

NATURAL DEDUCTION RULES FOR MODAL LOGICS

THOMAS W. SATRE

Natural deduction rules for systems of modal logic have been formulated in the style of Gentzen by Ohnishi, Matsumoto, Kanger, and Curry. The purpose of this paper is to formulate natural deduction rules for the S-, E-, and D-families of systems studied by Lemmon in [6]. Some of the rules for modal operators contained in this paper have been employed by others. The remainder are, to the author's knowledge, new. The style of presentation of the rules shall be that of Lemmon's natural deduction rules for propositional calculus in [5]; similar formulations of rules are to be found in [15]. In section 1 the standard proof of the deduction theorem is shown to hold in the three families of systems to be studied. In section 2 each system of these families is given a deductively equivalent formalization by means of natural deduction rules. Finally, some suggestions concerning interpretation of modal logics are offered in section 3.

1. In [6] Lemmon showed that the Lewis systems S1-S5, together with S0.5, S0.9, and the families of systems D1-D5 and E1-E5, can be axiomatized as extensions of classical propositional calculus. The propositional calculus basis common to all these systems shall be referred to as **PC** and shall have the axioms and rules:

PC1: $p \supset (q \supset p)$

PC2: $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$

PC3: $(\neg q \supset \neg p) \supset ((\neg q \supset p) \supset q)$

PC4: *If $\vdash A$ and B comes from A by uniform substitution for propositional variables of A , then $\vdash B$.*

PC5: *From A and $A \supset B$, B may be inferred.*

The usual definitions of $\&$, \vee , and \equiv are assumed. Further, it is assumed that the reader is acquainted with the modal rules (a)-(D), the modal axioms (1)-(5), and the definitions of \boxrightarrow , \diamond , and \boxequiv of [6].¹

1. In the notation of [6] these are respectively C' , M , and E' . The primitive modal operator there is L which is represented by \square in this paper.

To illustrate the deduction theorem for systems of modal logic let K be the system obtained by adding to PC axiom (1') and rule (a). The definition of a wff of PC is assumed to be extended to cover wffs of modal logic.² The standard definition of proof in a system S from hypotheses is adopted here, viz., a wff B is provable in S from hypotheses A_1, \dots, A_n (written $A_1, \dots, A_n \vdash B$ in S) if and only if there is a sequence of wffs B_1, \dots, B_m such that $B_m = B$ and for each i ($1 \leq i \leq m$) either:

- (i) B_i is an axiom of S ; or
- (ii) B_i is one of A_1, \dots, A_n ; or
- (iii) B_i follows directly from some B_{j_1}, \dots, B_{j_k} ($1 \leq j_k < i$) by some rule of S .

Where $n = 0$, B is said to be a theorem of S (written $\vdash B$ in S). The use of ' \vdash ' here thus corresponds to that of Kleene in [4], p. 88. It is a property of the notion of proof from hypotheses that if $\vdash B$ in S then, trivially, $A_1, \dots, A_n \vdash B$ in S . This property is used without mention in the proofs that follow.

Deduction theorem for K . *If $A_1, \dots, A_n \vdash B$ in K , then $A_1, \dots, A_{n-1} \vdash A_n \supset B$ in K .*

Proof: By supposition and the definition of proof from hypotheses, there is a sequence B_1, \dots, B_m of wffs such that for each B_i ($1 \leq i \leq m$) either (i) B_i is some A_j ($1 \leq j \leq n$); or (ii) B_i is one of $PC1$ - $PC3$, or (1'); or (iii) B_i comes from B_j ($j < i$) by $PC4$; or (iv) B_i comes from some B_j and $B_k = (B_j \supset B_i)$ ($j, k < i$) by $PC5$; or (v) B_i is $\Box B_j$ and comes from B_j by rule (a). By proving the theorem for all $i \leq m$, the result follows by putting $i = m$.

Case (i) (a): If $j = n$, then B_i is A_n . But $\vdash A_n \supset A_n$ in K in virtue of PC . So $\vdash A_n \supset B_i$ in K from which it follows that $A_1, \dots, A_{n-1} \vdash A_n \supset B_i$ in K . (b): If $j \neq n$, then, since $\vdash A_j \supset A_j$ in K , we have $A_1, \dots, A_{n-1} \vdash A_j$ in K which is $A_1, \dots, A_{n-1} \vdash B_i$ in K . By $PC1$ and $PC5$ it follows that $A_1, \dots, A_{n-1} \vdash A_n \supset B_i$ in K .

Case (ii): By definition $\vdash B_i$ in K . By appropriate substitution in $PC1$ we have $\vdash B_i \supset (A_n \supset B_i)$ in K . Thus $\vdash A_n \supset B_i$ in K by $PC5$. Hence, $A_1, \dots, A_{n-1} \vdash A_n \supset B_i$ in K .

Case (iii): Then, by the restriction on $PC4$, it follows that $\vdash B_i$ in K . So $A_1, \dots, A_{n-1} \vdash A_n \supset B_i$ in K by the same reasoning as case (ii).

Case (iv): As inductive hypotheses, suppose that the theorem holds for B_j and B_k . Thus, we have both $A_1, \dots, A_{n-1} \vdash A_n \supset B_j$ in K and $A_1, \dots, A_{n-1} \vdash A_n \supset (B_j \supset B_i)$ in K . As a case of $PC2$ $\vdash (A_n \supset (B_j \supset B_i)) \supset ((A_n \supset B_j) \supset (A_n \supset B_i))$ in K . Two steps of $PC4$ yield $A_1, \dots, A_{n-1} \vdash A_n \supset B_i$ in K .

Case (v): By the restriction on rule (a) we have $\vdash B_j$ in K , and by rule

2. The system K , as formulated here, was presented by Lemmon in lectures during the spring term of 1966.

(a) $\vdash \Box B_i$ in K which is $\vdash B_i$ in K . Again, the proof is the same as that in case (ii).

It will be noted that cases (i)-(iv) give just the standard proof of the deduction theorem for classical propositional calculus (*vide* [13], pp. 16-18). Case (iii) could be omitted if PC1-3 had been stated as axiom schemata. However, PC4 is needed when we consider the addition of modal operators, with appropriate axioms and rules, to **PC**. The addition of the modal axiom (1') made no difference at case (ii), and no restrictions are needed on the deduction theorem due to rule (a) since the rule will apply only to theorems.

