Nash equilibrium in compact-continuous games with a potential

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January 17, 2011

Abstract

If the preferences of the players in a strategic game satisfy certain continuity conditions, then the acyclicity of individual improvements implies the existence of a (pure strategy) Nash equilibrium. Moreover, starting from any strategy profile, an arbitrary neighborhood of the set of Nash equilibria can be reached after a finite number of individual improvements. *JEL* Classification Number: C 72.

Key words: finite improvement property; weak finite improvement property; compact-continuous game; potential game.

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1 Introduction

By definition, if a strategic game admits a generalized ordinal potential as defined by Monderer and Shapley (1996) and that potential attains its maximum, then the game possesses a Nash equilibrium. No doubt, this condition for equilibrium existence is not very widely applicable; however, we are concerned with another weak point here. Unless the game in question is finite, our second supposition is only remotely connected with the basics – strategies and preferences. For instance, although an *exact* potential of a continuous game must be continuous, it is by no means clear that anything like that holds for a generalized ordinal potential. (Concerning an *ordinal* potential, Voorneveld (1997, Theorem 4.1) obtained a negative answer.)

Our main result sounds somewhat similar to the opening statement, but bypasses the problem of (semi)continuity of potentials: If a compact-continuous game admits a generalized ordinal potential, then it possesses a Nash equilibrium.

To be more precise, we assume that each strategy set is a compact topological space, while each utility function is upper semicontinuous in the total strategy profile and continuous in the strategy profile of the partners/rivals; there is no finite individual improvement cycle, but the existence of a numeric potential is not needed. Finally, we obtain more than the mere existence of a Nash equilibrium: Given an arbitrary strategy profile, there is a finite individual improvement path which starts at the profile and ends arbitrarily close to a Nash equilibrium.

In Section 2 the basic definitions are given. Section 3 contains the formulation and proof of the main result. A discussion of some related questions in Section 4 concludes the paper.

2 Preliminaries

Our basic model is a *strategic game with ordinal preferences.* It is defined by a finite set of players N, and strategy sets X_i and ordinal utility functions $u_i\colon X_N \to \mathbb{R}$, where $X_N = \prod_{i\in N} X_i$, for all $i \in N$. We denote $X_{-i} = \prod_{j\in N\setminus\{i\}} X_j$ for each $i \in N$. Given a strategy profile $x_N \in X_N$ and $i \in N$, we denote x_i and x_{-i} its projections to X_i and X_{-i}, respectively.

With every strategic game, we associate this *individual improvement relation* $\rhd^{\text{Ind}}$ on X_N ($i \in N$, y_N, $x_N \in X_N$):

$$y_N \rhd_i^{\text{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; u_i(y_N) > u_i(x_N)]; \tag{1a}$$

$$y_N \rhd^{\text{Ind}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\text{Ind}} x_N]. \tag{1b}$$

By definition, a Nash equilibrium is a *maximizer* of the relation $\rhd^{\text{Ind}}$ on X_N, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd^{\text{Ind}} x_N$ holds for no $y_N \in X_N$.

An *individual improvement path* is a (finite or infinite) sequence $\langle x_N^k \rangle_{k=0,1,\ldots}$ such that $x_N^{k+1} \triangleright^{\mathrm{Ind}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined. Since we consider no other kind of improvements here, the adjective "individual" is dropped henceforth.

Following Monderer and Shapley (1996), we say that a strategic game Γ has the *finite improvement property* (*FIP*) if it admits no infinite improvement path; then every improvement path, if continued whenever possible, ends at a Nash equilibrium after a finite number of steps. Γ has the *weak FIP* (Friedman and Mezzetti, 2001) if a Nash equilibrium can be reached after a finite number of improvements starting from any strategy profile.

The relation $\triangleright^{\mathrm{Ind}}$ is *acyclic* if there is no *finite improvement cycle*, i.e., no improvement path for which $x_N^0 = x_N^m$ with $m > 0$. For a finite game, the acyclicity of $\triangleright^{\mathrm{Ind}}$ is equivalent to the FIP, and equivalent to the existence of a *generalized ordinal potential*, i.e., a function $P \colon X_N \to \mathbb{R}$ such that $P(y_N) > P(x_N)$ whenever $y_N \triangleright^{\mathrm{Ind}} x_N$ (Monderer and Shapley, 1996, Lemma 2.5). When Γ need not be finite, either FIP or the existence of a generalized ordinal potential still implies the acyclicity of $\triangleright^{\mathrm{Ind}}$, but is not implied by it (Voorneveld and Norde, 1996, Example 4.1).

