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Nash Equilibria of Games When Players' Preferences Are Quasi-Transitive

Kaushik Basu
Prasanta K. Pattanaik

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Abstract

Much of game theory is founded on the assumption that individual players are endowed with preferences that can be represented by a real-valued utility function. However, in reality human preferences are often not transitive. This is especially true for the indifference relation, which can lead an individual to make a series of choices which in their totality would be viewed as erroneous by the same individual. There is a substantial literature that raises intricate questions about individual liberty and the role of government intervention in such contexts. The aim of this paper is not to go into these ethical matters but to provide a formal structure for such analysis by characterizing games where individual preferences are quasi-transitive. The paper identifies a set of axioms which are sufficient for the existence of Nash equilibria in such 'games.'

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Nash Equilibria of Games When Players' Preferences Are Quasi-Transitive*

Kaushik Basu
World Bank
1818 H Street NW, Mail Stop MC4-404 Washington, DC 20433 USA
Tel: (202) 458-1076; Fax: (202) 522-0906; E-mail: kbasu@worldbank.org;

Department of Economics, Cornell University, Ithaca, NY 14853

Prasanta K. Pattanaik

Department of Economics, University of California, Riverside, CA 92521

Telephone & Fax: (951) 653 0979; E-mail: prasanta.pattanaik@ucr.edu

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1. Introduction

Much of game theory is founded on the assumption that individuals are endowed with well-defined payoff or utility functions. This is not quite as innocuous as may appear at first sight. One implication of having a real-valued function representing a person's preferences is that her preferences are necessarily transitive. A little bit of introspection makes it clear that this assumption is untenable when it comes to the indifference relation. Most of us are indifferent between having k and k+1 grains of sugar with coffee, for all k, but have a strict preference one way or the other between no sugar and a spoonful of sugar. There is a substantial analytical literature on this (see, among others, Georgescu-Roegen, 1936; Armstrong, 1939, 1951; Luce 1956; Mazumdar, 1957; Fishburn, 1970; Sen, 1970; Pattanaik, 1970; Quinn 1990; and Anand 1993); and the recognition of this problem actually goes back to ancient Greece in the form of the sorites paradox or heap paradox.

The recognition of this feature of human preference has important implications in interactive and game-theoretic situations. It raises important ethical questions about individual autonomy and collective norms. It can be argued for instance that each transaction between consenting adults should be permitted on grounds of Pareto improvement but a set of such transactions may leave everybody worse off, thereby making room for some restrictions of voluntary transactions (Parfit, 1984; Basu, 2003; 2007).

The present paper is concerned not with these ethical matters but some foundational game-theoretic questions. Suppose we have a game where the players' preferences are quasi-transitive (i.e., a game where the strict preference of each player is transitive) but the preferences of some of the players are not transitive, so that indifference relations are intransitive for some players. When can we be sure that a Nash equilibrium of such a game exists? In itself, this is an abstract exercise, but we hope that it will provide the groundwork for further investigation concerning government interventions, policy making and the ethics of individual autonomy.

The plan of the paper is as follows. In Section 2, we introduce finite games with quasi-transitive but not necessarily transitive preferences of players, and their mixed extensions. In Section 3, we prove a result, which shows that, if an agent's quasi-transitive preferences over lotteries over mixed strategy profiles satisfy certain axioms, then there exists a utility function of the von Neumann-Morgenstern type, which preserves the agent's strict preferences. Utilizing this result, in Section 4 we demonstrate the existence of a Nash equilibrium in a mixed strategy game where the players' preferences satisfy our axioms.

2. A finite game and its mixed extension

Let g be a finite game characterized by $N = \{1, 2, \dots, n\}, S_1, S_2, \dots, S_n$, and $\trianglerighteq_1, \trianglerighteq_2, \dots, \trianglerighteq_n$, where N is the (finite) set of players, for all $i \in N$, S_i is the (finite) set of strategies of player i, and, for all $i \in N$, $\trianglerighteq_i$ is the binary weak preference relation ("at least as good as") of player i defined over $S \equiv \times_{j \in N} S_j$. Let G be a given mixed extension of g, G being a game characterized by N, $T_1, T_2, \dots, T_n$, and $\succcurlyeq_1, \succcurlyeq_2, \dots, \succcurlyeq_n$, where N is the set of players, for all $i \in N$, T_i is the set of all simple lotteries (i.e., probability distributions) on S_i, and, for all $i \in N$, $\succcurlyeq_i$ is i's binary weak preference relation ("at least as good as") defined over $T \equiv \times_{j \in N} T_j$, such that

(2.1) for all $x = (x_1, \dots, x_n) \in S$, all $y = (y_1, \dots, y_n) \in S$, all $p = (p_1, \dots, p_n) \in T$, and all $p' = (p'_1, \dots, p'_n) \in T$, if, for all $j \in N, p_j(x_j) = p'_j(y_j) = 1$, then $x \trianglerighteq_i y$ iff $p \succcurlyeq_i p'$.

