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NASH EQUILIBRIA FOR NONCOOPERATIVE *n*-PERSON GAMES IN NORMAL FORM*

STEVE H. TIJS†

Abstract. This paper concentrates on the problem of the existence of ε-equilibrium points in the Nash sense for noncooperative n-person games in normal form. Firstly, a survey of known ε-equilibrium point theorems and proof techniques is given. Then various new ε-equilibrium point theorems are derived, in which one of the players has a big strategy space and where nonbounded, payoff functions are allowed. To prove these theorems, results of Arzelą–Ascoli are extended.

1. Introduction. Many social situations have in common that there are a number of decision makers with various objectives involved, who for some reason do not cooperate and where the final outcome of the situation depends only on the independently chosen actions or strategies of the different decision makers. When, moreover, the number of decision makers is n and when the rewards for the players, corresponding with an outcome, can be expressed in real numbers, then such a situation can be modeled as a noncooperative n-person game in normal form.

A very important role in noncooperative game theory is played by equilibria and ε-equilibria, introduced by John Nash, which are stable outcomes in the sense that a unilateral deviation from an equilibrium point (ε-equilibrium point) by one of the players does not increase (increases at most ε) the payoff of that player.

In this paper the problem of the existence of equilibria and ε-equilibria of n-person games is central, especially for situations where one of the players has a "big" strategy space and where unbounded payoff functions are allowed.

As a possible application we note that many oligopoly situations can be transformed into n-person games in normal form and that the problem of the existence of ε-Cournot-equilibria is equivalent to the problem of the existence of ε-Nash-equilibria (cf. Friedman [10, pp. 168–172]).

For work relating economic situations and noncooperative games we refer to the papers of Lee and Teo [20], Levitan and Shubik [22], Weddepohl [49] and the books of Friedman [10], Marschak and Selten [25] and Farquharson [9].

The organization of this paper is as follows. In § 2 we give formal definitions and introduce the notation, used in this paper. Section 3 is devoted to a survey of known (ε-)equilibrium point theorems, where in the discussion the emphasis is laid on used proof techniques. Also, the relation with minimax theorems for two-person games is discussed.

In § 4, a number of new ε-equilibrium point theorems are derived. To prove some of these theorems we also generalize in § 4 results of Arzela–Ascoli.

2. Definitions and notation. Let n be a natural number. In the following, $N = \{1, 2, \cdots, n\}$ will denote the set of *players* in an n-person game.

An *n-person game* (*in normal form*) can be described by an ordered $2n$-tuple $\langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n\rangle$, where for each player $i \in N$ the nonempty set X_i is his (*pure*) *strategy space* and where K_i is his *payoff function*, which assigns to each element $(x_1, x_2, \cdots, x_n)$ of the *outcome space* $X = \prod_{k\in N} X_k$ a real number $K_i(x_1, x_2, \cdots, x_n)$.

Such a game is played as follows.

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† Department of Mathematics, Catholic University, 6525 ED Nijmegen, the Netherlands.

(1) Each of the players i chooses a strategy $x_i \in X_i$ which results in the point $(x_1, x_2, \cdots, x_n)$ of the outcome space.

(2) Subsequently, each player $i \in N$ obtains a payoff $K_i(x_1, x_2, \cdots, x_n)$.

Corresponding to a fixed n-person game $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ we will use the following notation.

(a) For each $i \in N$, the Cartesian product $\prod_{k \in N-\{i\}} X_k$ will be denoted by X_{-i} and elements of X_{-i} will often be denoted by x_{-i}. If $x = (x_1, x_2, \cdots, x_n) \in X$, then the element $(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$ will also be denoted by x_{-i}.

(b) For $i \in N$, $a_i \in X_i$ and $x_{-i} = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$, the element $(x_1, x_2, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n)$ will be denoted by $(a_i \uparrow x_{-i})$. Hence $(a_i \uparrow x_{-i})$ is the element of X which can be constructed from $a_i \in X_i$ and $x_{-i} \in X_{-i}$ in the obvious way. Note that for $x \in X$ we have $K_i(x_i \uparrow x_{-i}) = K_i(x)$.

(c) Let $i \in N$, $\varepsilon_i \geqq 0$, $z \in X_{-i}$. The set of *ε_i-best replies to z for player i* is the set $\{a \in X_i: K_i(a \uparrow z) \geqq \sup_{x_i \in X_i} K_i(x_i \uparrow z) - \varepsilon_i\}$, which will be denoted by $R_i(\varepsilon_i; z)$.

Now we define the important concept for n-person games in normal form, introduced by J. F. Nash.

DEFINITION. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be nonnegative real numbers and let $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1; K_2, \cdots, K_n \rangle$ be an n-person game. Then we call a point $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) \in X$ an *$(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$-equilibrium point* of the game Γ, if for each $i \in N$ and each $x_i \in X_i$ we have

$$K_i(x_i \uparrow \hat{x}_{-i}) \leqq K_i(\hat{x}) + \varepsilon_i,$$

or equivalently, if for each $i \in N$, $\hat{x}_i \in R_i(\varepsilon_i; \hat{x}_{-i})$. A $(0, 0, \cdots, 0)$-equilibrium point is also called an *equilibrium point* or a *Nash equilibrium* for the game.

For a nonempty set X let us denote the probability measure (on the σ-algebra of all subsets of X) with mass 1 in $a \in X$ by $e(a)$. Let $\tilde{X}$ be the convex hull of $\{e(a): a \in X\}$; i.e., $\mu \in \tilde{X}$ iff there is an $s \in \mathbb{N}$ and there are $a_1, a_2, \cdots, a_s \in X$ and nonnegative real numbers $p_1, p_2, \cdots, p_s$ with sum $\sum_{k=1}^{s} p_k = 1$ such that $\mu = \sum_{k=1}^{s} p_k e(a_k)$.

DEFINITION. Let $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ be an n-person game. Let $\tilde{\Gamma}$ be the n-person game $\langle \tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_n, \tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_n \rangle$, with

$$\tilde{K}_i(\mu_1, \mu_2, \cdots, \mu_n) = \int K_i(x_1, x_2, \cdots, x_n)\, d\mu_1(x_1)\, d\mu_2(x_2) \cdots d\mu_n(x_n)$$

for all $i \in N$ and $(\mu_1, \mu_2, \cdots, \mu_n) \in \tilde{X} = \prod_{i \in N} \tilde{X}_i$.

