NATURAL EQUILIBRIUM STATES FOR MULTIMODAL MAPS

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Abstract. This paper is devoted to the study of the thermodynamic formalism for a class of real multimodal maps. This class contains, but it is larger than, Collet-Eckmann. For a map in this class, we prove existence and uniqueness of equilibrium states for the geometric potentials $-t \log |Df|$, for the largest possible interval of parameters t. We also study the regularity and convexity properties of the pressure function, completely characterising the first order phase transitions. Results concerning the existence of absolutely continuous invariant measures with respect to the Lebesgue measure are also obtained.

1. Introduction

The class of dynamical systems whose ergodic theory is best understood is the class of *hyperbolic dynamical systems*, or, more generally, systems where the interesting dynamics behaves in a uniformly hyperbolic way: Axiom A maps. This is due to several reasons, one of them is the fact that these systems often have a compact symbolic model whose dynamics is well known [Bo, Ru2]. For real one-dimensional maps, Axiom A maps are defined to be the class of maps where all points are either uniformly expanded or map into an attracting basin. This class is large even within families of maps with critical points such as the quadratic family, in which case it is a dense set, see [Ly2, GS]. Note that these maps do have a compact symbolic model (see [KH, Chapter 16]). In the example of the quadratic family, maps which are not Axiom A are nowhere dense, but nevertheless have positive Lebesgue measure, see [J, BC]. Due to the rich dynamics of these systems, the expansion properties of such systems, can be very delicate.

In recent years a great deal of attention has been paid to non-Axiom A systems which are expanding on most of the phase space, but not in all of it. The simplest example of these type of maps, namely *non-uniformly hyperbolic* dynamical systems, are interval maps with a parabolic fixed point (e.g. the Manneville-Pomeau map [MP]). The ergodic theory for these maps is fairly well understood [MP, PreSl, S3] and qualitatively different from the one observed in the hyperbolic case.

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We will study the ergodic theory of class of maps for which the lack of hyperbolicity can be even stronger: interval maps with critical points. The techniques we develop are different from the ones used to study hyperbolic systems and systems with a parabolic fixed point.

In this paper we will be devoted to study a particular branch of ergodic theory, namely *thermodynamic formalism*. This is a set of ideas and techniques which derive from statistical mechanics [Dobr, Si, Bo, K2, Ru2, Wa]. It can be thought of as the study of certain procedures for the choice of invariant measures. Let us stress that the dynamical systems we will consider have many invariant measures, hence the problem is to choose relevant ones. The main object in the theory is the topological pressure:

Definition 1.1. *Let $f : X \to X$ be a Borel function of a compact metric space X and denote by $\mathcal{M}_f$ the set of f-invariant Borel probability measures. Let $\varphi : X \to [-\infty, \infty]$ be a Borel potential. Assuming that $\mathcal{M}_f \neq \emptyset$, the topological pressure of φ with respect to f is defined, via the Variational Principle, by*

$$P_f(\varphi) = P(\varphi) = \sup\left\{ h(\mu) + \int \varphi \, d\mu : \mu \in \mathcal{M}_f \text{ and } -\int \varphi \, d\mu < \infty \right\},$$

where $h(\mu)$ denotes the measure theoretic entropy of f with respect to μ. We refer to the quantity in the curly brackets as the free energy *of μ with respect to (X, f, φ). Note that this is sometimes thought of as being minus the free energy; see for example [K2] for a discussion of this terminology.*

Note that we do not specify the regularity properties we require on the potential φ. If it is a continuous function, then the above definition coincides with classical notions of topological pressure (see [Wa, Chapter 9]). In this paper we will be interested in the geometric potential $x \mapsto -t \log |Df(x)|$ for some parameter $t \in \mathbb{R}$. This function is continuous in the uniformly hyperbolic case, but is not upper/lower semicontinuous for t positive/negative for the class of dynamical systems that we will consider.

A measure $\mu_\varphi \in \mathcal{M}_f$ is called an *equilibrium state* for φ if it satisfies:

$$h(\mu_\varphi) + \int \varphi \, d\mu_\varphi = P(\varphi).$$

In such a way, the topological pressure provides a natural way to pick up measures. Questions about existence, uniqueness and ergodic properties of equilibrium states are at the core of the theory. For instance, if the dynamical system f is transitive, uniformly hyperbolic and the potential φ is Hölder continuous then there exist a unique equilibrium state μ_φ for φ and it has strong ergodic properties [Bo, Ru2]. Moreover, the hyperbolicity of the system is reflected on the regularity of the pressure function $t \mapsto P(t\varphi)$. Indeed, the function is real analytic. When the system is no longer hyperbolic, as in the case of the Manneville-Pomeau map, then uniqueness of equilibrium states may break down [PreSl] and the pressure function might exhibit points where it is not analytic, the so called *phase transitions* [S3].

As mentioned above, we will consider maps for which the lack of hyperbolicity is strong: not only do the maps have critical points, but the orbit of these points can

be dense. We consider a family of real multimodal maps, that is smooth interval maps with a finite number of critical points. More precisely, let $\mathcal{F}$ be the collection of C^2 multimodal interval maps $f : I \to I$, where $I = [0, 1]$, satisfying:

a) the critical set $\mathcal{C}r = \mathcal{C}r(f)$ consists of finitely many critical points c with critical order $1 < \ell_c < \infty$, i.e., there exists a neighbourhood U_c of c and a C^2 diffeomorphism $g_c : U_c \to g_c(U_c)$ with $g_c(c) = 0$ $f(x) = f(c) \pm |g_c(x)|^{\ell_c}$;

b) f has negative Schwarzian derivative, i.e., $1/\sqrt{|Df|}$ is convex;

c) f is topologically transitive on I;

d) $f^n(\mathcal{C}r) \cap f^m(\mathcal{C}r) = \emptyset$ for $m \neq n$.

Conditions c) and d) are for ease of exposition, but not crucial. In particular, Condition c) excludes that f has any attracting cycles, or homtervals (a homterval is an interval U such that $U, f(U), f^2(U), \ldots$ are disjoint and the omega limit set is not a periodic orbit). Condition d) in particular excludes that one critical point is mapped onto another. If that happened, it would be possible to consider these critical points as a 'block', but to simplify the exposition, we will not do that here. Condition d) also, in particular, excludes that critical points are preperiodic, a case which is easier to handle (for example by combining [KH, Chapter 16] and [Bo]) and does not require the theory we present here, see Section 10.3. Together c) and d) exclude the renormalisable case.

Remark 1.1. *General C^2 multimodal maps satisfying a) and b) have no homtervals and the non-wandering set Ω (the set of points $x \in I$ such that for arbitrarily small neighbourhoods U of x there exists $n(U) \geqslant 1$ such that $f^n(U) \cap U \neq \emptyset$) can be broken down into finitely many elements Ω_k, on each of which f is topologically transitive, see [MvS, Section III.4]. However, for the maps we consider, assumption c) means this fact follows automatically without the C^2 assumption. We note that in the case where there is more than one transitive element in Ω, for example the renormalisable case, the analysis presented in this paper can be applied to any one of the transitive elements consisting of a union of intervals permuted by f.*

Now let Ω_{int} denote the union of all elements of Ω which consist of intervals permuted by f. If, contrary to the assumptions on $\mathcal{F}$ above, Ω_{int} did not cover I then there would be a (hyperbolic) Cantor set consisting of points which are always outside Ω_{int}. Dobbs [D3] showed that for renormalisable maps these hyperbolic Cantor sets can give rise to phase transitions in the pressure function not accounted for by the behaviour of critical points themselves.

Remark 1.2. *The smoothness of our maps is important for two further reasons: to allow us to bound distortion on iterates, and to guarantee the existence of 'local unstable manifolds'. For the first, the tool we use is the Koebe Lemma, see [MvS, Section IV]. The negative Schwarzian condition we impose still allows us to use this for C^2 maps. For a detailed explanation of this issue see [C].*

Given a measure $\mu \in \mathcal{M}_f$, the existence of local unstable manifolds was used in [B1, BT1] to show the existence of some natural 'inducing schemes' (see Section 3). As shown by Ledrappier [L], and later generalised by Dobbs [D4] (see the appendix), we only need a $C^{1+\alpha}$ condition on f to guarantee the existence of local unstable manifolds.

Note that our class $\mathcal{F}$ includes transitive Collet-Eckmann maps, that is maps where $|Df^n(f(c))|$ grows exponentially fast. Therefore the set of quadratic maps in $\mathcal{F}$ has positive Lebesgue measure in the parameter space of quadratic maps (see [J, BC]). In the appendix we show that our theory can be extended to a slightly more general class of maps, similar to the above, but only piecewise continuous.

As mentioned above, we will be particularly interested in the thermodynamic formalism for the geometric potentials $x \mapsto -t \log |Df|$. The study of these potentials has various motivations, for example the relevant equilibrium states and the pressure function are related to the Lyapunov spectrum, see for example [T]. Moreover, important geometric features are captured by this potential. Indeed, in several settings, the equilibrium states for this family are associated to conformal measures on the interval. This allows the study of the fractal geometry of dynamically relevant subsets of the space. Moreover, by [L] any equilibrium state μ for the potential $x \mapsto -\log |Df|$ is an absolutely continuous invariant probability measure (acip) provided $\int \log |Df| \, d\mu > 0$.

For $\mu \in \mathcal{M}_f$, we define the *Lyapunov exponent of μ* as

$$\lambda(\mu) := \int \log |Df| \, d\mu.$$

We let

$$\lambda_M := \sup\{\lambda(\mu) : \mu \in \mathcal{M}_f\}, \ \lambda_m := \inf\{\lambda(\mu) : \mu \in \mathcal{M}_f\}.$$

Remark 1.3. *Our assumptions on $f \in \mathcal{F}$, particularly non-flatness of critical points and a lack of attracting periodic cycles, means that by [Pr], $\lambda_m \geqslant 0$.*

We let

$$p(t) := P(-t \log |Df|)$$

and define

$$t^- := \inf\{t : p(t) > -\lambda_M t\} \text{ and } t^+ := \sup\{t : p(t) > -\lambda_m t\}. \tag{1}$$

Note that if $t^- \in \mathbb{R}$ (resp $t^+ \in \mathbb{R}$) then p is linear for all $t \leqslant t^-$ (resp $t \geqslant t^+$). We will later prove that for maps in $\mathcal{F}$, $t^- = -\infty$. We prove in Proposition 8.1 that $t^+ > 0$. In some cases $t^+ = \infty$. As we will show later, for non-Collet Eckmann maps with quadratic critical point, $\lambda_m = 0$ and $t^+ = 1$. [MS] suggests that there should also be Collet-Eckmann maps with $t^+ \in (1, \infty)$. In Proposition 9.2 we prove that under certain assumptions $t^+ \geqslant 1$: we expect that to be true for any map $f \in \mathcal{F}$.

The following is our main theorem.

Theorem A. *For $f \in \mathcal{F}$ and $t \in (-\infty, t^+)$ there exists a unique equilibrium measure μ_t for the potential $-t \log |Df|$. Moreover, the measure μ_t has positive entropy.*

A classical way to show the existence of equilibrium states is to use upper semicontinuity of entropy and the potential φ (see [K2, Chapter 4]), and in particular the upper semicontinuity of $\mu \mapsto \int \varphi \, d\mu$. However, in our setting even though, as noted in [BK], for $f \in \mathcal{F}$ the entropy map is upper semicontinuous, the existence

of equilibrium measures in the above theorem is not guaranteed since the potential $-t \log |Df|$ is not upper semicontinuous for $t > 0$. So for example, by [BK, Proposition 2.8] for unimodal maps satisfying the Collet-Eckmann condition, $\mu \mapsto -\lambda(\mu)$ is not upper semicontinuous. Theorem A generalises [BK] which applies to unimodal Collet-Eckmann maps for a small range of t near 1; [PS] which applies to a subset of Collet-Eckmann maps, but for all t in a neighbourhood of $[0, 1]$; and [BT2, Theorem 1] which applies to a class of non-Collet Eckmann multimodal maps with t in a left-sided neighbourhood of 1.

In order to prove Theorem A we use the theory of inducing schemes developed in [B1, BT1, BT2, T]. Let us note that the thermodynamic formalism is understood for certain complex rational maps. For example, Przytycki and Rivera-Letelier [PrR] proved that if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map of degree at least two, is expanding away from the critical points and has 'arbitrarily small nice couples' then the pressure function p is real analytic in a certain interval. These conditions are met for a wide class of rational maps including topological Collet-Eckmann rational maps, any at most finitely renormalisable polynomial with no indifferent periodic orbits, as well as every real quadratic polynomial. Also see [DU], where they show the existence and uniqueness of equilibrium states for all rational maps with degree greater than or equal to two, for all Hölder potentials φ with $\sup \varphi < P(\varphi)$.