3 Main Result

Henceforth, we assume that each X_i is a topological space and X_N is endowed with the product topology. We say that Γ has the *very weak FIP* if, for every strategy profile $x_N^0 \in X_N$, there is a Nash equilibrium $y_N \in X_N$ such that for every open neighborhood V of y_N there exists a finite improvement path $x_N^0, x_N^1, \ldots, x_N^m$ with $x_N^m \in V$.

Remark. There are plenty of alternative ways to "approximate" the weak FIP; e.g., we could demand the existence of a finite improvement path ending in arbitrary open neighborhood of the *set* of Nash equilibria, or the possibility to reach a Nash equilibrium either after a finite number of improvements or in the limit of a convergent infinite improvement path. The current version describes the strongest property of this kind I have been able to derive from the assumptions of the theorem.

We assume that each u_i is upper semicontinuous in x_N and continuous in x_{-i}; the assumption has an immediate corollary for individual improvements:

$$\forall i \in N \, \forall y_N, x_N \in X_N \left[y_N \triangleright_i^{\mathrm{Ind}} x_N \Rightarrow \exists U \subseteq X_N \big[x_N \in U \;\& \; [U \text{ is open}] \;\& \; \forall x'_N \in U \, [(y_i, x'_{-i}) \triangleright_i^{\mathrm{Ind}} x'_N] \big] \right]. \quad (2)$$

Actually, what is needed for our main result is just condition (2).

Theorem. *Let each X_i in a strategic game Γ be compact; let $\triangleright^{\mathrm{Ind}}$ satisfy condition* (2) *and be acyclic. Then Γ has the very weak FIP.*

Remark. By the definition of the very weak FIP, Γ then possesses a (pure strategy) Nash equilibrium.

Proof. Given $x_N^0 \in X_N$, we denote $Y \subseteq X_N$ the set of strategy profiles that can be reached from x_N^0 with finite improvement paths. Then we define $Z = \operatorname{cl} Y$; being a closed subset of the compact space X_N, Z is compact. We have to prove that Z contains a Nash equilibrium, i.e., a maximizer of $\rhd^{\mathrm{Ind}}$ on X_N.

Claim 1. *If $z_N \in Z$, $i \in N$, and $y_N \rhd_i^{\mathrm{Ind}} z_N$, then $y_N \in Z$ too.*

Proof. By (2), we have $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$ whenever x_N belongs to an appropriate neighborhood U of z_N. Let V be an arbitrary open neighborhood of y_N. We pick an open neighborhood V_{-i} of y_{-i} such that $\{y_i\} \times V_{-i} \subseteq V$; it is possible because V is open in the product topology. Since $y_{-i} = z_{-i}$, the set $U \cap (X_i \times V_{-i})$ is an open neighborhood of z_N, hence there is a finite improvement path $\langle x_N^0, x_N^1, \dots, x_N^m \rangle$ such that $x_N^m \in U \cap (X_i \times V_{-i})$. We define $x_N^{m+1} = (y_i, x_{-i}^m)$. Since $\langle x_N^0, x_N^1, \dots, x_N^m, x_N^{m+1} \rangle$ remains a finite improvement path, $x_N^{m+1} \in Y$. Since $x_N^{m+1} \in V$ and V was arbitrary, we have $y_N \in Z$. □

Claim 2. *There exists a maximizer of $\rhd^{\mathrm{Ind}}$ on Z.*

Proof. Supposing the contrary, we have $y_N(x_N) \in Z$ and $i(x_N) \in N$, for every $x_N \in Z$, such that $y_N(x_N) \rhd_{i(x_N)}^{\mathrm{Ind}} x_N$; therefore, there holds $(y_{i(x_N)}, x'_{-i(x_N)}) \rhd_{i(x_N)}^{\mathrm{Ind}} x'_N$ for every x'_N from an appropriate neighborhood of x_N by (2). Since Z is compact, there are open subsets $U^1, \dots, U^m \subseteq X_N$, strategy profiles $y_N^1, \dots, y_N^m \in Z$, and $i(h) \in N$ for each $h \in \{1, \dots, m\}$ such that $Z \subseteq \bigcup_{h=1}^m U^h$ and $(y_{i(h)}^h, x_{-i(h)}) \rhd_{i(h)}^{\mathrm{Ind}} x_N$ whenever $x_N \in U^h$ ($h \in \{1, \dots, m\}$).