Then $\tilde{\Gamma}$ is called the *(finite) mixed extension of the game* Γ and the elements of $\tilde{X}_i$ are called *mixed strategies* for player i.

We note that in $\tilde{\Gamma}$ the strategy spaces of Γ are extended by allowing the players to use mixtures of a finite number of pure strategies. We note further that we have here no measurability problems; the integral in the above definition is, essentially, a finite sum. If the players $1, 2, \cdots, n$ use the mixed strategies $\mu_1, \mu_2, \cdots, \mu_n$, respectively, then $\tilde{K}_i(\mu_1, \mu_2, \cdots, \mu_n)$ is the expected payoff for player i. It will be obvious, what the meaning of $\tilde{X}_{-i}$ and $\mu_{-i} \in \tilde{X}_{-i}$ is, if $i \in N$ and $\mu \in \tilde{X}$.

Much attention in the literature is paid to an important subclass of n-person games in normal form, namely the class of two-person zero-sum games, consisting of those two-person games $\langle X_1, X_2, K_1, K_2 \rangle$ for which $K_1 + K_2$ is the zero-function on the outcome space $X_1 \times X_2$. For a two-person zero-sum game $\langle X_1, X_2, K_1, K_2 \rangle$, important notions are the *lower value* $\underline{v}(X_1, X_2, K_1, K_2) = \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} K_1(x_1, x_2)$ and the *upper value* $\bar{v}(X_1, X_2, K_1, K_2) = \inf_{x_2 \in X_2} \sup_{x_1 \in X_1} K_1(x_1, x_2)$. If for a game $\langle X_1, X_2, K_1, K_2 \rangle$ the lower value is equal to the upper value, then we say that the game has a *value*, and then the elements of

$$O_1^\varepsilon(X_1, X_2, K_1, K_2) = \{\hat{x}_1 \in X_1: \inf_{x_2 \in X_2} K_1(\hat{x}_1, x_2) \geqq \underline{v}(X_1, X_2, K_1, K_2) - \varepsilon\}$$

and of

$$O_2^\varepsilon(X_1, X_2, K_1, K_2) = \{\hat{x}_2 \in X_2: \sup_{x_1 \in X_1} K_1(x_1, \hat{x}_2) \leqq \bar{v}(X_1, X_2, K_1, K_2) + \varepsilon\},$$

where $\varepsilon \geqq 0$, are called *ε-optimal strategies for player* 1 *and for player* 2, respectively. Furthermore, 0-optimal strategies are also called *optimal strategies.*

3. Minimax theorems and $\underline{\varepsilon}$-equilibrium point theorems. In minimax theorems one deals with the existence of values and optimal strategies for classes of two-person zero-sum games. The first minimax theorem was proved in 1928 by John von Neumann [29], and this theorem states that the mixed extension of a zero-sum two-person game with finite strategy spaces (also called a matrix game) possesses a value and optimal mixed strategies for both players. Since that time, many minimax theorems have been derived with a great variety of proof techniques and also under a variety of conditions of a topological and algebraic kind, put on strategy spaces and payoff functions. We do not want to go into details here but we invite the reader to look at the papers of Fan [8], Hirschfeld [15], König [18], Owen [31], Sion [40], Teh [42], Terkelsen [43] and Tijs [44] to get an impression of sufficient conditions for the existence of a value and of proof techniques available. For a systematic treatment of minimax theorems, we also refer to the survey paper of Yanovskaya [52], to Chapter 5 in the book [33] of Parthasarathy and Raghavan and to Chapter 6 in the book [41] of Stoer and Witzgall.

In $\underline{\varepsilon}$-equilibrium point theorems, sufficient conditions for strategy spaces and payoff functions are given to guarantee the existence of $\underline{\varepsilon}$-equilibria for all $\underline{\varepsilon} > 0$ or for the existence of equilibrium points.

The first equilibrium point theorem was due to J. F. Nash (see Theorem 3.1). In the rest of this section we will try to give an impression of the state of the art of this part of game theory by discussing proof techniques used by various authors, and by recalling some of the equilibrium point theorems which will be used in § 4. In that section some new $\underline{\varepsilon}$-equilibrium point theorems will be derived.

First of all let us note that the equilibrium problem is a natural extension of the minimax problem, as the following proposition shows. We leave the proof of this proposition to the reader (cf. Tijs [45, pp. 756–757]).

PROPOSITION 3.1. *Let $\Gamma = \langle X_1, X_2, K_1, K_2\rangle$ be a two-person zero-sum game. Then we have*:

(1) *Γ possesses a value and optimal strategies for both players iff Γ possesses an equilibrium point.*

(2) *Γ possesses a real value (i.e., $\underline{v}(\Gamma) = \bar{v}(\Gamma) \in \mathbb{R}$) iff Γ possesses $(\varepsilon_1, \varepsilon_2)$-equilibrium points for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.*

(3) *If Γ possesses a real value and ε_1 and ε_2 are nonnegative real numbers, then $(\hat{x}_1, \hat{x}_2)$ is an $(\varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_2)$-equilibrium point of Γ, if $\hat{x}_1$ is ε_1-optimal and $\hat{x}_2$ is ε_2-optimal.*

(4) *If Γ possesses a real value and if $(\hat{x}_1, \hat{x}_2)$ is an $(\varepsilon_1, \varepsilon_2)$-equilibrium point, then $\hat{x}_1$ is $(\varepsilon_1+\varepsilon_2)$-optimal for player* 1 *and $\hat{x}_2$ is $(\varepsilon_1+\varepsilon_2)$-optimal for player* 2.

In view of Proposition 3.1 it is obvious that zero-sum game theory is a source of inspiration for the general noncooperative theory. However, the equilibrium problem is much more complicated and many proof techniques in minimax theorems cannot be used. Also stronger conditions are often needed in equilibrium point theorems. The following two theorems will be needed later on, but at this moment they can serve to give the reader an impression of the conditions which appear in equilibrium point theorems.

THEOREM 3.1. (Equilibrium point theorem of J. Nash [27], [28]). *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game, where the strategy spaces* $X_1, X_2, \cdots, X_n$ *are finite sets. Then the mixed extension* $\tilde{\Gamma}$ *possesses at least one equilibrium point.*

We note that the strategy spaces of $\tilde{\Gamma}$ in Theorem 3.1 can be identified with compact convex subsets of finite dimensional spaces (simplices), and that the payoff functions of $\tilde{\Gamma}$ are continuous and multilinear.

