NATURAL STRESS RATE*

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Abstract. A three-dimensional definition of the natural stress rate is suggested. The behavior of a hypoelastic material of grade zero based on the natural stress rate is analyzed for the case of simple extension. The results of the theoretical analysis agree with existing experimental data for urethane rubber. An interesting relation between the natural stress rate and stress rates suggested by Truesdell and Hill is given. The paper is written in full tensorial notation which is especially suitable when rates of tensor are considered.

1. Notation.

$\mathbf{A}, d\mathbf{A}, \mathbf{1}_A, \mathbf{F}, d\mathbf{F}, \mathbf{1}_F, \mathbf{R}, \mathbf{V}$	vectors.
$\mathbf{C}, \mathbf{I}$	fourth-order tensors.
$\mathbf{D}, \mathbf{G}, \mathbf{H}, \mathbf{L}, \nabla\mathbf{V}, \mathbf{V}\nabla, \boldsymbol{\sigma}, \boldsymbol{\Omega}$	second-order tensors.
$D_\alpha{}^\alpha$	the first scalar invariant of the second-order tensor $\mathbf{D}$.
$\mathbf{e}_i$	constant unit base vectors.
$\mathbf{G}^\alpha, \mathbf{G}_\beta$	base vectors, moving with the material points.
$\nabla\mathbf{V}$	$= \mathbf{G}^\alpha(\partial\mathbf{V}/\partial\xi^\alpha)$.
$\mathbf{V}\nabla$	$= (\partial\mathbf{V}/\partial\xi^\beta)\mathbf{G}^\beta$.
$\boldsymbol{\sigma}\cdot d\mathbf{A} = (\sigma^{\alpha\beta}\mathbf{G}_\alpha\mathbf{G}_\beta)\cdot(dA_\gamma\mathbf{G}^\gamma)$	$= \sigma^{\alpha\beta}\, dA_\beta\mathbf{G}_\alpha$.
$\mathbf{C}\cdot\cdot\,\mathbf{D} = (C^{\alpha\beta\gamma\delta}\mathbf{G}_\alpha\mathbf{G}_\beta\mathbf{G}_\gamma\mathbf{G}_\delta)\cdot\cdot\,(D_{\phi\psi}\mathbf{G}^\phi\mathbf{G}^\psi)$	$= C^{\alpha\beta\gamma\delta}D_{\delta\gamma}\mathbf{G}_\alpha\mathbf{G}_\beta$.

2. Introduction. The problem of the "proper" stress rate has been considered by several authors, e.g., [1–3]. This problem, as well as the related problem of the measure of an incremental change of a continuum, is important when a rate-constitutive equation is to be constructed.

It is known that, even after introducing the two basic requirements of symmetry and objectivity, there is an infinite number of possible definitions of the stress rate. Some of these definitions appear to be more popular, in view of certain physical and geometrical considerations. Jaumann's definition, for instance, is preferable to the other definitions for use in constitutive equations of plasticity [1, 3–5].

A new definition of a "natural" stress rate is introduced in the present paper. This definition is strongly motivated by a recent work [6] where the interesting idea of a

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natural stress has been suggested. This natural stress is defined, for the unidirectional case, by

$$\tau = \int_0^t \frac{(d/dt)(dF)}{dA}\, dt = \int_0^t \frac{d\dot{F}}{dA}\, dt \tag{1}$$

where dA is the current elementary area and $d\dot{F}$ is the rate of the elementary force. The integration is performed over the whole process (t denotes time). The relation between the natural stress and the true stress $\sigma = dF/dA$ can be found by

$$\dot{\sigma} = \frac{d}{dt}\left(\frac{dF}{dA}\right) = \frac{(d/dt)(dF)}{dA} - \frac{dF}{(dA)^2}\cdot\frac{d}{dt}(dA) = \frac{d\dot{F}}{dA} - \sigma\frac{d\dot{A}}{dA}. \tag{2}$$

Substitution of (2) into (1) yields

$$\tau = \sigma + \int_0^t \sigma \frac{d\dot{A}}{dA}\, dt. \tag{3}$$

The definition (1) is analogous to the well-known definition of the natural strain. It is demonstrated in [6] that the natural stress-natural strain relation is linear for several urethane rubbers up to high levels of strain. It is therefore interesting to try to make a further investigation about the nature of (1).

The basic question which arises here is: what is the three-dimensional tensorial generalization of (1)? In an attempt to answer this question we note that a more fundamental concept is hidden in (1). In fact, (1) defines the natural unidirectional stress-rate,

$$d\dot{F}/dA = \dot{\tau} \tag{4}$$

where again the dot denotes differentiation with respect to time.

