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# NATURAL LIFTINGS OF FOLIATIONS TO THE TANGENT BUNDLE 

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Summary. A classification of natural liftings of foliations to the tangent bundle is given.
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## 0. Introduction

All manifolds are assumed to be finite dimensional, Hausdorff, without boundary and $C^{\infty}$. Mappings are assumed to be $C^{\infty}$ and foliations are assumed to be $C^{\infty}$ and without singularities. (For equivalent definitions of foliations see [2].)

From now on we fix two natural numbers $n$ and $p$ such that $p<n$. Suppose that to any $p$-dimensional foliation $F$ defined on an $n$-manifold $M$ there corresponds a foliation $L(F)$ on $T M$ projecting (by the tangent bundle projection) onto $F$. According to the general theory of natural transformations, see [1], we introduce the following definition.

Definition 0.1. A correspondence $L$ as above is called a natural lifting of foliations to the tangent bundle iff the following naturality condition is satisfied: for any foliation $F$ of dimension $p$ on an $n$-manifold $M$ and any diffeomorphism $\varphi$ from an $n$-manifold $N$ onto an open subset of $M$ we have $L\left(\varphi^{-1} F\right)=(\mathrm{d} \varphi)^{-1} L(F)$, where $\varphi^{-1} F$ is the inverse image of $F$ and $\mathrm{d} \varphi$ denotes the differential of $\varphi$.

We have the following examples of natural liftings of foliations to the tangent bundle.

Example 0.1. Let $F$ be a $p$-dimensional foliation on an $n$-manifold $M$. It is well-known that the tangent bundle $T M$ admits canonically defined foliations $L_{1}(F)$ of dimension $2 p$ and $L_{2}(F)$ of dimension $p+n$ projecting (by the tangent bundle
projection $\pi_{M}: T M \stackrel{\bullet}{\rightarrow} M$ ) onto the initial foliation $F$. More precisely, $L_{2}(F)=$ $\pi_{M}^{-1} F$, the inverse image, and $L_{1}(F)$ is defined by a cocycle $\left(\pi_{M}^{-1}\left(U_{i}\right), \mathrm{d} f_{i}, d g_{i j}\right)$, where $\left(U_{i}, f_{i}, g_{i j}\right)$ is a cocycle defining $F$. It is easy to verify that the correspondence $F \rightarrow L_{i}(F), i=1,2$, are natural liftings of foliations to the tangent bundle.

The main theorem in this paper is the following one.

Theorem 0.1. Any natural lifting of foliations to the tangent bundle belongs to the set $\left\{L_{1}, L_{2}\right\}$ described in Example 0.1.

## 1. Notation

From now on we use the following notation. We denote by $\partial_{1}, \ldots, \partial_{n}$ the canonical vector fields on $\mathbf{R}^{n}$, by $\partial$ the vector $\partial_{n} \mid 0$, by $\pi: T \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ the tangent bundle projection and by $F^{p}$ the standard $p$-dimensional foliation on $\mathbf{R}^{n}$ spanned by $\partial_{1}, \ldots$, $\partial_{p}$. By $e_{i}$ we denote the vector $(0, \ldots, 1, \ldots, 0) \in \mathbf{R}^{n}, 1$ in the $i$-th position.

From now on we identify a foliation with its tangent distribution.

## 2. Reducibility Lemma

The following Lemma plays an essential role in the proof of the main theorem.

Lemma 2.1. Let $L_{a}$ and $L_{b}$ be two natural liftings of foliations to the tangent bundle. Suppose that $L_{a}\left(F^{p}\right)_{\partial} \subset L_{b}\left(F^{p}\right)_{\partial}$. Then $L_{a}(F) \subset L_{b}(F)$ for any $p$-dimensional foliation $F$ on an $n$-manifold. In particular, if $L_{a}\left(F^{p}\right)_{\partial}=L_{b}\left(F^{p}\right)_{\partial}$, then $L_{a}=L_{b}$.