The deduction theorem, as well as rules (a)-(Eb'), is a subsidiary deduction rule, in the sense of Kleene [4], §22. It is not in general true that a subsidiary deduction rule will continue to hold when a system is extended by addition of new axioms or rules, since there may not be resulting deductions available in the extended system. Examination of the above proof will show that the restriction on rule (a) provides that the only new subsidiary deductions will be of theorems; thus, the corresponding resulting deductions will always be available in virtue of PC1, 2, 4, and 5. (PC3 was not used in the proof of the deduction theorem for K .)

We can generalize the result obtained as follows: Let S be any formal system having the symbols and formation rules of **PC** and obtained by adding to **PC** proper axioms and rules, together with formation rules for any new symbols introduced in the proper axioms and rules. Then the deduction theorem above will hold for S provided that each proper rule is applicable only to theorems of S , i.e., formulas provable from zero hypotheses.³ It will be observed that all of the proper rules (a)-(D) are of this type. The restrictions on (Eb') and (D) concern only the presence or absence of modal operators in component wffs of the theorems which result, and (a') can easily be stated so as to show that it applies only to theorems of any system to which it is added. Hence, we may conclude that the deduction theorem above holds for all of Lemmon's S-, E-, and D-systems.

2. As a basic set of natural deduction rules the following, taken from Lemmon [5], shall be used:

- Hyp: Any wff may be introduced at any line of a proof.
 MP: Given A and $A \supset B$, we may derive B .
 DN: Given A , we may derive $\neg\neg A$, and vice versa.
 &I: Given A and B , we may derive $A \& B$.
 &E: Given $A \& B$, we may derive either A or B separately.
 \vee I: Given either A or B separately, we may derive $A \vee B$.
 CP: Given a proof of B depending upon A as a hypothesis, we may derive $A \supset B$ depending upon the remaining hypotheses.

3. In [8], p. 58, it is noted that the deduction theorem will continue to hold in extensions of **PC** formed by the addition of new axioms alone.

- $\vee E$: Given $A \vee B$ together with a proof of C from A as a hypothesis and a proof of C from B as a hypothesis, we may derive C depending upon any hypotheses upon which $A \vee B$ depends or upon which C depends in its derivation from A (except for A) or upon which C depends in its derivation from B (except for B).
- RAA: Given a proof of B & $\neg B$ depending upon A as a hypothesis, we may derive $\neg A$ depending upon the remaining hypotheses.

In the case of rule Hyp, the conclusion derived depends upon itself as a hypothesis. For the rules MP- $\vee I$, the conclusion depends upon the combined hypotheses (if any) of the premise(s). For the remaining rules, the conclusion rests upon the hypotheses stipulated by the rule. Schematically, these rules may be written as follows: let Γ, Δ be sequences (possibly empty) of wffs, or conjunctions (possibly empty) of wffs, according as Γ, Δ occur to the left or right side of ' \vdash '. To the left of ' \vdash ' occur only hypotheses, if any, from which the wff occurring to the right is said to be deducible. Wffs to the right of ' \vdash ' and above a horizontal line are premises from which the wff to the right of ' \vdash ' and below the horizontal line is derivable in accordance with the rule in question. To the left of ' \vdash ' and below the horizontal line are the hypotheses upon which a conclusion drawn in accordance with the rule will depend. By rule Hyp any hypothesis is also available as a premise in a proof, though, of course, a premise need not have been introduced by the rule Hyp.

| | |
|--|--|
| Hyp: $A \vdash A$ | RAA: $\frac{A_1, \dots, A_n \vdash B \ \& \ \neg B}{A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_n \vdash \neg A_j \ (1 \leq j \leq n)}$ |
| MP: $\frac{\Gamma \vdash A; \ \Delta \vdash A \supset B}{\Gamma, \Delta \vdash B}$ | CP: $\frac{A_1, \dots, A_n \vdash B}{A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_n \vdash A_j \supset B \ (1 \leq j \leq n)}$ |
| &I: $\frac{\Gamma \vdash A; \ \Delta \vdash B}{\Gamma, \Delta \vdash A \ \& \ B}$ | &E: $\frac{\Gamma \vdash A \ \& \ B \quad \Gamma \vdash A \ \& \ B}{\Gamma \vdash A \quad \Gamma \vdash B}$ |
| $\vee I$: $\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \vdash A \vee B \quad \Delta \vdash A \vee B}$ | $\vee E$: $\frac{\Gamma \vdash A \vee B; \ \Delta, A \vdash C; \ \Delta', B \vdash C}{\Gamma, \Delta, \Delta' \vdash C}$ |
| DN: $\frac{\Gamma \vdash A}{\Gamma \vdash \neg \neg A}$ | $\frac{\Gamma \vdash \neg \neg A}{\Gamma \vdash A}$ |

For convenience, the rule of substitution for propositional variables shall be taken over for natural deduction also, though it could be derived as well. The above rules, together with the rule of substitution for propositional variables, will be referred to as NPC.

In this section systems of modal logic are constructed by adding various rules to NPC. The result will be families of systems deductively equivalent to the S-, E-, and D-families and sharing NPC as a basis just as those systems share PC as a basis on Lemmon's formalization. The deductively equivalent natural deduction systems will be indicated by prefixing 'N' to the name of the system. As a stock of natural deduction rules for modal operators the following are adopted:

$\Box E_1$: Given $\Box A$ we may derive A depending upon the same hypotheses as the premise.

$\Box E_2$: Given $\Box A$ we may derive $\Diamond A$ depending upon the same hypotheses as the premise.

$\Box E_3$: Given $\Box A$ we may derive separately $\Diamond A$, or, if A is fully modalized, A . The conclusion depends upon the same hypotheses as the premise. (A wff is said to be fully modalized if each occurrence of a propositional variable within it is within the scope of some modal operator also within it.)

Let Γ, Δ be any sequences (possibly empty) of wffs, or conjunctions of wffs as before. Let $\Box \Delta$ be the result of prefixing ' \Box ' to each wff of Δ .