THEOREM 3.2. (Equilibrium point theorem of Nikaido–Isoda [30]). *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game such that*

(1) $X_1, X_2, \cdots, X_n$ *are compact and convex subsets of topological vector spaces,*

(2) $K_1, K_2, \cdots, K_n$ *are continuous functions,*

(3) *for all* $i \in N$ *and all* $x_{-i} \in X_{-i}$ *the function on* X_i *defined by* $x_i \mapsto K_i(x_i \uparrow x_{-i})$ *is a concave function.*

Then Γ *possesses at least one equilibrium point.*

Now we want to discuss some proof techniques.

1. In many proofs the *$\underline{\varepsilon}$-best reply multifunction* $R(\underline{\varepsilon}, \cdot)$ plays a role. This map assigns to an $x \in X$ the subset $\prod_{k \in N} R_k(\varepsilon_k; x_{-k})$ of X ($\underline{\varepsilon} \geqq (0, \cdots, 0)$). The usefulness of this function follows from the following proposition.

PROPOSITION 3.2. *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game and* $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \mathbb{R}^n_+$. *Then we have*

$\hat{x} \in X$ is an $\underline{\varepsilon}$-equilibrium point iff $\hat{x}$ is a fixed point of the multifunction $R(\varepsilon, \cdot)$ (i.e., $\hat{x} \in R(\varepsilon, \hat{x})$).

In view of this proposition, the equilibrium point existence problem can be replaced by the problem of the existence of fixed points of certain multifunctions.

The first proof of Theorem 3.1 in [27] used the best reply multifunction and the fixed-point theorem of Kakutani [17] for multifunctions. Other equilibrium point theorems with proofs based on the ($\underline{\varepsilon}$-) best reply multifunction were given by Glicksberg [11], Browder [4], Rupp [37] and Bloemberg [2]. The first three mentioned authors and also Halpern [12] derived new fixed-point theorems extending Kakutani's theorem. Bloemberg gave an elementary proof for the existence of fixed points of the $\underline{\varepsilon}$-best reply multifunction, when $N = \{1, 2\}$, $X_1 = [0, 1]$ and $X_2 = [0, \infty)$.

Because Bloemerg's master's thesis [2] is not published, we will recall here his main result.

THEOREM 3.3. *Let* $\Gamma = \langle [0, 1], [0, \infty), K_1, K_2 \rangle$ *be a two-person game with the following properties*:

(1) K_2 *is an upper bounded function.*

(2) *For each* $x_2 \in [0, \infty)$ *the function* $x_1 \mapsto K_1(x_1, x_2)$ *is quasi-concave; for each* $x_1 \in [0, 1]$ *the function* $x_2 \mapsto K_2(x_1, x_2)$ *is quasi-concave.*

(3) *The family* $\{x_1 \mapsto K_1(x_1, x_2): x_2 \in [0, \infty)\}$ *of functions on* $[0, 1]$ *is an equicontinuous family.*

Then the game Γ *has* $\underline{\varepsilon}$*-equilibrium points for each* $\underline{\varepsilon} > (0, 0)$.

Ponstein [34] derived the existence of equilibrium points of constrained games with the same technique, using the Glicksberg-Fan extension [11], [8] of Kakutani's theorem. Debreu [5] also derived a very general existence theorem for constrained n-person games using the fixed point theorem of Eilenberg-Montgomery [7].

2. In the papers of Nikaido and Isoda [30] and of Rosen [35] a role is played by the function $L: X \times X \to \mathbb{R}$ and the multifunction $M: X \to X$ defined by

$$L(x, y) = \sum_{i \in N} K_i(y_i \uparrow x_{-i}),$$

$$M(x) = \{y \in X : L(x, y) = \max_{z \in X} L(x, z)\}.$$

This multifunction M has the following interesting property.

PROPOSITION 3.3. *Let Γ, L and M be as above. Then $\hat{x} \in X$ is an equilibrium point of the game Γ iff $\hat{x}$ is a fixed point of the multifunction M.*

Hence Proposition 3.3 also reduces the equilibrium problem to a fixed-point problem.

3. In his second proof in [28], J. F. Nash reduced the problem of the existence of equilibrium points to the problem of the existence of a fixed point of a (univalent) function. The ingenious function $f: \tilde{X} \to \tilde{X}$, which he constructed, we like to call the *correction function* of Nash. Extensions of Nash's result, also using a correction function, were given by Borges [3], Mahn [24], Marchi [26] and Jiang Jia-He [16]. Fixed-point theorems of Brouwer or of Schauder–Tykhonov play a role. Vrieze also used, in [47], a correction function (and the fixed-point theorem of Schauder–Tykhonov) to show that a class of games with a countable infinite number of players has an equilibrium point.

4. Another approach to the existence problem was followed in the papers of Tijs [44, pp. 95–96], Hazen [14] and Owen [32], where a sequence of finite subgames of the original game was considered and where an (ε-)equilibrium point of the original game was found, using the sequence of equilibrium points of the subgames considered. In [44], this approach was used to prove that $m \times \infty$-bimatrix games (A, B) with (upper) bounded B have ε-equilibrium points for each $\varepsilon > 0$, and in [14] and [32] mixed extensions of noncooperative games were considered, where the strategy spaces were equal to the closed interval $[0, 1]$.

5. In Tijs [45], ε-equilibrium points for two-person games Γ were derived by looking at subgames Γ' of Γ for which ε-equilibria exist and where the strategy spaces of Γ' arise from the corresponding strategy spaces of Γ by the removal of ε-dominated strategies. With the aid of this technique some of the results of [45] will be extended in § 4 of this paper to games with more than two persons.

