International Journal of Applied Mathematics

Volume 31 No. 5 2018, 669-678

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v31i5.11

NABLA INTEGRAL FOR FUZZY FUNCTIONS ON TIME SCALES

R. Leelavathi[1 §], G. Suresh Kumar[2], M.S.N. Murty[3]

[1,2]Department of Mathematics
Koneru Lakshmaiah Education Foundation
Vaddeswaram, 522502, Andhra Pradesh, INDIA
[3]D. No. 21-47, Sainivas, Opp. State Bank of India
Bank Street, Nuzvid, Andhra Pradesh, INDIA

Abstract: In this paper, we define fuzzy nabla integrals for fuzzy functions on time scales and obtain some of its fundamental properties and also we establish the relationship between nabla differentiation and integration.

AMS Subject Classification: 03E72, 34A07, 26E07

Key Words: fuzzy differentials and integrals, time scales, Hukuhara nabla derivative, Hukuhara nabla integral

1. Introduction

Fuzzy integral is an important tool to study fuzzy differential equations. Integration on time scales was studied by (see [3]). Recently, Fard and Bidgoli (see [2]) introduced and studied Henstock-Kurzweil integrals of fuzzy valued functions on time scales, using generalized Hukuhara difference. Vasavi et. al. (see [9, 10, 11]) introduced Hukuhara delta integrals using Hukuhara difference and studied fuzzy dynamic equations on time scales. With the importance and advantages of nabla derivatives in recent applications, we proposed to develop the theory of fuzzy nabla dynamic equations on time scales. In this context, we introduce Hukuhara nabla integrals for fuzzy valued functions on time scales

Received: July 14, 2018

§Correspondence author

and study their properties. We organized this paper as follows. In Section 2, we present some definitions, properties basic results relating to fuzzy sets, calculus of fuzzy functions and time scales calculus. Finally in Section 3, we introduces Fuzzy nabla integrals of fuzzy functions on time scales and establish some fundamental properties. We refer (see [5, 1, 6]) for basic results in fuzzy differential equations and time scales.

2. Nabla-Hukuhara Integrability

In this section we introduce and study the properties of ∇-integrals for fuzzy functions on time scales.

Definition 1. A mapping $g : \mathbb{T}^{[a,b]} \to \Re^n$ is said to be measurable nabla-sector of a fuzzy function $G : \mathbb{T}^{[a,b]} \to E_n$ if to each $\theta \in \mathbb{T}^{[a,b]}$, $g(\theta) \in G(\theta)$.

Definition 2. If a regulated measurable nabla-sectors exists, then the fuzzy function $G : \mathbb{T}^{[a,b]} \to E_n$ is said to be regulated. If a ld-continuous measurable nabla-sectors exists, then the fuzzy function G is said to be ld-continuous.

Definition 3. If G has a ld-continuous measurable nabla-sector on $\mathbb{T}^{[a,b]}$ then the fuzzy function $G : \mathbb{T}^{[a,b]} \to E_n$ is ∇-integrable on $\mathbb{T}^{[a,b]}$. We define the nabla integral of G on $\mathbb{T}^{[a,b]}$, by $\int_{\mathbb{T}^{[a,b]}} G(\tau)\nabla\tau$ and is defined levelwise by the equation

$$\begin{aligned}\left[\int_{\mathbb{T}^{[a,b]}} G(\tau)\nabla\tau\right]^{\lambda} &= \int_{\mathbb{T}^{[a,b]}} G_{\lambda}(\tau)\nabla\tau \\ &= \left\{\int_{\mathbb{T}^{[a,b]}} g(\tau)\nabla\tau : g \in S_{G_{\lambda}}(\mathbb{T}^{[a,b]})\right\},\end{aligned}$$

where the set of all nabla integrable sectors of G_{λ} on $\mathbb{T}^{[a,b]}$ is denoted by $S_{G_{\lambda}}(\mathbb{T}^{[a,b]})$.