A three-dimensional tensorial generalization of (4) is suggested in the present work. The proposed definition, given by (15), degenerates to (4) in the unidirectional case. The natural stress tensor is defined as the time integral of the natural stress rate tensor.

A hypoelastic material of grade zero based on the natural stress rate is proposed and its behavior in simple extension is analysed. The results of this analysis agree very well with experimental results given in [6].

It is also shown that the natural stress rate is the arithmetic mean of the two stress rates introduced by Truesdell and Hill.

3. The natural stress rate. Consider a deforming continuum. Let ξ^α ($\alpha = 1, 2, 3$) be the material coordinates and $\mathbf{R}(\xi^\alpha, t)$ the radius vector of each material point. $\mathbf{G}_\alpha$ and $\mathbf{G}^\alpha$ denote, as usual, the covariant and contravarient base vectors. At a given time the elementary force $\mathbf{dF} = dF^\alpha\mathbf{G}_\alpha = dF_\alpha\mathbf{G}^\alpha$, acting on an elementary area $d\mathbf{A} = dA^\alpha\mathbf{G}_\alpha = dA_\alpha\mathbf{G}^\alpha$, is given by

$$\mathbf{dF} = \boldsymbol{\sigma}\cdot\mathbf{dA} = \mathbf{dA}\cdot\boldsymbol{\sigma} \tag{5}$$

where $\boldsymbol{\sigma}$ is the symmetric second-order stress tensor

$$\boldsymbol{\sigma} = \sigma^{\alpha\beta}\mathbf{G}_\alpha\mathbf{G}_\beta = \cdots = \sigma_{\alpha\beta}\mathbf{G}^\alpha\mathbf{G}^\beta. \tag{6}$$

From (5) it follows that

$$\mathbf{dF}\cdot\mathbf{dF} = \mathbf{dA}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\mathbf{dA}. \tag{7}$$

Taking now the time derivative of (7) yields

$$2(dF)(d\dot{F}) = (\mathbf{d}\dot{\mathbf{A}}\cdot\boldsymbol{\sigma} + \mathbf{dA}\cdot\dot{\boldsymbol{\sigma}})\cdot\mathbf{dF} + \mathbf{dF}\cdot(\dot{\boldsymbol{\sigma}}\cdot\mathbf{dA} + \boldsymbol{\sigma}\cdot\mathbf{d}\dot{\mathbf{A}}). \tag{8}$$

It is recalled that the rate of change of $\mathbf{dA}$ is given by ([4, 5])

$$\mathbf{d}\dot{\mathbf{A}} = D_\alpha{}^\alpha\mathbf{dA} - \mathbf{L}\cdot\mathbf{dA} \tag{9}$$

where $\mathbf{L}$ is the left gradient of the velocity $\mathbf{V}$ (obviously $\mathbf{V} = \dot{\mathbf{R}}$)

$$\mathbf{L} = \nabla\mathbf{V}, \tag{10}$$

$\nabla = \mathbf{G}^\alpha(\partial/\partial\xi^\alpha)$ and $\mathbf{D}$ is the linear strain rate:

$$\mathbf{D} = \tfrac{1}{2}(\nabla\mathbf{V} + \mathbf{V}\nabla). \tag{11}$$

The symbol $D_\alpha{}^\alpha$ stands for the first invariant of a second-order tensor which is obtained by inner scalar multiplication [7]; e.g.$\mathbf{D}_\alpha{}^\alpha = \nabla\cdot\mathbf{V}$. Substituting (9) into (8) and rearranging yields

$$d\dot{F}/dA = \tfrac{1}{2}[\mathbf{1}_A\cdot(\dot{\boldsymbol{\sigma}} - \mathbf{H}\cdot\boldsymbol{\sigma} + D_\alpha{}^\alpha\boldsymbol{\sigma})\cdot\mathbf{1}_F + \mathbf{1}_F\cdot(\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot\mathbf{L} + D_\alpha{}^\alpha\boldsymbol{\sigma})\cdot\mathbf{1}_A] \tag{12}$$

where $\mathbf{H} = \mathbf{V}\nabla = \mathbf{L}^T$ and $\mathbf{1}_A$, $\mathbf{1}_F$ are unit vectors in the directions of $\mathbf{dA}$ and $\mathbf{dF}$ respectively.

