_2\|=1$, we can prove by direct computations that

$$\left\|\boldsymbol{u}_{\{1\},2}\right\|=\left\|\boldsymbol{u}_{\{2\},1}\right\|=1,\qquad \left|\boldsymbol{u}_1\cdot\boldsymbol{u}_{\{1\},2}^\dagger\right|=\left|\boldsymbol{u}_{\{2\},1}\cdot\boldsymbol{u}_2^\dagger\right|,$$
$$\left|\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right|=\left|\boldsymbol{u}_{\{2\},1}\cdot\boldsymbol{u}_{\{1\},2}^\dagger\right|.$$

Although we omit here the tiresome proof, the most important relation $\|\boldsymbol{u}_{\{1\},2}\| = \|\boldsymbol{u}_{\{2\},1}\| = 1$ is shown in §5 in a more general context. This relation shows that the collision is elastic if we observe it with conserved density, $\|\boldsymbol{q}\|^2 = \sum_{j=1}^{m} |q_j|^2$.

- As a result of the collision, the polarization vectors rotate nontrivially on the unit sphere in $\mathbb{C}^m$. Thus, if we observe the collision with respect to each component q_k, it appears as if it were inelastic.
- We have expressed ϕ_{12}, $\boldsymbol{u}_{\{1\},2}$ and $\boldsymbol{u}_{\{2\},1}$ in terms of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Then, the collision laws are symmetric with respect to interchange of the subscripts 1 and 2. This form of the collision laws is very useful in studying the factorization of an N-soliton collision into pair collisions. For any fixed unit vector $\boldsymbol{u}_1$, we can invert the mapping $\boldsymbol{u}_2 \mapsto \boldsymbol{u}_{\{1\},2}$ using the following relation for the projection operator $\boldsymbol{u}_1^\dagger \boldsymbol{u}_1$ (cf. $(\boldsymbol{u}_1^\dagger \boldsymbol{u}_1)^2 = \boldsymbol{u}_1^\dagger \boldsymbol{u}_1$):

$$\left(I - \frac{\zeta_1 - \zeta_1^*}{\zeta_1 - \zeta_2^*} \boldsymbol{u}_1^\dagger \boldsymbol{u}_1 \right) \left(I + \frac{\zeta_1 - \zeta_1^*}{\zeta_1^* - \zeta_2^*} \boldsymbol{u}_1^\dagger \boldsymbol{u}_1 \right) = I.$$

Then, we can express ϕ_{12}, $\boldsymbol{u}_2$ and $\boldsymbol{u}_{\{2\},1}$ in terms of $\boldsymbol{u}_1$ and $\boldsymbol{u}_{\{1\},2}$ as Manakov did in Ref. 4):

$$\mathrm{e}^{-2\phi_{12}} = \left| \frac{\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2^*} \right|^2 \left\{ 1 - \frac{(\zeta_1 - \zeta_1^*)(\zeta_2 - \zeta_2^*)}{\left|\zeta_1 - \zeta_2\right|^2} \left| \boldsymbol{u}_1 \cdot \boldsymbol{u}_{\{1\},2}^\dagger \right|^2 \right\}^{-1}, \quad (3\cdot10\text{a})$$

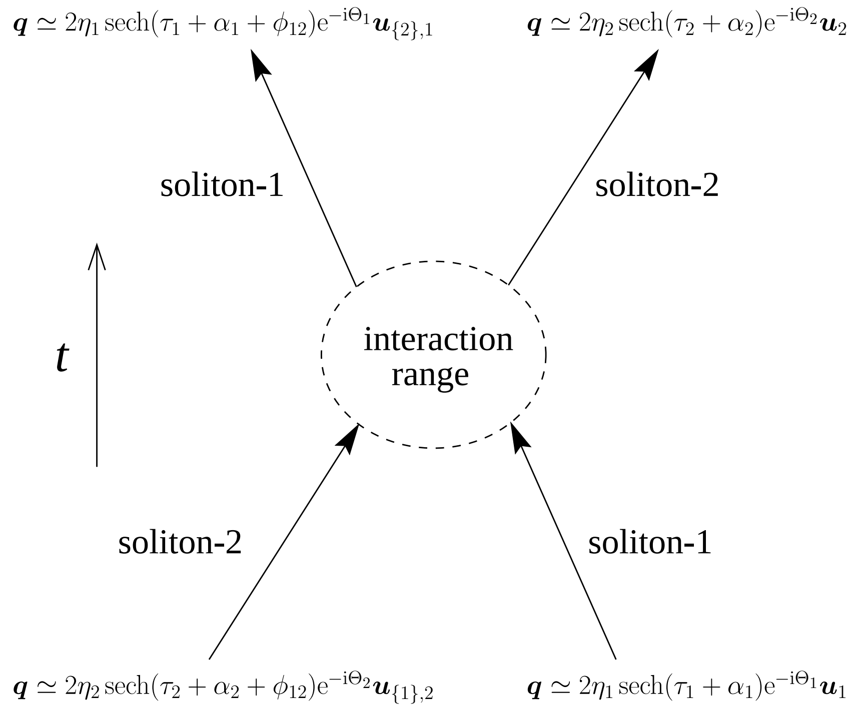

Fig. 1. Two-soliton collision.

$$\boldsymbol{u}_2 = \mathrm{e}^{-\phi_{12}} \frac{\zeta_1 - \zeta_2^*}{\zeta_1^* - \zeta_2^*} \left\{ \boldsymbol{u}_{\{1\},2} + \frac{\zeta_1 - \zeta_1^*}{\zeta_1^* - \zeta_2^*} \left(\boldsymbol{u}_{\{1\},2} \cdot \boldsymbol{u}_1^\dagger \right) \boldsymbol{u}_1 \right\}, \quad (3\cdot10\text{b})$$

$$\boldsymbol{u}_{\{2\},1} = \mathrm{e}^{-\phi_{12}} \frac{\zeta_2^* - \zeta_1}{\zeta_2 - \zeta_1} \left\{ \boldsymbol{u}_1 - \frac{\zeta_2 - \zeta_2^*}{\zeta_2 - \zeta_1} \left(\boldsymbol{u}_1 \cdot \boldsymbol{u}_{\{1\},2}^\dagger \right) \boldsymbol{u}_{\{1\},2} \right\}. \quad (3\cdot10\text{c})$$

Owing to the lack of symmetry under the exchange of 1 and 2 (even after a change of the vector notation), this new form of the collision laws is not useful in studying the factorization problem. On the other hand, it is of prime importance in explicitly showing that a two-soliton collision can be expressed as a mapping from the initial state to the final state. Moreover, using (3·10), we can easily check that if $\|\boldsymbol{u}_1\| = \left\|\boldsymbol{u}_{\{1\},2}\right\| = 1$, then $\|\boldsymbol{u}_2\| = 1$. This verifies that for any unit vector $\boldsymbol{u}_1$, the mapping $\boldsymbol{u}_2 \mapsto \boldsymbol{u}_{\{1\},2}$ in Theorem 3.1 is a bijection on the unit sphere. We use this fact in §5 to prove the Yang-Baxter property of the mapping (3·10).

§4. Asymptotic behavior of the *N*-soliton solution

In this section, we compute the asymptotic forms of the N-soliton solution in the limits $t \to \mp\infty$ and simplify them as much as possible. Extensions of some techniques applied to the KdV equation are used.[33)–35)]

We first rewrite the N-soliton solution (2·9) before considering the limits $t \to \mp\infty$. We use the tilde to denote cofactors. For instance, the cofactor $\tilde{U}_{kj}$ is obtained by deleting the k-th row and the j-th column from the determinant of U and multiplying it by $(-1)^{k+j}$. Using the definition of U given in (2·10) and the multilinearity of determinants, we can rewrite (2·9) as

$$\begin{aligned}
&\boldsymbol{q}(x,t) \\
&= \frac{2}{\det U} \sum_{j=1}^{N} \sum_{k=1}^{N} \tilde{U}_{kj} \mathrm{e}^{-\mathrm{i}\Theta_k} \boldsymbol{u}_k \\
&= \frac{2}{\det U} \left[(\tilde{U}_{11} + \cdots + \tilde{U}_{1N}) \mathrm{e}^{-\mathrm{i}\Theta_1} \boldsymbol{u}_1 + \cdots + (\tilde{U}_{N1} + \cdots + \tilde{U}_{NN}) \mathrm{e}^{-\mathrm{i}\Theta_N} \boldsymbol{u}_N \right] \\
&= \frac{2}{\det U} \left[\begin{vmatrix} 1 & 1 & \cdots & 1 \\ U_{21} & U_{22} & \cdots & U_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ U_{N1} & U_{N2} & \cdots & U_{NN} \end{vmatrix} \mathrm{e}^{-\mathrm{i}\Theta_1} \boldsymbol{u}_1 \right. \\
&\qquad \left. + \cdots + \begin{vmatrix} U_{11} & U_{12} & \cdots & U_{1N} \\ \vdots & \vdots & & \vdots \\ U_{N-1\,1} & U_{N-1\,2} & \cdots & U_{N-1\,N} \\ 1 & 1 & \cdots & 1 \end{vmatrix} \mathrm{e}^{-\mathrm{i}\Theta_N} \boldsymbol{u}_N \right] \\
&= \frac{2}{\det U} \left[\sum_{n=0}^{N-1} \sum_{2 \le j_1 < \cdots < j_n \le N} \left(\prod_{\substack{k=2 \\ k \ne j_1, \cdots, j_n}}^{N} \frac{\mathrm{e}^{\tau_k + \alpha_k}}{2\eta_k} \right) \sum_{l_1, \cdots, l_n = 1}^{N} \mathrm{e}^{-(\tau_{l_1} + \alpha_{l_1}) + \mathrm{i}(\Theta_{l_1} - \Theta_{j_1})} \right.
\end{aligned}$$