Theorem 4. *Suppose $G, H : \mathbb{T}^{[a,b]} \to E^n$ is nabla integrable, then we have:*

(a) $\int_a^b [G(\tau) \oplus H(\tau)]\nabla\tau = \int_a^b G(\tau)\nabla\tau \oplus \int_a^b H(\tau)\nabla\tau.$

(b) $\int_a^b \alpha G(\tau)\nabla\tau = \alpha \int_a^b G(\tau)\nabla\tau,\ \alpha \in R.$

(c) $\int_a^b G(\tau)\nabla\tau = \int_a^c G(\tau)\nabla\tau \oplus \int_c^b G(\tau)\nabla\tau$.

(d) $\int_a^a G(\tau)\nabla\tau = \{0\}$.

(e) *If* $g \in S_G(\mathbb{T}^{[a,b]})$*, then* $D_H(G(.), \hat{0}) : \mathbb{T}^{[a,b]} \to R^+$ *is nabla integrable and*

$$D_H\left(\int_a^b G(\tau)\nabla\tau, \hat{0}\right) \leq \int_a^b D_H(G(\tau), \hat{0})\nabla\tau.$$

(f) *If* $g \in S_G(\mathbb{T}^{[a,b]})$ *and* $h \in S_H(\mathbb{T}^{[a,b]})$ *implies that* $g, h \in C_{ld}(\mathbb{T}^{[a,b]})$ *respectively, then* $D_H(G(.), H(.)) : \mathbb{T}^{[a,b]} \to \Re^+$ *is nabla integrable and*

$$D_H\left(\int_a^b G(\tau)\nabla\tau, \int_a^b H(\tau)\nabla\tau\right) \leq \int_a^b D_H\left(G(\tau), H(\tau)\right)\nabla\tau,$$

where the set of all fuzzy ld-continuous functions $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ *is defined by*

$$C_{ld} = C_{ld}(\mathbb{T}^{[a,b]}) = C_{ld}(\mathbb{T}^{[a,b]}, \mathbb{E}_n).$$

Proof. Suppose that $G, H : \mathbb{T}^{[a,b]} \to E_n$ and $a, b \in \mathbb{T}^{[a,b]}$ is nabla integrable.

(a) Since G, H are nabla integrable and for $\lambda \in [0, 1]$, then G_λ and H_λ have measurable nabla-sectors. Thus, for every $p \in \int_a^b [G(\tau) \oplus H(\tau)]_\lambda \nabla\tau$, $\exists\, g \in S_G(\mathbb{T}^{[a,b]})$, $h \in S_H(\mathbb{T}^{[a,b]})$ such that

$$\begin{aligned} p &= \int_a^b [g(\tau) + h(\tau)]\nabla\tau \\ &= \int_a^b g(\tau)\nabla\tau + \int_a^b h(\tau)\nabla\tau \in \int_a^b G_\lambda(\tau)\nabla\tau \oplus \int_a^b H_\lambda(\tau)\nabla\tau, \end{aligned}$$

it implies $\int_a^b [G(\tau) \oplus H(\tau)]_\lambda \nabla\tau \subset \int_a^b [G_\lambda(\tau)]\nabla\tau \oplus \int_a^b [H_\lambda(\tau)]\nabla\tau$. In the same way, it is easy to prove $\int_a^b [G_\lambda(\tau)]\nabla\tau \oplus \int_a^b [H_\lambda(\tau)]\nabla\tau \subset \int_a^b [G(\tau) \oplus H(\tau)]_\lambda \nabla\tau$. Therefore,

$$\int_a^b [G(\tau) \oplus H(\tau)]\nabla\tau = \int_a^b G(\tau)\nabla\tau \oplus \int_a^b H(\tau)\nabla\tau.$$

(b) Let $p \in \int_a^b [\alpha G(\tau)]_\lambda \nabla\tau$, $\exists\, g \in S_G(\mathbb{T}^{[a,b]})$, $\ni$

$$p = \int_a^b [\alpha g(\tau)\nabla\tau] = \alpha \int_a^b g(\tau)\nabla\tau \in \alpha \int_a^b G_\lambda(\tau)\nabla\tau,$$

it implies $\int_a^b [\alpha G(\tau)]_\lambda \nabla\tau \subset \alpha \int_a^b [G(\tau)]_\lambda \nabla\tau$. In the same manner, it is easy to prove $\alpha \int_a^b [G(\tau)]_\lambda \nabla\tau \subset \int_a^b [\alpha G(\tau)]_\lambda \nabla\tau$ Hence, $\int_a^b \alpha G(\tau)\nabla\tau = \alpha \int_a^b G(\tau)\nabla\tau, \alpha \in R$.