Related to the above are the regularity properties of the pressure function.

Definition 1.2. *Let* $\varphi : [0, 1] \to \mathbb{R}$ *be a Borel potential. The pressure function has a* first order phase transition *at* $t_0 \in \mathbb{R}$ *if p is not differentiable at $t = t_0$.*

The pressure function, being the supremum of convex functions, is convex (see [Roc, p.35]) and when finite is continuous (see [Roy, p. 113]). This implies that the left and right derivatives $D^- p(t)$ and $D^+ p(t)$ at each t exist. Moreover, the pressure, when finite, can have at most a countable number of points t_i where it is not differentiable (i.e, $Dp^-(t_i) \neq D^+ p(t_i)$), hence of first order phase transitions. The regularity of the pressure is related to several dynamical properties of the system. For example, it has deep connections to large deviations [E] and to different modes of recurrence [S3, S5]. In Section 8 we prove that the pressure function restricted to the interval $(-\infty, t^+)$ not only does not have first order phase transitions, but it is C^1.

Theorem B. *For $f \in \mathcal{F}$, the pressure function p is C^1, strictly convex and strictly decreasing in $t \in (-\infty, t^+)$.*

First order phase transitions are also related to the existence of absolutely continuous invariant probability measures. If $p(t) = 0$ for all $t \geqslant 1$ and there is an acip, then the pressure function is not differentiable at $t = 1$. This occurs for example if $f \in \mathcal{F}$ is unimodal and non-Collet Eckmann, but has an acip (see [NS]). The following proposition gives the converse result.

Proposition 1.1. *Let $f \in \mathcal{F}$ be such that $p(1) = 0$. If the pressure function has a first order phase transition at $t = 1$ then the map f has an acip.*

We summarise some of the other results we present here for the potential $x \mapsto -t \log |Df(x)|$ in the simpler case of unimodal maps with quadratic critical point in the following proposition.

Proposition 1.2. *If $f \in \mathcal{F}$ is unimodal, non-Collet Eckmann and $\ell_c = 2$ then p is C^1, strictly convex and decreasing throughout $(-\infty, 1)$ and $p(t) = 0$ for all $t \geqslant 1$. Moreover,*

(a) if f has no acip then p is C^1 throughout $\mathbb{R}$;
(b) if f has an acip then p has a first order phase transition at $t = 1$.

The paper is organised as follows. In Section 2 we give an introduction to the theory of thermodynamic formalism for countable Markov shifts, which was developed by Mauldin and Urbańki and by Sarig. In Section 3 we give some preliminary results on inducing schemes, which will allow us to code any of our systems by a countable Markov shift. In Section 4 we show that the inducing schemes in Section 3 have some of the properties which will allow us to produce equilibrium states for our systems. In Section 5 we prove the most technically complex part of our paper which gives us the existence of equilibrium states for our systems. Section 6 gives details of the uniqueness of these equilibrium states which then allows us to prove Theorem A in Section 7. In Section 8 we prove Theorem B and in Section 9 we prove Propositions 1.1 and 1.2. In Section 10 we discuss statistical properties of the measures constructed, the ergodic optimisation problem and the case in which the critical points are preperiodic. Finally in the appendix we show how the results of this paper extend to a class of Lorenz-like maps, of the kind studied by Rovella [Rov] and Keller and St Pierre [KStP].

Note that many of the results we quote in this paper are proved using the theory of Markov extensions introduced by Hofbauer. To prove our main theorems it is not necessary to explain this theory in any detail since it is sufficient to quote results from elsewhere. However, for a short description of this construction, see the appendix.

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2. Preliminaries: countable Markov shifts

In this section we present the theory of countable Markov shifts: an extension of the finite case, and the relevant model for many non-uniformly hyperbolic systems, including maps in $\mathcal{F}$.

Let $\sigma \colon \Sigma \to \Sigma$ be a one-sided Markov shift with a countable alphabet S. That is, there exists a matrix $(t_{ij})_{S \times S}$ of zeros and ones (with no row and no column made entirely of zeros) such that

$$\Sigma = \{x \in S^{\mathbb{N}_0} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N}_0\},$$

and the shift map is defined by $\sigma(x_0 x_1 \cdots) = (x_1 x_2 \cdots)$. We say that (Σ, σ) is a *countable Markov shift*. We equip Σ with the topology generated by the cylinder sets

$$C_{i_0 \cdots i_n} = \{x \in \Sigma : x_j = i_j \text{ for } 0 \leqslant j \leqslant n\}.$$

Given a function $\varphi \colon \Sigma \to \mathbb{R}$, for each $n \geqslant 1$ we set

$$V_n(\varphi) = \sup \{|\varphi(x) - \varphi(y)| : x, y \in \Sigma, \ x_i = y_i \text{ for } 0 \leqslant i \leqslant n - 1\}.$$

We say that φ has *summable variations* if $\sum_{n=2}^{\infty} V_n(\varphi) < \infty$. We will sometimes refer to $\sum_{n=2}^{\infty} V_n(\varphi)$ as the *distortion bound* for φ. Clearly, if φ has summable variations then it is continuous. We say that φ is *weakly Hölder continuous* if $V_n(\varphi)$ decays exponentially. If this is the case then it has summable variations. In what follows we assume (Σ, σ) to be topologically mixing (see [S1, Section 2] for a precise definition).

It is a subtle matter to define a notion of topological pressure for countable Markov shifts. Indeed, the classical definition for continuous maps on compact metric spaces is based on the notion of (n, ε)-separated sets (see [Wa, Chapter 9]). This notion depends upon the metric of the space. In the compact setting, since all metrics generating the same topology are uniformly equivalent, the value of the pressure does not depend upon the metric. However, in non-compact settings this is no longer the case. Based on work of Gurevich [Gu1, Gu2], Sarig [S1] introduced a notion of pressure for countable Markov shifts which does not depend upon the metric of the space and which satisfies a Variational Principle. Let (Σ, σ) be a topologically mixing countable Markov shift, fix a symbol i_0 in the alphabet S and let $\varphi \colon \Sigma \to \mathbb{R}$ be a potential of summable variations. We let

$$Z_n(\varphi, C_{i_0}) := \sum_{x : \sigma^n x = x} \exp\left(S_n \varphi(x)\right) \chi_{C_{i_0}}(x) \tag{2}$$

where $\chi_{C_{i_0}}$ is the characteristic function of the cylinder $C_{i_0} \subset \Sigma$, and

$$S_n \varphi(x) := \varphi(x) + \cdots + \varphi \circ \sigma^{n-1}(x).$$

Moreover, the so-called *Gurevich pressure* of φ is defined by

$$P^G(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, C_{i_0}).$$

Since σ is topologically mixing, one can show that $P^G(\varphi)$ does not depend on i_0. We define

$$\mathcal{M}_\sigma(\varphi) := \left\{\mu \in \mathcal{M}_\sigma : -\int \varphi \, d\mu < \infty\right\}.$$

If (Σ, σ) is the full-shift on a countable alphabet then the Gurevich pressure coincides with the notion of pressure introduced by Mauldin and Urbański [MU1]. Furthermore, the following property holds (see [S1, Theorem 3]):

Proposition 2.1 (Variational Principle)**.** *If $\varphi : \Sigma \to \mathbb{R}$ has summable variations and $P^G(\varphi) < \infty$ then*

$$P^G(\varphi) = \sup \left\{h_\mu(\sigma) + \int_\Sigma \varphi \, d\mu : \mu \in \mathcal{M}_\sigma(\varphi)\right\}.$$

Let us stress that the right hand side of the above inequality only depends on the Borel structure of the space and not on the metric. Therefore, a notion of pressure which is to satisfy the Variational Principle need not depend upon the metric of the space.

The Gurevich pressure also has the property that it can be approximated by its restriction to compact sets. More precisely [S1, Corollary 1]:

Proposition 2.2 (Approximation property)**.** *If $\varphi : \Sigma \to \mathbb{R}$ has summable variations then*

$$P^G(\varphi) = \sup\{P_{\sigma|K}(\varphi) : K \subset \Sigma : K \neq \emptyset \text{ compact and } \sigma\text{-invariant}\},$$

where $P_{\sigma|K}(\varphi)$ is the classical topological pressure on K.

We consider a special class of invariant measures. As in [MU2] (see also [S4]), we say that $\mu \in \mathcal{M}_\sigma$ is a *Gibbs measure* for the function $\varphi\colon \Sigma \to \mathbb{R}$ if for some constants P, $C > 0$ and every $n \in \mathbb{N}$ and $x \in C_{i_0 \cdots i_n}$ we have

$$\frac{1}{C} \leqslant \frac{\mu(C_{i_0 \cdots i_n})}{\exp\left(-nP + S_n \varphi(x)\right)} \leqslant C.$$

This definition is analogous to that in the finite Markov shift case considered by Bowen [Bo]. We refer to any such C as a *distortion constant* for the Gibbs measure. It was proved by Mauldin and Urbański [MU2] that if (Σ, σ) is a full-shift and the function φ is of summable variations with finite Gurevich pressure $P^G(\varphi)$ then it has an invariant Gibbs measure. Moreover $P = P^G(\varphi)$, and if $-\int \varphi \, d\mu < \infty$ then μ is an equilibrium state for φ. Furthermore, this is the unique equilibrium state for φ by [MU2, Theorem 3.5] (note that this was later generalised for any topologically mixing countable Markov shift in [BuS]).

3. Inducing schemes

In order to prove Theorem A we will use the machinery of inducing schemes. We will use the fact that inducing schemes for the system (I, f) can be coded by the full-shift on countably many symbols.

Given $f \in \mathcal{F}$, we say that $(X, \{X_i\}_i, F, \tau)$ is an *inducing scheme* for (I, f) if

- X is an interval and $\{X_i\}_i$ is a finite or countable collection of disjoint intervals such that F maps each X_i diffeomorphically onto X, with bounded distortion on all iterates (i.e. there exists $K > 0$ so that if there exist $i_0, \ldots, i_{n-1}$ and x, y such that $F^j(x), F^j(y) \in X_{i_j}$ for $j = 0, 1, \ldots, n-1$ then $1/K \leqslant DF^n(x)/DF^n(y) \leqslant K$);
- $\tau|_{X_i} = \tau_i$ for some $\tau_i \in \mathbb{N}$ and $F|_{X_i} = f^{\tau_i}$. If $x \notin \cup_i X_i$ then $\tau(x) = \infty$.

The function $\tau : \cup_i X_i \to \mathbb{N}$ is called the *inducing time*. It may happen that $\tau(x)$ is the first return time of x to X, but that is certainly not the general case. For ease of notation, we will write $(X, F, \tau) = (X, \{X_i\}_i, F, \tau)$ and moreover, frequently write $(X, F) = (X, F, \tau)$. We denote the set of points $x \in I$ for which there exists $k \in \mathbb{N}$ such that $\tau(F^n(f^k(x))) < \infty$ for all $n \in \mathbb{N}$ by $(X, F)^\infty$.

Given an inducing scheme (X, F, τ), we say that a probability measure μ_F is a *lift* of μ if for any μ-measurable subset $A \subset I$,

$$\mu(A) = \frac{1}{\int_X \tau \, d\mu_F} \sum_i \sum_{k=0}^{\tau_i - 1} \mu_F(X_i \cap f^{-k}(A)). \tag{3}$$

Conversely, given a measure μ_F for (X, F), we say that μ_F *projects* to μ if (3) holds. Note that if (3) holds then μ_F is F-invariant if and only if μ is f-invariant. We call a measure μ *compatible with* the inducing scheme (X, F, τ) if

- $\mu(X) > 0$ and $\mu(X \setminus (X, F)^\infty) = 0$; and
- there exists a measure μ_F which projects to μ by (3): in particular $\int_X \tau \, d\mu_F < \infty$ (equivalently $\mu_F \in \mathcal{M}_F(-\tau)$).

Remark 3.1. *Given $f \in \mathcal{F}$ and an ergodic measure $\mu \in \mathcal{M}_f$ with positive Lyapunov exponent, there exists an inducing scheme (X, F, τ) with a corresponding $F-$invariant measure μ_F, see for example [BT2, Theorem 3].*

Definition 3.1. *Let (X, F, τ) be an inducing scheme for the map f. Then for a potential $\varphi : I \to \mathbb{R}$, the induced potential Φ for (X, F, τ) is given by*

$$\Phi(x) = \Phi^F(x) := S_{\tau(x)}\varphi(x).$$

Note that in particular for the potential $\log |Df|$, the induced potential for a scheme (X, F) is $\log |DF|$. Moreover, the map $x \mapsto \log |DF(x)|$ has summable variations (see for example [BT2, Lemma 8]).