6. In the papers [37] and [38] Rupp reduces the existence problem to the existence of ε-equilibria for certain subgames called λ-*marginal games* (λ-Rand Spiele). Let $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ and $\lambda > 0$. Then the λ-marginal game of Γ is the game

$$\Gamma(\lambda) = \langle X_1(\lambda), X_2(\lambda), \cdots, X_n(\lambda), K_1, K_2, \cdots, K_n \rangle,$$

where $X_i(\lambda)$ is the subset $\bigcup\{R_i(\lambda; x_{-i}): x_{-i} \in X_{-i}\}$ of X_i consisting of λ-best replies for player i to some x_{-i} in X_{-i}, and where the payoff functions of $\Gamma(\lambda)$ are restrictions of the payoff functions of Γ. We note that λ-marginal games were introduced by Bierlein [1] for two-person games. The connection between Γ and $\Gamma(\lambda)$ with respect to $\underline{\varepsilon}$-equilibrium points is described in the following

PROPOSITION 3.4. *Let $\lambda > 0$ and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n \in [0, \lambda]$. Then*:

(1) *If $\hat{x}$ is an $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$-equilibrium point of Γ, then $\hat{x} \in \Pi X_i(\lambda)$ and $\hat{x}$ is an $(\varepsilon_1, \cdots, \varepsilon_n)$-equilibrium point of $\Gamma(\lambda)$.*

(2) *If $\hat{x}$ is an $\underline{\varepsilon}$-equilibrium point of $\Gamma(\lambda)$, then $\hat{x}$ is also an $\underline{\varepsilon}$-equilibrium point of Γ.*

7. An elementary algebraic proof of the two-person version of the equilibrium point theorem of Nash is included in the paper of Lemke–Howson [21], where the existence problem is transformed into a linear complementarity problem (cf. Shapley [39]). Also a method is described there (the Lemke algorithm) of finding equilibrium points for bimatrix games. For a quite different proof of the Nash theorem, using algebraic geometry, we refer to Harsanyi [13].

We conclude this section with two remarks.

(i) In the foregoing, the emphasis was on proof techniques and we paid less attention to the precise conditions in the various mentioned ε-equilibrium point theorems. But, roughly speaking, we can say that in the proofs using fixed-point theorems, the strategy spaces of the considered games are small (compact, precompact metric) sets and the payoff functions satisfy heavy continuity and concavity conditions. The methods, where a game is approximated by subgames make it possible to derive $\underline{\varepsilon}$-equilibrium point theorems for games where not all strategy spaces need to be small.

(ii) For works in which computational aspects of equilibrium points are considered we refer to Rosen [35], Kuhn [19], Lemke–Howson [21], Rosenmüller [36], Lüthi [23], Wilson [50] and Winkels [51]. Much research can still be done here; promising are fixed-point algorithms for functions and multifunctions (cf. [23]).

4. Some new $\underline{\varepsilon}$-equilibrium point theorems for n-person games in normal form. When the author was presenting in Heidelberg the paper [45] concerning the existence of ε-equilibrium points for two-person games in normal form, the question arose, whether results in that paper could be extended to games with more than two persons. In this section we show that most of the theorems in [45] can be extended in a certain way to n-person games. We derive, also, some other ε-equilibrium point theorems for n-person games and mixed extensions with big outcome sets and unbounded payoff functions. In the proofs the games will be compared with suitable small subgames for which one already knows that ε-equilibrium points exist.

DEFINITIONS. Let $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ be an n-person game and let $Y_1, Y_2, \cdots, Y_n$ be nonempty subsets of $X_1, X_2, \cdots, X_n$, respectively. Then the game $\Gamma' = \langle Y_1, Y_2, \cdots, Y_n, K_1, \cdots, K_n \rangle$, in which the payoff functions are restrictions of the payoff functions in Γ to the set $Y = \prod_{i=1}^{n} Y_i$ (and in which restrictions are denoted by the same symbol), is called a *subgame* of Γ.

Let $\delta_1, \delta_2, \cdots, \delta_n$ be n nonnegative real numbers. We shall say that the subgame Γ' of the game Γ is a $(\delta_1, \delta_2, \cdots, \delta_n)$-*approximation of* Γ if for each player i the following holds:

APP(i): For each $x_i \in X_i$ there exists a $y_i \in Y_i$, such that for all $z \in X_{-i}$ we have $K_i(x_i \uparrow z) \leqq K_i(y_i \uparrow z) + \delta_i$.

Much use will be made of the following two lemmas, describing a relation between ε-equilibrium points of a game and of its approximations.

LEMMA 4.1. *Let* $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ *and* $\underline{\delta} = (\delta_1, \delta_2, \cdots, \delta_n)$ *be nonnegative vectors in* $\mathbb{R}^n$. *Let* Γ *and* Γ' *be as above and suppose that the n-person game* Γ' *is a* $\underline{\delta}$-*approximation of the game* Γ. *Then each* $\underline{\varepsilon}$-*equilibrium point of* Γ' *is also an* $(\underline{\varepsilon} + \underline{\delta})$-*equilibrium point of* Γ.

Proof. Let $\hat{y} = (\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n)$ be an $\underline{\varepsilon}$-equilibrium point of Γ'. Take $i \in N$ and $x_i \in X$. Since APP(i) holds, there is a $y_i \in Y_i$ such that

$$K_i(x_i \uparrow \hat{y}_{-i}) \leqq K_i(y_i \uparrow \hat{y}_{-i}) + \delta_i. \tag{1}$$

Furthermore,

$$K_i(y_i \uparrow \hat{y}_{-i}) \leqq K_i(\hat{y}) + \varepsilon_i, \tag{2}$$

because $\hat{y}$ is an $\underline{\varepsilon}$-equilibrium point of Γ'.

Combining (1) and (2), we may conclude that

$$K_i(x_i \uparrow \hat{y}_{-i}) \leqq K_i(\hat{y}) + \delta_i + \varepsilon_i \quad \text{for all } x_i \in X_i \text{ and } i \in N.$$

Hence $\hat{y}$ is an $(\underline{\varepsilon} + \underline{\delta})$-equilibrium point of Γ. □

LEMMA 4.2. *Let $\underline{\varepsilon}, \underline{\delta} \in \mathbb{R}^n_+$ and let Γ' be a $\underline{\delta}$-approximation of the n-person game Γ. For the mixed extensions $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ of Γ and Γ', respectively, we have:*

(1) *$\tilde{\Gamma}'$ is a $\underline{\delta}$-approximation of $\tilde{\Gamma}$.*

(2) *An $\underline{\varepsilon}$-equilibrium point of $\tilde{\Gamma}'$ is an $(\underline{\varepsilon} + \underline{\delta})$-equilibrium point of $\tilde{\Gamma}$.*

Proof. In view of Lemma 4.1, we need only to prove conclusion (1), because (2) is a direct consequence of (1).