Now we recursively construct an infinite sequence $\langle x_N^k \rangle_{k \in \mathbb{N}}$ in Z, starting with x_N^0 already given. Having $x_N^k \in Z$ defined, we pick h such that $x_N^k \in U^h$ and define $x_N^{k+1} = (y_{i(h)}^h, x_{-i(h)}^k)$. By (2), we have $x_N^{k+1} \rhd_{i(h)}^{\mathrm{Ind}} x_N^k$, hence $x_N^{k+1} \in Z$ by Claim 1. Therefore, $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path in Z. The way our path is constructed ensures that, for every $i \in N$ and $k \in \mathbb{N}$, x_i^k is either x_i^0 or one of y_i^h ($h \in \{1, \dots, m\}$), i.e., there is a finite number of possible values. Therefore, we must have $x_N^{k'} = x_N^{k''}$ with $k' \neq k''$, which contradicts the supposed acyclicity of $\rhd^{\mathrm{Ind}}$. □

To finish with the proof of the theorem, we pick a maximizer z_N of $\rhd^{\mathrm{Ind}}$ on Z, existing by Claim 2. By Claim 1, it is a maximizer of $\rhd^{\mathrm{Ind}}$ on X_N, i.e., a Nash equilibrium. Let V be an arbitrary open neighborhood of z_N. By the definition of Z, there is a finite improvement path $\langle x_N^0, x_N^1, \dots, x_N^m \rangle$ such that $x_N^m \in V$. Therefore, Γ has the very weak FIP indeed. □

4 Concluding remarks

4.1. The upper semicontinuity of u_i in x_N alone is not sufficient for our theorem to remain valid, even under the existence of an ordinal potential rather than just acyclicity of $\triangleright^{\mathrm{Ind}}$ (Kukushkin, 1999, Example 2). [For the analysis there to be correct, "$\pi(x) = 0$" in the definition of the ordinal potential should be replaced with "$\pi(x) = -\infty$"; the infinity can be avoided by the replacement of all other values of π with, say, their exponentials.]

4.2. There is no counterexample to a conjecture that the assumptions of our theorem imply the existence of a generalized ordinal potential. In particular, the game constructed in the proof of Theorem 4.1 from Voorneveld (1997), which satisfies our assumptions and has the very weak FIP, even admits an upper semicontinuous generalized ordinal potential. If $\mathbb{R}$ as the strategy set of player 2 in Example 4.1 of Voorneveld and Norde (1996) is replaced with, say, a closed interval, the game will have the weak FIP (even without "very") although still admit no generalized ordinal potential; however, the game is not continuous.

By the way, it is unclear whether a (numeric or not) "very weak potential" could be defined producing an analog of Proposition 6.2 in Kukushkin (2004).

4.3. Following Milchtaich (1996) and Kukushkin (2004), we may consider *best response improvement* paths. However, our theorem cannot be extended that far.

Example 1. Let $N = \{1, 2\}$ and $X_1 = X_2$ be circles in the plane with polar coordinates, $\{(\rho_i, \varphi_i) \mid \rho_i = 1\}$ $(0 \leq \varphi_i < 2\pi)$, while utility functions be $u_1(x_1, x_2) = -d(\varphi_1, \varphi_2)$ and $u_2(x_1, x_2) = -d(\varphi_1 \oplus \varphi^0, \varphi_2)$, where $d(\varphi, \psi)$ is the distance between points $(1, \varphi)$ and $(1, \psi)$ in the plane, $\oplus$ denotes addition modulo 2π, and $\varphi^0 \neq q \cdot \pi$ for any rational q. Both utility functions are continuous; best response improvements never cycle. However, there is no Nash equilibrium, to say nothing of the very weak FIP.

4.4. One may wonder whether the (very) weak FIP is implied by popular sufficient conditions for the existence of a Nash equilibrium. The answer is "yes" for a finite game with perfect information (Kukushkin, 2002, Theorem 3) or strategic complementarities (Kukushkin et al. 2005, Theorem 1). On the other hand, the applicability of Tarski's fixed point theorem to the best responses does not, by itself, ensure the weak FIP even in a finite two person game (Kukushkin et al., 2005, Example 1).

Let us show that the applicability of the Brouwer fixed point theorem also does not ensure the very weak FIP even in a continuous two person game.