The structure of Eq. (12) is similar to that of Eq. (4). It is therefore reasonable to consider the following symmetric stress rate:

$$\tfrac{1}{2}[(\dot{\boldsymbol{\sigma}} - \mathbf{H}\cdot\boldsymbol{\sigma} + D_\alpha{}^\alpha\boldsymbol{\sigma}) + (\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot\mathbf{L} + D_\alpha{}^\alpha\boldsymbol{\sigma})] = \dot{\boldsymbol{\sigma}} - \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{L} + \mathbf{H}\cdot\boldsymbol{\sigma}) + D_\alpha{}^\alpha\boldsymbol{\sigma} \tag{13}$$

as a logical generalization of $\dot{\tau}$ from (4).

Unfortunately, the basic requirement of objectivity is not met by (13). This point has been overlooked in [4, 5] and will be clarified here. When the continuum performs a rigid body motion, $\mathbf{D} = \mathbf{0}$ and (13) becomes

$$\overset{\circ}{\boldsymbol{\sigma}} - \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}) = \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}) \tag{14}$$

where $\overset{\circ}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$ is the rigid body derivative of the stress tensor [5] and $\boldsymbol{\Omega} = \tfrac{1}{2}(\nabla\mathbf{V} - \mathbf{V}\nabla)$ is the spin tensor. Since (14) is in general different from zero it follows that (13) does not vanish during a rigid body motion and hence is not objective.

We suggest now the definition of the natural stress rate tensor, denoted by $\overset{*}{\boldsymbol{\sigma}}$, as the difference between Eqs. (13) and (14), namely

$$\overset{*}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot(\tfrac{1}{2}\mathbf{D} + \boldsymbol{\Omega}) - (\tfrac{1}{2}\mathbf{D} - \boldsymbol{\Omega})\cdot\boldsymbol{\sigma} + D_\alpha{}^\alpha\boldsymbol{\sigma}. \tag{15}$$

It is clear from the above discussion that $\overset{*}{\boldsymbol{\sigma}}$ is symmetric and objective and is, at least from that point of view, a possible stress rate. The relation between $\overset{*}{\boldsymbol{\sigma}}$ and Jaumann's stress rate $\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \overset{\circ}{\boldsymbol{\sigma}}$ is given by

$$\overset{*}{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} - \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma}) + D_\alpha{}^\alpha\boldsymbol{\sigma}. \tag{16}$$

It seems that $\overset{*}{\boldsymbol{\sigma}}$ is the three-dimensional tensorial generalization of $\dot{\tau}$. We may now define the natural stress tensor $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = \int_0^t \overset{*}{\boldsymbol{\sigma}}\,dt = \boldsymbol{\sigma} - \int_0^t [\boldsymbol{\sigma}\cdot(\tfrac{1}{2}\mathbf{D} + \boldsymbol{\Omega}) + (\tfrac{1}{2}\mathbf{D} - \boldsymbol{\Omega})\cdot\boldsymbol{\sigma} - D_\alpha{}^\alpha\boldsymbol{\sigma}]\,dt. \tag{17}$$

It is easy to see that for the one-dimensional case Eq. (17) reduces exactly to Eq. (3).

4. A constitutive equation. Consider a material whose constitutive equation is given by

$$\overset{\bullet}{\boldsymbol{\sigma}} = \mathbf{C} \cdot\cdot \mathbf{D} \tag{18}$$

where $\mathbf{C}$ is the classical elastic material tensor

$$\mathbf{C} = 2\mu\mathbf{I} + \lambda\mathbf{G}\mathbf{G}; \tag{19}$$

μ, λ are the Lamé coefficients and $\mathbf{G}$, $\mathbf{I}$ are the second- and fourth-order unit tensors respectively:

$$\mathbf{G} = \mathbf{G}^\alpha\mathbf{G}_\alpha\,, \qquad \mathbf{I} = \tfrac{1}{2}\mathbf{G}^\alpha\mathbf{G}^\beta(\mathbf{G}_\alpha\mathbf{G}_\beta + \mathbf{G}_\beta\mathbf{G}_\alpha). \tag{20}$$

Substituting (19) into (18) yields

$$\overset{\bullet}{\boldsymbol{\sigma}} = 2\mu\mathbf{D} + \lambda D_\alpha{}^\alpha\mathbf{G}. \tag{21}$$

Using modern terminology of continuum mechanics, a material whose behavior is governed by (21) defines some kind of hypoelastic material of grade zero.

It is mentioned that Eq. (18) should by no means be confused with the equation $\boldsymbol{\tau} = \mathbf{C} \cdot\cdot \mathbf{E}_L$ where $\mathbf{E}_L$ is the finite logarithmic strain tensor. In fact, such an equation is not admissible since by taking its time derivative, noting that $\dot{\mathbf{C}} = \mathbf{0}$, we obtain $\overset{\bullet}{\boldsymbol{\sigma}} = \mathbf{C} \cdot\cdot \dot{\mathbf{E}}_L$ which contains the non-objective tensor $\dot{\mathbf{E}}_L$.