$$\times \cdots \times \mathrm{e}^{-(\tau_{l_n}+\alpha_{l_n})+\mathrm{i}(\Theta_{l_n}-\Theta_{j_n})} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_1 1 l_1} & \lambda_{j_1 j_1 l_1} & \cdots & \lambda_{j_1 j_n l_1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_n 1 l_n} & \lambda_{j_n j_1 l_n} & \cdots & \lambda_{j_n j_n l_n} \end{vmatrix} \mathrm{e}^{-\mathrm{i}\Theta_1} \boldsymbol{u}_1$$

$$+ \cdots + \sum_{n=0}^{N-1} \sum_{1 \le j_1 < \cdots < j_n \le N-1} \left(\prod_{\substack{k=1 \\ k \neq j_1, \cdots, j_n}}^{N-1} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{l_1,\cdots,l_n=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})}$$

$$\left. \times \cdots \times \mathrm{e}^{-(\tau_{l_n}+\alpha_{l_n})+\mathrm{i}(\Theta_{l_n}-\Theta_{j_n})} \begin{vmatrix} \lambda_{j_1 j_1 l_1} & \cdots & \lambda_{j_1 j_n l_1} & \lambda_{j_1 N l_1} \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_{j_n j_1 l_n} & \cdots & \lambda_{j_n j_n l_n} & \lambda_{j_n N l_n} \\ 1 & \cdots & 1 & 1 \end{vmatrix} \mathrm{e}^{-\mathrm{i}\Theta_N} \boldsymbol{u}_N \right] . \tag{4·1}$$

Similarly, we can rewrite the determinant of U as

$$\det U$$

$$= \prod_{k=1}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} + \sum_{j_1=1}^{N} \left(\prod_{\substack{k=1 \\ k \neq j_1}}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{l_1=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})} \lambda_{j_1 j_1 l_1}$$

$$+ \sum_{1 \le j_1 < j_2 \le N} \left(\prod_{\substack{k=1 \\ k \neq j_1, j_2}}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{l_1, l_2=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})} \mathrm{e}^{-(\tau_{l_2}+\alpha_{l_2})+\mathrm{i}(\Theta_{l_2}-\Theta_{j_2})}$$

$$\times \begin{vmatrix} \lambda_{j_1 j_1 l_1} & \lambda_{j_1 j_2 l_1} \\ \lambda_{j_2 j_1 l_2} & \lambda_{j_2 j_2 l_2} \end{vmatrix}$$

$$+ \cdots + \sum_{l_1,\cdots,l_N=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_1)} \cdots \mathrm{e}^{-(\tau_{l_N}+\alpha_{l_N})+\mathrm{i}(\Theta_{l_N}-\Theta_N)}$$

$$\times \begin{vmatrix} \lambda_{1 1 l_1} & \cdots & \lambda_{1 N l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{N 1 l_N} & \cdots & \lambda_{N N l_N} \end{vmatrix}$$

$$= \sum_{n=0}^{N} \sum_{1 \le j_1 < \cdots < j_n \le N} \left(\prod_{\substack{k=1 \\ k \neq j_1, \cdots, j_n}}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{l_1,\cdots,l_n=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})}$$

$$\times \cdots \times \mathrm{e}^{-(\tau_{l_n}+\alpha_{l_n})+\mathrm{i}(\Theta_{l_n}-\Theta_{j_n})} \begin{vmatrix} \lambda_{j_1 j_1 l_1} & \cdots & \lambda_{j_1 j_n l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{j_n j_1 l_n} & \cdots & \lambda_{j_n j_n l_n} \end{vmatrix} . \tag{4·2}$$

Here, we note [cf. the definition of λ_{jkl} (2·11)] that the quantity

$$\frac{(\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger)}{\zeta_l-\zeta_{j'}^*}\lambda_{jkl} = -\frac{(\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger)2\eta_l(\boldsymbol{u}_j\cdot\boldsymbol{u}_l^\dagger)}{(\zeta_l-\zeta_{j'}^*)(\zeta_l-\zeta_k^*)(\zeta_l-\zeta_j^*)}$$

is invariant under interchange of the subscripts j and j'. Thus, we have

$$\frac{(\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger)}{\zeta_l-\zeta_{j'}^*}\lambda_{jkl} - \frac{(\boldsymbol{u}_{j}\cdot\boldsymbol{u}_l^\dagger)}{\zeta_l-\zeta_{j}^*}\lambda_{j'kl} = 0.$$

This shows that if $\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger \neq 0$ or $\boldsymbol{u}_{j}\cdot\boldsymbol{u}_l^\dagger \neq 0$, the two vectors $(\lambda_{j1l}, \lambda_{j2l}, \cdots, \lambda_{jNl})$ and $(\lambda_{j'1l}, \lambda_{j'2l}, \cdots, \lambda_{j'Nl})$ are linearly dependent. In the case that $\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger = \boldsymbol{u}_{j}\cdot\boldsymbol{u}_l^\dagger = 0$, according to (2·11), both vectors become zero. Therefore, the determinants in (4·1) or (4·2) contribute only if $l_1, \cdots, l_n$ are all distinct. This fact is a generalization of the relations (3·3).

Next, we assume that

$$\xi_1(=\operatorname{Re}\zeta_1) > \xi_2(=\operatorname{Re}\zeta_2) > \cdots > \xi_N(=\operatorname{Re}\zeta_N),$$

and investigate the asymptotic behavior of $\boldsymbol{q}(x,t)$ in the limits $t\to\mp\infty$. This is accomplished by identifying the dominant terms in the numerator of (4·1) and its denominator (4·2). We here note the relations $\tau_j/\eta_j = \tau_k/\eta_k + 8(\xi_j-\xi_k)t$.

In the limit $t\to-\infty$, we have

$$\frac{\tau_1}{\eta_1} \ll \frac{\tau_2}{\eta_2} \ll \cdots \ll \frac{\tau_N}{\eta_N}.$$

In this case, we have to consider the following N regions (1^-)–(N^-) separately. It is easily seen that $\boldsymbol{q}\simeq\boldsymbol{0}$ in all other regions.

(1^-) finite τ_1, $\tau_2, \cdots, \tau_N \to +\infty$
In this case, the dominant terms are those which contain the factor $\mathrm{e}^{\tau_2+\cdots+\tau_N}$. Then, using the relation $\lambda_{111} = 1/(2\eta_1)$, we obtain

$$\begin{aligned}\boldsymbol{q} &\simeq \frac{2\left(\prod_{k=2}^N \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1}{\prod_{k=1}^N \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} + \left(\prod_{k=2}^N \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\mathrm{e}^{-(\tau_1+\alpha_1)}\lambda_{111}} \\ &= 2\eta_1 \operatorname{sech}(\tau_1+\alpha_1)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1. \end{aligned} \tag{4·3}$$

(n^-) $\tau_1, \cdots, \tau_{n-1}\to-\infty$, finite τ_n, $\tau_{n+1}, \cdots, \tau_N\to+\infty$, $n = 2, \cdots, N-1$
Here, the dominant terms are those which contain the factor $\mathrm{e}^{-\tau_1-\cdots-\tau_{n-1}+\tau_{n+1}+\cdots+\tau_N}$. Then, those in the numerator of (4·1) are

$$2\sum_{j=1}^n\left(\prod_{k=n+1}^N \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\cdots,l_{n-1}\}\\=\{1,\cdots,n-1\}}} \mathrm{e}^{-(\tau_1+\alpha_1)}\cdots\mathrm{e}^{-(\tau_{n-1}+\alpha_{n-1})}\mathrm{e}^{\mathrm{i}(\Theta_j-\Theta_n)}$$

$$
\times \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1nl_1} \\ \vdots & & \vdots \\ \lambda_{j-1\,1\,l_{j-1}} & \cdots & \lambda_{j-1\,n\,l_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{j+1\,1\,l_j} & \cdots & \lambda_{j+1\,n\,l_j} \\ \vdots & & \vdots \\ \lambda_{n1l_{n-1}} & \cdots & \lambda_{nnl_{n-1}} \end{vmatrix} \mathrm{e}^{-\mathrm{i}\Theta_j} \boldsymbol{u}_j,
$$

while those in the denominator (4·2) are

$$
\left(\prod_{k=n}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{\substack{\{l_1,\cdots,l_{n-1}\} \\ =\{1,\cdots,n-1\}}} \mathrm{e}^{-(\tau_1+\alpha_1)} \cdots \mathrm{e}^{-(\tau_{n-1}+\alpha_{n-1})}
\times \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1\,n-1\,l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1\,1\,l_{n-1}} & \cdots & \lambda_{n-1\,n-1\,l_{n-1}} \end{vmatrix}
$$

$$
+ \left(\prod_{k=n+1}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{\substack{\{l_1,\cdots,l_n\} \\ =\{1,\cdots,n\}}} \mathrm{e}^{-(\tau_1+\alpha_1)} \cdots \mathrm{e}^{-(\tau_n+\alpha_n)} \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1nl_1} \\ \vdots & \ddots & \vdots \\ \lambda_{n1l_n} & \cdots & \lambda_{nnl_n} \end{vmatrix}.
$$