Throughout this paper, we assume that the finite game g and its mixed extension G are given and fixed. Let $\succ_i$ and $\sim_i$ denote, respectively, the asymmetric factor and the symmetric factor of $\succcurlyeq_i$.

The next assumption captures the central idea that human beings often violate the transitivity axiom when it comes to indifference. This stems from the fact that no one has perfectly fine perception. So when the change is minor, a person may be indifferent but multiple minor shifts can add up to something more perceptible and people can then express a strict preference. We take account of this by restricting transitivity to the asymmetric part of a player's preference relation.

Assumption 2.1. For all $i \in N$, $\succcurlyeq_i$ is reflexive (i.e., for all $p \in T$, $p \succcurlyeq_i p$), connected (i.e., for all distinct $p, p' \in T, p \succcurlyeq_i p'$ or $p' \succcurlyeq_i p$), and quasi-transitive (i.e., for all $p, p', p" \in T$, if $p \succ_i p'$ and $p' \succ_i p"$, then $p \succ_i p"$) but not necessarily transitive.

Given (2.1), Assumption 2.2 implies that, for all $i \in N$, $\trianglerighteq_i$ is reflexive, connected, and quasi-transitive but not necessarily transitive.

A *Nash equilibrium* of G is $p \in T$, such that for all $i \in N$ and all $p'_i \in T_i$, $p \succcurlyeq_i (p'_i, p_{-i})$.

3. A partial representation of $\succcurlyeq_i$

In this section, we derive a result that is crucial for our proof of the existence of a Nash equilibrium of G.

A simple lottery on a non-empty set Ω is a probability distribution on Ω with a finite support. Let Y be the set of all simple lotteries on S. Let Z be the union of Y and the set of all simple lotteries on $S \cup Y$. For every $L \in Z$, let sL be the simple lottery on S to which L can be reduced (if L itself is a simple lottery on S, then $sL = L$). For all $L \in Y$ and all $x \in S$, let $L(x)$ be the probability assigned by L to x. For all x, let $C(x)$ be the simple lottery L such that $L(x) = 1$. Recalling that, for all $i \in N$, S_i is a finite set, which implies that S is a finite set, and, denoting the elements of S as $x^1, \dots, x^m$, for the purpose of explicitness, we shall write a simple lottery L on S as $(L(x^1),\ x^1;\ L(x^2),\ x^2; \dots;\ L(x^m),\ x^m)$. When some of the probabilities assigned by a lottery on S are zeroes, for convenience we shall drop the corresponding components from the notation for the lottery. Thus, if $S = \{x^1, x^2, x^3, x^4\}$ and $L = (\frac{1}{2}, x^1;\ 0, x^2; 0, x^3; \frac{1}{2}, x^4)$, then we shall write the lottery as $(\frac{1}{2}, x^1;\ \frac{1}{2}, x^4)$, and, without any ambiguity, we shall also call it a lottery on $\{x^1, x^4\}$.

We shall make several assumptions about the players' preferences defined over Z.

Assumption 3.1. For all $i \in N, i$ has a reflexive, connected, and quasi-transitive

weak preference relation R_i over Z such that:

(3.1) for all $p, p' \in T$ and all $L, L' \in Y$, if [for all $x \in S$, $L(x) = \times_{j \in N}\, p(x_j)$ and $L'(x) = \times_{j \in N}\, p'_j(x_j)$], then ($L \succcurlyeq_i L'$ iff pR_ip').

For all $i \in N$, let P_i and I_i denote, respectively, the asymmetric and the symmetric factors of R_i.

