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Natural transformations of higher order cotangent bundle functors

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Abstract. We determine all natural transformations of the rth order cotangent bundle functor T^{r*} into T^{s*} in the following cases: $r = s$, $r < s$, $r > s$. We deduce that all natural transformations of T^{r*} into itself form an r-parameter family linearly generated by the pth power transformations with $p = 1, \ldots, r$.

Using general methods developed in [2]–[5], we deduce that all natural transformations of the rth order cotangent bundle functor T^{r*} into itself form an r-parameter family generated by the pth power transformations $A_p^{r,r}$ with $p = 1, \ldots, r$.

Then we deduce that all natural transformations of T^{r*} into $T^{(r+q)*}$ form an r-parameter family generated by the generalized pth power transformations $A_p^{r,r+q}$ with $p = q + 1, \ldots, q + r$.

Moreover, we deduce that all natural transformations of T^{r*} into $T^{(r-q)*}$ form an $(r - q)$-parameter family generated by the generalized pth power transformations $A_p^{r,r-q}$ with $p = 1, \ldots, r - q$.

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1. Let M be a smooth n-dimensional manifold. Let $T^{r*}M = J^r(M, \mathbb{R})_0$ be the space of all r-jets $j_x^r f$ of smooth functions $f : M \to \mathbb{R}$ with source at $x \in M$ and target at $0 \in \mathbb{R}$. The fibre bundle $\pi_M : T^{r*}M \to M$ with source r-jet projection $\pi_M : j_x^r f \mapsto x$ has a canonical structure of a vector bundle with

$$(1.1) \qquad j_x^r f + j_x^r g = j_x^r (f + g)\,, \qquad k \cdot j_x^r f = j_x^r (k \cdot f)$$

for $x \in M$ and $k \in \mathbb{R}$ [1].

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The vector bundle $\pi_M : T^{r*}M \to M$ is called the *r-th cotangent bundle* over M.

Every local diffeomorphism $\varphi : M \to N$ is extended to a vector bundle morphism $T^{r*}\varphi : T^{r*}M \to T^{r*}N$, $j^r_x f \mapsto j^r_{\varphi(x)}(f\circ\varphi^{-1})$, where φ^{-1} is locally defined. Hence, the rth cotangent bundle functor T^{r*} is defined on the category $\mathcal{M}f_n$ of smooth n-dimensional manifolds with local diffeomorphisms as morphisms and has values in the category $\mathcal{VB}$ of vector bundles.

If (x^i) are local coordinates on M, then we have the induced fibre coordinates $(u_i, u_{i_1i_2}, \ldots, u_{i_1\ldots i_r})$ on $T^{r*}M$ (symmetric in all indices):

$$u_i(j^r_x f) = \frac{\partial f}{\partial x^i}\bigg|_{(x)}, \qquad u_{i_1i_2}(j^r_x f) = \frac{\partial^2 f}{\partial x^{i_1}\partial x^{i_2}}\bigg|_{(x)}, \ldots,$$
$$u_{i_1\ldots i_r}(j^r_x f) = \frac{\partial^r f}{\partial x^{i_1}\ldots\partial x^{i_r}}\bigg|_{(x)}. \tag{1.2}$$

Since the functor T^{r*} takes values in the category $\mathcal{VB}$ of vector bundles, we may define natural transformations $A^{r,r}_p$ of T^{r*} into itself for $p = 1, \ldots, r$.

Definition 1. The natural transformation $A^{r,r}_p$ of the rth cotangent bundle functor T^{r*} into itself defined by

$$A^{r,r}_p : j^r_x f \mapsto j^r_x(f)^p\,, \tag{1.3}$$

where $(f)^p$ denotes the pth power of f, is called the *p-th power transformation*.

Definition 2. The natural transformation $A^{r,s}_p$ of T^{r*} into T^{s*} defined by

$$A^{r,s}_p : j^r_x f \mapsto j^s_x(f)^p \tag{1.4}$$

is called the *generalized p-th power transformation*.

We note that in the case $s = r + q$ this definition is correct only for $p = q + 1, \ldots, q + r$.

Definition 3. The natural transformation $P^{r,r-q}$ of T^{r*} into $T^{(r-q)*}$ defined by

$$P^{r,r-q} : j^r_x f \mapsto j^{r-q}_x f \tag{1.5}$$

is called a *projection*.