As a natural extension of (3·6), we define $\phi_{i_1 i_2 \cdots i_p}$ for distinct positive integers $i_1, i_2, \cdots, i_p$ by

$$
\mathrm{e}^{-2\phi_{i_1 i_2 \cdots i_p}} \equiv \frac{1}{\lambda_{i_1 i_1 i_1} \cdots \lambda_{i_p i_p i_p}} \sum_{\substack{\{l_1,\cdots,l_p\} \\ =\{i_1,\cdots,i_p\}}} \begin{vmatrix} \lambda_{i_1 i_1 l_1} & \cdots & \lambda_{i_1 i_p l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{i_p i_1 l_p} & \cdots & \lambda_{i_p i_p l_p} \end{vmatrix}. \tag{4·4}
$$

We should note that $\phi_{i_1 i_2 \cdots i_p}$ is symmetric with respect to permutations of the subscripts $i_1, i_2, \cdots, i_p$. We prove below that $\phi_{i_1 i_2 \cdots i_p}$ can always be taken as real. In terms of $\phi_{i_1 i_2 \cdots i_p}$, we can express the asymptotic form of $\boldsymbol{q}$ in this region as

$$
\boldsymbol{q} \simeq 2\eta_n \,\mathrm{sech}(\tau_n + \alpha_n + \phi_{12\cdots n} - \phi_{12\cdots n-1}) \mathrm{e}^{-\mathrm{i}\Theta_n} \mathrm{e}^{\phi_{12\cdots n} + \phi_{12\cdots n-1}}
\times \frac{1}{\lambda_{111} \cdots \lambda_{n-1\,n-1\,n-1}} \sum_{j=1}^{n} \sum_{\substack{\{l_1,\cdots,l_{n-1}\} \\ =\{1,\cdots,n-1\}}} \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1nl_1} \\ \vdots & & \vdots \\ \lambda_{j-1\,1\,l_{j-1}} & \cdots & \lambda_{j-1\,n\,l_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{j+1\,1\,l_j} & \cdots & \lambda_{j+1\,n\,l_j} \\ \vdots & & \vdots \\ \lambda_{n1l_{n-1}} & \cdots & \lambda_{nnl_{n-1}} \end{vmatrix} \boldsymbol{u}_j. \tag{4·5}
$$

(N^-) $\tau_1, \cdots, \tau_{N-1} \to -\infty$, finite τ_N
Here, the dominant terms are those which contain the factor $\mathrm{e}^{-\tau_1-\cdots-\tau_{N-1}}$. With calculations similar to those in the case (n^-), we obtain the asymptotic form of $\boldsymbol{q}$ given by (4·5) with $n = N$.

In the limit $t \to +\infty$, we have

$$\frac{\tau_1}{\eta_1} \gg \frac{\tau_2}{\eta_2} \gg \cdots \gg \frac{\tau_N}{\eta_N}.$$

In this case, we have to consider the following N regions (N^+)–(1^+) separately. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in all other regions.

(N^+) $\tau_1, \cdots, \tau_{N-1} \to +\infty$, finite τ_N
In this case, the dominant terms are those which contain the factor $\mathrm{e}^{\tau_1+\cdots+\tau_{N-1}}$. Then, using the relation $\lambda_{NNN} = 1/(2\eta_N)$, we obtain

$$\begin{aligned}\boldsymbol{q} &\simeq \frac{2\left(\prod_{k=1}^{N-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\mathrm{e}^{-\mathrm{i}\Theta_N}\boldsymbol{u}_N}{\prod_{k=1}^{N}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} + \left(\prod_{k=1}^{N-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\mathrm{e}^{-(\tau_N+\alpha_N)}\lambda_{NNN}} \\ &= 2\eta_N \operatorname{sech}(\tau_N + \alpha_N)\mathrm{e}^{-\mathrm{i}\Theta_N}\boldsymbol{u}_N. \end{aligned} \tag{4·6}$$

(n^+) $\tau_1, \cdots, \tau_{n-1} \to +\infty$, finite τ_n, $\tau_{n+1}, \cdots, \tau_N \to -\infty$, $n = 2, \cdots, N-1$
In this case, the dominant terms are those which contain the factor $\mathrm{e}^{\tau_1+\cdots+\tau_{n-1}-\tau_{n+1}-\cdots-\tau_N}$. Then, those in the numerator of (4·1) are

$$2\sum_{j=n}^{N}\left(\prod_{k=1}^{n-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\cdots,l_{N-n}\}\\=\{n+1,\cdots,N\}}}\mathrm{e}^{-(\tau_{n+1}+\alpha_{n+1})}\cdots\mathrm{e}^{-(\tau_N+\alpha_N)}\mathrm{e}^{\mathrm{i}(\Theta_j-\Theta_n)}$$
$$\times\begin{vmatrix}\lambda_{nnl_1} & \cdots & \lambda_{nNl_1}\\ \vdots & & \vdots \\ \lambda_{j-1\,n\,l_{j-n}} & \cdots & \lambda_{j-1\,N\,l_{j-n}} \\ 1 & \cdots & 1 \\ \lambda_{j+1\,n\,l_{j-n+1}} & \cdots & \lambda_{j+1\,N\,l_{j-n+1}} \\ \vdots & & \vdots \\ \lambda_{Nnl_{N-n}} & \cdots & \lambda_{NNl_{N-n}}\end{vmatrix}\mathrm{e}^{-\mathrm{i}\Theta_j}\boldsymbol{u}_j,$$

while those in the denominator (4·2) are

$$\left(\prod_{k=1}^{n}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\cdots,l_{N-n}\}\\=\{n+1,\cdots,N\}}}\mathrm{e}^{-(\tau_{n+1}+\alpha_{n+1})}\cdots\mathrm{e}^{-(\tau_N+\alpha_N)}$$
$$\times\begin{vmatrix}\lambda_{n+1\,n+1\,l_1} & \cdots & \lambda_{n+1\,N\,l_1}\\ \vdots & \ddots & \vdots \\ \lambda_{N\,n+1\,l_{N-n}} & \cdots & \lambda_{NNl_{N-n}}\end{vmatrix}$$

$$+\left(\prod_{k=1}^{n-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\cdots,l_{N-n+1}\}\\=\{n,\cdots,N\}}}\mathrm{e}^{-(\tau_n+\alpha_n)}\cdots\mathrm{e}^{-(\tau_N+\alpha_N)}$$
$$\times\begin{vmatrix}\lambda_{nnl_1} & \cdots & \lambda_{nNl_1}\\ \vdots & \ddots & \vdots\\ \lambda_{Nnl_{N-n+1}} & \cdots & \lambda_{NNl_{N-n+1}}\end{vmatrix}.$$

In terms of $\phi_{i_1i_2\cdots i_p}$ defined by (4·4), we can express the asymptotic form of $\boldsymbol{q}$ in this region as

$$\boldsymbol{q}\simeq 2\eta_n\,\mathrm{sech}(\tau_n+\alpha_n+\phi_{n\,n+1\cdots N}-\phi_{n+1\,n+2\cdots N})\mathrm{e}^{-\mathrm{i}\Theta_n}\mathrm{e}^{\phi_{n\,n+1\cdots N}+\phi_{n+1\,n+2\cdots N}}$$
$$\times\frac{1}{\lambda_{n+1\,n+1\,n+1}\cdots\lambda_{NNN}}\sum_{j=n}^{N}\sum_{\substack{\{l_1,\cdots,l_{N-n}\}\\=\{n+1,\cdots,N\}}}\begin{vmatrix}\lambda_{nnl_1} & \cdots & \lambda_{nNl_1}\\ \vdots & & \vdots\\ \lambda_{j-1\,n\,l_{j-n}} & \cdots & \lambda_{j-1\,N\,l_{j-n}}\\ 1 & \cdots & 1\\ \lambda_{j+1\,n\,l_{j-n+1}} & \cdots & \lambda_{j+1\,N\,l_{j-n+1}}\\ \vdots & & \vdots\\ \lambda_{Nnl_{N-n}} & \cdots & \lambda_{NNl_{N-n}}\end{vmatrix}\boldsymbol{u}_j. \tag{4·7}$$

(1^+) finite τ_1, $\tau_2,\cdots,\tau_N\to-\infty$
In this case, the dominant terms are those which contain the factor $\mathrm{e}^{-\tau_2-\cdots-\tau_N}$. With calculations similar to those in the case (n^+), we obtain the asymptotic form of $\boldsymbol{q}$ given by (4·7) with $n=1$.