Let $C(S)$ denote the set, $\{C(x): x \in S\}$, of all trivial lotteries on S. Since $C(S)$ is a finite set and, for every $i \in N$, R_i is reflexive, connected and quasi-transitive, for all $i \in N$, there must exist an R_i-greatest and an R_i-least element in the set $C(S)$ (see Sen 1970 and Pattanaik 1971). For all $i \in N$, let ${}^{i}x$ be an R_i-greatest element in $C(S)$ and let ${}_{i}x$ be an R_i-least element in $C(S)$. Let Y^{*i} be the set of all $L \in Y$, such that $L(x) = 0$ for all $x \in S - \{{}^{i}x,\ {}_{i}x\}$; thus, Y^{*i} is the set of all probability mixtures of ${}^{i}x$ and ${}_{i}x$.

For all $L', L" \in Z$, we say that L' is *equivalent to* $L"$ *for* i (and we write $L'E_i\, L"$) iff for all $L \in Z$, (LR_iL' iff $LR_iL"$) and ($L'R_iL$ iff $L"R_iL$). Note that, if $L'E_i\, L"$, then $L'\, I_i\, L"$, but the converse is not necessarily true, given that I_i is not necessarily transitive.

Assumption 3.2. For all $i \in N$ and all $L \in Z$, $LE_is(L)$.

Assumption 3.2 says that, for every lottery $L \in Z$, a player considers L to be "essentially the same" as the simple lottery $s(L)$, so that her preference between L and any other lottery $L' \in Z$ is exactly analogous to her preference between L' and $s(L)$.

,Assumption 3.3. For all $i \in N$, and all $x \in S$,

(3.2) there exist $a^i(x), b^i(x) \in [0,1]$ such that $b^i(x) \geq a^i(x)$, $\left[a^i(x), b^i(x)\right] \subset [0,1]$, and, for all $L \in Y^{*i}$, [if $L\left({}^i x\right) > b^i(x)$, then $LP_iC(x)$], [if $L\left({}^i x\right) < a^i(x)$, then $C(x)P_iL$], and [if $L\left({}^i x\right) \in \left[a^i(x), b^i(x)\right]$, then $LI_iC(x)$].

Note that the interval $[a^i(x), b^i(x)]$ figuring in (3.2) is permitted, but not constrained, to be degenerate.

In a framework where R_i is assumed to be transitive and the restriction of R_i to Y^{*i} is assumed to be strictly monotonic in the sense that [for all $L', L'' \in Y^{*i}$, if $L'\left({}^i x\right) > L''\left({}^i x\right)$, then $L'P_iL''$], we cannot have two distinct $L', L'' \in Y^{*i}$, such that $L'I_iC(x)$ and $L''I_iC(x)$. But, given our intuition about indifference being often the result of imperfect discrimination, not only have we discarded the assumption that I_i is transitive, but we have also discarded the assumption of strict monotonicity of the restriction of R_i to Y^{*i}. We do, however, believe that, monotonicity of the restriction of R_i to Y^{*i} (in the sense that, for all $L', L'' \in Y^{*i}$, if $L'\left({}^i x\right) > L''\left({}^i x\right)$, then $L'R_iL''$) is an intuitively compelling property). It is this intuition, together with the intuition involving continuity, which underlies our next assumption.

Assumption 3.4. Suppose Assumption 3.3 holds. Let $i \in N$ and $L', L'' \in Z$ be such that $L'P_iL''$. Then, if there exists $L \in Y^{*i}$, such that LPL', then there exists $L^* \in Y^{*i}$, such that L^*I_iL' and L^*P_iL''. Further, if there exists $L \in Y^{*i}$, such that $L''PL$, then there exists $L^* \in Y^{*i}$, such that L^*I_iL'' and $L'P_iL^*$.

Suppose Assumption 3.3 holds, Suppose $i \in N$ and $L', L'' \in Z$ are such that $L'P_iL''$, and $L \in Y^{*i}$ is such that LPL' . Now starting with L, generate new lotteries in Y^{*i} by continuously reducing the probability attached to ${}^i x$ with corresponding increases in the probability of ${}_i x$. Note that, as we progress, intuitively the perceived attractiveness of the lottery never increases at any step in this process, though it may not necessarily decrease. Given Assumption 3.3, our intuition based on the notion of continuity tells us that, at some stage, we shall reach a lottery in Y^{*i} that is indifferent to L' and also that, at some stage, we shall reach a lottery in Y^{*i} that is indifferent to L''. But given that $L'P_iL''$, L' is perceptibly superior to L'' for i. Therefore, it seems plausible that, in the process of continuously shifting probability from ${}_i x$ to ${}_i x$, we shall first reach a lottery L^* in Y^{*i}, such that the agent cannot perceive the difference, in terms of desirability, between L^* and L' and, hence, L^*I_iL' , and yet L^* is perceptibly better than L'', and, hence, L^*P_iL''. The intuition for the second part of Assumption 3.3 is similar, though in this case, starting with $L \in Y^{*i}$, we consider continuous increases in the probability attached to ${}^i x$ with corresponding decreases in the probability of ${}_i x$.