Let $i \in N$ and $\mu_i \in \tilde{X}_i$. We have to prove that there is a $\nu_i \in \tilde{Y}_i$ such that

$$\tilde{K}_i(\mu_i \uparrow \mu) \leqq \tilde{K}_i(\nu_i \uparrow \mu) + \delta_i \quad \text{for all } \mu \in \tilde{X}_{-i} = \prod_{k \neq i} \tilde{X}_k. \tag{3}$$

Suppose that $\mu_i = \sum_{j=1}^{s} p_j e(x_i(j))$, where $s \in N$ and $p_1, \cdots, p_s$ are nonnegative real numbers with sum 1 and $x_i(1), \cdots, x_i(s) \in X_i$. It follows from APP($i$) that there exist $y_i(1), \cdots, y_i(s) \in Y_i$ such that for $j = 1, \cdots, s$ we have

$$\tilde{K}_i(e(x_i(j)) \uparrow \mu) \leqq \tilde{K}_i(e(y_i(j)) \uparrow \mu) + \delta_i \quad \text{for all } \mu \in X_{-i}. \tag{4}$$

Take $\nu_i = \sum_{j=1}^{s} p_j e(y_i(j))$. Then it follows immediately from (4), that (3) holds. □

DEFINITIONS. Let $\varepsilon > 0$ and let $\mathcal{F}$ and $\mathcal{G}$ be two families of real-valued functions on a set K. We will say that *$\mathcal{G}$ ε-dominates $\mathcal{F}$*, if for each $f \in \mathcal{F}$ there is a $g \in \mathcal{G}$ such that $f(x) \leqq g(x) + \varepsilon$ for all $x \in K$. The family $\mathcal{F}$ will be called an *upper bounded family*, if there is a $c \in \mathbb{R}$ such that $f(x) \leqq c$ for all $x \in K$ and $f \in \mathcal{F}$.

The following proposition was proved in [45, pp. 759–760].

PROPOSITION 4.1. *Let $\varepsilon > 0$. Let V be a nonempty upper bounded subset of $\mathbb{R}^m$ (i.e., there is a $c \in \mathbb{R}$ such that for all $(x_1, \cdots, x_n) \in V$ we have $x_1 \leqq c, x_2 \leqq c, \cdots, x_m \leqq c$). Then there exists a finite subset W of V such that W ε-dominates V; i.e., for each $v \in V$ there is a $w \in W$, such that $v_i \leqq w_i + \varepsilon$ for $i = 1, \cdots, m$.*

LEMMA 4.3. *Let $\varepsilon > 0$. Let E be a finite set and let $\mathcal{F}$ be an upper bounded family of real-valued functions on E. Then there exists a finite subfamily $\mathcal{G}$ of $\mathcal{F}$, which ε-dominates the family $\mathcal{F}$.*

Proof. Let us denote the elements of the finite set E by $a_1, a_2, \cdots, a_m$ ($m \in \mathbb{N}$). For each $f \in \mathcal{F}$, let $\alpha(f)$ be the vector $(f(a_1), f(a_2), \cdots, f(a_m)) \in \mathbb{R}^m$. Since $\mathcal{F}$ is an upper bounded family, the set $V = \{\alpha(f) : f \in \mathcal{F}\}$ is an upper bounded subset of $\mathbb{R}^m$. In view of Proposition 4.1, we can take a finite subset W, which ε-dominates V. But then $\mathcal{G} = \{f \in \mathcal{F} : \alpha(f) \in W\}$ is a finite subfamily of $\mathcal{F}$, which ε-dominates $\mathcal{F}$. □

The first ε-equilibrium point theorem, which we now are going to prove, deals with mixed extensions of n-person games, where all but one pure strategy space are finite sets and one of them is an arbitrary set.

THEOREM 4.1. *Let $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ be an n-person game in normal form such that*

(1) *$X_1, X_2, \cdots, X_{n-1}$ are finite sets,*

(2) *K_n is an upper bounded function on X.*

Then for each $\varepsilon > 0$ the mixed extension $\tilde{\Gamma}$ has $(0, 0, \cdots, 0, \varepsilon)$-equilibrium points.

Proof. Let $\varepsilon > 0$ and let E be the finite set $X_{-n} = \prod_{i=1}^{n-1} X_i$. By (2) the family

$$\mathcal{F} = \{(x_1, x_2, \cdots, x_{n-1}) \mapsto K_n(x_1, x_2, \cdots, x_{n-1}, x_n) : x_n \in X_n\}$$

of functions on E is an upper bounded family. In view of Lemma 4.3, there exists a finite subset Y_n of X_n, such that the subfamily

$$\mathcal{G} = \{(x_1, x_2, \cdots, x_{n-1}) \mapsto K_n(x_1, x_2, \cdots, x_{n-1}, x_n) : x_n \in Y_n\}$$

of $\mathscr{F}$ ε-dominates $\mathscr{F}$. This implies that the finite subgame $\langle X_1, X_2, \cdots, X_{n-1}, Y_n, K_1, K_2, \cdots, K_n\rangle$ is a $(0, 0, \cdots, 0, \varepsilon)$-approximation of Γ. By Nash's Theorem 3.1, for $\tilde{\Gamma}'$ equilibrium points exist. So it follows from Lemma 4.2 that such an equilibrium point is a $(0, 0, \cdots, 0, \varepsilon)$-equilibrium point of $\tilde{\Gamma}$. □

If two (or more) pure strategy spaces of an n-person game are nonfinite, then such a game may have no $\underline{\varepsilon}$-equilibrium points for small positive $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, as the following example shows.

Example 4.1. For $n \geqq 2$ let $\Gamma_n = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n\rangle$ be the game where $X_i = \{1\}$ for $i = 1, 2, \cdots, n-2$ and $X_{n-1} = X_n = \mathbb{N}$, where $K_i(x) = 0$ for all $x \in X$ and $i = 1, 2, \cdots, n-2$ and where

$$K_{n-1}(1, 1, \cdots, 1, x_{n-1}, x_n) = -K_n(1, 1, \cdots, 1, x_{n-1}, x_n)\begin{cases} =1 & \text{if } x_{n-1} \geqq x_n, \\ =0 & \text{if } x_{n-1} < x_n. \end{cases}$$

Then it is easy to show that $\tilde{\Gamma}_n$ has no $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$-equilibrium points if $\varepsilon_{n-1} + \varepsilon_n < 1$.