$\Box I_1$: Given $\Box \Delta$ and a proof of B from Δ alone as hypotheses, we may derive $\Box B$ depending upon the combined hypotheses of the premises $\Box \Delta$.

$\Box I_2$: Given $\Box \Delta$ and a proof of B from Δ alone as hypotheses, we may derive $\Box B$, provided that Δ is not empty. The conclusion depends upon the combined hypotheses of the premises $\Box \Delta$.

$\Box I_3$: Given $\Box \Delta$ and a proof of B from Δ alone as hypotheses, we may derive $\Box B$, provided that Δ is not empty and that all wffs of Δ and B are wffs of propositional calculus, i.e., Δ, B contain no occurrences of modal operators. The conclusion depends upon the combined hypotheses of the premises $\Box \Delta$.

$\Box I_4$: Given B we may derive $\Box B$ depending upon the same hypotheses, provided that each of those hypotheses is of the form $\Box C$ for some wff C .

$\Box I_5$: Given B depending upon some hypotheses, we may derive $\Box B$, provided that each of those hypotheses is of the form $\Box C$ for some wff C . The conclusion depends upon the same hypotheses as the premise.

$\Box I_6$: Given B we may derive $\Box B$, depending upon the same hypotheses, provided that each of those hypotheses is of the form of either $\Box C$ or $\Diamond C$ for some wff C .

$\Box I_7$: Given B depending upon some hypotheses, we may derive $\Box B$, provided that each of those hypotheses is of the form of either $\Box C$ or $\Diamond C$ for some wff C . The conclusion depends upon the same hypotheses as the premise.

$\Box I_8$: (Rule Scheme) Given $\Box \Delta$, where Δ is not empty, and a proof of B by rules of NPC alone from Δ alone as hypotheses, we may derive $\Box B$, depending upon the combined hypotheses of the premises; and, if Δ is empty and $\vdash B$, then $\vdash \Box B$, provided that B is in Λ .

$\Box I_9$: Given $\Box \Delta$ as premises and a proof of B by rules NPC alone from Δ alone as hypotheses, we may derive $\Box B$ depending upon the same hypotheses as the premises $\Box \Delta$.

$\Box I_{10}$: Given a proof of B from Δ alone as hypotheses, where Δ is not empty, then (i) given as premises $\Box \Delta$ we may derive $\Box B$ depending upon the hypotheses upon which $\Box \Delta$ depend; and (ii) if each hypothesis of Δ is of the form $\Box C$, for some wff C , then we may derive $\Box B$ depending upon Δ .

$\Box I_{11}$: Given a proof of B from Δ alone as hypotheses, where Δ is not empty, then (i) given as premises $\Box \Delta$ we may derive $\Box B$ depending upon

the hypotheses upon which $\Box\Delta$ depend; and (ii) if each hypothesis of Δ is of the form of either $\Box C$, or $\Diamond C$, for some wff C , then we may derive $\Box B$ depending upon Δ .

$\Box S_1$: Given as a premise $\Box(A \supset B)$ we may derive $\Box(\Box A \supset \Box B)$ depending upon the same hypotheses, if any.

$\Box S_2$: Given a proof of $\Box(A \supset B)$ as a theorem, i.e., $\vdash\Box(A \supset B)$, we may derive as a theorem $\Box(\Box A \supset \Box B)$.

$\Box S_3$: Given proofs of $\Box(A \supset B)$ and $\Box(B \supset A)$ as theorems, we may derive $\Box(\Box A \supset \Box B)$ as a theorem.

These rules may be presented schematically as follows:

- $$\Box E_1: \frac{\Gamma \vdash \Box A}{\Gamma \vdash A} \quad \Box E_2: \frac{\Gamma \vdash \Box A}{\Gamma \vdash \Diamond A}.$$
- $$\Box E_3: \frac{\Gamma \vdash \Box A}{\Gamma \vdash \Diamond A}; \text{ and } \frac{\Gamma \vdash \Box A}{\Gamma \vdash A}, \text{ provided } A \text{ is fully modalized.}$$
- $$\Box I_1: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B}{\Gamma \vdash \Box B} \quad \Box I_2: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B}{\Gamma \vdash \Box B}, \text{ provided } \Delta \text{ is not empty.}$$
- $$\Box I_3: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B}{\Gamma \vdash \Box B}, \text{ provided that } \Delta \text{ is not empty and that no wffs } \Delta \text{ or } B \text{ contain occurrences of modal operators.}$$
- $$\Box I_4: \frac{\Delta \vdash B}{\Delta \vdash \Box B}, \text{ provided that each wff of } \Delta \text{ is of the form } \Box C \text{ for some wff } C.$$
- $$\Box I_5: \frac{\Delta \vdash B}{\Delta \vdash \Box B}, \text{ provided that } \Delta \text{ is not empty and that each wff of } \Delta \text{ is of the form } \Box C \text{ for some wff } C.$$
- $$\Box I_6: \frac{\Delta \vdash B}{\Delta \vdash \Box B}, \text{ provided that each wff of } \Delta \text{ is of the form of either } \Box C \text{ or } \Diamond C \text{ for some wff } C.$$
- $$\Box I_7: \frac{\Delta \vdash B}{\Delta \vdash \Box B}, \text{ provided that } \Delta \text{ is not empty and that each wff of } \Delta \text{ is of the form of either } \Box C \text{ or } \Diamond C \text{ for some wff } C.$$
- $$\Box I_8: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B \text{ in NPC}}{\Gamma \vdash \Box B}, \text{ provided that } \Delta \text{ is not empty, and}$$
- $$\frac{\vdash B}{\vdash \Box B}, \text{ provided that } B \text{ is in } \Lambda.$$
- $$\Box I_9: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B \text{ in NPC}}{\Gamma \vdash \Box B}.$$
- $$\Box I_{10}: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B}{\Gamma \vdash \Box B}, \text{ and } \frac{\Delta \vdash B}{\Delta \vdash \Box B}, \text{ provided that all wffs of } \Delta \text{ are of the form } \Box C \text{ for some wff } C; \text{ in either case } \Delta \text{ is not empty.}$$
- $$\Box I_{11}: \frac{\Gamma \vdash \Box \Delta; \Delta \vdash B}{\Gamma \vdash \Box B}, \text{ and } \frac{\Delta \vdash B}{\Delta \vdash \Box B}, \text{ provided that all hypotheses of } \Delta \text{ are of the form of either } \Box C \text{ or } \Diamond C \text{ for some wff } C; \text{ in either case } \Delta \text{ is not empty.}$$
- $$\Box S_1: \frac{\Gamma \vdash \Box(A \supset B)}{\Gamma \vdash \Box(\Box A \supset \Box B)} \quad \Box S_2: \frac{\vdash \Box(A \supset B)}{\vdash \Box(\Box A \supset \Box B)}.$$
- $$\Box S_3: \frac{\vdash \Box(A \supset B); \vdash \Box(B \supset A)}{\vdash \Box(\Box A \supset \Box B)}.$$

