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Note

Natural Convection Flow of a Non-Newtonian Fluid Between Two Vertical Flat Plates

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With 2 Figures

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Summary

The natural convection of a homogeneous incompressible fluid of grade three is investigated between two infinite parallel vertical plates. The effect of the non-Newtonian nature of fluid on the skin friction and heat transfer are studied.

1. Introduction

In this note we consider the natural convection of a non-Newtonian fluid, namely the Rivlin-Ericksen fluid of grade three, between two infinite parallel vertical flat plates. The stress in such a fluid is related to the motion in the following manner (cf. Truesdell and Noll [1]):

$$\begin{aligned} \boldsymbol{T} = -p\boldsymbol{l} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2 + \beta_1 \boldsymbol{A}_3 \\ + \beta_2 [\boldsymbol{A}_1 \boldsymbol{A}_2 + \boldsymbol{A}_2 \boldsymbol{A}_1] + \beta_3 \, (\mathrm{tr} \boldsymbol{A}_1^2) \, \boldsymbol{A}_1 \end{aligned} \tag{1}$$

where μ is the coefficient of viscosity, α_1 and α_2 material moduli popularly referred to as normal stress moduli. The kinematical tensors $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are defined through (cf. Rivlin and Ericksen [2]):

$$\boldsymbol{A}_1 = \mathrm{grad}\, \boldsymbol{v} + (\mathrm{grad}\, \boldsymbol{v})^T, \tag{2}$$

and

$$\boldsymbol{A}_n = \frac{d}{dt} \boldsymbol{A}_{n-1} + \boldsymbol{A}_{n-1} \boldsymbol{L} + \boldsymbol{L}^T \boldsymbol{A}_{n-1}, \qquad n = 1, 2, \tag{3}$$

where $\frac{d}{dt}$ denotes material time differentiation, and $\boldsymbol{L} = \mathrm{grad}\, \boldsymbol{v}$.

The above model contains as a subclass the classical linearly viscous Newtonian fluid (when all material moduli except μ are zero) and the class of fluids of grade two (when the β's are zero).

The natural convection problem between vertical flat plates for a certain class of non-Newtonian fluids has been carried out by Bruce and Na [3]. Other laminar natural convection problems involving heat transfer have been studied and we refer the reader to [4] for details of the same. However, in these problems a complete thermodynamic analysis of the constitutive functions have not been carried out. For the model (1), a detailed thermodynamic study has been carried out in [5]. If the model is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality (which is usually considered as an interpretation of the second law of thermodynamics) and the assumption that the specific Helmholtz free energy be a minimum in equilibrium, then

$$\mu \geqq 0, \quad \alpha_1 \geqq 0, \quad |\alpha_1 + \alpha_2| \leqq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geqq 0. \tag{4}$$

Thus in the case of a thermodynamically compatible fluid of third grade (1) reduces to

$$\boldsymbol{T} = -p\boldsymbol{l} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_3(\mathrm{tr}\,\boldsymbol{A}_1^2)\,\boldsymbol{A}_1 . \tag{5}$$

For the problem under consideration, a fluid represented by (5), whose material coefficients satisfy (4), is in between two vertical flat plates a distance '$2b$' apart. The walls at $x = +b$ and $x = -b$ are held at constant temperatures θ_2 and θ_1 respectively, where $\theta_1 > \theta_2$. This difference in temperature causes the fluid near the wall at $x = -b$ to rise and the fluid near the wall at $x = b$ to fall. In the next section we shall determine the velocity profile due to this flow.

2. Analysis

For the problem in question, we shall seek velocity and temperature fields of the form

$$\boldsymbol{v} = v(x)\,\boldsymbol{j}, \qquad \theta = \theta(x). \tag{6.1, 2}$$