Proof. Let $F$ be a $\boldsymbol{p}$-dimensional foliation on an $n$-manifold M. Consider $z \in T M \backslash F$. By the Frobenius theorem there exists a diffeomorphism $\varphi$ from an open subset $U \subset R^{n}$ onto an open subset of $N$ such that $\varphi^{-1} F=i^{-1} F^{p}$ and $\mathrm{d} \varphi(\tilde{\partial})=z$; where $i: U \rightarrow \mathbf{R}^{n}$ is the inclusion and $\tilde{\partial} \in T U$ is the vector such that $\mathrm{d} i(\tilde{\partial})=\boldsymbol{\partial}$. Using the naturality condition we see that $L_{a}(F)_{z}=\mathrm{d}(\mathrm{d} \varphi)\left(L_{a}\left(i^{-1} F^{p}\right)_{\tilde{\partial}}\right)$ and $\mathrm{d}(\mathrm{d} i)\left(L_{a}\left(i^{-1} F^{p}\right)_{\tilde{\partial}}\right)=L_{a}\left(F^{p}\right)_{z}$, and similarly for $L_{b}$. Therefore the assumption of the Lemma implies that $L_{a}(F)_{z} \subset L_{b}(F)_{z}$. Since $T M \backslash F$ is dense in $T M$, we deduce that $L_{a}(F) \subset L_{b}(F)$.

## 3. Admissible subspaces

We introduce the following definition.
Definition 3.1. Let $z \in T_{0} \mathbf{R}^{n}$ be a vector. A global diffeomorphism $\varphi: \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ is called $z$-admissible iff $\varphi^{-1} F^{p}=F^{p}$ and $\mathrm{d} \varphi(z)=z$. A subspace $W \subset T_{z} T R^{n}$ is called $z$-admissible iff for any $z$-admissible diffeomorphism $\varphi$ we have $\mathrm{d}(\mathrm{d} \varphi)(W)=W$ and $\mathrm{d} \pi(W)=F_{0}^{p}$.

Using the naturality condition it is easy to verify the following Lemma.
Lemma 3.1. If $L$ is a natural lifting of foliations to the tangent bundle, then $L\left(F^{p}\right)_{z}$ is $z$-admissible for any $z \in T_{0} \mathbf{R}^{n}$.

Therefore, to prove Theorem 0.1 it is sufficient to verify the following proposition.
Proposition 3.1. Any $\partial$-admissible subspace contains $L_{1}\left(F^{p}\right)_{\theta}$. Any 0 -admissible subspace strictly containing $L_{1}\left(F^{p}\right)_{0}$ is equal to $L_{2}\left(F^{p}\right)_{0}$, where $L_{1}, L_{2}$ are described in Example 0.1.

From Proposition 3.1 and Lemmas 2.1 and 3.1 we deduce Theorem 0.1 in the following way. Consider a natural lifting $L \neq L_{1}$. By Lemmas 3.1 and 2.1 and Proposition 3.1 it follows that $L\left(F^{p}\right)_{\partial} \nsupseteq L_{1}\left(F^{p}\right)_{\partial}$. Then (by Lemma 2.1 and the dimension argument) $L\left(F^{p}\right)_{0} \supseteq L_{1}\left(F^{p}\right)_{0}$ and then (from Proposition 3.1) $L\left(F^{p}\right)_{0}$ has dimension $p+n$. Therefore $L\left(F^{p}\right)_{\partial}$ has dimension $p+n$, too. On the other hand, $L\left(F^{p}\right)_{\theta} \subset\left(d_{\theta} \pi\right)^{-1}\left(F_{0}^{p}\right)=L_{2}\left(F^{p}\right)_{\partial}$. Hence $L\left(F^{p}\right)_{\theta}=L_{2}\left(F^{p}\right)_{\theta}$. Therefore $L=L_{2}$ because of Lemma 2.1.