Note that in this example $n-2$ of the n pure strategy spaces of Γ_n are finite, and that the payoff functions are bounded. Note further that for $n = 2$ the corresponding game Γ_2 is the zero-sum game of A. Wald [48], which he used to show that not all mixed extensions of $\infty \times \infty$-matrix games have a value.

If we want to derive ε-equilibrium point theorems for games, where more than one pure strategy space is infinite, then that is possible if extra topological properties for strategy spaces and payoff functions hold. We will need some definitions.

DEFINITIONS. Let Z be a metric space with metric d and let $\mathscr{F}$ be a family of real-valued functions on Z.

1. We will say that $\mathscr{F}$ is an *equicontinuous family* of functions on Z if

$$\forall_{z \in Z} \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in Z} \forall_{f \in \mathscr{F}} [d(x, z) < \delta \Rightarrow |f(x) - f(z)| < \varepsilon].$$

2. We will say that $\mathscr{F}$ is an *equi-upper semicontinuous family* on Z if

$$\forall_{z \in Z} \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in Z} \forall_{f \in \mathscr{F}} [d(x, z) < \delta \Rightarrow f(x) < f(z) + \varepsilon].$$

3. We will say that $\mathscr{F}$ is an *equi-uniform continuous family* if

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{f \in \mathscr{F}} \forall_{x, z \in Z} [d(x, z) < \delta \Rightarrow |f(x) - f(z)| < \varepsilon].$$

THEOREM 4.2. *Let $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n\rangle$ be an n-person game with the following properties*:

(1) *for each $i \in \{1, 2, \cdots, n-1\}$, there is a metric d_i on X_i, such that (X_i, d_i) is a compact metric space and such that the family $\{x_i \mapsto K_i(x_i \uparrow x_{-i}) \colon x_{-i} \in X_{-i}\}$ is an equi-upper semicontinuous family on X_i,*

(2) *for each $x \in X_{-n}$, the function $x_n \mapsto K_n(x_n \uparrow x)$ is an upper bounded function on X_n.*

Then for each $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) > (0, 0, \cdots, 0)$ the game $\tilde{\Gamma}$ has $\underline{\varepsilon}$-equilibrium points.

Proof. Take $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) > (0, 0, \cdots, 0)$. Let $i \in \{1, \cdots, n-1\}$. Denote the open ball $\{x \in X_i : d_i(x, x_i) < \delta\}$ by $U(x_i, \delta)$. In view of condition (1), for each $x_i \in X_i$ there is a $\delta(x_i) > 0$, such that

$$K_i(x_i' \uparrow x_{-i}) < K_i(x_i \uparrow x_{-i}) + \varepsilon_i \quad \text{for all } x_i' \in U(x_i, \delta(x_i)) \text{ and } x_{-i} \in X_{-i}. \tag{5}$$

Since X_i is a compact space, we can find a finite subset E_i of X_i such that

$$\bigcup_{e_i \in E_i} U(e_i, \delta(e_i)) = X_i. \tag{6}$$

It follows from (5) and (6) that the game

$$\Gamma' = \langle E_1, E_2, \cdots, E_{n-1}, X_n, K_1, \cdots, K_n \rangle \tag{7}$$

$(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}, 0)$-dominates Γ. Now the first $n-1$ pure strategy spaces of Γ' are finite, and by condition (2) the nth payoff function of Γ' is upper bounded.

Hence the game Γ' satisfies the conditions of Theorem 4.1, and this implies that there is a $(0, 0, \cdots, 0, \varepsilon_n)$-equilibrium point $\hat{\mu}$ of $\tilde{\Gamma}'$. In view of (7) and Lemma 4.2 we may conclude that $\hat{\mu}$ is an $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$-equilibrium point of $\tilde{\Gamma}$. □

The proof of the following theorem runs along similar lines as that of the foregoing theorem and is left for the reader to perform.

THEOREM 4.3. *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game, in which the strategy spaces* $X_1, X_2, \cdots, X_{n-1}$ *are precompact metric spaces and where for each* $i \in \{1, 2, \cdots, n-1\}$ *the family* $\mathcal{F}_i = \{x_i \mapsto K_i(x_i \uparrow x_{-i}) : x_{-i} \in X_{-i}\}$ *is an equi-uniform continuous family on* X_i *and where for each* $x \in X_{-n}$ *the function* $x_n \mapsto K_n(x_n \uparrow x)$ *on* X_n *is upper bounded.*

Then $\tilde{\Gamma}$ *has* $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$*-equilibrium points for all* $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) > (0, 0, \cdots, 0)$.

The following theorem extends results in [32] and is a simple consequence of Theorem 4.2. Such games as are in this theorem can be interpreted as games of timing, where the first $n-1$ players are obliged to act in the time interval $[0, 1]$, and player n has the freedom to choose his action moment in A.

THEOREM 4.4. *Let* $\Gamma = \langle X_1, X_2, \cdots, X_{n-1}, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game such that*

(1) $X_1 = X_2 = \cdots = X_{n-1} = [0, 1]$ *and* $X_n = A \subset \mathbb{R}$,

(2) $K_i : X \to \mathbb{R}$ *is a uniformly continuous function for* $i = 1, 2, \cdots, n-1$,

(3) $K_n : X \leftarrow \mathbb{R}$ *is upper bounded.*

Then $\tilde{\Gamma}$ *has* $\underline{\varepsilon}$*-equilibrium points for each* $\underline{\varepsilon} > (0, 0, \cdots, 0)$.

A simple consequence of Theorem 4.3, which will be used later on, is the following theorem.

THEOREM 4.5. *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, \cdots, K_n \rangle$ *be a game such that*:

(1) $X_1, X_2, \cdots, X_n$ *are precompact metric spaces,*

(2) $K_1, K_2, \cdots, K_{n-1}$ *are uniformly continuous functions,*

(3) K_n *is an upper bounded function.*

Then $\tilde{\Gamma}$ *has* $\underline{\varepsilon}$*-equilibrium points for each* $\underline{\varepsilon} > (0, 0, \cdots, 0)$.

In the following theorem we give suffcient conditions for the existence of pure $\underline{\varepsilon}$-equilibrium points.

THEOREM 4.6. *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game with the properties* (1) *and* (2) *of Theorem* 4.2 *and with the following additional properties*:

(3) $X_1, X_2, \cdots, X_n$ *are convex subsets of certain vector spaces,*

(4) *for each* $i \in N$ *and* $x_{-i} \in X_{-i}$ *the function* $x_i \mapsto K_i(x_i \uparrow x_{-i})$ *is a concave function on* X_i,

(5) *for each* $i \in N$ *and each* $x_i \in X_i$ *the function* $x_{-i} \mapsto K_i(x_i \uparrow x_{-i})$ *is an affine function on* X_{-i}.