(c) Let $\lambda \in [0,1]$ and g be a measurable ∇-sector for G_λ. $\int_a^b g(\tau)\nabla\tau = \int_a^c g(\tau)\nabla\tau + \int_c^b g(\tau)\nabla\tau$, then

$$\int_a^b G_\lambda(\tau)\nabla\tau \subset \int_a^c G_\lambda(\tau)\nabla\tau \oplus \int_c^b G_\lambda(\tau)\nabla\tau. \tag{1}$$

Now, let $z = \int_a^c g_1(\tau)\nabla\tau + \int_c^b g_2(\tau)\nabla\tau$, where g_1 is a measurable ∇-sector for G_λ in $[a,c]$ and g_2 is a measurable ∇-sector for G_λ in $[c,b]$. Then g defined by $g(\theta) = \begin{cases} g_1(\theta), & \text{if} \quad \theta \in [a,c] \\ g_2(\theta), & \text{if} \quad \theta \in [c,b], \end{cases}$ is a measurable ∇-sector for G_λ in $\mathbb{T}^{[a,b]}$ and

$$\int_a^b g(\theta)\nabla\tau = \int_a^c g_1(\tau)\nabla\tau + \int_c^b g_2(\tau)\nabla\tau = z.$$

Thus,

$$\int_a^c G_\lambda(\tau)\nabla\tau \oplus \int_c^b G_\lambda(\tau)\nabla\tau \subset \int_a^b G_\lambda(\tau)\nabla\tau. \tag{2}$$

From (1) and (2), we have

$$\int_a^b G(\tau)\nabla\tau = \int_a^c G(\tau)\nabla\tau \oplus \int_c^b G(\tau)\nabla\tau.$$

(d) Suppose $\lambda \in [0,1]$ and g be a measurable ∇-sector for G_λ. From Theorem 1.77 in [1], we have $\int_a^a g(\tau)\nabla\tau = \{0\}$, then we have

$$\int_a^a G_\lambda(\tau)\nabla\tau = \int_a^a g(\tau)\nabla\tau = 0.$$

(e) It is enough to show (f) because (e) is a trivial case when $H(\theta) = \hat{0}$ in (e). For given any $g \in C_{ld}$, we have

$$d(g(\tau), H(\tau)) \le d(g(\tau), h(\tau)) \le d(g(\tau), g(s)) + d(g(s), h(\tau))$$

Taking infimum over $h(\tau) \in H_\lambda(\tau)$, we get

$$d(g(\tau), H_\lambda(\tau)) \le d(g(\tau), g(s)) + d(g(s), H_\lambda(\tau)).$$

It implies that,

$$d(g(\tau), H_\lambda(\tau)) - d(g(s), H_\lambda(\tau)) \le d(g(\tau), g(s)).$$

Further more, if we interchange s and τ then the inequality holds and ld-continuity of $d(g(.), H_\lambda(\tau))$ at $\tau \in \mathbb{T}^{[a,b]}$ follows for every $g \in C_{ld}$. Hence, $D_H[G(.), H(.)]$ is ld-continuous and the integral $\int_a^b D_H[G(\tau), H(\tau)]\nabla\tau$ is well defined.