Note that if (X, F, τ) is some inducing scheme for the map $f \in \mathcal{F}$ and if $\partial X \notin (X, F)^\infty$, then the system $F : (X, F)^\infty \to (X, F)^\infty$ is topologically conjugated to the full-shift on a countable alphabet.

For an inducing scheme (X, F, τ) and a potential $\varphi : X \to [-\infty, \infty]$ with summable variations, we can define the Gurevich pressure as in Section 2, and denote it by

$$P_F^G(\varphi),$$

where we drop the subscript if the dynamics is clear.

In fact the domains for the inducing schemes used above come from the natural cylinder structure of the map $f \in \mathcal{F}$. More precisely, the domains X are n-cylinders coming from the so-called *branch partition*: the set $\mathcal{P}_1^f$ consisting of maximal intervals on which f is monotone. So if two domains $C_1^i, C_1^j \in \mathcal{P}_1^f$ intersect, they do so only at elements of $\mathcal{C}r$. The set of corresponding n-cylinders is denoted $\mathcal{P}_n^f := \vee_{k=1}^n f^{-k} \mathcal{P}_1$. We let $\mathcal{P}_0^f := \{I\}$. For an inducing scheme (X, F) we use the same notation for the corresponding n-cylinders $\mathcal{P}_n^F$. Note the transitivity assumption on our maps f implies that $\mathcal{P}_1^f$ is a generating partition for any Borel probability measure.

4. Zero pressure schemes

For $t \in \mathbb{R}$, we let

$$\psi_t := -t \log |Df| - p(t).$$

Similarly, for an inducing scheme (X, F) the induced potential is Ψ_t. As in [PS, BT1, BT2] in order to apply the theory developed by Mauldin and Urbański and later by Sarig, we need to find an inducing scheme (X, F, τ) so that $P^G(\Psi_t) = 0$. Then [MU2, Corollary 2.10] gives a Gibbs measure for (X, F, Ψ_t), which if it projects to a measure in $\mathcal{M}_f$ by (3), must be an equilibrium state by the Abramov formula. The main purpose of this section is to show that there are inducing schemes with $P^G(\Psi_t) = 0$.

We note that a major difficulty when working with inducing schemes is that, in general, no single inducing scheme is compatible with all measures of positive Lyapunov exponent. As a direct consequence of work by Bruin and Todd [BT2, Remark 6] we obtain in Lemma 4.3 that for each $\epsilon > 0$, there exists $\eta > 0$ and a finite number of inducing schemes for which any measure of entropy greater than η is compatible with one of them. This will allow us to prove that for each $t \in (t^-, t^+)$ there exists an inducing scheme for which $P(\Psi_t) = 0$ and such that the pressure, $p(t)$, can be approximated with f-invariant measures of positive entropy compatible with the inducing scheme.

Proposition 4.1. *For each $t \in (t^-, t^+)$, there exist an inducing scheme (X, F) and a sequence $(\mu_n)_n \subset \mathcal{M}_f$ all compatible with (X, F) and such that*

$$h(\mu_n) - t\lambda(\mu_n) \to p(t) \ \text{and} \ \inf_n h(\mu_n) > 0.$$

Moreover, $P^G(\Psi_t) = 0$.

We need some lemmas and a definition for the proof.

Lemma 4.1. *For each $t \in \mathbb{R}$ and any inducing scheme (X, F), we have $P^G(\Psi_t) \leqslant 0$.*

Proof. We let (X^N, F_N, τ_N) denote the subsystem of (X, F, τ) where $X^N = \cup_{n=1}^N X_n$ and F_N, τ_N are the restrictions of F, τ to X^N. Similarly, $P_{F_N}^G(\Psi_t)$ is defined in the obvious way. By Proposition 2.2, $P_F^G(\Psi_t) > 0$ implies that for large enough N, $P_{F_N}^G(\Psi_t) > 0$. Hence there is an equilibrium state μ_{F_N} for this system so that $\int \tau_N \, d\mu_{F_N} < \infty$ and

$$h(\mu_{F_N}) - t \int \log |DF| \, d\mu_{F_N} - p(t) \int \tau_N \, d\mu_{F_N} > 0.$$

Similarly to the use of the Abramov formula above, the corresponding projected measure μ_{f_N} as in (3) has

$$h(\mu_{f_N}) - t \int \log |Df| \, d\mu_{f_N} > p(t).$$

This contradiction to the Variational Principle proves the lemma. □

Remark 4.1. *By [BT2, Lemma 8], the potentials Ψ_t we consider for the inducing schemes (X, F) in Lemma 4.3 are weakly Hölder continuous.*

Definition 4.1. *Given a function $g : [a, b] \to \mathbb{R}$, for $x_0 \in \mathbb{R}$, as in [Roy, p115], we refer to $s : [a, b] \to \mathbb{R}$ as a supporting line for g at x_0 if $s(x) = g(x_0) + b(x - x_0)$ for some $b \in \mathbb{R}$, and $g(x) \geqslant s(x)$ for all $x \in \mathbb{R}$.*

Lemma 4.2. *For each $t \in (t^-, t^+)$, there exists $\eta > 0$ such that any measure μ with free energy with respect to ψ_t close enough to 0 has $h(\mu) > \eta$.*

Proof. Let us first consider the case in which $t_0 > 0$. Suppose that there exists a sequence of invariant measures $(\mu_n)_n$ such that $\lim_{n \to \infty} h(\mu_n) = 0$ and $p(t_0) = -t_0 a$ where $a := \lim_{n \to \infty} \lambda(\mu_n)$. We will show that $t_0 \geqslant t^+$.

Let $L(t) := -at$. Since all measures have non-negative Lyapunov exponent, we have $a \geqslant 0$. Since we also know that $t_0 > 0$, this implies that $p(t_0) = -at_0 < 0$.

Claim. $p(t) = -at$ *for all* $t \geqslant t_0$.

Proof of the claim. Suppose the opposite, i.e. $p(t) > -at$ for some $t \geqslant t_0$. Then since the pressure function $p(t)$ can be found via a limit of supporting lines $h(\mu) - t\lambda(\mu)$ for $\mu \in \mathcal{M}$, we must have some $t_1 > t_0$ and $\mu \in \mathcal{M}$ such that

$$h(\mu) - t_1 \lambda(\mu) > -at_1. \tag{4}$$

We will show that this leads to a contradiction. Let $\tilde{L}(t) := h(\mu) - t\lambda(\mu)$. We may assume $L \neq \tilde{L}$.

By definition, the pressure always satisfies

$$p(t) \geqslant L(t) \text{ and } p(t) \geqslant \tilde{L}(t). \tag{5}$$

Since $\tilde{L}$ is affine and L is linear, both with negative slope, and both lines distinct, either these lines cross at a unique $t^* \in (0, t_0)$ or there is no such t^*. In the first case, $\tilde{L}$ must start above L and then go below it after t^*: since $\tilde{L}(0) \geqslant L(0)$, for all $t \geqslant t^*$ we must have $\tilde{L}(t) < L(t)$. This means that $L(t_1) > \tilde{L}(t_1)$, contradicting (4). In the second case, $\tilde{L}$ must be above the pressure function at t_0: we must have $\tilde{L}(t_0) > L(t_0)$, so by (5), $L(t_0)$ cannot have been the pressure at t_0, a contradiction. $\square$

If there is a measure $\mu \in \mathcal{M}$ such that $\lambda(\mu) < a$ then for some, possibly very large, $t > 0$ we must have $p(t) \geqslant h(\mu) - t\lambda(\mu) > L(t)$ by the same arguments as in the claim. But this then contradicts the claim. Hence $a = \lambda_m$, the infimum of the Lyapunov exponents. Therefore, by definition of t^+ we have $t_0 \geqslant t^+$.

If $t_0 < 0$ an analogous argument proves that we must have $t_0 \leqslant t^-$. So in either case, $t_0 \notin (t^-, t^+)$, as required. $\square$

Lemma 4.3. *For each $\epsilon > 0$ there exists $\theta > 0$ and a finite number of inducing schemes $\{(X^n, F_n, \tau_n)\}_{n=1}^{N}$ such that any ergodic measure with $h(\mu) > \epsilon$ is compatible with one of these schemes (X^n, F_n, τ_n) and $\int \tau_n \, d\mu_{F_n} < \theta$.*

Proof. This follows from [BT2, Remark 6]. We give a brief sketch of the ideas there. That remark gives, for $\epsilon > 0$, a set $\{(X^n, F_n, \tau_n)\}_{n=1}^{N}$ such that for each $\mu \in \mathcal{M}_f$ with $h(\mu) > \epsilon$, μ must be compatible with some (X^n, F_n, τ_n). These schemes are constructed from sets $\hat{X}^n$ on the so-called Hofbauer extension (see the appendix for details). The map F is derived from a first return map $\hat{F}$ in this tower. Measures $\mu \in \mathcal{M}_f$ with $h(\mu) > 0$ can be lifted to the tower, and if they have $h(\mu) > \epsilon$ they

must give one of the sets $\hat{X}_n$ mass greater than some $\eta = \eta(\epsilon) > 0$. Since $\hat{F}$ is a first return map with return time $\hat{\tau}_n$, we use Kac's lemma to get

$$\int \tau_n \, d\mu = \int \hat{\tau}_n \, d\hat{\mu} = \hat{\mu}(\hat{X}_n)^{-1} < \eta^{-1},$$

as required.   □

As in [BT2, Remark 6], we denote this set of inducing schemes by $Cover(\epsilon)$.

Proof of Proposition 4.1. By Lemmas 4.3 and 4.2, we can take a sequence of ergodic measures μ_p such that

$$h(\mu_p) + \int \psi_t \, d\mu_p = \epsilon_p \text{ where } \epsilon_p \to 0 \text{ as } p \to \infty,$$

$h(\mu_p) > \eta$ (some $\eta > 0$), all μ_p are compatible with some inducing scheme $(X, F, \tau) \in Cover(\epsilon)$ and $\int \tau \, d\mu_p < \theta$ for all $p \in \mathbb{N}$. This implies that $P^G(\Psi_t) \geqslant 0$ since we have a sequence of measures $\mu_{F,p}$ such that

$$h(\mu_{F,p}) + \int \Psi_t \, d\mu_{F,p} = \left(\int \tau \, d\mu_{F,p} \right) \left(h(\mu_p) + \int \psi_t \, d\mu_p \right) \geqslant \theta \epsilon_p$$

On the other hand $P^G(\Psi_t) \leqslant 0$ by Lemma 4.1. So the proposition is proved.   □

Since the inducing scheme (X, F) can be coded by the full-shift on countably many symbols we have, as explained in Section 2, a Gibbs measure μ_{Ψ_t} for Ψ_t. We need to show that this measure has integrable inducing time and thus that it projects to a measure in $\mathcal{M}_f$.

5. The Gibbs measure has integrable inducing times

This section is devoted to proving that the inducing time is integrable with respect to the Gibbs measure constructed in Section 4. In particular, this implies that the measure has finite entropy and that it is an equilibrium state for the induced potential. It also implies that it can be projected to a measure in $\mathcal{M}_f$.

Proposition 5.1. *Let $t \in (t^-, t^+)$ and $\psi = \psi_t$. Suppose that we have an inducing scheme $(X, \tilde{F})$. Then there exists $k \in \mathbb{N}$ such that replacing $(X, \tilde{F})$ by (X, F), where $F = \tilde{F}^k$, the following holds. There exist $\gamma_0 \in (0, 1)$ and, for any cylinder $\mathrm{C}_n^j \in \mathcal{P}_n^F$ any $n \in \mathbb{N}$, a constant $\delta_n^j < 0$ such that any measure $\mu_F \in \mathcal{M}_F$ with*

$$\mu_F(\mathrm{C}_n^j) \leqslant (1 - \gamma_0) m_\Psi(\mathrm{C}_n^j) \text{ or } \mu_F(\mathrm{C}_n^j) \geqslant \frac{m_\Psi(\mathrm{C}_n^j)}{1 - \gamma_0},$$

where m_Ψ denotes the conformal measure for the system (X, F, Ψ), must have $h(\mu_F) + \int \Psi \, d\mu_F \leqslant \delta_n^j$.