Then the game Γ *has* $\underline{\varepsilon}$*-equilibrium points for each* $\underline{\varepsilon} > (0, 0, \cdots, 0)$.

Proof. Take $\underline{\varepsilon} > 0$. In view of the properties (1) and (2) and Theorem 4.2, we can take an ε-equilibrium point $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2, \cdots, \hat{\mu}_n) \in \tilde{X}$ of the mixed extension $\tilde{\Gamma}$ of Γ. For each $i \in N$ the mixed strategy $\hat{\mu}_i$ is of the form $\hat{\mu}_i = \sum_{s=1}^{n_i} p_i(s) e(x_i(s))$, where $p_i(s) \geqq 0$ and $x_i(s) \in X_i$ for each $s \in \{1, 2, \cdots, n_i\}$ and $\sum_s p_i(s) = 1$.

Let $\hat{x}_i = \sum_{s=1}^{n_i} p_i(s)x_i(s)$. Then $\hat{x}_i \in X_i$ for each $i \in N$, because X_i is a convex set. We are going to prove that $\hat{x} = (\hat{x}_1, x_2, \cdots, \hat{x}_n)$ is an $\underline{\varepsilon}$-equilibrium point of the game Γ. For that purpose we note that for each $i \in N$ and each $x_i \in X_i$ we have:

(a) $K_i(x_i \uparrow \hat{x}_{-i}) = \tilde{K}_i(e(x_i) \uparrow \hat{\mu}_{-i})$ in view of property (5).

(b) $K_i(e(x_i) \uparrow \hat{\mu}_{-i}) \leqq \tilde{K}_i(\hat{\mu}_i \uparrow \hat{\mu}_{-i}) + \varepsilon_i$, since $\hat{\mu}$ is an $\underline{\varepsilon}$-equilibrium point of $\tilde{\Gamma}$.

(c) $\tilde{K}_i(\hat{\mu}_i \uparrow \hat{\mu}_{-i}) = \tilde{K}_i(\sum_s p_i(s) e(x_i(s)) \uparrow \hat{\mu}_{-i}) = \sum_{s=1}^{n_i} p_i(s) K_i(x_i(s) \uparrow \hat{x}_{-i})$ by property (5).

(d) $\sum_{s=1}^{n_i} p_i(s) K_i(x_i(s) \uparrow \hat{x}_{-i}) \leqq K_i(\hat{x}_i \uparrow \hat{x}_{-i})$ by property (4).

Combining (a), (b), (c) and (d) yields for each $i \in N$

$$K_i(x_i \uparrow \hat{x}_{-i}) \leqq K_i(\hat{x}_i \uparrow \hat{x}_{-i}) + \varepsilon_i = K_i(\hat{x}) + \varepsilon_i \quad \text{for all } x_i \in X_i.$$

Hence $\hat{x}$ is an $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$-equilibrium point of Γ. □

The following two propositions, 4.2 and 4.3, can be seen as extensions of well-known results of Arzela–Ascoli (cf. [6, pp. 266–267]) to nonbounded families of functions. They will be used to find suitable approximations for certain classes of games.

First we recall that a family $\mathcal{F}$ of real-valued functions on a topological space K is said to be an *equicontinuous family*, if for each $x \in K$ and each $\varepsilon > 0$, there exists a neighborhood $U(x)$ of x, such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U(x)$ and all $f \in \mathcal{F}$.

PROPOSITION 4.2. *Let K be a compact topological space, let $\mathcal{F}$ be an upper bounded and equicontinuous family of real-valued functions on K and let $\varepsilon > 0$. Then there exists a finite subfamily $\mathcal{G}$ and $\mathcal{F}$ such that $\mathcal{G}$ ε-dominates $\mathcal{F}$.*

Proof. For each $a \in K$, let $U(a)$ be an open neighborhood of a such that

$$|f(x) - f(a)| \leqq \tfrac{1}{3}\varepsilon \quad \text{for all } x \in U(a) \text{ and all } f \in \mathcal{F}. \tag{8}$$

Since K is compact, we can take a finite subset E of K such that $K = \bigcup_{a \in E} U(a)$. For each $f \in \mathcal{F}$, let $f^*: E \to \mathbb{R}$ be the function on E with $f^*(x) = f(x)$ for each $x \in E$.

Then the family $\mathcal{F}^* = \{f^* : f \in \mathcal{F}\}$ is an upper bounded family of real-valued functions on the finite set E. In view of Lemma 4.3, we can find a finite subfamily $\mathcal{G}^*$ of $\mathcal{F}^*$ such that $\mathcal{G}^* \frac{1}{3}\varepsilon$-dominates $\mathcal{F}^*$. Let $\mathcal{G}^* = \{h_1, h_2, \cdots, h_s\}$. For each $k \in \{1, \cdots, s\}$ take a $g_k \in \mathcal{F}$, such that $g_k^* = h_k$.

Then the family $\mathcal{G} = \{g_1, g_2, \cdots, g_s\}$ is a finite subfamily of $\mathcal{F}$ and we will show now that $\mathcal{G}$ ε-dominates $\mathcal{F}$. Let $f \in \mathcal{F}$. Then there is a $k \in \{1, \cdots, s\}$, such that

$$f(e) = f^*(e) \leqq g_k^*(e) + \tfrac{1}{3}\varepsilon = g_k(e) + \tfrac{1}{3}\varepsilon \quad \text{for each } e \in E. \tag{9}$$

We are finished with the proof, if we can show that $f(x) \leqq g_k(x) + \varepsilon$ for all $x \in X$. Let $x \in X$. Take $e \in E$ such that $x \in U(e)$. Then in view of (8),

$$f(x) \leqq f(e) + \tfrac{1}{3}\varepsilon, \qquad g_k(e) \leqq g_k(x) + \tfrac{1}{3}\varepsilon. \tag{10}$$

In view of (9) and (10) we obtain

$$f(x) = (f(x) - f(e)) + (f(e) - g_k(e)) + (g_k(e) - g_k(x)) + g_k(x) \leqq \varepsilon + g_k(x).$$