Thus for every $p \in \int_a^b G(\tau)\nabla\tau$, $\exists$ a measurable ∇-sector $g \in S_G(\mathbb{T}^{[a,b]}])$ and for any $q \in \int_a^b H(\tau)\nabla\tau$, $\exists$ a measurable nabla-sector $h \in S_H(\mathbb{T}^{[a,b]}) \ni$ $p = \int_a^b g(\tau)\nabla\tau$, $q = \int_a^b h(\tau)\nabla\tau$.

$$\begin{aligned} d(p,q) &= d\left(\int_a^b g(\tau)\nabla\tau, \int_a^b h(\tau)\nabla\tau\right) \\ &= \left\|\int_a^b g(\tau)\nabla\tau - \int_a^b h(\tau)\nabla\tau\right\| \\ &\le \int_a^b \|g(\tau)-h(\tau)\|\,\nabla\tau \le \int_a^b d(g(\tau),h(\tau))\nabla\tau. \end{aligned}$$

Since $g(\tau)$, $h(\tau)$ are arbitrary measurable sectors of $G_\lambda(\tau)$, $H_\lambda(\tau)$, we have

$$d_H\left(\int_a^b G_\lambda(\tau)\nabla\tau, \int_a^b H_\lambda(\tau)\nabla\tau\right) \le \int_a^b d_H\left(G_\lambda(\tau), H_\lambda(\tau)\right)\nabla\tau$$

And hence,

$$D_H\left(\int_a^b G(\tau)\nabla\tau, \int_a^b H(\tau)\nabla\tau\right) \le \int_a^b D_H\left(G(\tau), H(\tau)\right)\nabla\tau.$$

□

Now we discuss the relation between nabla differentiation and integration.

Definition 5. ([7]) Suppose $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is a fuzzy function and $\theta \in \mathbb{T}_k^{[a,b]}$. Let $G^{\nabla_h}(\theta)$ be an element of $\mathbb{E}_n$ exists provided for any given $\epsilon > 0$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and for some $\delta > 0$ such that

$$D_H[(G(\theta+\hbar) \ominus_h G(\rho(\theta)), (\hbar+\nu(\theta)) \odot G^{\nabla_h}(\theta)] \le \epsilon|\hbar+\nu(\theta)|,$$
$$D_H[(G(\rho(\theta)) \ominus_h G(\theta-\hbar), (\hbar-\nu(\theta)) \odot G^{\nabla_h}(\theta)] \le \epsilon|\hbar-\nu(\theta)|,$$

for all $\theta-\hbar, \theta+\hbar \in N_{\mathbb{T}^{[a,b]}}$ with $0 < h < \delta$ where $\nu(\theta) = \theta - \rho(\theta)$. Then G is called nabla Hukuhara form-I differentiable (nabla-h differentiable) at θ and is denoted by $G^{\nabla_h}(\theta)$.

Definition 6. ([8]) Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is a fuzzy function and $\theta \in \mathbb{T}_k^{[a,b]}$. Let $G^{\nabla^{sh}}(\theta)$ be an element of $\mathbb{E}_n$ exists provided for any given $\epsilon > 0$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and for some $\delta > 0$ such that

$$D_H[(G(\rho(\theta)) \ominus G(\theta+\hbar)), -(\hbar+\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon|-(\hbar+\nu(\theta))|,$$

$$D_H[(G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon|-(\hbar-\nu(\theta))|,$$

for all $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$ with $0 < h < \delta$ where $\nu(\theta) = \theta - \rho(\theta)$. Then G is called second type nabla Hukuhara form-II differentiable (∇^{sh}-differentiable) at θ and is denoted by $G^{\nabla^{sh}}(\theta)$.

We consider only right limit at left scattered points and one-sided limit at the end points of $\mathbb{T}_k^{[a,b]}$.

Note. If both $\mathbb{T}$-limits exists at left scattered point, then the nabla-h or nabla-sh derivative is in $\Re^n$ (crisp). It will restrict the nabla-h or nabla-sh differentiability of fuzzy functions on time scales having left scattered points. To avoid this, we consider only right limit at left scattered points.

From the above definitions, we can easily prove the following lemma.