Note that $\delta_n^j \to 0$ as $m_\Psi(\mathrm{C}_n^j) \to 0$. Also note that if $K = \exp\left(\sum_{k=1}^\infty V_k(\tilde{\Psi}) \right)$ is a distortion constant for the potential $\tilde{\Psi}$ for the inducing scheme $(X, \tilde{F})$ then it is also a distortion constant the potential Ψ on (X, F).

The following lemma will allow us to choose k in the proof of Proposition 5.1. It is true for $\Psi = \Psi_t$, but also for more general potentials of summable variation.

Lemma 5.1. *Suppose that we have an inducing scheme (X, F) and potential $\Psi = \Psi_t$ with distortion constant $K = \exp\left(\sum_{k=1}^{\infty} V_k(\Psi)\right)$ and $P^G(\Psi) = 0$. We let m_Ψ denote the conformal measure for the system (X, F, Ψ). Then for any $C_n \in \mathcal{P}_n^F$ and $n \in \mathbb{N}$,*

$$m_\Psi(C_n) \leqslant e^{-\lambda n}$$

where $\lambda := -\log\left(K \sup_{C_1 \in \mathcal{P}_1^F} m_\Psi(C_1)\right)$.

Proof. Since m_Ψ is a conformal measure, for $C_n^i \in \mathcal{P}_n^F$ we have

$$1 = m_\Psi(F^n(C_n^i)) = \int_{C_n^i} e^{-S_n \Psi} \, dm_\Psi.$$

So by the Intermediate Value Theorem we can choose $x \in C_n^i$ so that $e^{S_n \Psi(x)} = m_\Psi(C_n^i)$. For future use we will write $S_n^i \Psi := S_n \Psi(x)$. Therefore,

$$m_\Psi(C_n^i) = e^{S_n^i \Psi} \leqslant e^{n \sup \Psi}.$$

By the Gibbs property,

$$e^{\sup \Psi} \leqslant K \sup_{C_1 \in \mathcal{P}_1^F} m_\Psi(C_1).$$

Therefore

$$\sup \Psi \leqslant \log\left(K \sup_{C_1 \in \mathcal{P}_1^F} m_\Psi(C_1)\right).$$

We can choose this as our value for $-\lambda$. $\qquad\square$

In the following proof we use the notation $A = \theta^{\pm C}$ to mean $\theta^{-C} \leqslant A \leqslant \theta^C$.

Proof of Proposition 5.1. Suppose that the distortion of the potential $\tilde{\Psi}$ for the scheme $(X, \tilde{F})$ is bounded by $K \geqslant 1$. We first prove that measures giving cylinders very small mass compared to m_Ψ must have low free energy. Note that for any $k \in \mathbb{N}$, the potential Ψ for the scheme (X, F) where $F = \tilde{F}^k$ also has distortion bounded by K. We will choose k later so that $\lambda = \lambda(K, \sup_i m_\Psi(X_i))$ for (X, F) as defined in Lemma 5.1, is large enough to satisfy the conditions associated to (7), (8) and (10). Note that as in [S3, Lemma 3] we also have $P^G(\Psi) = 0$.

In Lemma 5.3 below, we will use the Variational Principle to bound the free energy of measures for the scheme which, for some γ, have $\mu(C_n^i) \leqslant K m_\Psi(C_n^i)(1-\gamma)/(1-m_\Psi(C_n^i))^n$ in terms of the Gurevich pressure. However, instead of using Ψ, which, in the computation of Gurevich pressure weights points $x \in C_n^i$ by $e^{\Psi(x)}$, we use a potential which weights points in C_n^i by $(1-\gamma)e^{\Psi}(x)$. That is, we consider $(X, F, \Psi^\flat)$ where

$$\Psi^\flat(x) = \begin{cases} \Psi(x) + \log(1-\gamma) & \text{if } x \in C_n^i, \\ \Psi(x) & \text{if } x \in C_n^j, \text{ for } j \neq i. \end{cases}$$

Firstly we will compute $P^G(\Psi^\flat)$.

Lemma 5.2. $P^G(\Psi^\flat) = \log\left(1 - \gamma m_\Psi(\mathrm{C}_n^i)\right).$

Proof. We prove the lemma assuming that $n = 1$ since the general case follows similarly. We will estimate $Z_j(\Psi^\flat, \mathrm{C}_1^i)$, where Z_j is defined in (2). The ideas we use are similar to those in the proof of Claim 2 in the proof of [BT2, Proposition 2]. As can be seen from the definition,

$$Z_j(\Psi^\flat, \mathrm{C}_1^i) = e^{\pm \sum_{k=0}^{j-1} V_k(\Psi)} \sum_{\mathrm{C}_j \in \mathcal{P}_j^F \cap \mathrm{C}_1^i} \sum_{\text{any } x \in \mathrm{C}_j} e^{S_j \Psi^\flat(x)}.$$

As in the proof of Lemma 5.1, the conformality of m_Ψ and the Intermediate Value Theorem imply that for each k there exists $x_{\mathrm{C}_1^k} \in \mathrm{C}_1^k$ such that $m_\Psi(\mathrm{C}_1^k) = e^{\Psi(x_{\mathrm{C}_1^k})}$. For the duration of this proof we write $\Psi_k := \Psi(x_{\mathrm{C}_1^k})$. As above, we have $e^{\Psi_i^\flat} := (1 - \gamma)e^{\Psi_i}$. Therefore,

$$\sum_i e^{\Psi_i^\flat} = 1 - \gamma e^{\Psi_i}.$$

For each $\mathrm{C}_j \in \mathcal{P}_j^F$ and for any $k \in \mathbb{N}$, there exists a unique $\mathrm{C}_{j+1} \subset \mathrm{C}_j$ such that $F^j(\mathrm{C}_{j+1}) = \mathrm{C}_1^k$. Moreover, there exists $x_{\mathrm{C}_{j+1}} \in \mathrm{C}_{j+1}$ such that $F^j(x_{\mathrm{C}_{j+1}}) = x_{\mathrm{C}_1^k}$. Then for $\mathrm{C}_j \subset \mathrm{C}_1^i$,

$$\sum_{\mathrm{C}_{j+1} \subset \mathrm{C}_j} e^{S_{j+1} \Psi^\flat(x_{\mathrm{C}_{j+1}})} = e^{\pm V_{j+1}(\Psi)} e^{S_j \Psi^\flat(x_{\mathrm{C}_j})} \left(\sum_i e^{\Psi_i^\flat} \right)$$

$$= e^{\pm V_{j+1}(\Psi)} e^{S_j \Psi^\flat(x_{\mathrm{C}_j})} (1 - \gamma e^{\Psi_i}).$$

Therefore,

$$Z_{j+1}(\Psi^\flat, \mathrm{C}_1^i) = (1 - \gamma e^{\Psi_i}) e^{\pm\left(V_{j+1}(\Psi) + \sum_{k=0}^{j-1} V_k(\Psi)\right)} Z_j(\Psi^\flat, \mathrm{C}_1^i),$$

hence

$$Z_{j+1}(\Psi^\flat, \mathrm{C}_1^i) = (1 - \gamma e^{\Psi_i})^j e^{\pm \sum_{k=0}^{j} (k+1) V_k(\Psi)}.$$

As in Remark 4.1, Ψ is weakly Hölder, so $\sum_{k=0}^{j} (k+1) V_k(\Psi) < \infty$. Therefore we have $P^G(\Psi^\flat) = \log(1 - \gamma e^{\Psi_i}) = \log(1 - \gamma m_\Psi(\mathrm{C}_1^i))$, proving the lemma. $\qquad\square$

For the next step in the proof of the upper bound on the free energy of measures giving C_n^i small mass, we relate properties of (X, F, Ψ) and $(X, F, \Psi^\flat)$.

Lemma 5.3. $\mathcal{M}_F(\Psi) = \mathcal{M}_F(\Psi^\flat)$ *and for any* $\mathrm{C}_n^i \in \mathcal{P}_n^F$ *we have*

$$\sup\left\{ h_F(\mu) + \int \Psi \, d\mu : \mu \in \mathcal{M}_F(\Psi), \ \mu(\mathrm{C}_n^i) < \frac{K(1-\gamma)}{(1 - m_\Psi(\mathrm{C}_n^i))^n} m_\Psi(\mathrm{C}_n^i) \right\}$$

$$\leqslant \sup\left\{ h_F(\mu) + \int \Psi^\flat \, d\mu : \mu \in \mathcal{M}_F(\Psi^\flat), \ \mu(\mathrm{C}_n^i) < \frac{K(1-\gamma)}{(1 - m_\Psi(\mathrm{C}_n^i))^n} m_\Psi(\mathrm{C}_n^i) \right\}$$

$$- \left[\frac{K(1-\gamma)\log(1-\gamma)}{(1 - m_\Psi(\mathrm{C}_n^i))^n} \right] m_\Psi(\mathrm{C}_n^i)$$

$$\leqslant P^G(\Psi^\flat) - \left[\frac{K(1-\gamma)\log(1-\gamma)}{(1 - m_\Psi(\mathrm{C}_n^i))^n} \right] m_\Psi(\mathrm{C}_n^i).$$

Note that we can prove that the final inequality is actually an equality, but since we don't require this here we will not prove it.

Proof. The fact that $\mathcal{M}_F(\Psi) = \mathcal{M}_F(\Psi^\flat)$ is clear from the definition.

Suppose that $\mu \in \mathcal{M}_F(\Psi)$ and $\mu(\mathrm{C}_n^i) \leqslant m_\Psi(\mathrm{C}_n^i)K(1-\gamma)/(1-m_\Psi(\mathrm{C}_n^i))^n$. Then

$$\left(h_F(\mu) + \int \Psi \, d\mu\right) - \left(h_F(\mu) + \int \Psi^\flat \, d\mu\right) = \int \Psi - \Psi^\flat \, d\mu$$

$$= \mu(\mathrm{C}_n^i)(-\log(1-\gamma)) \leqslant - \left[\frac{K(1-\gamma)\log(1-\gamma)}{(1-m_\Psi(\mathrm{C}_n^i))^n}\right] m_\Psi(\mathrm{C}_n^i),$$

proving the first inequality in the Lemma. The final inequality follows from the definition of pressure. $\qquad\square$

Lemmas 5.2 and 5.3 imply that any measure μ_F with $\mu_F(\mathrm{C}_n^i) < K(1-\gamma)m_\Psi(\mathrm{C}_n^i)/(1-m_\Psi(\mathrm{C}_n^i))^n$ must have

$$h(\mu_F) + \int \Psi \, d\mu_F \leqslant P^G(\Psi^\flat) - \left[\frac{K(1-\gamma)\log(1-\gamma)}{(1-m_\Psi(\mathrm{C}_n^i))^n}\right] m_\Psi(\mathrm{C}_n^i) \tag{6}$$

$$\leqslant \log\left(1 - \gamma m_\Psi(\mathrm{C}_n^i)\right) - \left[\frac{K(1-\gamma)\log(1-\gamma)}{(1-m_\Psi(\mathrm{C}_n^i))^n}\right] m_\Psi(\mathrm{C}_n^i). \tag{7}$$

If $m_\Psi(\mathrm{C}_n^i)$ is very small then $\log\left(1 - \gamma m_\Psi(\mathrm{C}_n^i)\right) \approx -\gamma m_\Psi(\mathrm{C}_n^i)$ and so choosing $\gamma \in (0,1)$ close enough to 1 the above is strictly negative. By Lemma 5.1, $m_\Psi(\mathrm{C}_n^i) < e^{-\lambda n}$ so C_n^i is small if λ large. Hence if λ is sufficiently large then we can set $\gamma = \tilde{\gamma}^\flat \in (0,1)$ so that

$$\log\left(1 - \tilde{\gamma}^\flat e^{-\lambda n}\right) - \left[\frac{K(1-\tilde{\gamma}^\flat)\log(1-\tilde{\gamma}^\flat)}{(1-e^{-\lambda n})^n}\right] e^{-\lambda n}$$

is strictly negative for all $n \in \mathbb{N}$. This implies that (7) with $\gamma = \tilde{\gamma}^\flat$ is strictly negative for any $\mathrm{C}_n^i \in \mathcal{P}_n^F$ and any n, so we set (7) to be the value $\delta_n^{i,\flat}$.