Hence we have proved that $\mathcal{G}$ ε-dominates $\mathcal{F}$. □

PROPOSITION 4.3. *Let K be a precompact metric space with metric d, let $\mathcal{F}$ be an upper bounded equi-uniform continuous family of real-valued functions on K and let $\varepsilon > 0$. Then there exists a finite subfamily $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{G}$ ε-dominates $\mathcal{F}$.*

Proof. Take a $\delta > 0$ such that for all $x, y \in K$ with $d(x, y) < \delta$ we have $|f(x) - f(y)| \leqq \frac{1}{3}\varepsilon$. Since K is precompact, we can find a finite subset E of K, such that $K = \bigcup_{a \in E} U(a)$, where $U(a)$ is the ball $\{x \in K : d(x, a) < \delta\}$. Now similarly as in the proof of Proposition 4.2, we can introduce a family $\mathcal{F}^*$ of functions on E and find a $\frac{1}{3}\varepsilon$-dominating subfamily $\mathcal{G}^*$ of $\mathcal{F}^*$ and construct with the aid of $\mathcal{G}^*$ an ε-dominating subfamily $\mathcal{G}$ of $\mathcal{F}$. □

THEOREM 4.7. *Let* $\Gamma = \langle X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game with the following properties*:

(1) $X_1, X_2, \cdots, X_n$ *are convex subsets of linear topololgical spaces.*

(2) $X_1, X_2, \cdots, X_{n-1}$ *are compact sets.*

(3) $K_1, K_2, \cdots, K_n$ *are continuous functions on* X.

(4) *For each* $i \in N$ *and* $x_{-i} \in X_{-i}$ *the function* $x_i \to K_i(x_i \uparrow x_{-i})$ *is a concave function on* X_i.

(5) *The family* $\mathcal{F} = \{x_{-n} \mapsto K_n(x_n \uparrow x_{-n}): x_n \in X_n\}$ *is an upper bounded and equicontinuous family of functions on* X_{-n}.

Then Γ *has* $(0, 0, \cdots, 0, \varepsilon)$*-equilibrium points for each* $\varepsilon > 0$.

Proof. Take $\varepsilon > 0$. It follows from property (2), that the set X_{-n} is compact. So, in view of property (5) and Proposition 4.2, we can take a finite subset E_n of X_n such that the finite family

$$\mathcal{G} = \{x_{-n} \mapsto K_n(x_n \uparrow x_{-n}): x_n \in E_n\}$$

ε-dominates $\mathcal{F}$. This implies that the game $\Gamma' = \langle X_1, X_2, \cdots, X_{n-1}, E_n, K_1, \cdots, K_n \rangle$ $(0, 0, \cdots, 0, \varepsilon)$-dominates Γ. Then also $\Gamma'' \langle X_1, X_2, \cdots, X_{n-1}, \text{conv}(E_n), K_1, \cdots, K_n \rangle$ $(0, \cdots, 0, \varepsilon)$-dominates Γ, where conv (E_n) is the convex hull of E_n. Since conv (E_n) is convex and compact, the game Γ'' satisfies all conditions in the equilibrium point Theorem 3.2 of Nikaido–Isoda. Hence Γ'' has equilibrium points and then Γ has $(0, 0, \cdots, \varepsilon)$-equilibrium points in view of Lemma 4.1. □

THEOREM 4.8. *Let* $\Gamma = < X_1, X_2, \cdots, X_n, K_1, K_2, \cdots, K_n \rangle$ *be an n-person game with the following properties*:

(1) $X_1, X_2, \cdots, X_{n-1}$ *are precompact metric spaces with metrics* $d_1, d_2, \cdots, d_{n-1}$, *respectively.*

(2) *For each* $i \in \{1, 2, \cdots, n-1\}$ *and each* $x_n \in X_n$, *the function on* X_{-n}

$$(x_1, x_2, \cdots, x_{n-1}) \mapsto K_i(x_1, x_2, \cdots, x_{n-1}, x_n)$$

is uniformly continuous.

(3) *The family* $\mathcal{F} = \{x_{-n} \mapsto K_n(x_n \uparrow x_{-n}): x_n \in X_n\}$ *is an upper bounded equi-uniform continuous family of functions on* X_{-n}.

Then $\tilde{\Gamma}$ *has* $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$*-equilibrium points for each* $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) > (0, 0, \cdots, 0)$.

Proof. Take $\varepsilon > 0$. In view of (1), the set X_{-n} is a precompact metric space with metric $d(x_{-n}, y_{-n}) = \max_{k=1,\cdots,n-1} d_k(x_k, y_k)$, where $x_{-n} = (x_1, x_2, \cdots, x_{n-1})$ and $y_{-n} = (y_1, y_2, \cdots, y_{n-1})$. Hence, it follows from (3) and Proposition 4.3 that there is a finite subset Y_n of X_n such that

$$\Gamma' = \langle X_1, X_2, \cdots, X_{n-1}, Y_n, K_1, \cdots, K_n \rangle$$

$(0, 0, \cdots, 0, \frac{1}{2}\varepsilon)$-dominates Γ. It follows from (2) and the finiteness of Y_n that $K_i : X_1 \times X_2 \times \cdots \times X_{n-1} \times Y_n \to \mathbb{R}$ is uniformly continuous for $i = 1, \cdots, n-1$. Hence Γ' satisfies the conditions in Theorem 4.5. This implies that $\tilde{\Gamma}'$ has $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}, \frac{1}{2}\varepsilon_n)$-equilibrium points. By Lemma 4.2, $\tilde{\Gamma}'$ $(0, \cdots, 0, \frac{1}{2}\varepsilon_n)$-dominates $\tilde{\Gamma}$. Hence $\tilde{\Gamma}$ has $(\varepsilon_1, \cdots, \varepsilon_n)$-equilibrium points □.

It will be obvious now, that with the aid of other well-known ε-equilibrium point theorems and the technique of approximation, many other new ε-equilibrium point theorems may be derived.

5. Final comments. In § 4 we have seen that the proof technique using δ-approximations of games yields ε-equilibrium point theorems, where rather weak conditions are posed on strategy spaces and payoff functions. In a forthcoming paper [46], we will use a similar technique to derive ε-equilibrium point theorems for stochastic games, where one of the players at each stage may have a big action space.

In this paper we considered only finite mixed extensions of n-person games, to avoid measurability problems. But in a standard manner many of the results in this paper for finite mixed extensions can be generalized to other mixed extensions.

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