Example 2. Let $N = \{1, 2\}$ and $X_1 = X_2$ be unit discs in the plane with polar coordinates, $\{(\rho, \varphi) \mid 0 \leq \rho \leq 1\}$; let the utility functions be defined with the following construction. We define $V(\rho_1, \rho_2) = \min\{\rho_1, 4\rho_2 - \rho_1\}$ and $r(\rho) = \min\{2\rho, 1\}$. Then we pick functions $\eta'(\rho_1, \rho_2)$ and $\eta''(\varphi_1, \varphi_2)$ satisfying these requirements: $\eta'(\rho_1, \rho_2) = 1$ if $\rho_1 = r(\rho_2)$, $0 < \eta'(\rho_1, \rho_2) < 1$ whenever $0 < |\rho_1 - r(\rho_2)| < \min\{\rho_2, 1/3\}$ and $\eta'(\rho_1, \rho_2) = 0$ otherwise;

$\eta''(\varphi_1, \varphi_2) = 1$ if $\varphi_1 = \varphi_2$ and $0 \le \eta''(\varphi_1, \varphi_2) < 1$ otherwise. To be more precise, we pick η'' continuous everywhere, while η' continuous on $[0,1]^2 \setminus \{(0,0)\}$. We also pick $\varphi^0 \in]0, 2\pi[$. Finally, we set $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = V(\rho_1, \rho_2) + \rho_2 \cdot \eta'(\rho_1, \rho_2) \cdot \eta''(\varphi_1, \varphi_2)$ and $u_2((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = u_1((\rho_2, \varphi_2), (\rho_1, \varphi_1 \oplus \varphi^0))$, where $\oplus$ denotes addition modulo 2π.

Both utility functions are continuous; the symmetry allows us to restrict attention to the viewpoint of player 1. Given $x_2 = (\rho_2, \varphi_2)$, both V and η' are maximized when $\rho_1 = r(\rho_2)$; if $\rho_2 > 0$, η'' is maximized when $\varphi_1 = \varphi_2$. Thus, the unique best response is $x_1 = (r(\rho_2), \varphi_2)$. Similarly, the unique best response to $x_1 = (\rho_1, \varphi_1)$ is $x_2 = (r(\rho_1), \varphi_1 \oplus \varphi^0)$. Therefore, the existence of a Nash equilibrium is ensured by the Brouwer theorem; indeed, the origin is the unique equilibrium.

Suppose that $\rho_2 \ge 1/3$. Then $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) \ge V(\rho_1, \rho_2) \ge 1/3$ whenever $\rho_1 \ge 1/3$. Meanwhile, if $\rho_1 < 1/3$, then $V(\rho_1, \rho_2) < 1/3$ whereas $\rho_1 < r(\rho_2) - 1/3$, hence $\eta'(\rho_1, \rho_2) = 0$; thus, $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = V(\rho_1, \rho_2) < 1/3$. We see that any improvement path starting in the region where $\rho_i \ge 1/3$ for both i remains in the region forever, hence never reaches, nor even approaches, a Nash equilibrium. It may be noted that the players have no reason to regret this failure because their utility levels at the equilibrium are $\langle 0, 0 \rangle$.

4.5. Our approach is purely ordinal to the extent that the preferences of the players can be described with binary relations $\succ_i$ rather than utility functions u_i. It is enough to replace $u_i(y_N) > u_i(x_N)$ in (1a) with $y_N \succ_i x_N$. Condition (2) remains a meaningful "quasi-continuity" assumption; it holds, e.g., if each set $\{(y_N, x_N) \in X_N^2 \mid y_N \succ_i x_N\}$ $(i \in N)$ is open. The theorem remains valid; no modification of the proof is needed. We do not even need any *a priori* restriction on the preference relations such as transitivity, acyclicity, etc.

Under this broad interpretation of preferences, a maximizer of any binary relation can be seen as a Nash equilibrium in a game with one player. (2) then becomes the "open lower contours" assumption, and our theorem implies the main result of Walker (1977). If there are two (or more) non-dummy players, (2) does not imply open lower contours of $\rhd^{\mathrm{Ind}}$, so our theorem (even if restricted to mere existence) does not follow from Walker's.

Acknowledgments

Financial support from the Russian Foundation for Basic Research (project 08-07-00158), the Russian Foundation for Humanities (project 08-02-00347), and the Spanish Ministry of Education and Science (project SEJ 2007-67135) is acknowledged. I thank Francisco Marhuenda for the organization and coordination of the last project.

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