## 4. Transformation rules

We trivialize $T \mathbf{R}^{n}$ by the diffeomorphism

$$
\begin{equation*}
I: T \mathbf{R}^{n} \rightarrow \mathbf{R}^{2 n}, \quad I\left(\frac{\mathrm{~d}}{\mathrm{~d} t} \gamma(t)_{t=0}\right)=\left(\gamma(0), \gamma^{\prime}(0)\right), \tag{4.1}
\end{equation*}
$$

where $\frac{d}{d t} \gamma(t)_{t=0}$ is the vector generated by $\gamma$. Denote by $\left(\bar{\delta}_{i}\right)$ the canonical vector fields on $\mathbf{R}^{2 n}$. In the vector spaces $T_{\partial} T R^{n}$ and $T_{0} T \mathbf{R}^{n}\left(0 \in T_{0} R^{n}\right)$ we fix the following bases:

$$
\begin{equation*}
X_{i}=\mathrm{d} I^{-1}\left(\bar{\partial}_{i} \mid\left(0, e_{n}\right)\right), \quad V_{i}=\mathrm{d} I^{-1}\left(\bar{\partial}_{i+n} \mid\left(0, e_{n}\right)\right) \tag{4.2}
\end{equation*}
$$

and

$$
\begin{equation*}
X_{i}^{0}=\mathrm{d} I^{-1}\left(\bar{\partial}_{i} \mid 0\right), \quad V_{i}^{0}=\mathrm{d} I^{-1}\left(\bar{\partial}_{i+n} \mid 0\right), \tag{4.3}
\end{equation*}
$$

$i=1, \ldots, n$. We have the following transformation rules.
Lemma 4.1. Let $\varphi=\left(\varphi^{1}, \ldots, \varphi^{n}\right): \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ be a diffeomorphism such that $\mathrm{d} \varphi(\delta)=\partial$. Then for any $i=1, \ldots, n$ we have

$$
\begin{equation*}
\mathrm{d}(\mathrm{~d} \varphi)\left(V_{i}\right)=\partial_{i} \varphi^{j}(0) V_{j} \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{d}(\mathrm{~d} \varphi)\left(X_{i}\right)=\partial_{i}\left(\partial_{n} \varphi^{j}\right)(0) V_{j}+\partial_{i} \varphi^{j}(0) X_{j}, \tag{4.5}
\end{equation*}
$$

$j=1, \ldots, n$ (we use Einstein summation convention). Similarly, if $\psi: \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ is a linear isomorphism, then for any $i=1, \ldots, n$ we have

$$
\begin{equation*}
\mathrm{d}(\mathrm{~d} \psi)\left(V_{i}^{0}\right)=\partial_{i} \psi^{j}(0) V_{j}^{0}, \quad j=1, \ldots, n . \tag{4.6}
\end{equation*}
$$

Proof. We prove only formula (4.5). It is obvious that

$$
\begin{aligned}
\mathrm{d} I \circ \mathrm{~d}(\mathrm{~d} \varphi)\left(X_{i}\right) & =\mathrm{d} I \circ \mathrm{~d}(\mathrm{~d} \varphi) \mathrm{d} I^{-1}\left(\frac{\mathrm{~d}}{\mathrm{~d} t}\left(t e_{i}, e_{n}\right)_{t=0}\right) \\
& =\frac{\mathrm{d}}{\mathrm{~d} t}\left(I \circ \mathrm{~d} \varphi\left(\frac{\mathrm{~d}}{\mathrm{~d} \tau}\left(t e_{i}+\tau e_{n}\right)_{\tau=0}\right)\right)_{t=0} \\
& =\frac{\mathrm{d}}{\mathrm{~d} t}\left(\varphi\left(t e_{i}\right), \partial_{n} \varphi\left(t e_{i}\right)\right)_{t=0} \\
& =\partial_{i} \varphi^{j}(0) \bar{\partial}_{j}\left|\left(0, e_{n}\right)+\partial_{i} \partial_{n} \varphi^{j}(0) \bar{\partial}_{j+n}\right|\left(0, e_{n}\right) .
\end{aligned}
$$

This implies formula (4.5) immediately. We get (4.4) and (4.6) similarly.

## 5. The admissible subspaces given by $L_{1}$ and $L_{2}$

Let $L_{1}$ and $L_{2}$ be as in Example 0.1 and let $X_{i}, V_{i}, X_{i}^{0}, V_{i}^{0}$ be as in Section 4. We see that

$$
\begin{equation*}
\mathrm{d} \pi\left(V_{i}\right)=\mathrm{d} \pi\left(V_{i}^{0}\right)=0 \quad \text { and } \quad \mathrm{d} \pi\left(X_{i}\right)=\mathrm{d} \pi\left(X_{i}^{0}\right)=\partial_{i} \mid 0 \tag{5.1}
\end{equation*}
$$