5. Simple extension. The uniaxial tension experiment is one of the more important sources available for constructing or justifying a constitutive equation of a solid. We shall consider therefore the behavior of the material described by (21) in simple extension. The result will be compared with the experimental data given in [6].

Let $\mathbf{e}_i$ ($i = 1, 2, 3$) be a system of orthonormal spatial vectors. We choose the material coordinates ξ^α so that the initial undeformed radius vector is given by

$$\mathbf{R}_{(0)} = \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3\,. \tag{22}$$

The finite displacement vector is written as

$$\mathbf{U} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3 \tag{23}$$

where, in accordance with the character of the deformation, it is assumed that $u = u(\xi^1, t)$, $v = v(\xi^2, t)$ and $w = w(\xi^3, t)$.

The ∇ operator and the velocity vector $\mathbf{V}$ follow from (23):

$$\nabla = \frac{\mathbf{e}_1}{1 + u_{,1}}\frac{\partial}{\partial\xi^1} + \frac{\mathbf{e}_2}{1 + v_{,2}}\frac{\partial}{\partial\xi^2} + \frac{\mathbf{e}_3}{1 + w_{,3}}\frac{\partial}{\partial\xi^3}\,, \tag{24}$$

$$\mathbf{V} = \dot{u}\mathbf{e}_1 + \dot{v}\mathbf{e}_2 + \dot{w}\mathbf{e}_3\,, \tag{25}$$

where $(\ )_{,\alpha} = \partial(\ )/\partial\xi^\alpha$.

Substituting now (24)–(25) into (11) yields

$$\mathbf{D} = [\dot{\ln}\,(1 + u_{,1})]\mathbf{e}_1\mathbf{e}_1 + [\dot{\ln}\,(1 + v_{,2})]\mathbf{e}_2\mathbf{e}_2 + [\dot{\ln}\,(1 + w_{,3})]\mathbf{e}_3\mathbf{e}_3 \tag{26a}$$

$$D_\alpha{}^\alpha = \dot{\ln}\,[(1 + u_{,1})(1 + v_{,2})(1 + w_{,3})] \tag{26b}$$

while the spin tensor vanishes:

$$\boldsymbol{\Omega} = \mathbf{0}. \tag{27}$$

Assume that the external force is applied in the direction of $\mathbf{e}_3$. Accordingly, the stress tensor is assumed to be ($P\mathbf{e}_3$ is the applied traction)

$$\boldsymbol{\sigma} = P\mathbf{e}_3\mathbf{e}_3 . \tag{28}$$

Substituting now Eqs. (26)–(28) into (15), we find that in this case the natural stress rate is given by

$$\overset{*}{\boldsymbol{\sigma}} = \{\dot{P} + [D_\alpha{}^\alpha - \ln (1 + w_{,3})]P\}\mathbf{e}_3\mathbf{e}_3 . \tag{29}$$

By limiting the discussion to quasi-static loading, it is easily verified that the equilibrium equation $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ is satisfied by (28). Therefore it remains to consider only the constitutive equation (21). Substituting (26) and (29) into (21), and remembering that $\mathbf{G} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$, yields a tensorial equation which separates into three scalar equations:

$$2\mu \, \dot{\ln} \, (1 + u_{,1}) + \lambda D_\alpha{}^\alpha = 0, \tag{30a}$$

$$2\mu \, \dot{\ln} \, (1 + v_{,2}) + \lambda D_\alpha{}^\alpha = 0, \tag{30b}$$

$$(2\mu + P) \, \dot{\ln} \, (1 + w_{,3}) + (\lambda - P)D_\alpha{}^\alpha = \dot{P}, \tag{30c}$$

or

$$\dot{\ln} \, (1 + u_{,1}) = \dot{\ln} \, (1 + v_{,2}) = -\nu \, \dot{\ln} \, (1 + w_{,3}), \tag{31a}$$

$$\dot{\ln} \, (1 + w_{,3}) = \frac{1}{2\nu} \, \dot{\ln} \left(1 + 2\nu \frac{P}{E}\right). \tag{31b}$$

Eqs. (31) can be integrated completely, after adding the usual initial conditions of zero stress at $t = 0$ and fixing the point (0, 0, 0). The integral of (31b) is

$$w = \left[\left(1 + 2\nu \frac{P}{E}\right)^{1/2\nu} - 1\right]\xi^3, \tag{32}$$

which can be written also as

$$\frac{P}{E} = \frac{1}{2\nu}\left[\left(\frac{L}{L_0}\right)^{2\nu} - 1\right] \tag{33}$$

where L_0 and L are the initial and final length respectively. Thus, the behavior of the material (21) is described by (33).