The motive behind the formalization here is to provide as constant factors in all of the theories to be studied the classical propositional calculus as well as the notion of deduction in accordance with rules alone. The differences between modal systems are shown by differences only in the rules governing modal operators. This is just what Lemmon-style formalization does from an axiomatic approach. A rule much like $\Box I_1$ was suggested by Lemmon in lectures in 1966. From it a rule like Ohnishi and Matsumoto's $\rightarrow\Box$ for M^* in [10] could be obtained given appropriate translation from Gentzen-style notation. $\Box I_2$ is likewise related to rule $\rightarrow\Box$ for system $Q2$ of [10]. $\Box I_4$ is essentially rule $\rightarrow\Box$ for $S4^*$ of [10], and $\Box I_5$ is just $\Box I_4$ with the restriction that Δ ($\Box\theta$ in [10]) not be empty. $\Box I_8$ should be compared with Ohnishi's rules $\rightarrow\Box$ for $S2^*$ and $S3^*$ of [9]. Ohnishi's use of 'tautology' there is not exactly that of Lemmon in [6], and Ohnishi's restriction, p. 126 of [9], does not seem to restrict use of his rule RT when α is a theorem other than an instance of a tautology of propositional calculus or an instance of (1') or (2). $\Box I_8$ obtains these restrictions by a class of wffs Λ , fixed for a given system, and by the restriction that B be derived from Δ by NPC alone. Ohnishi obtains a counterpart of the latter restriction by not allowing rule $\rightarrow\Box$ to occur below RT in a proof figure. When applied in system $S3$, $\Box I_8$ seems easier to employ than would be a counterpart of Ohnishi's $\rightarrow\Box$ for $S3^*$ of [9]. $\Box I_9$ and its use in $NS0.5$ should be compared with Routley's rule in [14] for $*S0.5$, which is a Gentzen-style calculus. Finally, $\Box E_1$ is a counterpart of $\Box\rightarrow$ of [10].

Let NK be NPC with the added rule $\Box I_1$. The deductive equivalence of K and NK , i.e., $\vdash A$ in NK if and only if $\vdash A$ in K , is proven as follows: suppose that $\vdash A$ in NK . The rules of NK hold in K .

Case (1): The rules of NPC hold in K in virtue of PC and the deduction theorem for K .

Case (2): Rule $\Box I_1$. If Δ is empty, in which case Γ is also, then $\Box I_1$ is just rule (a) of K . So let Δ be A_1, \dots, A_n and suppose that $A_1, \dots, A_n \vdash B$ in K . By the deduction theorem for K , applied n times, $\vdash A_1 \supset (\dots \supset (A_n \supset B) \dots)$ in K . By (a), (1'), and PC we have $\vdash \Box A_1 \supset (\dots \supset (\Box A_n \supset \Box B) \dots)$ in K . Hence, if Γ is any set of hypotheses such that $\Gamma \vdash \Box \Delta$ in K , then the proof of $\Gamma \vdash \Box \Delta$ in K may be extended by use of PC together with the above theorem to yield a proof of $\Gamma \vdash \Box B$ in K .

Conversely, suppose that $\vdash A$ in K . It suffices to show that the axioms and rules of K are derivable in NK . The axioms and rules of PC are in NPC and so are in NK ; so, Case (1) is proven.

Case (2): Let A be (1'). In NK we have the proof:

| | |
|---------------------------|---|
| 1 (1) $\Box(p \supset q)$ | Hyp |
| 2 (2) $\Box p$ | Hyp |
| 3 (3) $p \supset q$ | Hyp |
| 4 (4) p | Hyp |
| 3, 4 (5) q | 3, 4 MP |
| 1, 2 (6) $\Box q$ | 1, 2 ($=\Box\Delta = \Gamma$), 3, 4 ($=\Delta$), $\Box I_1$ |

$$\begin{array}{ll} 1 (7) \Box p \supset \Box q & 2, 6 \text{ CP} \\ (8) \Box(p \supset q) \supset (\Box p \supset \Box q) & 1, 7 \text{ CP} \end{array}$$

Case (3): Let A be derived by rule (a). Since rule (a) is just the special case of $\Box I_1$ where Δ is empty, $\vdash A$ in NK. So, K and NK are deductively equivalent.⁴ We may now easily obtain the natural deduction systems corresponding to T, S4, and S5, which are extensions of K, as extensions of NK. Corresponding to Lemmon's formulations of these systems we have:

$$\begin{array}{ll} \text{T: PC; (a); (1'); (2).} & \text{NT: NPC; } \Box E_1; \Box I_1. \\ \text{S4: PC; (a); (1'); (2); (4).} & \text{NS4: NPC; } \Box E_1; \Box I_4. \\ \text{S5:⁵ PC; (a); (1'); (2); (5).} & \text{NS5: NPC; } \Box E_1; \Box I_6. \end{array}$$

Since NT (T) is just NK (K) with the added rule (axiom) $\Box E_1$ (axiom(2)), the deductive equivalence of NT and T is proven by the previous proof except that we now add cases for $\Box E_1$ and (2). From (2) of T we easily derive $\Box E_1$, since, given $\Box A$ as a premise, A always follows depending upon the same hypotheses by appropriate substitution on axiom (2) and MP. Conversely, given $\Box E_1$ of NT, (2) is proven as follows:

$$\begin{array}{ll} 1 (1) \Box p & \text{Hyp} \\ 1 (2) p & 1, \Box E_1 \\ (3) \Box p \supset p & 1, 2 \text{ CP} \end{array}$$

Thus, NT and T are deductively equivalent. Since $\Box E_1$ and (2) are equivalent in the presence of NPC or PC, proofs of this equivalence will be omitted in what follows.