Note that

$$A^{r,r-q}_p = A^{r-q,r-q}_p \circ P^{r,r-q} \qquad \text{for } p = 1, \ldots, r-q\,. \tag{1.6}$$

2. In this part we determine, by induction on r, all natural transformations of T^{r*} into itself.

THEOREM 1. *All natural transformations $A : T^{r*} \to T^{r*}$ form the r-parameter family*

$$A = k_1 A_1^{r,r} + k_2 A_2^{r,r} + \ldots + k_r A_r^{r,r} \tag{2.1}$$

with any real parameters $k_1, k_2, \ldots, k_r \in \mathbb{R}$.

Proof. The functor T^{r*} is defined on the category $\mathcal{M}f_n$ of n-dimensional smooth manifolds with local diffeomorphisms as morphisms and is of order r. Thus, its standard fibre $S = (T^{r*}\mathbb{R}^n)_0$ is a G_n^r-space, where G_n^r is the group of all invertible r-jets from $\mathbb{R}^n$ into $\mathbb{R}^n$ with source and target at 0.

According to general standard methods [2]–[5], the natural transformations $A : T^{r*} \to T^{r*}$ are in bijection with the G_n^r-equivariant maps $f^{r,r} : (T^{r*}\mathbb{R}^n)_0 \to (T^{r*}\mathbb{R}^n)_0$ of the standard fibres.

Let $\widetilde{a} = a^{-1}$ denote the inverse element in G_n^r and let

$$t_{(i_1\ldots i_r)} = \frac{1}{r!} \sum_{\sigma\in S_r} t_{i_{\sigma(1)}\ldots i_{\sigma(r)}} \tag{2.2}$$

denote the symmetrization of a tensor with components $t_{i_1\ldots i_r}$.

By (1.2) the action of an element $(a^i_{j_1}, a^i_{j_1 j_2}, \ldots, a^i_{j_1\ldots j_r}) \in G_n^r$ on $(u_{i_1}, u_{i_1 i_2}, \ldots, u_{i_1\ldots i_r}) \in (T^{r*}\mathbb{R}^n)_0$ is given by

$$\begin{aligned}
&\overline{u}_{i_1} = u_{j_1}\widetilde{a}^{j_1}_{i_1}, \qquad \overline{u}_{i_1 i_2} = u_{j_1 j_2}\widetilde{a}^{j_1}_{i_1}\widetilde{a}^{j_2}_{i_2} + u_{j_1}\widetilde{a}^{j_1}_{i_1 i_2}, \ldots,\\
&\overline{u}_{i_1\ldots i_r} = u_{j_1\ldots j_r}\widetilde{a}^{j_1}_{i_1}\ldots\widetilde{a}^{j_r}_{i_r} + u_{j_1\ldots j_{r-1}}\frac{r!}{(r-2)!2!}\widetilde{a}^{j_1}_{(i_1}\ldots\widetilde{a}^{j_{r-2}}_{i_{r-2}}\widetilde{a}^{j_{r-1}}_{i_{r-1}i_r)} + \ldots\\
&\qquad + u_{j_1\ldots j_{r-s}}\left[\frac{r!}{(r-s-1)!(s+1)!}\widetilde{a}^{j_1}_{(i_1}\ldots\widetilde{a}^{j_{r-s-1}}_{i_{r-s-1}}\widetilde{a}^{j_{r-s}}_{i_{r-s}\ldots i_r)} + \ldots\right]\\
&\qquad + u_{j_1 j_2}\left[\frac{r!}{(r-1)!1!}\widetilde{a}^{j_1}_{(i_1}\widetilde{a}^{j_2}_{i_2\ldots i_r)} + \ldots\right] + u_{j_1}\widetilde{a}^{j_1}_{i_1\ldots i_r}\,.
\end{aligned} \tag{2.3}$$

I. Consider the case $r = 2$. The equivariance of a G_n^2-equivariant map $f^{2,2} = (f_i, f_{ij})$ of $(T^{2*}\mathbb{R}^n)_0$ into itself with respect to homotheties in G_n^2: $\widetilde{a}^i_j = k\delta^i_j$, $\widetilde{a}^i_{j_1 j_2} = 0$, gives the homogeneity conditions

$$k f_i(u_i, u_{ij}) = f_i(k u_i, k^2 u_{ij})\,, \qquad k^2 f_{ij}(u_i, u_{ij}) = f_{ij}(k u_i, k^2 u_{ij})\,. \tag{2.4}$$

By the homogeneous function theorem [2]–[5], we deduce that, first, the f_i are linear in u_i and independent of u_{ij}, and secondly, the f_{ij} are linear in u_{ij} and quadratic in u_i. Using the invariant tensor theorem for G_n^1 [2]–[5], we obtain $f^{2,2}$ in the form

$$f_i = k_1 u_i\,, \qquad f_{ij} = k_2 u_i u_j + k_3 u_{ij} \tag{2.5}$$

with any real parameters $k_1, k_2, k_3 \in \mathbb{R}$.