Assumption 3.5. Let $i \in N$. Let $L = (L(x^1),\ x^1; \dots;\ L(x^m), x^m\ ) \in Y$; let $L^1, \dots, L^m \in Y^{*i}$ be such that, for all $k \in \{1, 2, \dots, m\}$, $C(x^k) I_i L^k$; and let $L' = (L'(L^1), L^1; \dots;\ L'(L^m), L^m) \in Z$ be such that, for all $k \in \{1, \dots, m\}, L(x^k) = L'(L^k)$. Then,

(i) if for all $k \in \{1, \dots, m\}$, such that $x^k \in supp(L)$, $L^k\ I_i\ C(x^k)$, then $L'\ I_i L$;

(ii) if for all $k \in \{1, \dots, m\}$, such that $x^k \in supp(L), L^k\ R_i\ C(x^k)$, and, for some $k \in \{1, \dots, m\}$, such that $x^k \in supp(L), L^k\ P_i\ C(x^k)$, then $L'\ P_i L$;

and

(iii) if for all $k \in \{1, \dots, m\}$, such that $x^k \in supp(L)$, $C(x^k)\ R_i\ L^k$, and, for some k$\in \{1, \dots, m\}$, such that $x^k \in supp(L), C(x^k)\ P_i\ L^k$, then $L\ P_i L'$.

The next proposition is crucial for the proof of the central result of this paper.

Proposition 3.6. Suppose Assumptions 3.1 through 3.5 are satisfied. Then, for all $i \in N$, there exists a real-valued function u^i over Z and a real-valued function v^iover S such that

(3.3) for all $L \in Z$, $u^i(L) = sL(x^1).v^i(x^1) + \dots + sL(x^m).v^i(x^m)$;

and

(3.4) for all L', L''∈Z, $L' P_i\ L''$ implies $u^i(L') > u^i(L'')$.

We proceed to the proof of Proposition 3.6 through some additional notation and lemmas.

Note that, by Assumption 3.3, (3.2) holds for every $x \in S$. For all $L \in Z$, let $a^i(L)$ denote $sL(x^1).a^i(x^1) + \dots + sL(x^m).a^i(x^m)$ and let $b^i(L)$ denote $sL(x^1).b^i(x^1) + \dots + sL(x^m).b^i(x^m)$.

Lemma 3.7. Let Assumptions 3.1 through 3.5 hold. Then for all $i \in N$, for all $L \in Z$, and, for all $e \in [0, 1]$,

(i) if $e \in [a^i(L), b^i(L)]$, then there exist $e^k, k = 1, \dots, m$, such that, (for all $k \in \{1, \dots, m\}, e^k \in [a^i(x^k), b^i(x^k)]$) and $e = sL(x^1).e^1 + \dots + sL(x^m).e^m$;

(ii) if $1 \geq e > b^i(L)$ then there exist $e^k, k = 1, \dots, m$, such that, [for all $k \in \{1, \dots, m\}, e^k \geq a(x^k)$], [for some $k \in \{1, \dots, m\}$, such that $x^k \in supp(sL), e^k > b^i(x^k)$], and $e = sL(x^1).e^1 + \dots + sL(x^m).e^m$;

and

(iii) if $0 \leq e < a^i(L)$ then there exist $e^k, k = 1, \dots, m$, such that, [for all $k \in \{1, \dots, m\}, e^k \leq b^i(x^k)$], [for some $k \in \{1, \dots, m\}$, such that $x^k \in supp(sL), e^k < a^i(x^k)$], and $e = sL(x^1). e^1 + \cdots + sL(x^m). e^m$.

Given the specifications of $a^i(L)$ and $b^i(L)$, the proof of Lemma 3.7 is straightforward and is omitted.

Lemma 3.8. Let Assumptions 3.1 through 3.5 hold. Then for all $i \in N$, all $L \in Z$, and all $e \in [0, 1]$,

(i) if $e \in [a^i(L), b^i(L)]$, then $LI_i\big(e, \ {}^ix; \ 1 - e, \ {}_ix\big)$;

(iii) if $e > b^i(L)$, then $\big(e, \ {}^ix; \ 1 - e, \ {}_ix\big)P_iL$;

and

(iii) if $e < a^i(L)$, then $LP_i\big(e, \ {}^ix; \ 1 - e, \ {}_ix\big)$.