It follows from (5), (6) and the balance of linear momentum, that

$$(2\alpha_1 + \alpha_2)\,\frac{d}{dx}\left(\frac{dv}{dx}\right)^2 = \frac{\partial p}{\partial x}, \tag{7.1}$$

$$\mu\,\frac{d^2v}{dx^2} + 6\beta_3\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2} - \varrho_0\,[1 - \gamma(\theta - \theta_m)]\,g = \frac{\partial p}{\partial y}, \tag{7.2}$$

$$0 = \frac{\partial p}{\partial z}. \tag{7.3}$$

In deriving the Eqs. (7.1, 2, 3) the usual Boussinesq law is assumed for the body force, i.e.,

$$\varrho \boldsymbol{b} = -\varrho_0[1 - \gamma(\theta - \theta_m)]\, g\boldsymbol{j},$$

where g denotes gravity, γ is the coefficient of thermal expansion and ϱ_0 is a constant and θ_m a reference temperature which we shall pick as $\theta_m = \frac{1}{2}(\theta_1 + \theta_2)$. Defining a modified pressure through

$$\hat{p} = p - (2\alpha_1 + \alpha_2)\left(\frac{dv}{dx}\right)^2, \tag{8}$$

(7.1, 2, 3) can be re-written as

$$0 = \frac{\partial \hat{p}}{\partial x}, \tag{9.1}$$

$$\mu \frac{d^2v}{dx^2} + 6\beta_3\left(\frac{dv}{dx}\right)^2 \frac{d^2v}{dx^2} - \varrho_0\,[1 - \gamma(\theta - \theta_m)]\, g = \frac{\partial \hat{p}}{\partial y}, \tag{9.2}$$

$$0 = \frac{\partial \hat{p}}{\partial z}. \tag{9.3}$$

Equations (9.1, 2, 3) imply that $\frac{\partial \hat{p}}{\partial y}$ is at most a constant. On appropriately extending the usual approximations, the equation of motion reduces to

$$\mu \frac{d^2v}{dx^2} + 6\beta_3\left(\frac{dv}{dx}\right)^2 \frac{d^2v}{dx^2} + \varrho_0\gamma(\theta - \theta_m)\, g = 0. \tag{10}$$

We now proceed to derive the energy equation appropriate for the problem under consideration. We start with the energy equation

$$\varrho \frac{d\varepsilon}{dt} = \boldsymbol{T} \cdot \boldsymbol{L} - \operatorname{div} \boldsymbol{q} + \varrho r, \tag{11}$$

where ε is the specific internal energy, $\boldsymbol{L}$ is the gradient of velocity, $\boldsymbol{q}$ is the heat flux vector and r the radiant heating. It follows from (1) that

$$\boldsymbol{T} \cdot \boldsymbol{L} = \frac{\mu}{2}\,|\boldsymbol{A}_1|^2 + \frac{\alpha_1}{4}\frac{d}{dt}\,|\boldsymbol{A}_1|^2 + \frac{(\alpha_1 + \alpha_2)}{2}\,\operatorname{tr} \boldsymbol{A}_1^3 + \frac{\beta_3}{2}\,|\boldsymbol{A}_1|^4. \tag{12}$$

For the problem under consideration in virtue of (6.1), $\boldsymbol{T} \cdot \boldsymbol{L}$ reduces to

$$\boldsymbol{T} \cdot \boldsymbol{L} = \mu\left(\frac{dv}{dx}\right)^2 + 2\beta_3\left(\frac{dv}{dx}\right)^4. \tag{13}$$

It has been shown in [5] that if the model (1) is to be compatible with thermodynamics then the specific Helmholtz free energy which characterizes the fluid has to take the form

$$\psi = \psi(\theta, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3) = \overline{\varphi}(\theta, \boldsymbol{A}_1) = \overline{\varphi}(\theta, 0) + \frac{\alpha_1}{4\varrho} |\boldsymbol{A}_1|^2, \tag{14}$$

and further the specific entropy is defined through

$$\eta = -\overline{\varphi}_\theta, \tag{15}$$

where the subscript denotes partial differentiation with respect to that variable. Since the specific internal energy is related to the specific Helmholtz free energy through

$$\varepsilon = \psi + \theta\eta, \tag{16}$$

it follows from (14), (15), (16) that

$$\varepsilon = \hat{\varphi}(\theta) + \frac{\alpha_1}{4\varrho} |\boldsymbol{A}_1|^2 - \theta\hat{\varphi}_\theta, \tag{17}$$

where

$$\hat{\varphi}(\theta) \equiv \overline{\varphi}(\theta, 0). \tag{18}$$

Thus,

$$\varrho \frac{d\varepsilon}{dt} = \varrho \left\{\frac{d}{dt} \left(\hat{\varphi}(\theta) - \theta\hat{\varphi}_\theta\right) + \frac{\alpha_1}{4\varrho} \frac{d}{dt} |\boldsymbol{A}_1|^2\right\}. \tag{19}$$