We now simplify the above asymptotic forms by using the following definitions:

$$c_{jk}\equiv\frac{\mathrm{i}}{\zeta_k-\zeta_j^*},\qquad d_{jk}\equiv\frac{\mathrm{i}}{\zeta_k-\zeta_j^*}\left(\boldsymbol{u}_j\cdot\boldsymbol{u}_k^\dagger\right). \tag{4·8}$$

According to the definition of λ_{jkl} (2·11), we have $\lambda_{jkl}=2\eta_l c_{kl}d_{jl}$. Then, we can rewrite the definition of $\phi_{i_1i_2\cdots i_n}$ [(4·4) with $p\to n$] in a factorized form [cf. the paragraph below (4·2)]:

$$\mathrm{e}^{-2\phi_{i_1i_2\cdots i_n}}=\prod_{l=1}^{n}(2\eta_{i_l})\times\sum_{l_1=i_1,\cdots,i_n}\cdots\sum_{l_n=i_1,\cdots,i_n}\begin{vmatrix}2\eta_{l_1}c_{i_1l_1}d_{i_1l_1} & \cdots & 2\eta_{l_1}c_{i_nl_1}d_{i_1l_1}\\ \vdots & \ddots & \vdots\\ 2\eta_{l_n}c_{i_1l_n}d_{i_nl_n} & \cdots & 2\eta_{l_n}c_{i_nl_n}d_{i_nl_n}\end{vmatrix}$$
$$=\prod_{l=1}^{n}(2\eta_{i_l})^2\times\begin{vmatrix}d_{i_1i_1} & \cdots & d_{i_1i_n}\\ \vdots & \ddots & \vdots\\ d_{i_ni_1} & \cdots & d_{i_ni_n}\end{vmatrix}\times\begin{vmatrix}c_{i_1i_1} & \cdots & c_{i_ni_1}\\ \vdots & \ddots & \vdots\\ c_{i_1i_n} & \cdots & c_{i_ni_n}\end{vmatrix}. \tag{4·9}$$

Here $i_1,i_2,\cdots,i_n$ are distinct positive integers. For any nonzero vector $(y_{i_1},\cdots,y_{i_n})$, we have

$$
\begin{aligned}
&(y_{i_1}, \cdots, y_{i_n}) \underline{\begin{pmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_n} \end{pmatrix}} \begin{pmatrix} y_{i_1}^* \\ \vdots \\ y_{i_n}^* \end{pmatrix} \\
&\qquad = \sum_{j,k=i_1,\cdots,i_n} \frac{\mathrm{i}}{\zeta_k - \zeta_j^*} \left(\boldsymbol{u}_j \cdot \boldsymbol{u}_k^\dagger \right) y_j y_k^* \\
&\qquad = \sum_{j,k=i_1,\cdots,i_n} \int_0^\infty \mathrm{e}^{\mathrm{i}(\zeta_k - \zeta_j^*)z} \left(\boldsymbol{u}_j \cdot \boldsymbol{u}_k^\dagger \right) y_j y_k^* \mathrm{d}z \\
&\qquad = \int_0^\infty \left\| \sum_{j=i_1,\cdots,i_n} \mathrm{e}^{-\mathrm{i}\zeta_j^* z} y_j \boldsymbol{u}_j \right\|^2 \mathrm{d}z \\
&\qquad > 0.
\end{aligned}
$$

Thus, the eigenvalues of the underlined Hermitian matrix are all positive. This proves that the second term on the right-hand side of (4·9) is positive. Considering the special case in which all the vectors $\boldsymbol{u}_{i_1}, \cdots, \boldsymbol{u}_{i_n}$ are identical, we can prove the same for the third term in (4·9). Therefore, the right-hand side of (4·9) is positive, and $\phi_{i_1 i_2 \cdots i_n}$ can always be taken as real. In the same way as for (4·9), we can rewrite the second line of (4·5) or (4·7) in the following factorized form:

$$
\begin{aligned}
&\frac{1}{\lambda_{i_1 i_1 i_1} \cdots \lambda_{i_{n-1} i_{n-1} i_{n-1}}} \sum_{j=1}^{n} \sum_{\substack{\{l_1,\cdots,l_{n-1}\} \\ =\{i_1,\cdots,i_{n-1}\}}} \begin{vmatrix} \lambda_{i_1 i_1 l_1} & \cdots & \lambda_{i_1 i_n l_1} \\ \vdots & & \vdots \\ \lambda_{i_{j-1} i_1 l_{j-1}} & \cdots & \lambda_{i_{j-1} i_n l_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{i_{j+1} i_1 l_j} & \cdots & \lambda_{i_{j+1} i_n l_j} \\ \vdots & & \vdots \\ \lambda_{i_n i_1 l_{n-1}} & \cdots & \lambda_{i_n i_n l_{n-1}} \end{vmatrix} \boldsymbol{u}_{i_j} \\
&= \prod_{l=1}^{n-1} (2\eta_{i_l}) \times \sum_{j=1}^{n} \sum_{l_1=i_1,\cdots,i_{n-1}} \\
&\qquad \cdots \sum_{l_{n-1}=i_1,\cdots,i_{n-1}} \begin{vmatrix} 2\eta_{l_1} c_{i_1 l_1} d_{i_1 l_1} & \cdots & 2\eta_{l_1} c_{i_n l_1} d_{i_1 l_1} \\ \vdots & & \vdots \\ 2\eta_{l_{j-1}} c_{i_1 l_{j-1}} d_{i_{j-1} l_{j-1}} & \cdots & 2\eta_{l_{j-1}} c_{i_n l_{j-1}} d_{i_{j-1} l_{j-1}} \\ 1 & \cdots & 1 \\ 2\eta_{l_j} c_{i_1 l_j} d_{i_{j+1} l_j} & \cdots & 2\eta_{l_j} c_{i_n l_j} d_{i_{j+1} l_j} \\ \vdots & & \vdots \\ 2\eta_{l_{n-1}} c_{i_1 l_{n-1}} d_{i_n l_{n-1}} & \cdots & 2\eta_{l_{n-1}} c_{i_n l_{n-1}} d_{i_n l_{n-1}} \end{vmatrix} \boldsymbol{u}_{i_j}
\end{aligned}
$$

$$
= \prod_{l=1}^{n-1} (2\eta_{i_l})^2 \times \sum_{j=1}^{n} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & 0 \\ \vdots & & \vdots & \vdots \\ d_{i_{j-1} i_1} & \cdots & d_{i_{j-1} i_{n-1}} & 0 \\ 0 & \cdots & 0 & 1 \\ d_{i_{j+1} i_1} & \cdots & d_{i_{j+1} i_{n-1}} & 0 \\ \vdots & & \vdots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_{n-1}} & 0 \end{vmatrix} \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}} \\ 1 & \cdots & 1 \end{vmatrix} \boldsymbol{u}_{i_j}
$$

$$
= \prod_{l=1}^{n-1} (2\eta_{i_l})^2 \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}} \\ 1 & \cdots & 1 \end{vmatrix} \times \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & & \vdots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_{n-1}} & \boldsymbol{u}_{i_n} \end{vmatrix}. \quad (4\cdot10)
$$

The last determinant, which contains vectors in its last column, represents a vector defined in terms of the Laplace expansion with respect to the last column.

We can simplify the asymptotic forms further by noting some relations between the conventional determinants in (4·9) and (4·10). We have the following lemma:

Lemma 4.1. *The following equalities involving determinants hold:*

$$
\begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}} \\ 1 & \cdots & 1 \end{vmatrix} = \prod_{l=1}^{n-1} \frac{\zeta_{i_l}^* - \zeta_{i_n}^*}{\zeta_{i_l} - \zeta_{i_n}^*} \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_{n-1} i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_{n-1} i_{n-1}} \end{vmatrix}, \quad (4\cdot11)
$$

$$
\begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_n} & \cdots & c_{i_n i_n} \end{vmatrix} = \frac{\mathrm{i} \prod_{l=1}^{n-1} (\zeta_{i_n} - \zeta_{i_l})}{\prod_{l=1}^{n} (\zeta_{i_n} - \zeta_{i_l}^*)} \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}} \\ 1 & \cdots & 1 \end{vmatrix}, \quad (4\cdot12)
$$

$$
\begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_n} & \cdots & c_{i_n i_n} \end{vmatrix} = \frac{\mathrm{i}}{\zeta_{i_n} - \zeta_{i_n}^*} \prod_{l=1}^{n-1} \left| \frac{\zeta_{i_l} - \zeta_{i_n}}{\zeta_{i_l} - \zeta_{i_n}^*} \right|^2 \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_{n-1} i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_{n-1} i_{n-1}} \end{vmatrix}. \quad (4\cdot13)
$$

Proof. We can prove (4·11) by subtracting on the left-hand side the last column from each of the other columns and using the relation

$$
c_{jk} - c_{nk} = \frac{\zeta_j^* - \zeta_n^*}{\zeta_k - \zeta_n^*} c_{jk}.
$$

Similarly, (4·12) is proved by subtracting on the left-hand side the last row from each of the other rows and using the relation

$$
c_{jk} - c_{jn} = \frac{\zeta_n - \zeta_k}{\zeta_n - \zeta_j^*} c_{jk}.
$$

(4·13) is a direct consequence of (4·12) and (4·11). □

Taking the sum of (4·3) and (4·5) ($n = 2, \cdots, N$), or (4·6) and (4·7) ($n = 1, \cdots, N-1$), with the help of (4·9), (4·10) and Lemma 4.1, we finally arrive at the following proposition.