Lemma 2.1. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be a fuzzy function. If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous at θ and θ is left scattered, then:,*

(a) $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla-h differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ with

$$G^{\nabla_h}(\theta) = \frac{G(\theta) \ominus_h G(\rho(\theta))}{\nu(\theta)},$$

provided $G(\theta) \ominus_h G(\rho(\theta))$ exists,

or

(b) $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla-sh differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ with

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\nu(\theta)} \odot \left(G(\rho(\theta)) \ominus_h G(\theta)\right),$$

provided $G(\rho(\theta)) \ominus_h G(\theta)$ exists,

or

(c) $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ with

$$G^{\nabla}(\theta) = \frac{G(\theta) \ominus_h G(\rho(\theta))}{\nu(\theta)} = \frac{-1}{\nu(\theta)} \odot \left(G(\rho(\theta)) \ominus_h G(\theta)\right) \in \Re^n.$$

provided $G(\rho(\theta)) \ominus_h G(\theta)$ and $G(\theta) \ominus_h G(\rho(\theta))$ both exists.

Theorem 7. *Suppose $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ld-continuous and if $\theta_0 \in \mathbb{T}$, then:*

(a) $\mathcal{G}$ defined by

$$\mathcal{G}(\theta) = X_0 \oplus \int_{\theta_0}^{\theta} G(\tau)\nabla\tau, \quad \text{for } \theta \in \mathbb{T}^{[a,b]} \ \text{ and } \ X_0 \in \mathbb{E}_n,$$

is nabla-h differentiable and $\mathcal{G}^{\nabla_h}(\theta) = G(\theta)$ a.e. on $\mathbb{T}^{[a,b]}$.

(b) $\mathcal{G}$ defined by

$$\mathcal{G}(\theta) = X_0 \ominus_h (-1) \int_{\theta_0}^{\theta} G(\tau)\nabla\tau, \quad \text{for } \theta \in \mathbb{T}^{[a,b]} \ \text{ and } \ X_0 \in \mathbb{E}_n,$$

is nabla-sh differentiable and $\mathcal{G}^{\nabla^{sh}}(\theta) = G(\theta)$ a.e. on $\mathbb{T}^{[a,b]}$.

Proof. (a) If θ is left scattered. Clearly $\mathcal{G} : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous and from Lemma 2.1(a), we have $\mathcal{G}$ is ∇_h-differentiable at θ and

$$\begin{aligned}
\mathcal{G}^{\nabla_h} &= \frac{1}{\nu(\theta)} \odot (\mathcal{G}(\theta) \ominus_h \mathcal{G}(\rho(\theta))) \\
&= \frac{1}{\nu(\theta)} \odot \left[\int_{\theta_0}^{\theta} G(\tau)\nabla\tau \ominus_h \int_{\theta_0}^{\rho(\theta)} G(\tau)\nabla\tau \right] \\
&= \frac{1}{\nu(\theta)} \odot \left(\int_{\rho(\theta)}^{\theta} G(\tau)\nabla\tau \right) = G(\theta).
\end{aligned}$$

If θ is ld-point, since G is ld-continuous at θ. For every $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}_k^{[a,b]}}$, we have

$$\begin{aligned}
\mathcal{G}(\rho(\theta)) \ominus_h \mathcal{G}(\theta - \hbar) &= \int_{\theta_0}^{\rho(\theta)} G(\tau)\nabla\tau \ominus_h \int_{\theta_0}^{\theta - \hbar} G(\tau)\nabla\tau \\
&= \int_{\theta - \hbar}^{\rho(\theta)} G(\tau)\nabla\tau.
\end{aligned}$$

Let $\epsilon_1 > 0$ be arbitrary,

$$\begin{aligned}
&D_H\left(G(\theta), \frac{\mathcal{G}(\rho(\theta)) \ominus_h \mathcal{G}(\theta - \hbar)}{\hbar - \nu(\theta)} \right) \\
&\quad = \frac{1}{\hbar - \nu(\theta)} \odot [D_H((\hbar - \nu(\theta)) \odot G(\theta), \mathcal{G}(\rho(\theta)) \ominus_h \mathcal{G}(\theta - \hbar))] \\
&\quad = \frac{1}{\hbar - \nu(\theta)} \odot D_H\left(\int_{\theta - \hbar}^{\rho(\theta)} G(\theta)\nabla\tau, \int_{\theta - \hbar}^{\rho(\theta)} G(\tau)\nabla\tau \right)
\end{aligned}$$

$$\leq \frac{1}{\hbar - \nu(\theta)} \odot \int_{\theta-\hbar}^{\rho(\theta)} D_H(G(\theta), G(\tau))\nabla\tau < \epsilon_1.$$