For the upper bound on the free energy of measures giving C_n^i relatively large mass, we follow a similar proof, but with

$$\Psi^\sharp(x) = \begin{cases} \Psi(x) - \log(1-\gamma) & \text{if } x \in \mathrm{C}_n^i, \\ \Psi(x) & \text{if } x \in \mathrm{C}_n^j, \text{ for } j \neq i. \end{cases}$$

Similarly to above, one can show that $\mathcal{M}_F(\Psi) = \mathcal{M}_F(\Psi^\sharp)$ and

$$\sup\left\{ h_F(\mu) + \int \Psi\, d\mu : \mu \in \mathcal{M}_F(\Psi),\ \mu(\mathrm{C}_n^i) > \frac{m_\Psi(\mathrm{C}_n^i)}{K(1-\gamma)\left[1 + m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right)\right]^n} \right\}$$

$$\leqslant \sup\left\{ h_F(\mu) + \int \Psi^\sharp\, d\mu : \mu \in \mathcal{M}_F(\Psi^\sharp),\ \mu(\mathrm{C}_n^i) > \frac{m_\Psi(\mathrm{C}_n^i)}{K(1-\gamma)\left[1 + m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right)\right]^n} \right\}$$

$$+ \frac{\log(1-\gamma)m_\Psi(\mathrm{C}_n^i)}{K(1-\gamma)\left[1 + m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right)\right]^n}$$

$$\leqslant P^G(\Psi^\sharp) + \frac{\log(1-\gamma)m_\Psi(\mathrm{C}_n^i)}{K(1-\gamma)\left[1 + m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right)\right]^n}.$$

Moreover, we can show that

$$P^G(\Psi^\sharp) = \log\left(1 + m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right)\right) \leqslant m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right).$$

Therefore, if $\mu(\mathrm{C}_n^i) > \frac{m_\Psi(\mathrm{C}_n^i)}{K(1-\gamma)[1 + m_\Psi(\mathrm{C}_n^i)(\frac{\gamma}{1-\gamma})]^n}$, we have

$$h_F(\mu) + \int \Psi\, d\mu \leqslant m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right) + \frac{\log(1-\gamma)m_\Psi(\mathrm{C}_n^i)}{K(1-\gamma)\left[1 + m_\Psi(\mathrm{C}_n^i)\left(\frac{\gamma}{1-\gamma}\right)\right]^n}. \quad (8)$$

If λ is sufficiently large then we can choose $\gamma = \tilde{\gamma}^\sharp \in (0,1)$ so that this is strictly negative and can be fixed to be our value $\delta_n^{i,\sharp}$. This can be seen as follows: let and $\gamma = p/(p+1)$ for some p to be chosen later. Then the right hand side of (8) becomes

$$m_\Psi(\mathrm{C}_n^i)\,(p+1)\left[\frac{p}{p+1} - \frac{\log(p+1)}{K(1 + pe^{-\lambda n})^n}\right]. \quad (9)$$

If λ is sufficiently large, then there exists some large $\lambda' \in (0,\lambda)$ such that $(1 + pe^{-\lambda n})^n \leqslant 1 + pe^{-\lambda' n}$ for all $n \in \mathbb{N}$. Hence with this suitable choice of λ we can choose p so that the quantity in the square brackets in (9) is negative for all n. So we can choose $\delta_n^{i,\sharp} < 0$ to be (8) with $\gamma = \tilde{\gamma}^\sharp$.

We let

$$\gamma^\sharp = 1 - (1 - \tilde{\gamma}^\sharp)\left(1 + e^{-\lambda n}\left(\frac{\tilde{\gamma}^\sharp}{1 - \tilde{\gamma}^\sharp}\right)\right)^n. \quad (10)$$

For appropriately chosen λ this is in $(0,1)$.

We set $\gamma_0' := \max\{\gamma^\flat, \gamma^\sharp\}$ and for each $\mathrm{C}_n^i \in \mathcal{P}_n^F$ we let $\delta_n^i := \max\{\delta_n^{i,\flat}, \delta_n^{i,\sharp}\}$. The proof of the proposition is completed by setting $\gamma_0 := 1 - K(1 - \gamma_0')$, which we may assume is in $(0,1)$. □

Proposition 5.2. *There exists an inducing scheme (X,F) such that for $t \in (t^-, t^+)$ and $\psi = \psi_t$, any sequence of measures $(\mu_n)_n$ with $h(\mu_n) - \int \psi\, d\mu_n \to 0$ as $n \to \infty$ has a limit measure μ_ψ which is an equilibrium state for ψ.*

Note that (X, F) and $(\mu_n)_n$ can be chosen as in Proposition 4.1.

Proof. By Proposition 4.1, we can find $\theta > 0$, an inducing scheme $(X, \tilde{F})$ and a sequence of measures $(\mu_n)_n$ with $h(\mu_n) + \int \psi \, d\mu_n \to 0$ each compatible with $(X, \tilde{F})$ and with $\int \tilde{\tau} \, d\mu_{\tilde{F}, n} < \theta$. Proposition 4.1 also implies $P^G(\tilde{\Psi}_t) = 0$. Taking $F = \tilde{F}^k$ for k as in Proposition 5.1, that proposition then implies that there exists $K' > 0$ such that for any $C_k \in \mathcal{P}_k^F$, for all large enough n,

$$\frac{1}{K'} \leqslant \frac{\mu_{F,n}(C_k)}{e^{S_k \Psi(x)}} \leqslant K'$$

for all $x \in C_k$ (note that as in Proposition 5.1, we can actually take $K' = K/(1 - \gamma_0)$ where K is the distortion bound for $\tilde{\Psi}_t$). Note that $(\mu_{F,n})_n$ is tight (see [Bi, Section 25] for a discussion of this notion) and that any limit of the sequence $\mu_{F,\infty}$ must satisfy the Gibbs property with distortion constant K'. By the uniqueness of Gibbs measures ([MU2, Theorem 3.5]), $\mu_{F,\infty} = \mu_\Psi$. We now show that $\int \tau \, d\mu_\Psi < \theta k$. First note that $\int \tau \, d\mu_{F,n} = \int \tilde{\tau}^k \, d\mu_{\tilde{F}, n} < \theta k$. For the purposes of this proof we let $\tau_N := \min\{\tau, N\}$. By the Monotone Convergence Theorem,

$$\int \tau \, d\mu_\Psi = \lim_{N \to \infty} \int \tau_N \, d\mu_\Psi \leqslant \lim_{N \to \infty} \limsup_{n \to \infty} \int \tau_N \, d\mu_{F,n} \leqslant \theta k.$$

Thus we can project μ_Ψ to μ_ψ by (3).

The fact that μ_ψ is a weak* limit of $(\mu_n)_n$ follows as in, for example [FT, Section 6]. The fact that we have a uniform bound $\mu_{F,n} \{\tau \geqslant j\} \leqslant \theta k / j$ for all $n \in \mathbb{N}$ is again crucial in proving this.

The Abramov formula implies that

$$\int \Psi \, d\mu_\Psi = \left(\int \tau \, d\mu_\Psi \right) \left(\int \psi \, d\mu_\psi \right) = \left(\int \tau \, d\mu_\Psi \right) (\lambda(\mu_\psi) - p(t)).$$

Since $\lambda(\mu) \in [\lambda_m, \lambda_M]$ and both $p(t)$ and $\int \tau \, d\mu_\Psi$ are finite, this implies that $-\int \Psi \, d\mu_\Psi < \infty$ and hence μ_Ψ is an equilibrium state for Ψ. Using the Abramov formula again we have that μ_ψ is an equilibrium state for ψ. □

Remark 5.1. *Here we give an example of a way our setting can be changed so that the arguments in Proposition 5.1 and 5.2 fail. In the case where f is the (appropriately scaled) quadratic Chebyshev polynomial, $t^- \in (-\infty, 0)$. In this case there is a periodic point p such that the Dirac measure δ_p on the orbit of p has $\lambda(\delta_p) = \lambda_M$. The point p is the image of the critical point which means that our class of inducing schemes can not be compatible with δ_0 (indeed the only inducing scheme for δ_0 has only one domain and the only measure compatible to it is δ_0).*

However, any measure $\mu \in \mathcal{M}_f$ orthogonal to δ_0 must have $h(\mu) - t\lambda(\mu) \leqslant h(\mu_1) - t\lambda(\mu_1)$ for all $t \in \mathbb{R}$ where μ_1 is the acip. In particular, $h(\mu) - t\lambda(\mu) < p(t)$ for $t < t^-$. If $P^G(\Psi_t) = 0$ then arguments similar to those in the proofs of Lemma 4.1 and Proposition 4.1 imply that there are measures with free energy w.r.t. ψ_t is arbitrarily close to zero and positive entropy. This contradiction implies that for $t < t^-$, $P^G(\Psi_t) < 0$ so we cannot begin to apply the arguments above to that case. So it is important that $t \in (t^-, t^+)$.

6. Uniqueness of equilibrium states

The result in Proposition 5.2 gives the existence of equilibrium states for $-t \log |Df|$ for each $t \in (t^-, t^+)$. In this section we obtain uniqueness. To do this we will use more properties of the inducing schemes described in [BT2]. They were produced in as first return maps to an interval in the so-called Hofbauer tower. This theory was further developed in [BT1] and [T]. The following theorem gives some of their properties.

Theorem 6.1. *There exists a countable collection $\{(X^n, F_n)\}_n$ of inducing schemes with $\partial X^n \notin (X^n, F_n)^\infty$ such that:*

a) *any ergodic invariant probability measure μ with $\lambda(\mu) > 0$ is compatible with one of the inducing schemes (X^n, F_n). In particular there exists and ergodic F_n-invariant probability measure μ_{F_n} which projects to μ as in (3);*

b) *any ergodic equilibrium state for $-t \log |Df|$ where $t \in \mathbb{R}$ with $\lambda(\mu) > 0$ is compatible with all inducing schemes (X^n, F_n).*

Remark 6.1. *Note that it is crucial in our applications of Theorem 6.1, for example in the proofs of Proposition 6.1 and Proposition 7.1, that in b) we are able to weaken the condition $h(\mu) > 0$ to $\lambda(\mu) > 0$ when we wish to lift measures. This is why we need to use a countable number of inducing schemes in Theorem 6.1 rather than the finite number in [BT2, Remark 6].*

Before proving Theorem 6.1, we prove the following easy lemma.

Lemma 6.1. *If $t \in (t^-, t^+)$ and an equilibrium state μ_t from Proposition 5.2 is compatible with an inducing scheme (X, F), then $P^G(\Psi_t) = 0$. Moreover the lifted measure $\mu_{t,F}$ is a Gibbs measure and an equilibrium state for Ψ_t.*

Proof. First note that by Lemma 4.1, $P^G(\Psi_t) \leqslant 0$.

Denote by μ_t an equilibrium measure for the potential $-t \log |Df|$ of positive Lyapunov exponent and let $\mu_{t,F}$ be the lifted measure. Note that by Proposition 2.1 and by the Abramov formula, see for example [PS, Theorem 2.3], we have

$$P^G(\Psi_t) \geqslant h(\mu_{t,F}) + \int \Psi_t \, d\mu_{t,F} = h(\mu_{t,F}) - t \int \log |DF| \, d\mu_{t,F} - p(t) \int \tau \, d\mu_{t,F}$$

$$= \left(\int \tau \, d\mu_{t,F} \right) \left(\frac{h(\mu_{t,F})}{\int \tau \, d\mu_{t,F}} - t \left(\frac{\int \log |DF| \, d\mu_{t,F}}{\int \tau \, d\mu_{t,F}} \right) - p(t) \right)$$

$$= \left(\int \tau \, d\mu_{t,F} \right) \left(h(\mu_t) - t \int \log |Df| \, d\mu_t - p(t) \right).$$

But recall that μ_t is an equilibrium measure:

$$p(t) = h(\mu_t) - t \int \log |Df| \, d\mu_t.$$

Therefore $P^G(\Psi_t) \geqslant 0$.

Since $P^G(\Psi_t) = 0$ there exists a unique Gibbs measure μ_F corresponding to (X, F, Ψ_t). By the Abramov formula,

$$h(\mu_{t,F}) + \int \Psi_t \, d\mu_{t,F} = 0,$$

so $\mu_{t,F}$ is an equilibrium state for (X, F, Ψ_t). Since, in this setting, equilibrium states are unique (see [MU2, Theorem 3.5]) we have that $\mu_{t,F} = \mu_F$. □

Proof of Theorem 6.1. Part (a) of the theorem follows from the proof of [BT2, Theorem 3]. Part (b) is proved similarly to [BT2, Proposition 2], but with added information from our Proposition 5.1. We sketch some details. Suppose that μ is compatible to (X^n, F_n). Then Lemma 6.1 implies that $P^G(\Psi_n) = 0$. Claim 1 of the proof of [BT2, Proposition 2] implies that for any other inducing scheme $(X^{n'}, F_{n'})$ is 'topologically connected' to (X^n, F_n). Proposition 5.1, which is an improved version of Claim 2 in the proof of [BT2, Proposition 2], then can be used as in that proof to give a 'metric connection' which means that an equilibrium state compatible with (X^n, F_n) must be compatible with $(X^{n'}, F_{n'})$. □

Proposition 6.1. *For any $t \in (-\infty, t^+)$ there is at most one equilibrium state for $-t \log |Df|$. Moreover, if $t^+ > 1$ then for any $t \in \mathbb{R}$ there is at most one equilibrium state for $-t \log |Df|$.*

Clearly the equilibrium states, when unique, must be ergodic.