Proposition 4.2. *The asymptotic forms of the N-soliton solution of the Manakov model* (1·1) *are as follows:*
as $t \to -\infty$,

$$\boldsymbol{q} \simeq \sum_{n=1}^{N} 2\eta_n \operatorname{sech}\bigl(\tau_n + \alpha_n + \phi_{\{1,\cdots,n-1\},n}\bigr) \mathrm{e}^{-\mathrm{i}\Theta_n} \boldsymbol{u}_{\{1,\cdots,n-1\},n};$$

as $t \to +\infty$,

$$\boldsymbol{q} \simeq \sum_{n=1}^{N} 2\eta_n \operatorname{sech}\bigl(\tau_n + \alpha_n + \phi_{\{n+1,\cdots,N\},n}\bigr) \mathrm{e}^{-\mathrm{i}\Theta_n} \boldsymbol{u}_{\{n+1,\cdots,N\},n}.$$

Here $\phi_{\{i_1,\cdots,i_{n-1}\},i_n}$ and $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}$ are defined for distinct positive integers $i_1, \cdots, i_{n-1}, i_n$ by

$$\begin{aligned} \mathrm{e}^{-2\phi_{\{i_1,\cdots,i_{n-1}\},i_n}} &\equiv \mathrm{e}^{-2(\phi_{i_1\cdots i_{n-1}i_n} - \phi_{i_1\cdots i_{n-1}})} \\ &= \prod_{l=1}^{n-1} \left| \frac{\zeta_{i_l} - \zeta_{i_n}}{\zeta_{i_l} - \zeta_{i_n}^*} \right|^2 \times \frac{\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_n} \end{vmatrix}}{d_{i_n i_n} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{vmatrix}} \ (> 0), \end{aligned} \tag{4·14}$$

and

$$\begin{aligned} &\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n} \\ &\equiv \frac{\mathrm{e}^{\phi_{i_1\cdots i_{n-1}i_n} + \phi_{i_1\cdots i_{n-1}}}}{\lambda_{i_1 i_1 i_1} \cdots \lambda_{i_{n-1} i_{n-1} i_{n-1}}} \sum_{j=1}^{n} \sum_{\substack{\{l_1,\cdots,l_{n-1}\} \\ =\{i_1,\cdots,i_{n-1}\}}} \begin{vmatrix} \lambda_{i_1 i_1 l_1} & \cdots & \lambda_{i_1 i_n l_1} \\ \vdots & & \vdots \\ \lambda_{i_{j-1} i_1 l_{j-1}} & \cdots & \lambda_{i_{j-1} i_n l_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{i_{j+1} i_1 l_j} & \cdots & \lambda_{i_{j+1} i_n l_j} \\ \vdots & & \vdots \\ \lambda_{i_n i_1 l_{n-1}} & \cdots & \lambda_{i_n i_n l_{n-1}} \end{vmatrix} \boldsymbol{u}_{i_j} \end{aligned}$$

$$= \mathrm{e}^{\phi_{\{i_1,\cdots,i_{n-1}\},i_n}} \prod_{l=1}^{n-1} \frac{\zeta_{i_l}^* - \zeta_{i_n}^*}{\zeta_{i_l} - \zeta_{i_n}^*} \times \frac{\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & & \vdots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_{n-1}} & \boldsymbol{u}_{i_n} \end{vmatrix}}{\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{vmatrix}}. \tag{4·15}$$

When the set $\{i_1, \cdots, i_{n-1}\}$ *is empty, the definitions* (4·14) *and* (4·15) *should read* $\mathrm{e}^{-2\phi_{\{\,\},i}} \equiv 1$ *and* $\boldsymbol{u}_{\{\,\},i} \equiv \boldsymbol{u}_i$*.*

We prove in the next section that $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}$ is always a unit vector, i.e. $\left\|\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}\right\| = 1$. This ensures that an N-soliton collision does not change the amplitudes of solitons. The vector $\boldsymbol{u}_{\{1,\cdots,n-1\},n}$ gives the polarization vector of soliton-n before an N-soliton collision, while $\boldsymbol{u}_{\{n+1,\cdots,N\},n}$ gives that after the collision. Using the definition

$$\boldsymbol{q}_{\{i_1,\cdots,i_{n-1}\},i_n} \equiv 2\eta_{i_n} \operatorname{sech}\left(\tau_{i_n} + \alpha_{i_n} + \phi_{\{i_1,\cdots,i_{n-1}\},i_n}\right) \mathrm{e}^{-\mathrm{i}\Theta_{i_n}} \boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}, \tag{4·16}$$

we can diagram the asymptotic behavior of the N-soliton solution in the simplest way (see Fig. 2). We should note that $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1}\},i_n}$ is symmetric with respect to

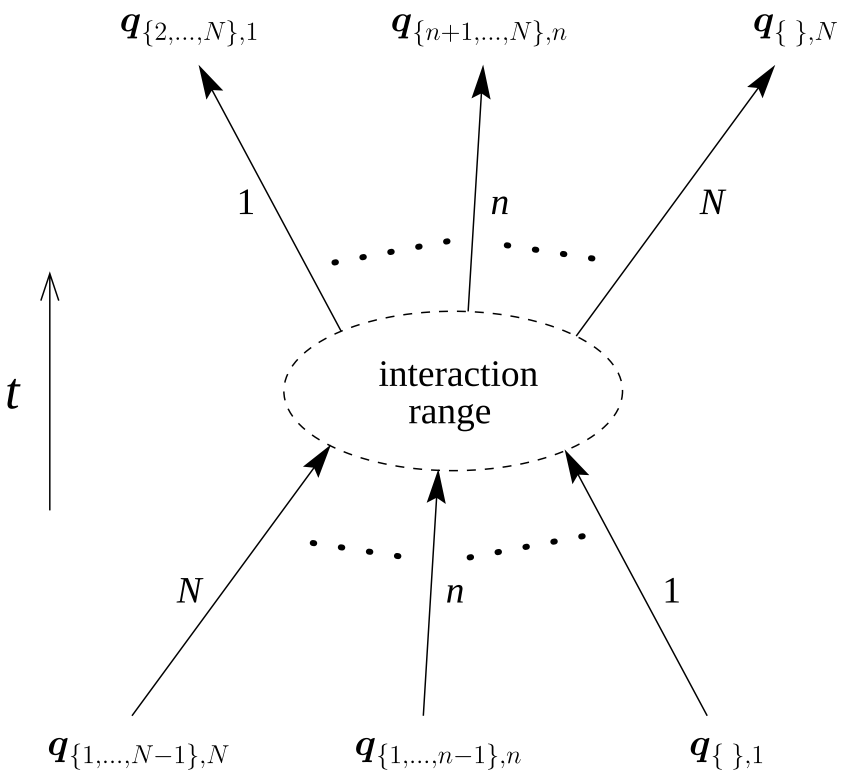

Fig. 2. Asymptotic behavior of the N-soliton solution.

permutations of $i_1, \cdots, i_{n-1}$. The last subscript i_n denotes the soliton's number, which is, of course, time independent. The significance of the other subscripts, $i_1, \cdots, i_{n-1}$, in $\{\ \}$ is clarified in the next section.

§5. Factorization of an *N*-soliton collision into a superposition of pair collisions

In this section, on the basis of the collision laws of two solitons presented in §3, we examine the asymptotic behavior of the N-soliton solution obtained in §4. We conclude that an N-soliton collision in the Manakov model (1·1) can be factorized into a nonlinear superposition of pair collisions in arbitrary order.

We first prove a lemma needed later to compute the Hermitian product of $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}$ and $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},k}$.

Lemma 5.1. *For any set of unit vectors* $\boldsymbol{u}_{i_1}, \cdots, \boldsymbol{u}_{i_{n-1}}, \boldsymbol{u}_j, \boldsymbol{u}_k$ *and* d_{il} *defined by* (4·8)*, the following equality holds:*

$$\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & \boldsymbol{u}_j \end{vmatrix} \cdot \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{k i_1} & \cdots & d_{k i_{n-1}} & \boldsymbol{u}_k \end{vmatrix}^{\dagger}$$

$$= \frac{\zeta_k - \zeta_j^*}{\mathrm{i}} \times \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{vmatrix} \times \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 k} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} k} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & d_{jk} \end{vmatrix}. \tag{5·1}$$

Proof. In the proof of this lemma, we use D to denote the last determinant in (5·1):

$$D \equiv \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 k} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} k} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & d_{jk} \end{vmatrix}.$$

We express minor determinants obtained by deleting one row and one column of D as

$$D\begin{bmatrix} i_l \\ k \end{bmatrix} = \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & & \vdots \\ d_{i_{l-1} i_1} & \cdots & d_{i_{l-1} i_{n-1}} \\ d_{i_{l+1} i_1} & \cdots & d_{i_{l+1} i_{n-1}} \\ \vdots & & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} \end{vmatrix},$$

$$D\begin{bmatrix} j \\ i_l \end{bmatrix} = \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{l-1}} & d_{i_1 i_{l+1}} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 k} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{l-1}} & d_{i_{n-1} i_{l+1}} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} k} \end{vmatrix},$$

$$D\begin{bmatrix} j \\ k \end{bmatrix} = \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{vmatrix}.$$