The equivariance of $f^{2,2}$ of the form (2.5) with respect to the kernel of the projection $G_n^2 \to G_n^1 : \widetilde{a}^i_j = \delta^i_j$ and $\widetilde{a}^i_{jk}$ arbitrary, gives

$$k_3 = k_1 \,. \tag{2.6}$$

This proves our theorem for $r = 2$.

II. Suppose that the theorem holds for $r-1$ and the G_n^{r-1}-equivariant maps $f^{r-1,r-1}$ of $(T^{(r-1)*}\mathbb{R}^n)_0$ into itself define the $(r-1)$–parameter family $\overline{A} = k_1 A_1^{r-1,r-1} + \ldots + k_{r-1} A_{r-1}^{r-1,r-1}$ with any real parameters $k_1, \ldots, k_{r-1} \in \mathbb{R}$.

Our aim is to obtain the general form of any G_n^r-equivariant map of $(T^{r*}\mathbb{R}^n)_0$ into itself.

Let $(u_1, u_2, \ldots, u_r) := (u_{i_1}, u_{i_1 i_2}, \ldots, u_{i_1 \ldots i_r})$ denote the fibre coordinates on $T^{r*}M$. We assume that a G_n^r-equivariant map $f^{r,r}$ is of the general form $f^{r,r} = (f_1, \ldots, f_{r-1}, f_r)$ and the given map $f^{r-1,r-1}$ defines the first $r-1$ components $(f_1, \ldots, f_{r-1})$ of $f^{r,r}$.

Considering the equivariance of $f^{r,r}$ with respect to the homotheties $\widetilde{a}^i_j = k\delta^i_j$, $\widetilde{a}^i_{j_1 j_2} = 0, \ldots, \widetilde{a}^i_{j_1 \ldots j_r} = 0$ in G_n^r, for the rth component f_r we obtain the homogeneity condition

$$k^r f_r(u_1, u_2, \ldots, u_r) = f_r(k u_1, k^2 u_2, \ldots, k^r u_r) \,. \tag{2.7}$$

By the homogeneous function theorem [2]–[5], f_r is of the general form

$$\begin{aligned} f_{i_1 \ldots i_r} = {} & h_r u_{i_1} \cdot \ldots \cdot u_{i_r} + h_{r-1} u_{(i_1} \ldots u_{i_{r-2}} u_{i_{r-1} i_r)} \\ & + \ldots + h_{2,1} u_{(i_1} u_{i_2 \ldots i_r)} + h_{2,2} u_{(i_1 i_2} u_{i_3 \ldots i_r)} + \ldots + h_1 u_{i_1 \ldots i_r} \end{aligned} \tag{2.8}$$

with any real parameters $h_1, h_{2,1}, h_{2,2}, \ldots, h_{r-1}, h_r \in \mathbb{R}$. The equivariance of $f^{r,r}$ with respect to the kernel of the projection $G_n^r \to G_n^{r-1} : \widetilde{a}^i_j = \delta^i_j$, $\widetilde{a}^i_{j_1 j_2} = 0, \ldots, \widetilde{a}^i_{j_1 \ldots j_{r-1}} = 0$ and $\widetilde{a}^i_{j_1 \ldots j_r}$ arbitrary, gives

$$h_1 = k_1 \,. \tag{2.9}$$

Thus, we obtain the 1st power transformation $A_1^{r,r}$.

Now, considering the equivariance of $A - k_1 A_1^{r,r}$ with respect to the kernel of the projection $G_n^{r-1} \to G_n^1$: $\widetilde{a}^i_j = \delta^i_j$ and $\widetilde{a}^i_{j_1 j_2}, \ldots, \widetilde{a}^i_{j_1 \ldots j_{r-1}}$ arbitrary, we obtain

$$h_{2,1} = \frac{r!}{1!(r-1)!} k_2 \,, \qquad h_{2,2} = \frac{r!}{2!(r-2)!} k_2 \,, \quad \ldots \tag{2.10}$$

Thus, we obtain the 2nd power transformation $A_2^{r,r}$.