Proof. The idea here is first to show that any equilibrium state can be decomposed into a sum of countably many measures, each of which is an equilibrium state and is compatible with an inducing scheme as in Theorem 6.1. [MU2, Theorem 3.5] implies that there is only one equilibrium state per inducing scheme. Lemma 6.1 then implies that this equilibrium state must be unique.

We suppose that μ is an equilibrium state for $-t \log |Df|$ for $t \in (-\infty, t^+)$. We first note that μ may be expressed in terms of its ergodic decomposition, see for example [K2, Section 2.3], $\mu(\cdot) = \int \mu_y(\cdot) \, d\mu(y)$ where $y \in I$ is a generic point of the ergodic measure $\mu_y \in \mathcal{M}_f$. Clearly, for any set $A \subset I$ such that $\mu(A) > 0$, the measure $\mu_A(\cdot) := \frac{1}{\mu(A)} \int_A \mu_y(\cdot) \, d\mu(y)$ must have

$$h(\mu_A) - t\lambda(\mu_A) = p(t),$$

i.e. it must be an equilibrium state itself (otherwise, removing μ_A from the integral for μ would increase $h_\mu - t\lambda(\mu)$). As in the proof of Lemma 4.2, $\lambda(\mu_A) > 0$.

Theorem 6.1(a) implies that any such μ_A must decompose into a sum $\mu = \sum_n \alpha_n \mu_n$ where μ_n is a probability measure compatible with the scheme (X^n, F_n) and $\alpha_n \in (0, 1]$. Then there are F_n-invariant probability measures μ_{F_n}, each of which projects to μ_n by (3).

By Lemma 6.1 and [BuS], μ_{F_n} must be the unique equilibrium state for the scheme (X^n, F_n, τ_n) with potential $-t \log |DF_n| - p(t)\tau_n$. Therefore, μ_n is the only equilibrium state for $-t \log |Df|$ which is compatible with (X^n, F_n).

We finish the proof by using Theorem 6.1 b) which implies that any of these equilibrium states compatible with an inducing scheme (X^n, F_n) as above must be compatible with each of the other inducing schemes (X_j, F_j). Hence $\mu_i = \mu_j$ for every $i, j \in \mathbb{N}$. Since μ was an arbitrary equilibrium state, this argument implies that μ is ergodic and is the unique equilibrium state for $-t \log |Df|$, as required.

Suppose that $t^+ > 1$. Since $\lambda_m \geqslant 0$ this means that $t \mapsto p(t)$ must be strictly decreasing in the interval $(1, t^+)$. Since Bowen's formula implies that $p(t) \leqslant 0$ this means that $p(t) < 0$. Ruelle's formula [Ru1] then implies that we must have $\lambda_m > 0$. Therefore, if $t^+ > 1$ then $\lambda(\mu) > 0$ for all $\mu \in \mathcal{M}_f$ and so we can apply Theorem 6.1 to the case $t \geqslant t^+$ also. $\square$

7. Proof of Theorem A

The previous sections give most of the information we need to prove Theorem A. In this section we prove the remaining part: that the critical parameter t^-, defined in equation (1), is not finite. We then put the proof of Theorem A together.

Lemma 7.1. *There exists a measure μ_M such that $\lambda(\mu_M) = \lambda_M$.*

Proof. This follows from the compactness of $\mathcal{M}_f$ and the upper semicontinuity of $x \mapsto \log |Df(x)|$. $\square$

Proposition 7.1. $t^- = -\infty$.

Proof. Suppose, for a contradiction, that $t^- > -\infty$. This implies that for $t \leqslant t^-$, the measure μ_M in Lemma 7.1 also maximises $h(\mu) - t\lambda(\mu)$ for $\mu \in \mathcal{M}_f$, and must have $h(\mu_M) = 0$.

By Theorem 6.1, we can choose an inducing scheme (X, F) compatible with μ_M.

Claim 1. $P^G(\Psi_t) = 0$ *for all $t \leqslant t^-$.*

Proof. $P^G(\Psi_t) \leqslant 0$ follows by Lemma 4.1. $P^G(\Psi_t) \geqslant 0$ follows since μ_M is compatible with our scheme. $\square$

Since by construction, μ_M is compatible with (X, F), the induced measure being denoted by $\mu_{F,M}$, and since $h(\mu_M) + \int \psi_t \, d\mu_M = 0$, we have

$$h(\mu_{F,M}) + \int \Psi_t \, \mu_{F,M} = 0,$$

and so $\mu_{F,M}$ is an equilibrium state for Ψ_t. However, by Theorems 1.1 and 1.2 of [BuS] any equilibrium state of Ψ_t must have positive entropy, a contradiction. $\square$

Proof of Theorem A. The existence of the equilibrium state for $-t \log |Df|$ and $t \in (t^-, t^+)$ follows from Proposition 5.2. Uniqueness follows from Proposition 6.1. Positivity of the entropy of μ_t comes from Lemma 4.2. Finally the fact that $t^- = -\infty$ comes from Proposition 7.1. $\square$

8. The pressure is of class C^1 and strictly convex in $(-\infty, t^+)$

As discussed in the introduction, for general systems the pressure function $t \mapsto p(t)$ is convex, therefore it can have at most a countable number of first order phase transitions. In [S2] an example is constructed with the property that the set of parameters at which the pressure function is not analytic has positive measure (in this case, there also exist *higher order phase transitions*, see [S5]). Nevertheless, for multimodal maps it has been shown that in certain intervals the pressure function is indeed real analytic, see [BT1, BT2]. Dobbs [D3, Proposition 9] proved that in the quadratic family $x \mapsto \gamma x(1-x)$, $\gamma \in (3,4)$ there exists uncountably many parameters for which the pressure function admits infinitely many phase transitions. However, these transitions are caused by the existence of an infinite sequence renormalisations of the map, so for these parameters the corresponding quadratic maps do not have a representative in the class $\mathcal{F}$. He also notes [D3, Proposition 4] that in the quadratic family there is a always a phase transition for negative t caused by the repelling fixed point at 0. Since, this fixed point is not in the transitive part of the system (which actually must be contained in $[f^2(c), f(c)]$), from our perspective this point is not dynamically relevant, so any representative of such a map in $\mathcal{F}$ would miss this part of the dynamics, and hence not exhibit this transition.

Proposition 8.1. *For $f \in \mathcal{F}$, the pressure function p is C^1 in the interval $(-\infty, t^+)$.*

Proof. We first show that p is differentiable. By Theorem 6.1, we can choose an inducing scheme (X, F, τ) which is compatible with μ_t for each $t \in (-\infty, t^+)$. Then we have the limits

$$\lim_{t' \to t} \int \log |DF| \, d\mu_{\Psi_{t'}} = \int \log |DF| \, d\mu_{\Psi_t} \text{ and } \lim_{t' \to t} \int \tau \, d\mu_{\Psi_{t'}} = \int \tau \, d\mu_{\Psi_t}.$$

We emphasise that these limits are the same if t' are taken to the left or to the right of t. Hence $\lambda(\mu_{\psi_t})$ is continuous in (t^-, t^+). Since the derivative of p is $-\lambda(\mu_{\psi_t})$, the derivative is continuous, proving the lemma. This standard fact can be seen as follows (see also, for example, [K2, Theorem 4.3.5]): given $\varepsilon > 0$, by the definition of pressure the free energy of μ_t with respect to $\psi_{t+\varepsilon}$ is no more than $p(t + \varepsilon)$. Similarly the free energy of $\mu_{t+\varepsilon}$ with respect to ψ_t is no more than $p(t)$. Hence

$$\frac{(-(t+\varepsilon)+t)\lambda(\mu_{t+\varepsilon})}{\varepsilon} \geqslant \frac{p(t+\varepsilon) - p(t)}{\varepsilon} \geqslant \frac{(-(t+\varepsilon)+t)\lambda(\mu_t)}{\varepsilon}.$$

So whenever $t \mapsto \lambda(\mu_t)$ is continuous, $Dp(t) = -\lambda(\mu_t)$. $\qquad\square$

Proposition 8.2. *For $f \in \mathcal{F}$, $t^+ > 0$ and the pressure function p is strictly convex in $(-\infty, t^+)$.*

Before proving this proposition, we need two lemmas: the first guarantees that $t^+ > 0$, while the second will be used to obtain strict convexity of the pressure function (both these facts are in contrast with the quadratic Chebyshev case).

Lemma 8.1. *For $f \in \mathcal{F}$, $\lambda(\mu_0) > \lambda_m$ where μ_0 is the measure of maximal entropy for f.*

Proof. The existence of a (unique) measure of maximal entropy μ_0 is guaranteed by [H]. Suppose for a contradiction that the lemma is false and hence $\lambda(\mu_0) = \lambda_m$.

Since when the derivative of p exists at a point t, it is equal to $-\lambda(\mu_t)$ (see [Ru2] as well as the computation in the proof of Proposition 8.1) and by convexity, the pressure function must be affine with constant slope $-\lambda_m$. i.e. $p(t) = h_{top}(f) - t\lambda_m$ for $t \in [0, \infty)$. This implies that μ_0 must be an equilibrium state for the potential $-t \log |Df|$ for every $t \in \mathbb{R}$. In particular this applies when $t = 1$. Moreover, by Ruelle's inequality [Ru1], we have $\lambda(\mu_0) > 0$, so μ_0 must be an acip by [L]. By [D2, Proposition 3.1], this implies that f has finite postcritical set, which is a contradiction. $\qquad\square$

Lemma 8.2. *For any $\varepsilon > 0$ there exists an inducing scheme (X, F) a sequence $i_k \to \infty$ such that the domains X_{i_k} have*

$$|X_{i_k}| \geqslant e^{-(\lambda_m + \varepsilon)\tau_{i_k}}.$$

Proof. It is standard to show that for any $\varepsilon > 0$, there exists a periodic point p with Lyapunov exponent $\leqslant \lambda_m + \varepsilon/3$, see for example [D3, Lemma 19]. We can choose (X, F) as in Theorem 6.1 so that the orbit of p is disjoint from X. We may further assume that (X, F) has distortion bounded by e^δ for some $\delta > 0$, i.e.

$$\frac{|DF(x)|}{|DF(y)|} \leqslant e^\delta$$

for all $x, y \in X_i$ for any $i \in \mathbb{N}$. In this case, by the transitivity of (I, f), which is reflected in our inducing scheme, there must exist an infinite sequence of domains X_{n_k} of (X, F) which shadow the orbit of p for longer and longer. One can use standard distortion arguments to prove that for all large k, $|X_i| \geqslant |X|e^{-\delta}e^{-(\lambda_m + \varepsilon/2)\tau_i}$. Choosing $\delta > 0$ appropriately completes the proof of the lemma. $\qquad\square$

Proof of Proposition 8.2. For the first part of the proposition, $t^+ > 0$ is guaranteed by Lemma 8.1.

For the second part of the proposition, since p is convex, we only have to rule out p being affine in some interval. Suppose first that p is affine in an interval $[t_1, t_2] \subset (-\infty, t^*)$ where $t^* := \inf\{t : Dp(t) = -\lambda_m\}$. I.e. for some $\beta > \lambda_m$, $t \in [t_1, t_2]$ implies $p(t) = p(-t_1) - (t - t_1)\beta$. We let $\varepsilon > 0$ be such that $\beta > \lambda_m + \varepsilon$. By Lemma 8.2, there exists $i_k \to \infty$ such that

$$|X_{i_k}| \geqslant e^{-(\lambda_m + \varepsilon)\tau_{i_k}}.$$

The fact that the pressure function is affine in $[t_1, t_2]$ implies that the equilibrium state is the same μ for every $t \in [t_1, t_2]$. Denote the induced version of μ by μ_F. By the Gibbs property of our inducing schemes, $\mu_F(X_i) \asymp |X_i|^t e^{-\tau_i p(t)}$ for all $t \in [t_1, t_2]$. Therefore,

$$\frac{|X_i|^{t_1} e^{-\tau_i p(t_1)}}{|X_i|^t e^{-\tau_i(p(t_1) - (t - t_1)\beta)}} \asymp 1,$$

which implies that

$$|X_i| \asymp e^{-\tau_i \beta}$$

for all i. Since

$$|X_{i_k}| \geqslant e^{-(\lambda_m + \varepsilon)\tau_{i_k}}$$

for an infinite sequence of domains X_{i_k}, and $\beta > \lambda_m + \varepsilon$, this yields a contradiction.