Using these abbreviations and the Laplace expansion of determinants, we can rewrite the left-hand side of (5·1) as

$$\begin{aligned}
\text{l.h.s.} &= \left\{ \sum_{p=1}^{n-1} (-1)^{n+p} D\begin{bmatrix} i_p \\ k \end{bmatrix} \boldsymbol{u}_{i_p} + D\begin{bmatrix} j \\ k \end{bmatrix} \boldsymbol{u}_j \right\} \\
&\quad \cdot \left\{ \sum_{q=1}^{n-1} (-1)^{n+q} D\begin{bmatrix} j \\ i_q \end{bmatrix} \boldsymbol{u}_{i_q}^\dagger + D\begin{bmatrix} j \\ k \end{bmatrix} \boldsymbol{u}_k^\dagger \right\} \\
&= \sum_{p=1}^{n-1} \sum_{q=1}^{n-1} (-1)^{p+q} D\begin{bmatrix} i_p \\ k \end{bmatrix} D\begin{bmatrix} j \\ i_q \end{bmatrix} \times \left\{ \frac{(\zeta_{i_q} - \zeta_j^*) + (\zeta_j^* - \zeta_{i_p}^*)}{\mathrm{i}} \right\} d_{i_p i_q} \\
&\quad + \sum_{q=1}^{n-1} (-1)^{n+q} D\begin{bmatrix} j \\ k \end{bmatrix} D\begin{bmatrix} j \\ i_q \end{bmatrix} \times \frac{(\zeta_{i_q} - \zeta_j^*)}{\mathrm{i}} d_{j i_q} \\
&\quad + \sum_{p=1}^{n-1} (-1)^{n+p} D\begin{bmatrix} i_p \\ k \end{bmatrix} D\begin{bmatrix} j \\ k \end{bmatrix} \times \left\{ \frac{(\zeta_k - \zeta_j^*) + (\zeta_j^* - \zeta_{i_p}^*)}{\mathrm{i}} \right\} d_{i_p k} \\
&\quad + D\begin{bmatrix} j \\ k \end{bmatrix} D\begin{bmatrix} j \\ k \end{bmatrix} \times \frac{(\zeta_k - \zeta_j^*)}{\mathrm{i}} d_{jk} \\
&= \sum_{q=1}^{n-1} (-1)^{n+q} D\begin{bmatrix} j \\ i_q \end{bmatrix} \frac{(\zeta_{i_q} - \zeta_j^*)}{\mathrm{i}} \left\{ \sum_{p=1}^{n-1} (-1)^{n+p} D\begin{bmatrix} i_p \\ k \end{bmatrix} d_{i_p i_q} + D\begin{bmatrix} j \\ k \end{bmatrix} d_{j i_q} \right\} \\
&\quad + \sum_{p=1}^{n-1} (-1)^{n+p} D\begin{bmatrix} i_p \\ k \end{bmatrix} \frac{(\zeta_j^* - \zeta_{i_p}^*)}{\mathrm{i}} \left\{ \sum_{q=1}^{n-1} (-1)^{n+q} D\begin{bmatrix} j \\ i_q \end{bmatrix} d_{i_p i_q} + D\begin{bmatrix} j \\ k \end{bmatrix} d_{i_p k} \right\} \\
&\quad + D\begin{bmatrix} j \\ k \end{bmatrix} \frac{(\zeta_k - \zeta_j^*)}{\mathrm{i}} \left\{ \sum_{p=1}^{n-1} (-1)^{n+p} D\begin{bmatrix} i_p \\ k \end{bmatrix} d_{i_p k} + D\begin{bmatrix} j \\ k \end{bmatrix} d_{jk} \right\} \\
&= \sum_{q=1}^{n-1} (-1)^{n+q} D\begin{bmatrix} j \\ i_q \end{bmatrix} \frac{(\zeta_{i_q} - \zeta_j^*)}{\mathrm{i}} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 i_q} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} i_q} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & d_{j i_q} \end{vmatrix}
\end{aligned}$$

$$+\sum_{p=1}^{n-1}(-1)^{n+p}D\left[\begin{array}{c} i_p \\ k \end{array}\right]\frac{(\zeta_j^*-\zeta_{i_p}^*)}{\mathrm{i}}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & d_{i_1k} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & d_{i_{n-1}k} \\ d_{i_pi_1} & \cdots & d_{i_pi_{n-1}} & d_{i_pk} \end{vmatrix}$$

$$+D\left[\begin{array}{c} j \\ k \end{array}\right]\frac{(\zeta_k-\zeta_j^*)}{\mathrm{i}}D.$$

It is easily seen that in the last expression, only the last term remains. This is the right-hand side of (5·1). □

Corollary 5.2. *The vector* $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}$ *defined for distinct positive integers* $i_1,\cdots,i_{n-1},i_n$ *by* (4·15) *with* (4·14) *is a unit vector, i.e.*

$$\left\|\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}\right\|=1.$$

Proof. Using Lemma 5.1 in the special case $j=k\ (\equiv i_n)$, we have

$$\begin{aligned}
&\left\|\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}\right\|^2 \\
&=\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}\cdot\boldsymbol{u}^\dagger_{\{i_1,\cdots,i_{n-1}\},i_n} \\
&=\prod_{l=1}^{n-1}\left|\frac{\zeta_{i_l}-\zeta_{i_n}^*}{\zeta_{i_l}-\zeta_{i_n}}\right|^2\times\frac{d_{i_ni_n}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix}}{\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_n} \\ \vdots & \ddots & \vdots \\ d_{i_ni_1} & \cdots & d_{i_ni_n} \end{vmatrix}}\times\prod_{l=1}^{n-1}\left|\frac{\zeta_{i_l}^*-\zeta_{i_n}^*}{\zeta_{i_l}-\zeta_{i_n}^*}\right|^2 \\
&\quad\times\frac{\dfrac{\zeta_{i_n}-\zeta_{i_n}^*}{\mathrm{i}}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_n} \\ \vdots & \ddots & \vdots \\ d_{i_ni_1} & \cdots & d_{i_ni_n} \end{vmatrix}}{\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix}^2} \\
&=1.
\end{aligned}$$

□

We are now able to apply the collision laws defined by Theorem 3.1 to the two-soliton collision in which soliton $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1},j\},k}$ overtakes soliton $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1}\},j}$ [cf. the definition (4·16)]. Here, $i_1,\cdots,i_{n-1},j,k$ are distinct positive integers.

Proposition 5.3. *The two-soliton collision in which* $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1},j\},k}$ *overtakes* $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1}\},j}$ *changes these solitons to* $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1}\},k}$ *and* $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1},k\},j}$*, as shown in*

Fig. 3. *According to Theorem* 3.1, *this is equivalent to the following set of equalities:*

$$\mathrm{e}^{-2\left(\phi_{\{i_1,\cdots,i_{n-1},j\},k}-\phi_{\{i_1,\cdots,i_{n-1}\},k}\right)} = \mathrm{e}^{-2\left(\phi_{\{i_1,\cdots,i_{n-1},k\},j}-\phi_{\{i_1,\cdots,i_{n-1}\},j}\right)} \tag{5·2a}$$

$$= \left|\frac{\zeta_j-\zeta_k}{\zeta_j-\zeta_k^*}\right|^2 \left\{1+\frac{(\zeta_j-\zeta_j^*)(\zeta_k-\zeta_k^*)}{\left|\zeta_j-\zeta_k^*\right|^2}\left|\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}\cdot\boldsymbol{u}^\dagger_{\{i_1,\cdots,i_{n-1}\},k}\right|^2\right\}, \tag{5·2b}$$

$$\boldsymbol{u}_{\{i_1,\cdots,i_{n-1},j\},k} = \mathrm{e}^{\phi_{\{i_1,\cdots,i_{n-1},j\},k}-\phi_{\{i_1,\cdots,i_{n-1}\},k}}\frac{\zeta_j^*-\zeta_k^*}{\zeta_j-\zeta_k^*}\left\{\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},k} - \frac{\zeta_j-\zeta_j^*}{\zeta_j-\zeta_k^*}\left(\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},k}\cdot\boldsymbol{u}^\dagger_{\{i_1,\cdots,i_{n-1}\},j}\right)\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}\right\}, \tag{5·3}$$

$$\boldsymbol{u}_{\{i_1,\cdots,i_{n-1},k\},j} = \mathrm{e}^{\phi_{\{i_1,\cdots,i_{n-1},k\},j}-\phi_{\{i_1,\cdots,i_{n-1}\},j}}\frac{\zeta_k^*-\zeta_j^*}{\zeta_k-\zeta_j^*}\left\{\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j} - \frac{\zeta_k-\zeta_k^*}{\zeta_k-\zeta_j^*}\left(\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}\cdot\boldsymbol{u}^\dagger_{\{i_1,\cdots,i_{n-1}\},k}\right)\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},k}\right\}. \tag{5·4}$$