To establish the deductive equivalence of NS4 and S4 it suffices to show that $\Box I_4$ is deductively equivalent to (a), (1'), and (4) in the presence of NPC and $\Box E_1$ (alternatively, in the presence of PC and (2)). Given (a), (1'), and (4), $\Box I_4$ is derived as follows: if $\vdash B$ in S4, then $\vdash \Box B$ in S4 by rule (a). So suppose that Δ is not empty, i.e., $A_1, \dots, A_n \vdash B$ in S4 where each A_i ($1 \leq i \leq n$) is $\Box C_i$ for some wff C_i . By the deduction theorem for S4 and PC we have $\vdash (A_1 \& \dots \& A_n) \supset B$ in S4 which, by supposition, is $\vdash (\Box C_1 \& \dots \& \Box C_n) \supset B$ in S4. By the theorem of S4 that $(\Box C_1 \& \dots \& \Box C_n) \equiv \Box(C_1 \& \dots \& C_n)$ and substitution of material equivalents we have $\vdash \Box(C_1 \& \dots \& C_n) \supset B$ in S4. By (a) and (1') this yields $\vdash \Box \Box(C_1 \& \dots \& C_n) \supset \Box B$ in S4, from which, by PC and appropriate substitution on axiom (4), we have $\vdash \Box(C_1 \& \dots \& C_n) \supset \Box B$ in S4. By the previously used equivalence we have $\vdash (\Box C_1 \& \dots \& \Box C_n) \supset \Box B$ in S4 which is $\vdash (A_1 \& \dots$

4. Case 2 of this proof also illustrates the formal proof procedure in NK. Aiming to prove (1') one tries to prove $\Box q$ from $\Box(p \supset q)$ and $\Box p$ as hypotheses. The strategy for employing $\Box I_1$ is to add as hypotheses $p \supset q$ and p and then prove q . In step (6) of $\Box I_1$, $p \supset q$ and p are discharged as hypotheses, and the conclusion is put to depend upon hypotheses of $\Box(p \supset q)$ and $\Box p$, which here happen to be themselves. The numbers at the extreme left provide a record of the hypotheses upon which any given line in the proof depends.

5. This formulation of S5 is taken from Prior, [12], who attributes it to Gödel.

& A_n) $\supset \Box B$ in S4. Hence, given as premises in a proof A_1, \dots, A_n depending upon Γ , the last theorem together with PC yields a proof of $\Gamma \vdash \Box B$ in S4. Conversely, given $\Box I_4$ (1') is proven in NS4 by the same proof given in NK since $\Box I_1$ is easily derived from $\Box I_4$ in the presence of $\Box E_1$. For (4) we have the proof in NS4:

| | |
|----------------------------------|---------------|
| 1 (1) $\Box p$ | Hyp |
| 1 (2) $\Box \Box p$ | 1, $\Box I_4$ |
| (3) $\Box p \supset \Box \Box p$ | 1, 2 CP. |

Note that ' $\Box p$ ' here is the appropriate B as well as the one member sequence Δ ; so the step at (2) is a correct use of $\Box I_4$. For rule (a), it suffices to note that (a) is just $\Box I_4$ for the case where B depends upon no hypotheses. So NS4 and S4 are equivalent.

The deductive equivalence of NS5 and S5 is proven similarly by showing that $\Box I_6$ is deductively equivalent to (a), (1'), and (5) in the presence of NPC and $\Box E_1$ (alternatively, in the presence of (2) and PC). $\Box I_6$ is derived in S5 as follows: if Δ is empty, then $\Box I_6$ is just (a). Suppose that Δ is not empty, i.e., that $A_1, \dots, A_n \vdash B$ in S5 where each A_i ($1 \leq i \leq n$) is either $\Box C_i$ or $\Diamond C_i$ for some wff C_i . By the deduction theorem for S5: $\vdash A_1 \supset (\dots \supset (A_n \supset B) \dots)$ in S5. By (a), (1'), and PC we have:

$$(i) \vdash \Box A_1 \supset (\dots \supset (\Box A_n \supset \Box B) \dots) \text{ in S5.}$$

By appropriate substitution on (5) and the definition of \Diamond , $\vdash \Diamond C_i \supset \Box \Diamond C_i$ in S5. Further, (4) is a theorem of S5; hence, $\vdash \Box C_i \supset \Box \Box C_i$ in S5. So, whether A_i is $\Box C_i$ or $\Diamond C_i$ we have

$$(ii) \vdash A_i \supset \Box A_i \text{ in S5.}$$

By PC, (i) and (ii) yield $\vdash A_1 \supset (\dots \supset (A_n \supset \Box B) \dots)$ in S5. Thus given as premises A_1, \dots, A_n the last theorem permits the proof in S5 of $\Box B$ depending upon the same hypotheses as the premises. Conversely, given $\Box I_6$, $\Box I_4$ is easily derived. Hence, the proof of (1') in NS5 is the same as that given in NK since $\Box I_1$ is derivable from $\Box I_4$ in the presence of $\Box E_1$. Rule (a) is the special case of $\Box I_6$ where B is a theorem. Finally (5) is proven in NS5 as follows:

| | |
|---|---------------|
| 1 (1) $\Diamond \neg p$ | Hyp |
| 1 (2) $\neg \Box \neg \neg p$ | 2, Df |
| (3) $\neg \Box \neg \neg p \supset \neg \Box p$ | Theorem of NK |
| 1 (4) $\neg \Box p$ | 2, 3 MP |
| 1 (5) $\Box \neg \Box p$ | 4 $\Box I_6$ |
| (6) $\Diamond \neg p \supset \Box \neg \Box p$ | 1, 5 CP |
| (7) $\neg \Box p \supset \Diamond \neg p$ | Theorem of NK |
| (8) $\neg \Box p \supset \Box \neg \Box p$ | 6, 7 by NPC |

Lines (3) and (7) are easily derived in NK which is contained in NS5, and their introduction as theorems in the proof can be taken as an abbreviated way of inserting complete proofs of them. Similarly, the step at line (8) can be taken as an abbreviated way of inserting a proof of (8) from lines (6) and (7).