Next, note that (17) implies that

$$\varepsilon_\theta = \frac{d}{d\theta} (\hat{\varphi} - \theta\hat{\varphi}_\theta) = -\theta\hat{\varphi}_{\theta\theta} \equiv c, \tag{20}$$

where c is called the specific heat. Thus

$$\frac{d\varepsilon}{dt} = c \frac{d\theta}{dt} = 0, \tag{21}$$

by virtue of the assumed form of the temperature field (6.2). Thus, the balance of energy (11), Eqs. (13) and (21) imply that

$$\mu \left(\frac{dv}{dx}\right)^2 + 2\beta_3 \left(\frac{dv}{dx}\right)^4 - \operatorname{div} \boldsymbol{q} + \varrho r = 0. \tag{22}$$

We shall assume that the heat flux vector $\boldsymbol{q}$ satisfies Fourier's law with a thermal conductivity constant k, i.e.,

$$\boldsymbol{q} = -k \operatorname{grad} \theta.$$

Then, (6.2) implies that

$$\operatorname{div} \boldsymbol{q} = -k \frac{d^2\theta}{dx^2}. \tag{23}$$

Thus, if one ignores the radiant heating, (22) and (23) yield

$$\mu \left(\frac{dv}{dx}\right)^2 + 2\beta_3 \left(\frac{dv}{dx}\right)^4 + k \frac{d^2\theta}{dx^2} = 0. \tag{24}$$

In the next section we shall solve the system of Eqs. (10) and (24). The equations are coupled and highly non-linear. The appropriate boundary conditions are

$$v = 0, \qquad \theta = \theta_1 \quad \text{at} \quad x = -b \tag{25.1}$$

$$v = 0, \qquad \theta = \theta_2 \quad \text{at} \quad x = +b. \tag{25.2}$$

3. Solution

Let us introduce non-dimensional parameters

$$\bar{v} = \frac{v}{V_0}, \quad \bar{x} = \frac{x}{b}, \quad \bar{\theta} = \frac{\theta - \theta_m}{\theta_1 - \theta_2}, \tag{26}$$

where V_0 is some reference velocity. Then, Eqs. (10) and (24) can be re-written as

$$\frac{d^2\bar{v}}{d\bar{x}^2} + \frac{6\beta_3 V_0^2}{\mu b^2} \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 \frac{d^2\bar{v}}{d\bar{x}^2} + \frac{\varrho_0 \gamma b^2}{\mu V_0} g(\theta_1 - \theta_2)\, \bar{\theta} = 0, \tag{27}$$

and

$$\frac{d^2\bar{\theta}}{dx^2} + \frac{\mu V_0^2}{k(\theta_1 - \theta_2)} \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 + 2\beta_3 \frac{V_0^4}{b^2 k(\theta_1 - \theta_2)} \left(\frac{d\bar{v}}{d\bar{x}}\right)^4 = 0. \tag{28}$$

Let us select

$$V_0 = \frac{\varrho_0 b^2 (\theta_1 - \theta_2)\, \gamma g}{\mu},$$

then (27) and (28) can be further simplified to

$$\frac{d^2\bar{v}}{d\bar{x}^2} + 6\delta \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 \frac{d^2\bar{v}}{d\bar{x}^2} + \bar{\theta} = 0, \tag{29}$$

$$\frac{d^2\bar{\theta}}{d\bar{x}^2} + E \cdot (Pr) \left(\frac{d\bar{v}}{d\bar{x}}\right)^2 + 2\delta E \cdot (Pr) \left(\frac{d\bar{v}}{d\bar{x}}\right)^4 = 0, \tag{30}$$

where

$$E \equiv \frac{V_0^2}{c(\theta_1 - \theta_2)}, \quad Pr = \frac{\mu c}{k},$$

and

$$\delta = \frac{6\beta_3 V_0^2}{\mu b^2},$$

where c is the specific heat of the fluid. The appropriate boundary conditions are

$$\bar{v} = 0, \qquad \bar{\theta} = \frac{1}{2} \quad \text{at} \quad x = -b \tag{31}$$

$$\bar{v} = 0, \qquad \bar{\theta} = -\frac{1}{2} \quad \text{at} \quad x = +b. \tag{32}$$

The Eqs. (29)—(32) have been solved numerically.