Proof. Throughout this proof, we employ the following notation in order to express determinants compactly:

$$d\begin{pmatrix} j_1,j_2,\cdots,j_l \\ k_1,k_2,\cdots,k_l\end{pmatrix} \equiv \begin{vmatrix} d_{j_1k_1} & d_{j_1k_2} & \cdots & d_{j_1k_l} \\ d_{j_2k_1} & d_{j_2k_2} & \cdots & d_{j_2k_l} \\ \vdots & \vdots & \ddots & \vdots \\ d_{j_lk_1} & d_{j_lk_2} & \cdots & d_{j_lk_l}\end{vmatrix}.$$

To prove (5·2), we first rewrite the left-hand side of (5·2a) as

$$\mathrm{e}^{-2\left(\phi_{\{i_1,\cdots,i_{n-1},j\},k}-\phi_{\{i_1,\cdots,i_{n-1}\},k}\right)} = \left|\frac{\zeta_j-\zeta_k}{\zeta_j-\zeta_k^*}\right|^2 \times \frac{d\begin{pmatrix} i_1,\cdots,i_{n-1},j,k \\ i_1,\cdots,i_{n-1},j,k\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1} \\ i_1,\cdots,i_{n-1}\end{pmatrix}}{d\begin{pmatrix} i_1,\cdots,i_{n-1},j \\ i_1,\cdots,i_{n-1},j\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k \\ i_1,\cdots,i_{n-1},k\end{pmatrix}}, \tag{5·5}$$

using the definition of $\phi_{\{i_1,\cdots,i_{n-1}\},i_n}$, (4·14). Obviously, the right-hand side of (5·5) is symmetric with respect to interchange of j and k. It follows that the equality (5·2a) holds.

Next, we prove the equality (5·2b). Using the definition of $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},i_n}$, (4·15), and Lemma 5.1, we obtain

$$\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}\cdot\boldsymbol{u}^\dagger_{\{i_1,\cdots,i_{n-1}\},k} = \mathrm{e}^{\phi_{\{i_1,\cdots,i_{n-1}\},j}+\phi_{\{i_1,\cdots,i_{n-1}\},k}} \times \prod_{l=1}^{n-1}\frac{(\zeta_{i_l}^*-\zeta_j^*)(\zeta_{i_l}-\zeta_k)}{(\zeta_{i_l}-\zeta_j^*)(\zeta_{i_l}^*-\zeta_k)} \times \frac{\zeta_k-\zeta_j^*}{\mathrm{i}} \times \frac{d\begin{pmatrix} i_1,\cdots,i_{n-1},j \\ i_1,\cdots,i_{n-1},k\end{pmatrix}}{d\begin{pmatrix} i_1,\cdots,i_{n-1} \\ i_1,\cdots,i_{n-1}\end{pmatrix}}. \tag{5·6}$$

Multiplying (5·6) by its complex conjugate on each side, we obtain, with the help of (4·14),

$$\left|\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\cdots,i_{n-1}\},k}\right|^2 = -\frac{\left|\zeta_k-\zeta_j^*\right|^2}{(\zeta_j-\zeta_j^*)(\zeta_k-\zeta_k^*)}$$
$$\times\frac{d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},k\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},j\end{pmatrix}}{d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},j\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},k\end{pmatrix}}.$$

Then, we can rewrite the right-hand side of (5·2b) as

$$\left|\frac{\zeta_j-\zeta_k}{\zeta_j-\zeta_k^*}\right|^2\left\{1+\frac{(\zeta_j-\zeta_j^*)(\zeta_k-\zeta_k^*)}{\left|\zeta_j-\zeta_k^*\right|^2}\left|\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\cdots,i_{n-1}\},k}\right|^2\right\}$$
$$=\left|\frac{\zeta_j-\zeta_k}{\zeta_j-\zeta_k^*}\right|^2\left\{1-\frac{d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},k\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},j\end{pmatrix}}{d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},j\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},k\end{pmatrix}}\right\}. \quad (5·7)$$

Here, thanks to the Jacobi formula for determinants, we have

$$d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},j\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},k\end{pmatrix}$$
$$-d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},k\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},j\end{pmatrix}$$
$$=d\begin{pmatrix} i_1,\cdots,i_{n-1}\\ i_1,\cdots,i_{n-1}\end{pmatrix} d\begin{pmatrix} i_1,\cdots,i_{n-1},j,k\\ i_1,\cdots,i_{n-1},j,k\end{pmatrix}. \quad (5·8)$$

Thus, (5·7) is equal to (5·5). This completes the proof of the equality (5·2b).

To prove the equality (5·3), we need to extend the Jacobi formula (5·8). We remark that, although the matrix elements d_{il} here are given by (4·8), the two sides of (5·8) are equal as a polynomial for general elements d_{il}. Therefore, maintaining the validity of (5·8), we can replace the columns with index k by columns consisting of the vectors $\boldsymbol{u}_{i_1},\cdots,\boldsymbol{u}_{i_{n-1}},\boldsymbol{u}_j$ or $\boldsymbol{u}_k$:

$$d\begin{pmatrix} i_1,\cdots,i_{n-1},j\\ i_1,\cdots,i_{n-1},j\end{pmatrix}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & \boldsymbol{u}_{i_1}\\ \vdots & \ddots & \vdots & \vdots\\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & \boldsymbol{u}_{i_{n-1}}\\ d_{ki_1} & \cdots & d_{ki_{n-1}} & \boldsymbol{u}_k\end{vmatrix}$$
$$-d\begin{pmatrix} i_1,\cdots,i_{n-1},k\\ i_1,\cdots,i_{n-1},j\end{pmatrix}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & \boldsymbol{u}_{i_1}\\ \vdots & \ddots & \vdots & \vdots\\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & \boldsymbol{u}_{i_{n-1}}\\ d_{ji_1} & \cdots & d_{ji_{n-1}} & \boldsymbol{u}_j\end{vmatrix}$$

$$= d\begin{pmatrix} i_1, \cdots, i_{n-1} \\ i_1, \cdots, i_{n-1} \end{pmatrix} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 j} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} j} & \boldsymbol{u}_{i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & d_{jj} & \boldsymbol{u}_j \\ d_{k i_1} & \cdots & d_{k i_{n-1}} & d_{kj} & \boldsymbol{u}_k \end{vmatrix}. \tag{5·9}$$

We rewrite the right-hand side of (5·3) using (4·14), (4·15) and (5·6) (with $j \leftrightarrow k$) as

$$\begin{aligned} & \mathrm{e}^{\phi_{\{i_1,\cdots,i_{n-1},j\},k} - \phi_{\{i_1,\cdots,i_{n-1}\},k}} \frac{\zeta_j^* - \zeta_k^*}{\zeta_j - \zeta_k^*} \Bigg\{ \boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},k} \\ & \qquad - \frac{\zeta_j - \zeta_j^*}{\zeta_j - \zeta_k^*} \left(\boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},k} \cdot \boldsymbol{u}^\dagger_{\{i_1,\cdots,i_{n-1}\},j} \right) \boldsymbol{u}_{\{i_1,\cdots,i_{n-1}\},j} \Bigg\} \\ = {} & \mathrm{e}^{\phi_{\{i_1,\cdots,i_{n-1},j\},k}} \times \frac{\zeta_j^* - \zeta_k^*}{\zeta_j - \zeta_k^*} \prod_{l=1}^{n-1} \frac{\zeta_{i_l}^* - \zeta_k^*}{\zeta_{i_l} - \zeta_k^*} \times \frac{1}{d\begin{pmatrix} i_1, \cdots, i_{n-1}, j \\ i_1, \cdots, i_{n-1}, j \end{pmatrix} d\begin{pmatrix} i_1, \cdots, i_{n-1} \\ i_1, \cdots, i_{n-1} \end{pmatrix}} \\ & \times \left\{ d\begin{pmatrix} i_1, \cdots, i_{n-1}, j \\ i_1, \cdots, i_{n-1}, j \end{pmatrix} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{k i_1} & \cdots & d_{k i_{n-1}} & \boldsymbol{u}_k \end{vmatrix} \right. \\ & \qquad \left. - d\begin{pmatrix} i_1, \cdots, i_{n-1}, k \\ i_1, \cdots, i_{n-1}, j \end{pmatrix} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & \boldsymbol{u}_j \end{vmatrix} \right\}. \end{aligned}$$

Owing to the extended Jacobi formula (5·9), this is equal to $\boldsymbol{u}_{\{i_1,\cdots,i_{n-1},j\},k}$ [cf. the definition (4·15)]. Now, the proof of the equality (5·3) is complete. The proof of the equality (5·4) is obtained by interchanging j and k in the proof of (5·3). □

Proposition 5.3 is applicable to two-soliton collisions satisfying the following condition:

- Before the collision, the set of subscripts in the left-hand soliton's $\{\ \}$ is equal to the entire set of subscripts of the right-hand soliton: $\{i_1, \cdots, i_{n-1}, j\} = \{i_1, \cdots, i_{n-1}\} \cup \{j\}$.