Therefore,

$$\mathbb{T} - \lim_{\hbar\to 0} \frac{\mathcal{G}(\rho(\theta)) \ominus_h \mathcal{G}(\theta - \hbar)}{\hbar - \nu(\theta)} = \lim_{\hbar\to 0} \frac{\mathcal{G}(\theta) \ominus_h \mathcal{G}(\theta - \hbar)}{\hbar} = G(\theta).$$

Similarly, we can prove

$$\mathbb{T} - \lim_{\hbar\to 0} \frac{\mathcal{G}(\theta + \hbar) \ominus_h \mathcal{G}(\rho(\theta))}{\hbar + \nu(\theta)} = \lim_{\hbar\to 0} \frac{\mathcal{G}(\theta + \hbar) \ominus_h \mathcal{G}(\theta)}{\hbar} = G(\theta).$$

and hence $\mathcal{G}^{\nabla_h}(\theta) = G(\theta)$ a.e on $\mathbb{T}_k^{[a,b]}$.

(b) If θ is left scattered. Clearly $\mathcal{G} : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous and from Lemma 2.1(b), we have $\mathcal{G}$ is ∇^{sh}-differentiable at θ and

$$\begin{aligned}
\mathcal{G}^{\nabla^{sh}} &= \frac{-1}{\nu(\theta)} \odot (\mathcal{G}(\rho(\theta)) \ominus_h \mathcal{G}(\theta)) \\
&= \frac{-1}{\nu(\theta)} \odot \left[\int_{\theta_0}^{\rho(\theta)} G(\tau)\nabla\tau \ominus_h \int_{\theta_0}^{\theta} G(\tau)\nabla\tau\right] \\
&= \frac{-1}{\nu(\theta)} \odot \left(\int_{\theta}^{\rho(\theta)} G(\tau)\nabla\tau\right) \\
&= \frac{-1}{\nu(\theta)} \odot (-1) \left(\int_{\rho(\theta)}^{\theta} G(\tau)\nabla\tau\right) = G(\theta).
\end{aligned}$$

If θ is ld-point, since G is ld-continuous at θ. For every $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}_k^{[a,b]}}$, we have

$$\begin{aligned}
\mathcal{G}(\theta - \hbar) \ominus_h \mathcal{G}(\rho(\theta)) &= \int_{\theta_0}^{\theta-\hbar} G(\tau)\nabla\tau \ominus_h \int_{\theta_0}^{\rho(\theta)} G(\tau)\nabla\tau \\
&= \int_{\rho(\theta)}^{\theta-\hbar} G(\tau)\nabla\tau.
\end{aligned}$$

Let $\epsilon_1 > 0$ be arbitrary,

$$\begin{aligned}
&D_H\left(G(\theta), \frac{\mathcal{G}(\theta - \hbar) \ominus_h \mathcal{G}(\rho(\theta))}{-(\hbar - \nu(\theta))}\right) \\
&\quad = \frac{-1}{\hbar - \nu(\theta)} \odot [D_H(-(\hbar - \nu(\theta)) \odot G(\theta), \mathcal{G}(\theta - \hbar) \ominus_h \mathcal{G}(\rho(\theta))]
\end{aligned}$$

$$= \frac{-1}{\hbar - \nu(\theta)} \odot D_H \left(\int_{\rho(\theta)}^{\theta-\hbar} G(\theta)\nabla\tau, \int_{\rho(\theta)}^{\theta-\hbar} G(\tau)\nabla\tau \right)$$
$$\leq \frac{-1}{\hbar - \nu(\theta)} \odot \int_{\rho(\theta)}^{\theta-\hbar} D_H(G(\theta), G(\tau))\nabla\tau < \epsilon_1.$$