for $i=1, \ldots, n$ and

$$
\mathrm{d}(\mathrm{~d} f)\left(V_{i}\right)=\mathrm{d}(\mathrm{~d} f)\left(V_{i}^{0}\right)=\mathrm{d}(\mathrm{~d} f)\left(X_{i}\right)=\mathrm{d}(\mathrm{~d} f)\left(X_{i}^{0}\right)=0
$$

for $i=1, \ldots, p$, where $f: R^{n} \rightarrow R^{n-p}, f(x)=\left(x^{p+1}, \ldots, x^{n}\right)$. Since $F^{p}$ is given by the cocycle ( $\left.\mathrm{R}^{n}, f, i d\right), L_{1}\left(F^{p}\right)$ is given by the cocycle ( $T R^{n}, \mathrm{~d} f, i d$ ), and then

$$
L_{1}\left(F^{p}\right)_{z}=\left(d_{z} \mathrm{~d} f\right)^{-1}(0)
$$

for any $z \in T_{0} \mathbf{R}^{n}$. Obviously

$$
L_{2}\left(F^{p}\right)_{z}=\left(d_{z} \pi\right)^{-1}\left(F_{0}^{p}\right)
$$

for any $z \in T_{0} \mathbf{R}^{n}$. The above facts complete the proof of the following formulas:

$$
\begin{align*}
L_{1}\left(F^{p}\right)_{\partial} & =\operatorname{span}\left(V_{1}, \ldots, V_{p}, X_{1}, \ldots, X_{p}\right)  \tag{5.2}\\
L_{1}\left(F^{p}\right)_{0} & =\operatorname{span}\left(V_{1}^{0}, \ldots, V_{p}^{0}, X_{1}^{0}, \ldots, X_{p}^{0}\right)  \tag{5.3}\\
L_{2}\left(F^{p}\right)_{\partial} & =\operatorname{span}\left(V_{1}, \ldots, V_{n}, X_{1}, \ldots, X_{p}\right) \text { and }  \tag{5.4}\\
L_{2}\left(F^{p}\right)_{0} & =\operatorname{span}\left(V_{1}^{0}, \ldots, V_{n}^{0}, X_{1}^{0}, \ldots, X_{p}^{0}\right) \tag{5.5}
\end{align*}
$$

## 6. Proof of the main theorem

It is sufficient to prove Proposition 3.1. By formulas (5.1) it follows that

$$
\begin{equation*}
W \subset\left(d_{\partial} \pi\right)^{-1}\left(F_{0}^{p}\right)=\operatorname{span}\left(V_{1}, \ldots, V_{n}, X_{1}, \ldots, X_{p}\right) \tag{6.1}
\end{equation*}
$$

for any $\partial$-admissible subspace $W$. Similarly, for any 0 -admissible subspace $W$ we have

$$
\begin{equation*}
W \subset \operatorname{span}\left(V_{1}^{0} \ldots, V_{n}^{0}, X_{1}^{0}, \ldots, X_{p}^{0}\right) \tag{6.2}
\end{equation*}
$$

First we prove the second part of Proposition 3.1. Let $W$ be a 0 -admissible subspace, such that $W \supseteq L_{1}\left(F^{p}\right)_{0}$. Then formulas (6.2) and (5.3) imply that there exists a vector $Y \in W \backslash\{0\}$ of the form

$$
Y=a^{p+1} V_{p+1}^{0}+\ldots+a^{n} V_{n}^{0}
$$

Let us consider a number $k \in\{p+1, \ldots, n\}$. There exists a linear isomorphism $\psi$ : $\mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ such that $\psi\left(e_{i}\right)=e_{i}$ for $i=1, \ldots, p$ and

$$
\psi\left(e_{k}\right)=a^{p+1} e_{p+1}+\ldots+a^{n} e_{n}
$$

Then $\psi^{-1}$ is 0 -admissible and $\mathrm{d}\left(\mathrm{d}\left(\psi^{-1}\right)\right)(Y)=V_{k}^{0}$ because of formula (4.6). Since $W$ is 0 -admissible and $Y \in W$, we have $V_{k}^{0} \in W$. Hence $W=L_{2}\left(F^{p}\right)_{0}$.