Proof: Suppose Assumptions 3.1 through 3.5 hold. Let $i \in N, L \in Z$, and $e \in [a^i(L), b^i(L)]$. Then, by Lemma 3.7 (i), there exist $e^k, k = 1, \dots, m$, such that, (for all $k \in \{1, \dots, m\}, e^k \in \big[a^i(x^k), b^i(x^k)\big]$) and $e = sL(x^1). e^1 + \cdots + sL(x^m). e^m$. For all $k \in \{1, \dots, m\}$, let L^k denote $(e^k, \ {}^ix; 1 - e^k, \ {}_ix)$. Then, noting $e^k \in [a(x^k), b(x^k)]$) for all $k \in \{1, \dots, m\}$, we have $C(x)I_iL(x)$ for all $x \in S$. Hence, by Assumption 3.5 (i), $LI_i(L(x^1), L^1; \dots; L(x^m), L^m)$. By Assumption 3.2, $(L(x^1), L^1; \dots; L(x^m), L^m) \ E_i \ s(L(x^1), L^1; \dots; L(x^m), L^m)$ = $\big(e, \ {}^ix; \ 1 - e, \ {}_ix\big)$. Hence, noting $LI_i(L(x^1), L^1; \dots; L(x^m), L^m)$, we have $LI_i(e, x^1; \ 1 - e, x^m)$. This completes the proof of Lemma 3.8 (i).

The proof of Lemma 3.8 (ii) [resp. Lemma 3.8 (iii)] is similar except that we shall need to use Lemma 3.7 (ii) [resp. Lemma 3.7 (iii)] and Assumption 3.5 (ii) [resp. assumption 3.5 (iii)]. ■

Lemma 3.9. Let Assumptions 3.1 through 3.5 hold. Then, for all $i \in N$ and all $L', L'' \in Z$, if $[a^i(L'), b^i(L')]$ = $[a^i(L''), b^i(L'']$, then $L'I_iL''$.

Proof: Suppose Assumptions 3.1 through 3.5 hold. Let $i \in N$, and let $L', L'' \in Z$ be such that $\big[a^i(L'), b^i(L')\big]$ = $[a^i(L''), b^i(L'')]$. Suppose not $L'I_iL''$. Without loss of generality, assume that $L'P_iL''$.

By Assumption 3.3, $\big[a^i(L'), b^i(L')\big]$ = $\big[a^i(L''), b^i\big(L''\big)\big] \subset [0, 1]$. Then either there exists $e > b^i(L') = b^i(L'')$ or there exists $e < a^i(L'') = a^i(L'')$. Then, either there exists $L \in Y^{*i}$ such that LP_iL' or there exists $L \in Y^{*i}$ such that $L"P_iL$. Suppose there exists $L \in Y^{*i}$ such that P_iL' . Then by Assumption 3.4, there exists $L^* \in Y^{*i}$, such that L^*I_iL' and L^*P_iL''. In that case,

$[a^i(L'), b^i(L')] \neq [a^i(L''), b^i(L'']$, which is a contradiction. The proof for the case where there exists $L \in Y^{*i}$ such that $L"P_iL$. ■

Proof of Proposition 3.6: Suppose Assumptions 3.1 through 3.5 are satisfied. Let $i \in N.$ For all $L \in Z$, let $u^i(L) \equiv \frac{1}{2}(a^i(L) + b^i(L))$. For all $x \in S$, let $v^i(x) \equiv u^i(C(x))$. Then, by the specification of $a^i(L)$ and $b^i(L)$, we have (3.3) (recall that, by construction, $u^i\big(C(x)\big) = \frac{1}{2}[a^i(\ ^ix) + \ b^i(\ ^ix)]$).

To show that (3.4) holds, suppose $L', L'' \in Y$ are such that $L'P_iL''$ but $u(L'') \geq u(L')$. We shall derive a contradiction. We first show that

(3.4) $a^i(L') \geq \ a^i(L")$
and
(3.5) $b^i(L') \geq b^i(L")$.