Therefore,

$$\mathbb{T} - \lim_{\hbar\to 0} \frac{\mathcal{G}(\theta-\hbar) \ominus_h \mathcal{G}(\rho(\theta))}{-(\hbar - \nu(\theta))} = \lim_{\hbar\to 0} \frac{\mathcal{G}(\theta-\hbar) \ominus_h \mathcal{G}(\theta)}{-\hbar} = G(\theta).$$

Similarly, we can prove

$$\mathbb{T} - \lim_{\hbar\to 0} \frac{\mathcal{G}(\rho(\theta)) \ominus_h \mathcal{G}(\theta+\hbar)}{-(\hbar + \nu(\theta))} = \lim_{\hbar\to 0} \frac{\mathcal{G}(\theta) \ominus_h \mathcal{G}(\theta+\hbar)}{-\hbar} = G(\theta).$$

and hence $\mathcal{G}^{\nabla^{sh}}(\theta) = G(\theta)$ a.e on $\mathbb{T}_k^{[a,b]}$.

□

Remark 3.2. If $G : \mathbb{T}^{[a,b]} \to E_n$ is ∇_h-differentiable and its derivative (G_h^{∇}-derivative) is nabla-integrable over $\mathbb{T}^{[a,b]}$, then to each $\theta \in \mathbb{T}_k^{[a,b]}$, we have

$$\mathcal{G}(\theta) = \mathcal{G}(\theta_0) \oplus \int_{\theta_0}^{\theta} G(\tau)\nabla\tau \ \text{ for } \ \theta \in \mathbb{T}_k^{[a,b]},$$

(or)

if G is ∇^{sh}-differentiable and its derivative ($G^{\nabla^{sh}}$-derivative) is nabla-integrable over $\mathbb{T}^{[a,b]}$, then to each $\theta \in \mathbb{T}_k^{[a,b]}$, we have

$$\mathcal{G}(\theta) = \mathcal{G}(\theta_0) \ominus_h (-1) \int_{\theta_0}^{\theta} G(\tau)\nabla\tau \ \ for \ \ \theta \in \mathbb{T}_k^{[a,b]}.$$

3. Conclusions

In this paper, we develop and study the properties of Hukuhara nabla integral for fuzzy functions on time scales. In the future, we will introduce and study generalizations of Hukuhara nabla differentials and integrals for fuzzy functions on time scales. Further, these concepts can apply to study the fuzzy dynamic equations on time scales.

References

[1] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston (2001).

[2] O.S. Fard, T.A. Bidgoli, Calculus of fuzzy functions on time scales (I), *Soft Computing*, **19**, No 2 (2015), 293-305.

[3] G.S. Guseinov, Integration on time scales, *Journal of Mathematical Analysis and Applications* **285** (2003), 107-127.

[4] M. Hukahara, Integration des applications measurables dont la valeur est. un compact convexe, *Funkcialaj Ekvacioj*, **10** (1967), 205-223.

[5] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, **24** (1987), 301-317.

[6] V. Lakshmikantham, R.N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis (2003).

[7] R. Leelavathi, G. Suresh Kumar, M.S.N. Murty, Nabla Hukuhara differentiability for fuzzy functions on time scales, *Communicated.*

[8] R. Leelavathi, G. Suresh Kumar, M.S.N. Murty, Second type Nabla Hukuhara differentiability for fuzzy functions on time scales, *Communicated.*

[9] C.H. Vasavi, G. Suresh Kumar, M.S.N. Murty, Fuzzy Hukahara delta differential and applications to fuzzy dynamic equations on time scales, *Journal of Uncertain Systems*, **10**, No 1 (2016), 163-180.

[10] C.H. Vasavi, G. Suresh Kumar, M.S.N. Murty, Fuzzy dynamic equations on time scales under second type Hukuhara delta derivative, *International Journal of Chemical Sciences*, **14**, No 1 (2016), 49-66.

[11] C.H. Vasavi, G. Suresh Kumar, M.S.N. Murty, Generalized differentiability and integrability for fuzzy set-valued functions on time scales, *Soft Computing*, **20** (2016), 1093-1104.