It remains to prove the first part of Proposition 3.1. Let $W$ be a $\partial$-admissible subspace. Then $\mathrm{d} \pi(W)=F_{0}^{p}$. Therefore formulas (5.1) imply that for any $j \in$ $\{1, \ldots, p\}$ there exist $Z_{j} \in \operatorname{span}\left(V_{1}, \ldots, V_{p}\right)$ and $Y_{j} \in \operatorname{span}\left(V_{p+1}, \ldots, V_{n}\right)$ such that

$$
\begin{equation*}
Z_{j}+Y_{j}+X_{j} \in W \tag{6.3}
\end{equation*}
$$

Hence the first part of Proposition 3.1 is a consequence of the following inclusions:

$$
\begin{gather*}
\operatorname{span}\left(V_{1}, \ldots, V_{p}\right) \subset W \text { and }  \tag{6.4}\\
W \subset \operatorname{span}\left(V_{1}, \ldots, V_{p}, X_{1}, \ldots, X_{p}\right) \cap W \oplus \operatorname{span}\left(V_{p+1}, \ldots, V_{n}\right) \cap W . \tag{6.5}
\end{gather*}
$$

(In fact, formulas (6.5) and (6.3) yield that $X_{j}+Z_{j} \in W$, and then $X_{j} \in W$ for $j=1, \ldots, p$ because of formula (6.4). Therefore $L_{1}\left(F^{p}\right)_{\partial} \subset W$ as follows from formulas (6.4) and (5.2).) First we prove inclusion (6.5). Let $\varphi: \mathbf{R}^{\boldsymbol{n}} \rightarrow \mathbf{R}^{\boldsymbol{n}}$ be the diffeomorphism given by $\varphi\left(y^{1}, \ldots, y^{n}\right)=\left(2 y^{1}, \ldots, 2 y^{p}, y^{p+1}, \ldots, y^{n}\right)$. Then $\varphi$ is $\partial-$ admissible. Consider an arbitrary $Y \in W$. Inclusion (6.1) implies that $Y=Y^{1}+Y^{2}$, where $Y^{1} \in \operatorname{span}\left(V_{p+1}, \ldots, V_{n}\right)$ and $Y^{2} \in \operatorname{span}\left(V_{1}, \ldots, V_{p}, X_{1}, \ldots, X_{p}\right)$ are vectors. Using Lemma 4.1 we see that $\mathrm{d}(\mathrm{d} \varphi)(Y)=Y^{1}+2 Y^{2}$. Since $Y \in W$ and $W$ is $\partial$-admissible we have $Y^{1}+2 Y^{2} \in W$ and then $Y^{1}, Y^{2} \in W$. Inclusion (6.5) is proved.

Now, we prove inclusion (6.4). Consider a number $k \in\{1, \ldots, p\}$. Let $\Phi: \mathbf{R}^{\boldsymbol{n}} \rightarrow \mathbf{R}^{\boldsymbol{n}}$ be given by

$$
\Phi\left(y^{1}, \ldots, y^{n}\right)=\left(y^{1}, \ldots, y^{k-1}, y^{k}+\frac{1}{2} y^{k} \sin \left(y^{n}\right), y^{k+1}, \ldots y^{n}\right)
$$

Then $\Phi$ is a global diffeomorphism. Evidently $\Phi$ is $\partial$-admissible. Let $X=Z_{k}+$ $Y_{k}+X_{k} \in W$ be as in formula (6.3). It follows from Lemma 4.1 that $\mathrm{d}(\mathrm{d} \Phi)(X)=$ $X+\frac{1}{2} V_{k}$. (For $\mathrm{d}(\mathrm{d} \Phi)\left(V_{i}\right)=V_{i}, i=1, \ldots, n$, i.e. $\mathrm{d}(\mathrm{d} \Phi)\left(Z_{k}+Y_{k}\right)=Z_{k}+Y_{k}$, and $\mathrm{d}(\mathrm{d} \Phi)\left(X_{k}\right)=X_{k}+\frac{1}{2} V_{k}$.) Since $W$ is $\partial$-admissible and $X \in W$, we get that $V_{k} \in W$. Inclusion (6.4) is proved.

Theorem 0.1 is proved.
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