The expected containments hold for NK-NS5. Further, the progression from NT to NS5 can be viewed as progressive liberalizations of the rule for introducing '□' (in the presence of □E₁) into proofs from hypotheses. Given that B is provable from hypotheses A_1, \dots, A_n , □I₁ permitted one to derive □ B only by strengthening his hypotheses to □ $A_1, \dots, \square A_n$. □I₄ liberalizes this to permit deriving □ B anyway, provided that each hypothesis was of the form of □ C to begin with. □I₆ further liberalizes this to the condition that each hypothesis has been of the form of either □ C or ◇ C .

Just as we obtain extensions of NK or K by either liberalizing rules for '□' or by adding axioms, we obtain contractions of NK or K by either dropping axioms or restricting rules. By adopting more restricted rules for introducing '□' we have weaker NS- or S-systems. Corresponding to Lemmon's formalizations we have

- S3: **PC**; (a'); (1); (2). NS3: NPC; □E₁; □I₈, where Λ is the set of all substitution instances of tautologies, of (1), and of (2); □S₁.
- S2: **PC**; (a'); (b); (1'); (2). NS2: NPC; □E₁; □I₈, where Λ is the set of all substitution instances of tautologies, of (1'), and of (2); □S₂.
- S1: **PC**; (a'); (b'); (3); (2). NS1: NPC; □E₁; □I₈, where Λ is the set of all substitution instances of tautologies, of (3), and of (2); □S₃.

For the deductive equivalence of NS1 and S1 it suffices to show that □I₈ for NS1 and □S₃ are deductively equivalent to (a') for S1, (b'), and (3) in the presence of NPC and □E₁ (or **PC** and (2)). □I₈ for NS1 is proven for S1 as follows: if Δ is empty (in which case Γ is also), then □I₈ is just (a') of S1. So let Δ be A_1, \dots, A_n and suppose that $A_1, \dots, A_n \vdash B$ in NPC. By the equivalence of NPC and **PC** and the deduction theorem, we have:

$$(i) \vdash A_1 \supset (\dots \supset (A_n \supset B) \dots) \text{ in } \mathbf{PC}.$$

Since (i) is an instance of a theorem of **PC**, it is an instance of a tautology. So by (a') and PC4 of S1 we get $\vdash \square(A_1 \supset (\dots \supset (A_n \supset B) \dots))$ in S1. (1') is derivable in S1; this together with **PC** yields

$$(ii) \vdash \square A_1 \supset (\dots \supset (\square A_n \supset \square B) \dots) \text{ in } \mathbf{S1}.$$

Thus, given as premises □ $A_1, \dots, \square A_n$ depending upon Γ , (ii) and **PC** will yield a proof of $\Gamma \vdash \square B$. □S₃ is derived by (b') and the definition of \equiv by substitution on the theorem $\square(\square p \supset \square p)$ of S1. Conversely, in NS1 we prove (3) as follows:

- | | |
|--------------------------|----------|
| 1 (1) □($p \supset q$) | Hyp |
| 2 (2) □($q \supset r$) | Hyp |
| 3 (3) $p \supset q$ | Hyp |
| 4 (4) $q \supset r$ | Hyp |
| 3, 4 (5) $p \supset r$ | 3, 4 NPC |

| | |
|---|---|
| 1, 2 (6) $\Box(p \supset r)$ | 5 $\Box I_8$, (1, 2 = $\Gamma = \Box \Delta$; 3, 4 = Δ) |
| 1 (7) $\Box(q \supset r) \supset \Box(p \supset r)$ | 2, 6 CP |
| (8) $\Box(p \supset q) \supset (\Box(q \supset r) \supset \Box(p \supset r))$ | 1, 7 CP |
| (9) $(\Box(p \supset q) \ \& \ \Box(q \supset r)) \supset \Box(p \supset r)$ | 8 NPC |

Again, in a step of $\Box I_8$ there are hypotheses discharged, where Δ is not empty. The restriction that only NPC rules be used in the subsidiary deduction is easily checked. Rule (a') is $\Box I_8$ where Δ is empty. Finally, rule (b') is obtained from (i) $(p \equiv q) \supset (-p \equiv -q)$; (ii) $(p \equiv q) \supset ((r \supset p) \equiv (r \supset q))$; (iii) $(p \equiv q) \supset ((p \supset r) \equiv (q \supset r))$; and (iv) the rule $\Box S_3$, which is a counterpart of the rule (Gb') of [16].

Rule $\Box S_2$ is just rule (b). Thus, it suffices for showing the deductive equivalence of NS2 and S2 to show that $\Box I_8$ for NS2 is deductively equivalent to (a') for S2 and (1') in the presence of NPC, $\Box E_1$, and $\Box S_2$ (or PC, (2), and (b)). Where Δ is not empty the proof that $\Box I_8$ for NS2 holds for S2 is the same as that given for S1. Of course, where Δ is empty, $\Box I_8$ is just (a') for S2. Conversely, (1') is proven in NS2 by the same proof as that given in NK except that step (6) is now justified by $\Box I_8$ rather than by $\Box I_1$. Since the subsidiary deduction from steps (3) through (5) of that proof is by NPC alone the restrictions of $\Box I_8$ are met. Finally, (a') for S2 is the special case of $\Box I_8$ for NS2 where Δ is empty. Hence, NS2 and S2 are deductively equivalent.

It is obvious that axiom (1) and rule $\Box S_1$ are deductively equivalent in the presence of NPC or PC with the deduction theorem, just as (2) and $\Box E_1$ are. Also, in S3, $\Box I_8$ for NS3 holds where Δ is empty since it is then just (a') for S3. Where Δ is not empty, the proof that $\Box I_8$ holds for S3 is the same as that for S1, since (1') is derivable in S3. Conversely, in NS3 (a') for S3 holds as a special case of $\Box I_8$ where Δ is empty; (1) and (2) hold by virtue of $\Box E_1$ and $\Box I_8$ for NS3. Thus we obtain the deductive equivalence of NS3 and S3.