Then, considering the equivariance of $A - k_1 A_1^{r,r} - k_2 A_2^{r,r}$ with respect to the kernel of the projection $G_n^{r-2} \to G_n^1$, we obtain the general form of the 3rd power transformation $A_3^{r,r}$. Continuing this procedure gives the next power transformations $A_3^{r,r}, \ldots, A_{r-2}^{r,r}$. The equivariance of $A - k_1 A_1^{r,r} - k_2 A_2^{r,r} - \ldots - k_{r-2} A_{r-2}^{r,r}$ with respect to the kernel of the projection $G_n^2 \to G_n^1$:

$\widetilde{a}^i_j = \delta^i_j$ and $\widetilde{a}^i_{jk}$ arbitrary, leads to the next relation

$$h_{r-1} = \frac{r!}{(r-2)!2!}\, k_{r-1}\,. \tag{2.11}$$

Thus, we obtain the $(r-1)$th power transformation $A^{r,r}_{r-1}$.

Finally, the G^r_n-equivariant map

$$A - k_1 A^{r,r}_1 - k_2 A^{r,r}_2 - \ldots - k_{r-1} A^{r,r}_{r-1} = h_r A^{r,r}_r \tag{2.12}$$

is defined by the rth power transformation with any real parameter $h_r \in \mathbb{R}$. If we put $h_r = k_r$, this proves our theorem.

3. In this part we determine all natural transformations $T^{r*} \to T^{s*}$ in two cases: $r < s$ and $r > s$.

THEOREM 2. *All natural transformations $A : T^{r*} \to T^{(r+q)*}$ form the r-parameter family*

$$A = k_{q+1} A^{r,r+q}_{q+1} + k_{q+2} A^{r,r+q}_{q+2} + \ldots + k_{q+r} A^{r,r+q}_{q+r} \tag{3.1}$$

with any real parameters $k_{q+1}, k_{q+2}, \ldots, k_{q+r} \in \mathbb{R}$.

Proof. We apply induction on q.

I. Consider the case $q = 1$. According to general standard methods [2]–[5], the natural transformations $A : T^{r*} \to T^{(r+1)*}$ are in bijection with the G^{r+1}_n-equivariant maps of the standard fibres $f^{r,r+1} : (T^{r*}\mathbb{R}^n)_0 \to (T^{(r+1)*}\mathbb{R}^n)_0$.

Considering the equivariance of $f^{r,r+1} = (f_1, \ldots, f_r, f_{r+1})$ with respect to homotheties: $\widetilde{a}^i_j = k\delta^i_j$, $\widetilde{a}^i_{j_1 j_2} = 0$, $\ldots$, $\widetilde{a}^i_{j_1 \ldots j_{r+1}} = 0$, we obtain the homogeneity conditions

$$\begin{aligned} &k f_1(u_1, u_2, \ldots, u_r) = f_1(k u_1, k^2 u_2, \ldots, k^r u_r), \ldots, \\ &k^r f_r(u_1, u_2, \ldots, u_r) = f_r(k u_1, k^2 u_2, \ldots, k^r u_r), \\ &k^{r+1} f_{r+1}(u_1, u_2, \ldots, u_r) = f_{r+1}(k u_1, k^2 u_2, \ldots, k^r u_r). \end{aligned} \tag{3.2}$$

Additionally, using the equivariance of $f^{r,r} = (f_1, \ldots, f_r)$ with respect to the kernel of the projection $G^r_n \to G^1_n$, we obtain, by Theorem 1, the r-parameter family of the form (2.1): $\overline{A} = k_1 A^{r,r}_1 + k_2 A^{r,r}_2 + \ldots + k_r A^{r,r}_r$ with any real parameters $k_1, k_2, \ldots, k_r \in \mathbb{R}$.

Moreover, by the homogeneous function theorem and the invariant tensor theorem [2]–[5], we deduce that the $(r+1)$th component f_{r+1} is of the general form

$$\begin{aligned} f_{i_1 \ldots i_{r+1}} = {} & l_{r+1} u_{i_1} u_{i_2} \ldots u_{i_{r+1}} + l_r u_{(i_1} \ldots u_{i_{r-1}} u_{i_r i_{r+1})} \\ & + \ldots + l_{2,1} u_{(i_1} u_{i_2 \ldots i_{r+1})} + l_{2,2} u_{(i_1 i_2} u_{i_3 \ldots i_{r+1})} + \ldots \end{aligned} \tag{3.3}$$

with any real parameters $l_{2,1}, l_{2,2}, \ldots, l_r, l_{r+1} \in \mathbb{R}$.