Suppose (3.4) does not hold, so that $a^i\left(L^{'}\right) < \ a^i(L")$. Then consider e such that $a^i(L') < e < a^i(L")$ and let $L \in Y^{*i}$ be such that $L\big(\ ^ix\big) = e$. Since $a^i(L') < e < \ a^i(L")$, we must have have $L''P_iL$ and LR_iL'. But, given $L'P_iL''$, $[L''P_iL$ and $LR_iL']$ contradicts quasi-transitivity of R_i. This proves (3.4). Now suppose that (3.5) does not hold, so that $b^i(L') < b^i(L")$Then consider $\hat{e}$ such that $b^i(L") > \hat{e} > \ b^i(L')$ and $\hat{L} \in Y^{*i}$, such that $\hat{L}(\ ^i(x)) = \hat{e}$ and $\hat{L}\big(\ _ix\big)$. Since $b^i(L") > \hat{e} > b^i(L')$, we much have $\hat{L}P_iL'$ and $L''R_i\hat{L}$. But, given $L'P_iL''$, ($\hat{L}P'_iL'$ and $L''R'\hat{L}$) contradicts quasi-transitivity of R_i. This contradiction proves (3.5).

Given $u^i(L'') \geq u^i(L')$ and noting the specification of u^i, we have

(3.6) $\frac{1}{2}\left[a^i\ (L")+ b^i(\text{L"})]\right] \geq \ \frac{1}{2}[\ a^i\left(L^{'}\right) + \ b^i\left(L^{'}\right)]$.

(3.4), (3.5) and (3.6), together, imply $a^i(L') = \ a^i(L")$ and $b^i(L') = b^i(L")$. In that case, by Lemma 3.9, we have $L'I_iL''$, which contradicts $L'P_iL''$. This contradiction completes the proof of (3.4). ■

4. The existence of a Nash-equilibrium of the game G

Proposition 4.1. If Assumptions 3.1 through 3.5 are satisfied, then there exists a Nash equilibrium of G.

Proof: Suppose Assumptions 3.1 through 3.5 hold. Then, by Proposition 3.6, for all $i \in N$, there exist a real-valued utility function u^idefined over Z and a real-valued utility function v^i

defined over S such that (3.3) and (3.4) are satisfied. For all $i \in N$, let R_i' be the ordering over T induced by u^i as follows:

(3.7) for all $i \in N$ and for all $p, p' \in T$, $pR_i'p'$ iff $u^i((\times_{j\in N}\, p_j(x_j^1), x^1;\ \dots;$

$\dots, \times_{j\in N}\, p_j^m(x_j^m), x^m)) \geq u^i((\times_{j\in N}\, p_j'(x_j^1), x^1;\ \dots, \times_{j\in N}\, p_j'(x_j^m);\ x^m))$.

Then, by the specification of u^i $(i = 1, 2, \dots, n)$,

(3.8) for all $i \in N$ and for all $p, p' \in T$, $pR_i'p'$ iff $\sum_{x\in S}(\times_{j\in N}\, p_j(x)).\, v^i(x) \geq$

iff $\sum_{x\in S}(\times_{j\in N}\, p_j'(x)).\, v^i(x)$.

Consider the game G'characterized by $N, T_1, T_2, \dots, T_n$, and $R_i', R_2', \dots, R_n'$. Note that this is a "usual" game where the players' binary weak preference relations are all orderings and, further, these orderings are such that (3.8) holds for the real-valued functions v^i $(i = 1, 2, \dots, n)$ defined over T. Then we know that G' must have a Nash equilibrium, i.e.,

(3.9) there must exist $p \in T$, such that for all $i \in N$ and all $p_i' \in T_i$, not [$({p'}_i, p_{-i})\ P_i' p$].

Note that (3.1) holds by Assumption (3.1) and (3.3) holds by Proposition 3.6. Given (3.1), (3.3), and (3.8), we have,

(3.10) for all $p, p' \in T$, $({p'}_i, p_{-i})\ P_i p$ implies $({p'}_i, p_{-i})\ P_i' p$.

Hence, by (3.9), there must exist $p \in T$, such that for all $i \in N$ and all $p_i' \in T_i$, not [$({p'}_i, p_{-i})\ P_i p$].

Hence the game G has a Nash equilibrium. ■

5. Concluding remarks

We have shown that a game, where players' preferences are quasi-transitive but not necessarily transitive, must have a Nash equilibrium in mixed strategies if the players' preferences satisfy certain assumptions. This seems to be of interest given the reasons, much discussed in the literature, why an agent's indifference relation may not be transitive though the agent's strict preferences may satisfy transitivity.

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