We next want to prove that $t^+ = t^*$. We suppose not in order to get a contradiction. In the first case suppose that $\lambda_m = 0$. Then $p(t) \geqslant 0$ for all $t \in \mathbb{R}$. Coupled with Bowen's formula this implies that $p(1) = 0$. So the convexity of p implies $t^+ = t^*$, as required. Now suppose that $\lambda_m > 0$. Since we assumed t^+ the graph of $p(t)$ must be above, and parallel to $t \mapsto -t\lambda_m$ on $[t^*, \infty)$. This implies that $t^+ = \infty$ and so Theorem A gives equilibrium states for all $t \in \mathbb{R}$. Hence we can mimic the argument above, with the inducing scheme as in Theorem 6.1 compatible with μ_{t^*}, but instead taking $[t_1, t_2] \subset [t^*, \infty)$ and $\beta = \lambda_m$. Noting that the argument of Lemma 8.2 ensures that we chose the scheme (X, F) so that there is a sequence of domains $|X_{i_k}| \leqslant e^{-(\lambda_M - \varepsilon)\tau_{i_k}}$, we can complete the argument. $\square$

Proof of Theorem B. The convexity of p follows from Proposition 8.2, the smoothness from 8.1 and the fact that the pressure is decreasing from [Pr]. $\square$

9. Phase transitions in the positive spectrum

In this section we study the relation between the existence of first order phase transitions at the point $t = 1$ and the existence of an acip. The following proposition has Proposition 1.1 as a corollary.

Proposition 9.1. *Suppose that $f \in \mathcal{F}$ has $\lambda_m = 0$. Then f has an acip if and only if p has a first order phase transition at $t = 1$.*

Remark 9.1. *Note that if $\lambda_m > 0$ then the situation is quite different. For example if $f \in \mathcal{F}$ satisfies the Collet-Eckmann condition (which by [BS] implies $\lambda_m > 0$), in which case the map also has an acip, then by [BT2, Theorem 3], p is real analytic in a neighbourhood of $t = 1$.*

The following lemma will be used to prove Proposition 9.1.

Lemma 9.1. *Suppose that $t^+ \in (0, \infty)$ and there is a first order phase transition at t^+. Then there exists an inducing scheme (X, F), an equilibrium state μ_Ψ for $\Psi = \Psi_{t^+}$, and an equilibrium state μ_ψ for $\psi = \psi_{t^+}$ with $h(\mu_\psi) > 0$.*

Proof. The fact that there is a first order phase transition implies that the left derivative of p at t^+ has $Dp^-(t^+) < -\lambda_m$. The convexity of the pressure function implies that the graph of the pressure lies above the line $t \mapsto D^-p(t^+)t - t^+(\lambda_m + D^-p(t^+))$. This means that we can take a sequence of equilibrium states μ_{ψ_t} for t arbitrarily close to, and less than, t^+ with free energy converging to $p(t^+)$ with

$$h(\mu_{\psi_t}) \geqslant -t^+(Dp^-(t^+) + \lambda_m) > 0.$$

Hence the arguments used to prove Proposition 5.2 give us an equilibrium state for μ_{t^+} with positive entropy. $\square$

Proof of Proposition 9.1. If there exists an acip μ then $Dp^-(1) = -\lambda(\mu) < 0$. Since $\lambda_m = 0$ implies $p(t) = 0$ for all $t \geqslant 1$, the existence of an acip implies that there is a first order phase transition at $t = 1$.

On the other hand, if there exists a first order phase transition at $t = 1$ then Lemma 9.1 implies that there is an equilibrium state μ_1 for $-\log|Df|$, with $h(\mu_1) > 0$. By [L] this must be an acip. $\qquad\square$

Remark 9.2. *If $\lambda_m > 0$ and there is a measure μ_m such that $\lambda(\mu_m) = \lambda_m$, then by Lemma 9.1 and the arguments in the proof of Proposition 7.1, we have that $t^+ = \infty$.*

Remark 9.3. *There are examples of maps in $\mathcal{F}$ with $\{\mu \in \mathcal{M}_f : \lambda(\mu) = \lambda_m\} = \emptyset$, for example [BK, Lemma 5.5], a quadratic map in $\mathcal{F}$ is defined so that $\lambda_m = 0$, but there are no measures with zero Lyapunov exponent. There are also examples of maps $f \in \mathcal{F}$ with $\{\mu \in \mathcal{M}_f : \lambda(\mu) = \lambda_m\} \neq \emptyset$, for example in [B2] examples of quadratic maps in $\mathcal{F}$ are given for which the omega-limit set of the critical point supports (multiple) ergodic measures with zero Lyapunov exponent. Moreover, Cortez and Rivera-Letelier [CRL] proved that given $\mathcal{E}$ a non-empty, compact, metrisable and totally disconnected topological space then there exists a parameter $\gamma \in (0, 4]$ such that set of invariant probability measures of $x \mapsto \gamma x(1 - x)$, supported on the omega-limit set of the critical point is homeomorphic to $\mathcal{E}$.*

It is plausible that there are maps $f \in \mathcal{F}$ for which

$$\inf \{t \in \mathbb{R} : p(t) \leqslant 0\} < 1.$$

However, the following argument shows that this is not true for unimodal maps with quadratic critical point in $\mathcal{F}$.

Given an interval map $f : I \to I$, we say that $A \subset I$ is a *metric attractor* if $B(A) := \{\omega(x) \subset A\}$ has positive Lebesgue measure and there is no proper subset of A with this property. On the other hand A is a *topological attractor* if $B(A)$ is residual and there is no proper subset of A with this property. We say that f has a *wild attractor* if there is a set A which is a metric attractor, but not a topological one.

Proposition 9.2. *If $f \in \mathcal{F}$ is a unimodal map with no wild attractor then for $t < 1$, $p(t) > 0$.*

Remark 9.4. *It was shown in [BKNS] that there are unimodal maps with wild attractors in $\mathcal{F}$. However, if $\ell_c = 2$ then this is not possible by [Ly1].*

Lemma 9.2. *If $f \in \mathcal{F}$ is a unimodal map with no wild attractor then for each $\varepsilon > 0$ there exists a measure $\mu \in \mathcal{M}_f$ so that*

$$\frac{h(\mu)}{\lambda(\mu)} > 1 - \varepsilon.$$

Proof. By [MvS, Theorem V.1.4], originally proved by Martens, there must be an inducing scheme (X, F) such that $Leb(X \setminus \cup_i X_i) = 0$. For any $\delta > 0$ we can truncate (X, F) to a finite scheme (X^N, F_N) where $X^N = \cup_{i=1}^N X_i$ so that $Leb(\cup_{i=1}^N X_i) > (1 - \delta)|X|$. We therefore have

$$\dim_H \left\{ x : \tau^k(x) < \infty \text{ for all } k \in \mathbb{N} \right\} > 1 - \delta'$$

where δ' depends on δ and the distortion of F (in particular $\to 0$ as $\delta \to 0$). It follows from the Variational Principle and the Bowen formula (see [P, Chapter 7])

that there is an F-invariant measure, μ_F, for this system with

$$\frac{h(\mu_F)}{\lambda(\mu_F)} > 1 - \delta'.$$

By the Abramov formula, for μ the projection of μ_F,

$$\frac{h(\mu)}{\lambda(\mu)} > 1 - \delta'$$

also. Choosing $\delta > 0$ so small that $\delta' \leqslant \varepsilon$ completes the proof. $\qquad\square$

Proof of Proposition 9.2. Let $t < 1$ and choose $\varepsilon = 1-t > 0$. Then the measure μ in Lemma 9.2 has $h(\mu) - t\lambda(\mu) > 0$. Hence by the definition of pressure, $p(t) > 0$. $\quad\square$

Proof of Proposition 1.2. By Proposition 9.2 and Remark 9.4 we can take $t^+ = 1$. Hence we can conclude that p is C^1 strictly convex decreasing in $(-\infty, t^+)$ by Theorem B. The fact that $p(t) = 0$ for all $t \geqslant 1$ follows from [NS].

Part (a) follows from Proposition 9.1 since this implies that both left and right derivatives of $p(t)$ at $t = 1$ are zero. Part (b) is the converse of this since the left derivative is strictly negative and the right derivative is zero. $\qquad\square$

10. REMARKS ON STATISTICAL PROPERTIES AND CHEBYSHEV POLYNOMIALS

In this section we collect some further comments on our results.

10.1. **Statistical properties.** Given $f \in \mathcal{F}$ and an equilibrium state μ as in Theorem A, one can ask about the statistical properties of the system (I, f, μ). For an equilibrium state μ_t from Theorem A, we expect that as described in [BT2, Section 6], it should be possible to prove exponential decay of correlations (as in [Y]) and large deviations (see [MN, RY]), along with many other statistical laws. These laws can be proved when an inducing scheme (X, F, τ) compatible with μ_t has exponential decay in n of the induced measure of $\{\tau > n\}$. However, we do not have sufficient information on this quantity here. Nevertheless, we can use [BT3] to show that the system (I, f, μ) has 'exponential return time statistics'. We give a sketch of this theory here, but for more definitions see for example [BT3].

Given $f \in \mathcal{F}$ and $A \subset I$, we let

$$r_A(x) := \min\{j \in \mathbb{N} \cup \{+\infty\} : f^j(x) \in A\}.$$

For $\mu \in \mathcal{M}_f$, letting μ_A be the conditional measure on A, by Kac's Lemma, the expected value of r_A with respect to μ_A is $\int_A r_A \, d\mu_A = 1/\mu(A)$. Given a sequence of sets $(U_n)_n$ so that $\mu(U_n) \to 0$, the system has exponential return time statistics for $(U_n)_n$ if for all $\tau \geqslant 0$

$$\mu_{U_n}\left(r_{U_n} \geq \frac{\tau}{\mu(U_n)}\right) \to e^{-\tau} \quad \text{as } n \to \infty. \tag{11}$$

Let $t \in (-\infty, t^+)$ and μ_t be the equilibrium state for $-t \log|Df|$ given by Theorem A. By [BT3, Theorem 3], for μ_t-a.e. $x_0 \in I$, and any set of open intervals $(U_n)_n$

such that $U_n \to x_0$ as $n \to \infty$, the system has exponential return time statistics for $(U_n)_n$.

10.2. **Ergodic Optimisation.** Let $f \in \mathcal{F}$ and $\varphi : [0,1] \to \mathbb{R}$ a function. The study of invariant probability measures whose ergodic φ−average is as large (or as small) as possible is known as *ergodic optimisation*. A measure $\mu \in \mathcal{M}_f$ is called φ−minimising/maximising if

$$\int \varphi \, d\mu = \inf \left\{ \int \varphi \, d\nu : \nu \in \mathcal{M}_f \right\} \text{ or } \int \varphi \, d\mu = \sup \left\{ \int \varphi \, d\nu : \nu \in \mathcal{M}_f \right\}$$

respectively. For a survey on the subject see [Je]. Let $t \in (-\infty, t^+)$ and denote by μ_t the unique equilibrium state corresponding to the potential $-t \log |Df|$. A consequence of the results in this paper is that: any accumulation point μ of a sequence of measures μ_{t_n}, given by Theorem A, where $t_n \to -\infty$ is a $\log |Df|$-maximising measure. This is because $\log |Df|$ is upper semicontinuous; $Dp(t) = -\lambda(\mu_t)$; and this derivative is asymptotic to $-\lambda_M$. Hence there is a subsequence of these measures $(\mu_{t_{n_k}})_k$ so that

$$\lim_{k \to \infty} \lambda(\mu_{t_{n_k}}) = \lambda(\mu) = \lambda_M.$$

Note that Lemma 7.1 guarantees the existence of a $\log |Df|$−maximising measure. (We do not assert anything about the uniqueness of this measure.) Actually, any measure μ, which is an accumulation point of μ_{t_n} as $t_n \to -\infty$, is a measure maximising entropy among all measures which maximise $\log |Df|$. Then in fact $p(t)$ is asymptotic to the line $h(\mu) - t\lambda_M$ as $t \to -\infty$.