The natural stress tensor $\boldsymbol{\tau}$ can be found from the definition (17) by integrating (29). Substituting (31b) and noting that

$$D_\alpha{}^\alpha = \frac{1 - 2\nu}{2\nu} \, \dot{\ln} \left(1 + 2\nu \frac{P}{E}\right) \tag{34}$$

yields, after integration,

$$\boldsymbol{\tau} = \frac{E}{2\nu}\left[\ln \left(1 + 2\nu \frac{P}{E}\right)\right]\mathbf{e}_3\mathbf{e}_3 = \tau\mathbf{e}_3\mathbf{e}_3 , \tag{35}$$

where τ is the one-dimensional component of the natural stress. Now, it follows from (33) and (35) that

$$\tau = E \ln (L/L_0). \tag{36}$$

The natural stress—natural strain (ln (L/L_0)) relation is linear. It is remarkable that the same result has been obtained in experiments performed on urethane rubber [6]. An additional experimental result reported in [6] (Fig. 2) and predicted by the present analysis is the linear relation between the natural stress and the natural transverse strain, which follows from (31), (36),

$$\tau = -\frac{\nu}{E} \ln (1 + u_{,1}). \tag{37}$$

It is also noted that the linearity of (36)–(37) is independent of ν. For an incompressible material, ($\nu = \frac{1}{2}$), Eq. (33) becomes (see also [6])

$$P = E((L/L_0) - 1). \tag{38}$$

6. A different interpretation of the natural stress rate. Although we suggest definition (15) as the three-dimensional tensorial generalization of (4), it is recalled that Truesdell's stress rate is based on similar reasoning. Truesdell obtained [8] an expression for the stress rate $\overset{T}{\boldsymbol{\sigma}}$ considering the equation (with the present notation)

$$d\dot{F}^\alpha \mathbf{G}_\alpha = \overset{T}{\boldsymbol{\sigma}} \cdot d\mathbf{A} \tag{39}$$

with the solution ([4, 5]),

$$\overset{T}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{H} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{L} + D_\alpha{}^\alpha \boldsymbol{\sigma} \tag{40}$$

which is a possible stress rate.

Another stress rate can be obtained by considering the time derivative of the covariant component of the vector $\mathbf{dF}$. It can be shown that ([4, 5])

$$d\dot{F}_\alpha \mathbf{G}^\alpha = (\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \mathbf{L} + \mathbf{L} \cdot \boldsymbol{\sigma} + D_\alpha{}^\alpha \boldsymbol{\sigma}) \cdot d\mathbf{A} = d\mathbf{A} \cdot (\dot{\boldsymbol{\sigma}} - \mathbf{H} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{H} + D_\alpha{}^\alpha \boldsymbol{\sigma}). \tag{41}$$

Neither of the expressions in the brackets is symmetric but an acceptable stress rate $\overset{H}{\boldsymbol{\sigma}}$ can be formed by taking their mean value:

$$\overset{H}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega} + D_\alpha{}^\alpha \boldsymbol{\sigma} \tag{42}$$

This stress rate has been suggested by Hill [9], by applying Jaumann's derivative on the Kirchhoff stress tensor.

Eqs. (39) and (41) resemble, in their structure, Eq. (4). The two associated stress rates (40), (42) are different from the previous definition of the natural stress rate (15), but the following relation is noted:

$$\overset{*}{\boldsymbol{\sigma}} = \tfrac{1}{2}(\overset{T}{\boldsymbol{\sigma}} + \overset{H}{\boldsymbol{\sigma}}) \tag{43}$$

which is a different and more aesthetic interpretation of the natural stress rate.

7. Concluding remarks. The fact that the behavior of urethane rubber in simple extension is given by the constitutive equation (21) is of course not a sufficient justification for the use of (21) as the constitutive equation of that rubber. Even for this simple deformation the theoretical results (36)–(37) are valid for moderately large strains only. Additional basic cases of loading should therefore be analyzed and compared with experiments. A theoretical analysis for biaxial loading is also available and will be given on another occasion.

However, it is interesting to note that the behavior, within certain limits, of a material that is usually regarded as a hyperelastic material can be analysed within the framweork of hypoelasticity.

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Note added in proof: Eq. (16) is included in Hill's family of stress rates [10, Eq. (1)], given by (in the present notation),

$$\overset{\nabla}{\boldsymbol{\sigma}} - m(\boldsymbol{\sigma}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma}) + D_{\alpha}{}^{\alpha}\boldsymbol{\sigma}$$

with $m = 1/2$.