The equivariance of $f^{r,r+1}$ with respect to the kernel of the projections $G_n^{r+1} \to G_n^r$ and $G_n^{r+1} \to G_n^1$ gives the relations

$$k_1 = 0\,, \tag{3.4}$$

$$l_{2,1} = \frac{(r+1)!}{r!1!}k_2, \quad l_{2,2} = \frac{(r+1)!}{(r-1)!2!}k_2, \dots, l_r = \frac{(r+1)!}{(r-1)!2!}k_r\,. \tag{3.5}$$

If we put $l_{r+1} = k_{r+1}$, this gives the r-parameter family $f^{r,r+1} = (f^{r,r}, f_{r+1})$ of the form

$$A = k_2 A_2^{r,r+1} + \ldots + k_{r+1} A_{r+1}^{r,r+1} \tag{3.6}$$

with any real parameters $k_2, \ldots, k_{r+1} \in \mathbb{R}$.

II. Suppose that the theorem holds for $q-1$ and the G_n^{r+q-1}-equivariant maps $f^{r,r+q-1} : (T^{r*}\mathbb{R}^n)_0 \to (T^{(r+q-1)*}\mathbb{R}^n)_0$ define the r-parameter family

$$\overline{A} = k_q A_q^{r,r+q-1} + k_{q+1} A_{q+1}^{r,r+q-1} + \ldots + k_{q+r-1} A_{q+r-1}^{r,r+q-1} \tag{3.7}$$

with any real parameters $k_q, k_{q+1}, \ldots, k_{q+r-1} \in \mathbb{R}$.

Consider a G_n^{r+q}-equivariant map $f^{r,r+q} : (T^{r*}\mathbb{R}^n)_0 \to (T^{(r+q)*}\mathbb{R}^n)_0$ of the form $f^{r,r+q} = (f^{r,r+q-1}, f_{r+q})$.

The equivariance of $f^{r,r+q}$ with respect to the homotheties in G_n^{r+q}: $\widetilde{a}_j^i = k\delta_j^i$, $\widetilde{a}_{j_1 j_2}^i = 0, \ldots, \widetilde{a}_{j_1\ldots j_{r+q}}^i = 0$, gives for the $(r+q)$th component f_{r+q} the homogeneity condition

$$k^{r+q} f_{r+q}(u_1, u_2, \ldots, u_r) = f_{r+q}(ku_1, k^2 u_2, \ldots, k^r u_r)\,. \tag{3.8}$$

By the homogeneous function theorem and the invariant tensor theorem [2]–[5], f_{r+q} is of the form

$$\begin{aligned} f_{i_1\ldots i_{r+q}} &= l_{r+q} u_{i_1} \ldots u_{i_{r+q}} + l_{r+q-1} u_{(i_1} \ldots u_{i_{r+q-2}} u_{i_{r+q-1} i_{r+q})} \\ &\quad + \ldots + l_{q+1} u_{(i_1} \ldots u_{i_q} u_{i_{q+1}\ldots i_{q+r})} \end{aligned} \tag{3.9}$$

with any real parameters $l_{q+1}, \ldots, l_{q+r-1}, l_{q+r} \in \mathbb{R}$.

The equivariance of $f^{r,r+q}$ with respect to the kernel of the projections $G_n^{r+q} \to G_n^r$ and $G_n^{r+q} \to G_n^1$ gives the relations

$$k_q = 0,\ \ldots, \tag{3.10}$$

$$l_{q+1} = \frac{(q+r)!}{r!q!} k_{q+1},\ \ldots,\ l_{q+r-1} = \frac{(q+r)!}{(q+r-2)!2!} k_{q+r-1}\,. \tag{3.11}$$

If we put $l_{q+r} = k_{q+r}$, this proves our theorem.

THEOREM 3. *All natural transformations $A : T^{r*} \to T^{(r-q)*}$ form the $(r-q)$-parameter family*

$$A = k_1 A_1^{r,r-q} + k_2 A_2^{r,r-q} + \ldots + k_{r-q} A_{r-q}^{r,r-q} \tag{3.12}$$

with any real parameters $k_1, k_2, \ldots, k_{r-q} \in \mathbb{R}$.

P r o o f. Applying the same general procedure, we conclude that each $A : T^{r*} \to T^{(r-q)*}$ is the composition of the projection $P^{r,r-q} : T^{r*} \to T^{(r-q)*}$

and a transformation $\overline{A}$ of $T^{(r-q)*}$ into itself: $A = \overline{A} \circ P^{r,r-q}$. This proves our theorem.

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