The skin friction S on the plate at $x = -b$ is directly proportional to $\frac{d\bar{v}}{d\bar{x}}$,

$$S \sim \frac{d\bar{v}}{d\bar{x}}(-1),$$

and the heat transfer h is directly proportional to $\frac{d\bar{\theta}}{d\bar{x}}$,

$$h \sim \frac{d\bar{\theta}}{d\bar{x}}(-1).$$

Representative values of $\frac{d\bar{v}}{d\bar{x}}(-1)$ and $\frac{d\bar{\theta}}{d\bar{x}}(-1)$ have been provided for various values of E, Pr and δ. The analysis seems to indicate that an increase in δ holding E and Pr fixed increase the heat transfer slightly but decreases the skin friction. On the other hand, an increase in E holding δ and Pr fixed tends to increase the skin friction while decreasing the heat transfer. Similarly and increase in Pr holding E and δ fixed increases the skin friction and decreases the heat transfer.

Table 1. *Variation of* $\frac{d\bar{v}}{d\bar{x}}(-1)$ *and* $\frac{d\bar{\theta}}{d\bar{x}}(-1)$

γ	E	Pr	$\frac{d\bar{v}}{d\bar{x}}(-1)$	$\frac{d\bar{\theta}}{d\bar{x}}(-1)$
0.50	1.0	1.0	0.1628	−0.4966
1.0	1.0	1.0	0.1592	−0.4966
1.0	2.0	1.0	0.1593	−0.4932
1.0	4.0	1.0	0.1596	−0.4863
2	1.0	0.1	0.1532	−0.4997
2	1.0	1.0	0.1533	−0.4967

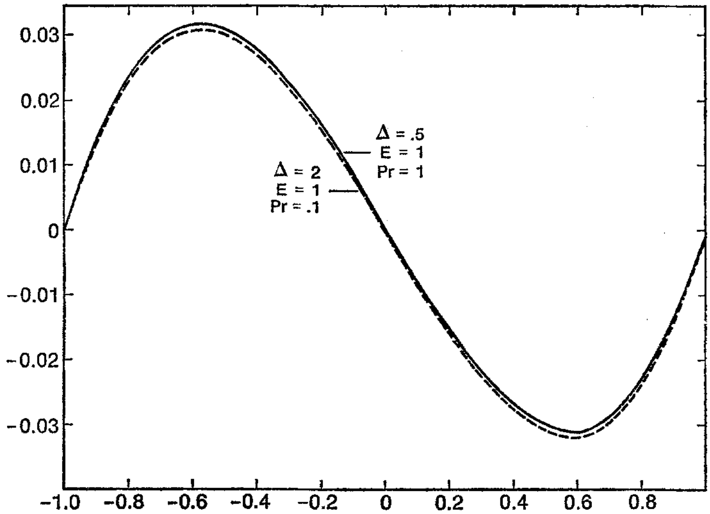

Fig. 1. Variation of velocity

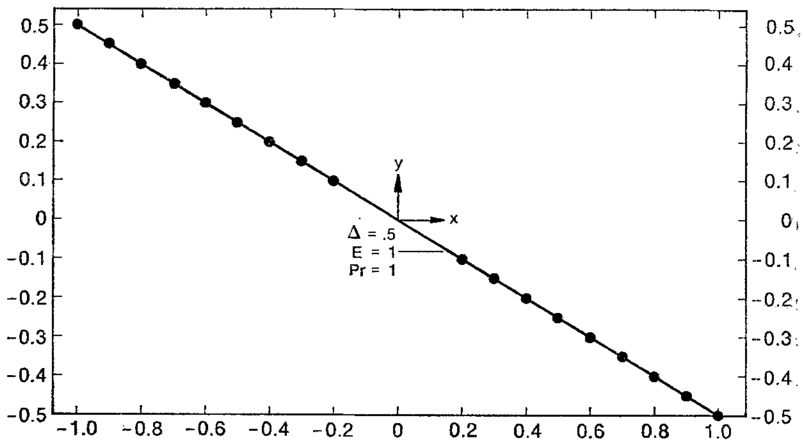

Fig. 2. Variation of temperature

Fig. 1 indicates the variation of the velocity profile $\bar{v}$ with E, Pr, and δ. Fig. 2 depicts the temperature field $\bar{\theta}$. It is found that for the values of the parameters which have been considered, $\bar{\theta}$ varies linearly.

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