We also note the following properties (cf. Fig. 3):

- After the collision, the set of subscripts in the left-hand soliton's $\{\ \}$ is still equal to the entire set of subscripts of the right-hand soliton: $\{i_1, \cdots, i_{n-1}, k\} = \{i_1, \cdots, i_{n-1}\} \cup \{k\}$.
- The entire set of subscripts of the left-hand soliton is unchanged: $\{i_1, \cdots, i_{n-1}, j\} \cup \{k\} = \{i_1, \cdots, i_{n-1}, k\} \cup \{j\}$.
- The set of subscripts in the right-hand soliton's $\{\ \}$ is unchanged: $\{i_1, \cdots, i_{n-1}\} = \{i_1, \cdots, i_{n-1}\}$.
- The overtaken soliton's number is removed from the overtaking soliton's $\{\ \}$,

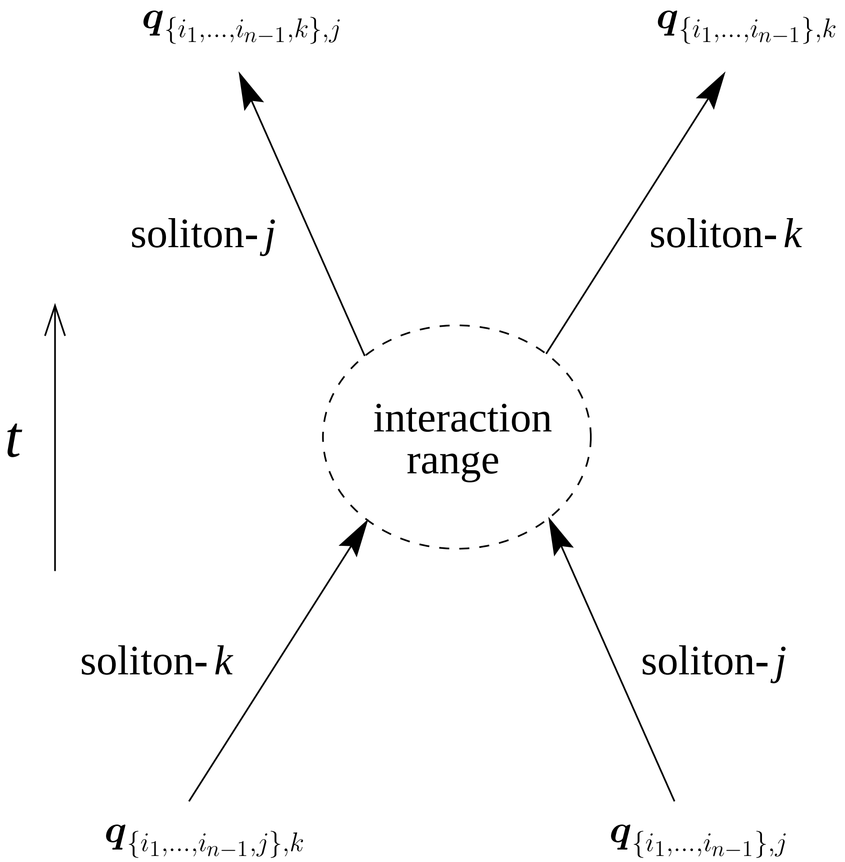

Fig. 3. Two-soliton collision in the presence of other solitons.

while the overtaking soliton's number is added to the overtaken soliton's { }.

We are now able to state the main result of this paper.

Theorem 5.4. *An N-soliton collision in the Manakov model* (1·1) *can be factorized into a nonlinear superposition of* $\binom{N}{2}$ *pair collisions in arbitrary order.*

Proof. According to the asymptotic behavior of the N-soliton solution as $t \to -\infty$ (see Fig. 2), solitons-$N, \cdots, 1$ are initially distributed along the x-axis as

$$\boldsymbol{q}_{\{1,\cdots,N-1\},N}, \cdots, \boldsymbol{q}_{\{1,\cdots,n-1\},n}, \cdots, \boldsymbol{q}_{\{\ \},1}.$$

We take this initial state as the point of departure and assume that the solitons collide pairwise in a given order. Then, a pair collision takes place $\binom{N}{2} = \frac{N(N-1)}{2}$ times. What will the final state be under this assumption? To answer this, we note the following two points:

- The set of subscripts in each soliton's { } is always equal to the entire set of subscripts of the next soliton to the right. This ensures that Proposition 5.3 is applicable to every pair collision.

- Soliton-n will overtake solitons-1, $\cdots$, $n-1$ and will be overtaken by solitons-$n+1$, $\cdots$, N.

We see that, regardless of the order of the pair collisions, solitons-1, $\cdots$, N are finally distributed along the x-axis as

$$\boldsymbol{q}_{\{2,\cdots,N\},1}, \cdots, \boldsymbol{q}_{\{n+1,\cdots,N\},n}, \cdots, \boldsymbol{q}_{\{\ \},N}.$$

This final state is exactly the same as the asymptotic behavior of the N-soliton solution in the $t \to +\infty$ limit (see Fig. 2).

Q.E.D.

Remark 1. Theorem 5.4, together with (3·10), demonstrates that the initial state of N solitons uniquely determines its final state. That is, an N-soliton collision described by Proposition 4.2 defines a mapping from the initial state to the final state. This fact is not evident from Proposition 4.2.

Remark 2. We can now prove that the mapping (3·10) satisfies the property (b) or, equivalently the Yang-Baxter property (b'), as stated in the introduction. It is sufficient to verify the following:

The initial polarization vectors of N solitons $(\boldsymbol{u}_{\{1,\cdots,N-1\},N}, \cdots, \boldsymbol{u}_{\{\ \},1})$ given in Proposition 4.2 can be made to coincide with any combination of N unit vectors in $\mathbb{C}^m$ by appropriately varying the bare polarization vectors $(\boldsymbol{u}_N, \cdots, \boldsymbol{u}_1)$.

This is easily verified if we consider the order of the pair collisions such that every soliton experiences the bare polarization $\boldsymbol{u}_{\{\ \},j} (= \boldsymbol{u}_j)$ once and inductively use the fact that the mapping $\boldsymbol{u}_2 \mapsto \boldsymbol{u}_{\{1\},2}$ in Theorem 3.1 is a bijection on the unit sphere.

Remark 3. The validity of the Yang-Baxter property allows us to extract a new "set-theoretical" solution to the quantum Yang-Baxter equation (cf. Refs. 15) and 16)). To be more specific, regarding (3·10) as the mapping $(\boldsymbol{u}_{\{1\},2}, \boldsymbol{u}_1) \mapsto (\boldsymbol{u}_2, \boldsymbol{u}_{\{2\},1})$, we obtain a nontrivial solution to the parameter-dependent Yang-Baxter equation for mappings that act on the direct product of two (complex) unit vectors. Naturally, we can extend this solution further by adding information concerning the center positions of solitons. However, this extension is not very intriguing, because the net change wrought by the mapping does not depend on the added information.

§6. Concluding remarks

In this paper, we have investigated soliton collisions in the Manakov model for the general case of m components (1·1) using a straightforward approach. We first derived the general N-soliton solution of the Manakov model from that of the matrix NLS equation (2·2) through a simple reduction.[14)] We considered the limits $t \to \mp\infty$ for the $N = 2$ case and obtained the collision laws of two solitons in the Manakov model. Next, we considered the same limits for the case of general N and obtained the asymptotic behavior of the N-soliton solution. We were able to diagram the asymptotic behavior in the simplest way in terms of the quantity $\boldsymbol{q}_{\{i_1,\cdots,i_{n-1}\},i_n}$

defined by (4·16) (see Fig. 2). Taking advantage of this, we proved with a simple combinatorial treatment that an N-soliton collision in the Manakov model can be factorized into a nonlinear superposition of $\binom{N}{2}$ pair collisions in arbitrary order. This clears up the longtime misunderstanding that multi-particle effects exist in the Manakov model.

This result is far from trivial in the $m \geq 2$ case. In the $m = 1$ case (scalar NLS), all the soliton parameters that play an essential role in the collision laws (in the notation of this paper, $\zeta_1, \zeta_2, \cdots, \zeta_N$) are invariant in time. A pair collision results only in a displacement of the soliton centers and a shift of the phases, which will not change the effects of future pair collisions. Thus, a superposition of $\binom{N}{2}$ pair collisions gives the same results for every order of the pair collisions. It is not difficult to prove in this case that an N-soliton collision reduces to a pair collision.[10)–12)] In contrast, in the $m \geq 2$ case, a pair collision results in a change of the polarization vectors. This changes the effects of future pair collisions completely. Therefore, it was not obvious before the present work that a nonlinear superposition of $\binom{N}{2}$ pair collisions gives the same results for every order of the pair collisions or that it exactly coincides with an N-soliton collision. The key to proving these facts is a highly nontrivial relation among determinants and extended determinants, given in Lemma 5.1. This implies the possibility that some new relations similar to Lemma 5.1 can be obtained through the investigation of soliton collisions in multi-component integrable systems.

Very recently, Steiglitz and coworkers[36),37)] proposed that soliton collisions in the Manakov model can be utilized to carry out any computation with beams in a nonlinear optical medium (see Ref. 38) for the experimental foundations). We believe that the results obtained in this paper will be useful to refine and reinforce their interesting idea. In particular, Theorem 5.4 supports[*)] their hypothesis on the pairwise nature of soliton collisions and Proposition 4.2, together with Proposition 5.3, provides a hint on how to design optical logic operations more simply and reliably than the method proposed in Ref. 37).

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[*)] To be precise, we need to extend our result to the case in which some solitons propagate at the same velocity and do not collide with one another. This is beyond the scope of this paper and is left to the reader as a future problem.

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