The relationships of S1-S3 to each other and to S4 and T are easily observed here. The progression from NS1 to NS3 is one of liberalizing restrictions on introducing ' \Box ' into a proof. NS2 contains NS1, and NS3 contains NS2 in virtue of the derivability of $\Box S_2$ from $\Box S_1$. Further, $\Box I_8$ for NS3 and $\Box S_1$ are derivable from $\Box I_4$; so, NS4 contains NS3. However, $\Box I_8$ for NS3 is not derivable from $\Box I_1$ since for NS3 Λ of $\Box I_8$ contains (1) which is not provable from $\Box I_1$. So NT does not contain NS3. Further NS3 does not contain NT since $\Box I_1$ is not derivable from $\Box I_8$ for NS3 given the restrictions for the case where Δ is empty. But $\Box I_8$ for NS2 is easily derivable from $\Box I_1$ and $\Box E_1$, since both (1') and (2) are provable by these rules. So NT contains NS2. Thus the relations of these systems noticed by Lemmon, p. 180 of [6], are obvious when the systems are given natural deduction formalizations also. Moreover, if we further restrict the introduction of ' \Box ' by replacing $\Box I_8$ in NS1 by $\Box I_9$, we then get a system deductively equivalent to S0.5. Thus we have:

S0.5: PC; (a''); (1'); (2). NS0.5: NPC; $\Box I_9$; $\Box E_1$.

In S0.5, NPC and $\Box E_1$ hold by PC, the deduction theorem, and (2) as usual. $\Box I_9$ is obtained as follows: if Δ is empty then we have just a special case of rule (a'') since all theorems of NPC are instances of tautologies. If Δ is not empty, then proof is the same as that given for $\Box I_8$ of NS1 for S1, citing (a'') in place of (a'). Conversely, in NS0.5 we have (2) by $\Box E_1$ as usual. (a'') is $\Box I_9$ where Δ is empty, and (1') follows by the same proof as that in NK, noting that at line (6) rule $\Box I_9$ would be cited correctly since all of the restrictions of that rule are met by the previous proof.

In the same manner we can construct corresponding natural deduction systems for the families of E- and D-systems. The deductive equivalence of $\Box E_1$ with (2) in the presence of NPC or PC has already been noted. Also, in the presence of PC and (2) either (1) or (1') together with (Eb) permits us to derive $\Box I_2$ as follows. Suppose that $A_1, \dots, A_n \vdash B$. By PC we have $\vdash A_1 \supset (A_2 \supset (\dots \supset (A_n \supset B) \dots))$. By (Eb) we have $\vdash \Box A_1 \supset \Box (A_2 \supset (\dots \supset (A_n \supset B) \dots))$, and so repeated use of (1') and PC yields $\vdash \Box A_1 \supset (\Box A_2 \supset (\dots \supset (\Box A_n \supset \Box B) \dots))$. Thus given a proof of $\Box A_1, \dots, \Box A_n$ from Γ , this theorem enables us to prove $\Gamma \vdash \Box B$. Since (1') is derivable from (1), given (2), the above proof holds for systems having axiom (1) in place of (1'), provided that they have (2). Conversely, $\Box I_2$ yields (1') by the same proof given in NK and yields (1) in the presence of the additional $\Box S_1$. (Eb) results as follows: suppose that $\vdash A \supset B$. Then we use the following proof scheme for any system containing NPC and $\Box I_2$:

| | |
|-----------------------------|------------------------|
| 1 (1) $\Box A$ | Hyp |
| 2 (2) A | Hyp |
| (3) $A \supset B$ | Theorem by supposition |
| 2 (4) B | 2, 3 MP |
| 1 (5) $\Box B$ | 1, 2, 4 $\Box I_2$ |
| (6) $\Box A \supset \Box B$ | 1, 5 CP |

The restrictions on $\Box I_2$ are met in this proof scheme since line (3) is not a hypothesis, though it is a premise. Thus the fact that no line of the proof scheme contains $\Box(A \supset B)$ is not a problem because $A \supset B$ is not in Δ anyway.

The above considerations yield natural deduction systems NE2 and NE3 deductively equivalent to E2 and E3 respectively:

| | |
|--------------------------|--|
| E2: PC; (Eb); (1'); (2). | NE2: NPC; $\Box I_2$; $\Box E_1$. |
| E3: PC; (Eb); (1); (2). | NE3: NPC; $\Box I_2$; $\Box E_1$; $\Box S_1$. |

If we restrict the wffs A, A_1, \dots, A_n, B of the previous paragraph to wffs of PC (NPC), i.e., wffs not containing modal operators, we have a proof of the deductive equivalence of (Eb') and (1') with $\Box I_3$. Thus we also have shown the deductive equivalence of:

| | |
|---------------------------|-------------------------------------|
| E1: PC; (Eb'); (1'); (2). | NE1: NPC; $\Box I_3$; $\Box E_1$. |
|---------------------------|-------------------------------------|

Corresponding to the remaining E-systems we have:

| | |
|-------------------------------|-------------------------------------|
| E4: PC; (Eb); (1'); (2); (4). | NE4: NPC; $\Box I_5$; $\Box E_1$. |
| E5: PC; (Eb); (1'); (2); (5). | NE5: NPC; $\Box I_7$; $\Box E_1$. |

By a proof similar to that deriving $\Box I_4$ ($\Box I_6$) from (a) and (4) ((a) and (5)) we can derive $\Box I_5$ ($\Box I_7$) from (Eb) and (4) ((Eb) and (5)) in the presence of (1'). Conversely, (4) holds in NE4 by the same proof given in NS4, and (5) is proved in NE5 by the same proof given in NS5, since both (i) $\neg\Box - \neg p \supset \neg\Box p$ and (ii) $\neg\Box p \supset \Diamond\neg p$ are easily derived as theorems of NE5. (1') is provable in NE4 and NE5 as follows, in the latter system citing $\Box I_7$ at line (6):

| | |
|---|--------------|
| 1 (1) $\Box(p \supset q)$ | Hyp |
| 2 (2) $\Box p$ | Hyp |
| 1 (3) $p \supset q$ | 1 $\Box E_1$ |
| 2 (4) p | 2 $\Box E_1$ |
| 1, 2 (5) q | 3, 4 MP |
| 1, 2 (6) $\Box q$ | 5 $\Box I_5$ |
| 1 (7) $\Box p \supset \Box q$ | 2, 6 CP |
| (8) $\Box(p \supset q) \supset (\Box p \supset \Box q)$ | 1,7 CP |

This proof is also available in both NS4 and NS5.