10.3. **The preperiodic critical point case.** For our class of maps $\mathcal{F}$ we assumed that the orbit of points in $\mathcal{C}r$ are infinite. Here we comment on an alternative case. In the case of the quadratic Chebychev polynomial $x \mapsto 4x(1-x)$ on I, it is well known that the two relevant measures are the acip μ_1, which has $\lambda(\mu_1) = \log 2 = \lambda_m$, and the Dirac measure δ_0 on the fixed point at 0, which has $\lambda(\delta_0) = \log 4 = \lambda_M$. So $t^- = -1$ and

$$p(t) = \begin{cases} (1-t) \log 2 & \text{if } t \geqslant -1, \\ -t \log 4 & \text{if } t \leqslant -1. \end{cases}$$

Note that the above piecewise affine form for the pressure function does not conflict with Theorem B, which might be expected to apply in the interval (t^-, t^+), since $t^+ = t^* = -1$, where t^* is defined in the proof of Proposition 8.2.

Appendix A. Cusp maps

In this section we outline how to extend the above results to some maps which are not smooth. This class includes the class of contracting Lorenz-like maps, see for example [Rov].

Definition A.1. $f : \cup_j I_j \to I$ *is a* cusp map *if there exist constants* $C, \alpha > 1$ *and a set* $\{I_j\}_j$ *is a finite collection of disjoint open subintervals of* I *such that*

(1) $f_j := f|_{I_j}$ is $C^{1+\alpha}$ on each $I_j =: (a_j, b_j)$ and $|Df_j| \in (0, \infty)$.

(2) $D^+f(a_j)$, $D^-f(b_j)$ exist and are equal to 0 or $\pm\infty$.

(3) For all $x, y \in \overline{I_j}$ such that $0 < |Df_j(x)|, |Df_j(y)| \leqslant 2$ we have $|Df_j(x) - Df_j(y)| < C|x - y|^\alpha$.

(4) For all $x, y \in \overline{I_j}$ such that $|Df_j(x)|, |Df_j(y)| \geqslant 2$, we have $|Df_j^{-1}(x) - Df_j^{-1}(y)| < C|x - y|^\alpha$.

We denote the set of points a_j, b_j by $\mathcal{C}r$.

Remark A.1. *Notice that if for some j, $b_j = a_{j+1}$, i.e. $I_j \cap I_{j+1}$ intersect, then f may not continuously extend to a well defined function at the intersection point b_j, since the definition above would then allow f to take either one or two values there. So in the definition above, the value of $f_j(a_j)$ is taken to be $\lim_{x \searrow a_j} f_j(x)$ and $f_j(b_j) = \lim_{x \nearrow b_j} f_j(x)$, so for each j, f_j is well defined on $\overline{I_j}$.*

Remark A.2. *In contrast to the class of smooth maps $\mathcal{F}$ considered previously in this paper, for cusp maps we can have $\lambda_M = \infty$ and/or $\lambda_m = -\infty$. The first possibility follows since we allow singularities (points where the one-sided derivative is ∞). The second possibility follows from the presence of critical points (although is avoided for smooth multimodal maps with non-flat critical points by [Pr]). Examples of both of these possibilities can be found in [D1, Section 3.4].*

We will ultimately be interested in cusp maps without singular points with negative Schwarzian derivative (in fact the latter rules out the former). Note that since we are only interested in the transitive parts the system, transitive multimodal maps as in the rest of the paper can be considered to fit into this class.

We show below that we can build a Hofbauer extension $(\hat{I}, \hat{f})$. We note that the possible issue of f not being well defined at the boundaries of I_j, discussed in Remark A.1, does not change anything in the definition of the Hofbauer tower.

We next define the Hofbauer extension. The setup we present here can be applied to general dynamical systems, since it only uses the structure of dynamically defined cylinders. An alternative way of thinking of the Hofbauer extension specifically for the case of multimodal interval maps, which explicitly makes use of the critical set, is presented in [BB].

We let $C_n[x]$ denote the member of $\mathcal{P}_n$, which defined as above, containing x. If $x \in \cup_{n \geqslant 0} f^{-n}(\mathcal{C}r)$ there may be more than one such interval, but this ambiguity will not cause us any problems here.

The *Hofbauer extension* is defined as

$$\hat{I} := \bigsqcup_{k \geqslant 0} \bigsqcup_{C_k \in \mathcal{P}_k} f^k(C_k) / \sim$$

where $f^k(C_k) \sim f^{k'}(C_{k'})$ as components of the disjoint union $\hat{I}$ if $f^k(C_k) = f^{k'}(C_{k'})$ as subsets in I. Let $\mathcal{D}$ be the collection of domains of $\hat{I}$ and $\pi : \hat{I} \to I$ be the natural inclusion map. A point $\hat{x} \in \hat{I}$ can be represented by (x, D) where $\hat{x} \in D$ for $D \in \mathcal{D}$ and $x = \pi(\hat{x})$. Given $\hat{x} \in \hat{I}$, we can denote the domain $D \in \mathcal{D}$ it belongs to by $D_{\hat{x}}$.

The map $\hat{f} : \hat{I} \to \hat{I}$ is defined by

$$\hat{f}(\hat{x}) = \hat{f}(x, D) = (f(x), D')$$

if there are cylinder sets $C_k \supset C_{k+1}$ such that $x \in f^k(C_{k+1}) \subset f^k(C_k) = D$ and $D' = f^{k+1}(C_{k+1})$. In this case, we write $D \to D'$, giving $(\mathcal{D}, \to)$ the structure of a directed graph. Therefore, the map π acts as a semiconjugacy between $\hat{f}$ and f:

$$\pi \circ \hat{f} = f \circ \pi.$$

We denote the 'base' of $\hat{I}$, the copy of I in $\hat{I}$, by D_0. For $D \in \mathcal{D}$, we define $\mathrm{lev}(D)$ to be the length of the shortest path $D_0 \to \cdots \to D$ starting at the base D_0. For each $R \in \mathbb{N}$, let $\hat{I}_R$ be the compact part of the Hofbauer tower defined by

$$\hat{I}_R := \sqcup \{D \in \mathcal{D} : \mathrm{lev}(D) \leqslant R\}.$$

For maps in $\mathcal{F}$, we can say more about the graph structure of $(\mathcal{D}, \to)$ since Lemma 1 of [BT2] implies that if $f \in \mathcal{F}$ then there is a closed primitive subgraph $\mathcal{D}_{\mathcal{T}}$ of $\mathcal{D}$. That is, for any $D, D' \in \mathcal{D}_{\mathcal{T}}$ there is a path $D \to \cdots \to D'$; and for any $D \in \mathcal{D}_{\mathcal{T}}$, if there is a path $D \to D'$ then $D' \in \mathcal{D}_{\mathcal{T}}$ too. We can denote the disjoint union of these domains by $\hat{I}_{\mathcal{T}}$. The same lemma says that if $f \in \mathcal{F}$ then $\pi(\hat{I}_{\mathcal{T}}) = \Omega$, the non-wandering set and $\hat{f}$ is transitive on $\hat{I}_{\mathcal{T}}$. Theorem A.1 gives these properties for transitive cusp maps.

Given an ergodic measure $\mu \in \mathcal{M}_f$, we say that μ *lifts to* $\hat{I}$ if there exists an ergodic $\hat{f}$-invariant probability measure $\hat{\mu}$ on $\hat{I}$ such that $\hat{\mu} \circ \pi^{-1} = \mu$. For $f \in \mathcal{F}$, if $\mu \in \mathcal{M}_f$ is ergodic and $\lambda(\mu) > 0$ then μ lifts to $\hat{I}$, see [K1, BK].

Property $(*)$ is that for any $\hat{x}, \hat{y} \notin \partial \hat{I}$ with $\pi(x) = \pi(y)$ there exists n such that $\hat{f}^n(\hat{x}) = \hat{f}^n(\hat{y})$. This follows for cusp maps by the construction of $\hat{I}$ using the branch partition.

We will only use the following result in the context of equilibrium states for cusp maps with no singularities. However, for interest we state the theorem in greater generality.

Theorem A.1. *Suppose that $f : I \to I$ is a transitive cusp map with $h_{top}(f) > 0$. Then:*

(1) *there is a transitive part $\hat{I}_{\mathcal{T}}$ of the tower such that $\pi(\hat{I}_{\mathcal{T}}) = I$;*
(2) *any measure $\mu \in \mathcal{M}_f$ with $0 < \lambda(\mu) < \infty$ lifts to $\hat{\mu}$ with $\mu = \hat{\mu} \circ \pi^{-1}$;*
(3) *for each $\varepsilon > 0$ there exists $\eta > 0$ and a compact set $\hat{K} \subset \hat{I}_{\mathcal{T}} \setminus \partial \hat{I}$ such that any measure $\mu \in \mathcal{M}_f$ with $h(\mu) > \varepsilon$ and $0 < \lambda(\mu) < \infty$ has $\hat{\mu}(\hat{K}) > \eta$.*

Proof. Part (1): The first part can be shown as in [BT2, Lemma 2], but we argue as in [H] (see also [KStP, Theorem 6]). Theorem 11 of that paper gives a decomposition of $\hat{I}$ into a countable union Γ of irreducible (maximal with these properties) closed (if there is a path $D \to D'$ for $D \in \mathcal{E}$ then $D' \in \mathcal{E}$) primitive (there is a path between any $D, D' \in \mathcal{E}$) subgraphs $\mathcal{E}$ along with some sets which carry no entropy. Since $h_{top}(\hat{f}) = h_{top}(f)$ and we have positive topological entropy, this means that $\Gamma \neq \emptyset$. Let $\mathcal{E} \in \Gamma$. Clearly $\pi(\mathcal{E})$ is open, so by the transitivity of f, there must

be a point $x \in \pi(\mathcal{E})$ which has a dense orbit in I. By definition, $\omega(x) \subset \pi(\mathcal{E})$. By property $(*)$, $\pi(\mathcal{E}) \cap \pi(\mathcal{E}') = \emptyset$ for any $\mathcal{E}, \mathcal{E}' \in \Gamma$ which implies that $\#\Gamma = 1$. That there is a dense orbit in $(\mathcal{E}, \hat{f})$ follows from the Markov property of this subgraph, so we let $\hat{I}_{\mathcal{T}} = \sqcup_{D \in \mathcal{E}} D$.

Part (2): Ledrappier, in [L, Propostion 3.2] proved the existence of non-trivial local unstable manifolds for a more general class of maps (so-called PC-maps) with an ergodic measure $\mu \in \mathcal{M}_f$ with $\lambda(\mu) > 0$. However, he also required a non-degeneracy condition. For cusp maps, Dobbs [D4, Theorem 13] was able to this but without the non-degeneracy requirement.

Keller showed in [K1, Theorem 6] that the existence of such unstable manifolds means that any non-atomic ergodic measure $\mu \in \mathcal{M}_f$ with $\lambda(\mu) > 0$ lifts to $\hat{\mu}$ on $(\hat{I}, \hat{f})$ and that $\mu = \hat{\mu} \circ \pi^{-1}$. Using Dobbs and assuming that μ is not supported on $\cup_{n \geqslant 0} f^n(\mathcal{C}r)$ we can drop the non-atomic assumption (see also [BK, Theorem 3.6]).

Part (3): The third part follows exactly as in [BT2, Lemma 4]. $\qquad\square$

Suppose now that f is a cusp map without singularities (i.e. $|Df|$ is bounded above), with negative Schwarzian and such that the non-wandering set Ω is an interval. We consider $f : \Omega \to \Omega$. For each $t \in (t^-, t^+)$, we can find a finite number of inducing schemes as in Proposition 6.1 with which all measures with large enough free energy w.r.t. ψ_t will be compatible. It is important here that we assume negative Schwarzian derivative since we need bounded distortion for our inducing schemes. This then allows us to prove Theorem A for this class of maps, but we may have $t^- > -\infty$. If we exclude maps with preperiodic critical points then we again have $t^- = -\infty$. Similarly we can prove Theorem B for this class of maps, although again we only get $t^- = -\infty$ if we exclude maps with preperiodic critical points. Note also that the fact that λ_m can be negative, and may even be $-\infty$, implies that t^+, which for the class $\mathcal{F}$ had to lie in $[1, \infty]$, could be any value in the range $[0, \infty]$ for cusp maps.

Note that for the maps considered by Rovella in [Rov], the critical values are periodic and so the measure supported on them is not seen by our inducing schemes. This is like the Chebyshev case, so as in that situation, the pressure function could be piecewise affine.

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