Systems D1 and D2 differ from E1 and E2 only in having the weaker axiom (2') in place of (2). In the presence of **PC** or NPC rule $\Box E_2$ and (2') are deductively equivalent just as $\Box E_1$ and (2) are. Also, the deductive equivalence of $\Box I_3$ with (Eb') and (1') and of $\Box I_2$ with (Eb) and (1') does not require that (2) or $\Box E_1$ be present, though the deductive equivalence of (Eb) and (1) with $\Box I_2$ did require the presence of (2) or $\Box E_1$. Thus, there are deductively equivalent natural deduction systems for D1 and D2, viz.:

| | |
|------------------------------------|-------------------------------------|
| D1: PC ; (Eb'); (1'); (2'). | ND1: NPC; $\Box I_3$; $\Box E_2$. |
| D2: PC ; (Eb); (1'); (2'). | ND2: NPC; $\Box I_2$; $\Box E_2$. |

The remaining D-systems have both the rule (D) and (2') in place of (2) of the corresponding E-systems. It is easily shown that (D) and (2') are deductively equivalent to $\Box E_3$ in the presence of **PC**. Further, for D3, (D) and (1) permit derivation of (1'). So the previously shown deductive equivalence of (Eb) and (1) with $\Box I_2$ and $\Box S_1$ continues to hold, now in the presence of (D) and (2') or $\Box E_3$. This yields the deductive equivalence of

| | |
|---------------------------------------|-------------------------------------|
| D3: PC ; (Eb); (D); (1); (2'). | ND3: NPC; $\Box I_2$; $\Box E_3$. |
|---------------------------------------|-------------------------------------|

Because of the lack of (2) in D4 and D5, the corresponding systems ND4 and ND5 require the hybrid rules $\Box I_{10}$ and $\Box I_{11}$, which allow proof of (1') and (Eb), in the absence of $\Box E_1$, as well as (4) and (5) respectively. We have:

| | |
|---|--|
| D4: PC ; (Eb); (D); (1'); (2'); (4). | ND4: NPC; $\Box I_{10}$; $\Box E_3$. |
| D5: PC ; (Eb); (D); (1'); (2'); (5). | ND5: NPC; $\Box I_{11}$; $\Box E_3$. |

For the deductive equivalence of D4 and ND4, it suffices to show that (Eb), (1') and (4) yield $\Box I_{10}$ and that $\Box I_{10}$ yields (Eb), (1') and (4) in ND4. $\Box I_{10}$ is derivable for D4 as follows: let $A_1, \dots, A_n \vdash B$ in D4 be given. By the deduction theorem for D4: $\vdash A_1 \supset (\dots \supset (A_n \supset B) \dots)$ in D4. By (Eb), (1'), and **PC**:

$$(a) \vdash \Box A_1 \supset (\dots \supset (\Box A_n \supset \Box B) \dots) \text{ in D4.}$$

Hence, (i) given a proof of $\Box A_1 \& \dots \& \Box A_n$ from Γ (a) yields a proof of $\Gamma \vdash \Box B$ in D4; (ii) if each A_i ($1 \leq i \leq n$) is $\Box C_i$ for some wff C_i then, by the theorem of D4: $\Box C_i \supset \Box \Box C_i$, we have $\vdash A_i \supset \Box A_i$ in D4. So, by PC: $\vdash A_1 \supset (\dots \supset (A_n \supset \Box B) \dots)$ in D4. Thus, $A_1, \dots, A_n \vdash \Box B$ in D4. Conversely, in ND4 (1') is derivable by either the proof in NK or that in NE4, citing $\Box I_{10}$ at line (6) in either case. (4) is easily derived by rules Hyp, $\Box I_{10}$, and CP. Finally, (Eb) follows by the previous proof of (Eb) for NE2, using $\Box I_{10}$ rather than $\Box I_2$. The deductive equivalence of (Eb), (1'), and (5) with $\Box I_{11}$ is shown similarly, and so the proof is omitted.

The containment relations pointed out by Lemmon between S- E- and D-systems are easily shown on the new rules here. For any S-system, the corresponding E-system is obtained by restricting the conditions for introducing and distributing ' \Box ' in proofs. The corresponding D-system is obtained by also restricting the conditions for eliminating ' \Box ' from proofs.

3. There is always a greater risk of error in drawing philosophical morals about a concept from a set of syntactic rules for a formal system than there is in drawing such morals from a semantic model of that system. Nevertheless, the following are drawn. A formal system S is, under interpretation, a system of reasoning in which one employs the concepts formalized. Given a formal system S , it is plausible to interpret $A_1, \dots, A_n \vdash B$ in S as " B is warranted by S as an assertion on the basis of A_1, \dots, A_n ." If one is warranted by S to assert B given certain hypotheses, then a \Box introduction rule states the further conditions, if any, required of one's hypotheses under which S would also warrant the assertion of $\Box B$. For example, interpreting ' \Box ' as 'Jones knows that', $\Box I_2$ warrants that Jones knows that B provided that he knows each of A_1, \dots, A_n and that $A_1, \dots, A_n \vdash B$ in the system of reasoning considered. Both E2 and E3 (or NE2 and NE3) are plausible formalizations for 'know' restricted to individuals. However, under this interpretation $\Box I_5$ would allow us to infer that Jones knows anything which follows from propositions to the effect that he knows certain things to be true. This perhaps makes E4 or NE4 unsuitable as a formalization of a given person's knowledge.

On the other hand, the fact that a proposition B follows from propositions stating that certain things have been verified constitutes a sort of verification of B . So NE4, with $\Box I_5$, might provide a plausible formalization of the concept 'it is verified that'. Further, $\Box I_5$ provides that nothing is verified simply by being a theorem of logic. Thus the concept formalized here would be applicable only to propositions contingent relative to NE4 or E4. However, the choice of an interpretation for a system of modal logic is difficult to make without considering set theoretic models for the system. These cannot be taken up here.

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Bradley University
Peoria, Illinois