

## NATURALLY REDUCTIVE METRICS OF NONPOSITIVE RICCI CURVATURE

CAROLYN GORDON AND WOLFGANG ZILLER<sup>1</sup>

**ABSTRACT.** The main theorem states that every naturally reductive homogeneous Riemannian manifold of nonpositive Ricci curvature is symmetric. As a corollary, every noncompact naturally reductive Einstein manifold is symmetric.

A homogeneous space  $G/H$  is called naturally reductive if there exists a decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$  with  $\text{ad}(\mathfrak{h})\mathfrak{p} \subset \mathfrak{p}$  and

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{p}.$$

The goal of this paper is to prove the following:

**THEOREM.** *Every naturally reductive Riemannian manifold of nonpositive Ricci curvature is symmetric.*

This strengthens a result of E. Deloff [D] asserting that every naturally reductive homogeneous manifold of nonpositive sectional curvature is symmetric. It also has the following consequence. A metric is called Einstein if there exists a constant  $E$  such that  $\text{Ric}(X, Y) = E\langle X, Y \rangle$ ;  $E$  is called the Einstein constant.

**COROLLARY.** *Every naturally reductive homogeneous Einstein manifold with nonpositive Einstein constant is symmetric.*

In particular, every noncompact naturally reductive Einstein manifold is symmetric. This is in sharp contrast to the compact case, where naturally reductive metrics provide a rich source of Einstein metrics [DZ, WZ].

To establish some preliminaries, let  $M$  be a connected homogeneous Riemannian manifold,  $G$  a transitive group of isometries of  $M$  and  $H$  the isotropy subgroup of  $G$  at a point  $p \in M$ . For convenience, we assume  $G$  acts effectively on  $M$ , i.e., only  $e$  acts as the identity transformation on  $M$ . Denote by  $\mathfrak{g}$  and  $\mathfrak{h}$  the Lie algebras of  $G$  and  $H$ , and by  $\mathfrak{p}$  an  $\text{Ad}(H)$ -invariant complement of  $\mathfrak{h}$  in  $\mathfrak{g}$ .  $M$  is naturally identified with the tangent space  $T_p M$ . Under this identification the Riemannian structure defines an  $\text{Ad}(H)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$ . The metric is called naturally reductive (with respect to  $G$  and  $\mathfrak{p}$ ) if  $\langle [X, Y]_{\mathfrak{p}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{p}} \rangle = 0$  for all  $X, Y, Z \in \mathfrak{p}$ , where  $[X, Y]_{\mathfrak{p}}$  is the  $\mathfrak{p}$  component of  $[X, Y]$ . Let

$$(1) \quad \begin{aligned} \bar{\mathfrak{g}} &= \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}], \quad \bar{\mathfrak{h}} = \mathfrak{h} \cap \bar{\mathfrak{g}} \quad \text{and} \\ \bar{G} &= \text{the subgroup of } G \text{ with Lie algebra } \bar{\mathfrak{g}}. \end{aligned}$$

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Then  $\bar{\mathfrak{g}}$  is an ideal in  $\mathfrak{g}$  and  $\bar{G}$  acts transitively on  $M$ . By a theorem of Kostant, see e.g. [DZ, p. 4], if  $M$  is naturally reductive there exists a unique symmetric nondegenerate bilinear form  $Q$  on  $\bar{\mathfrak{g}}$ , invariant under  $\text{Ad}(\bar{G})$ , such that  $Q(\bar{\mathfrak{h}}, \mathfrak{p}) = 0$  and  $Q|_{\mathfrak{p}}$  is equal to the given metric. Using  $Q$  one can express the Ricci curvature of the metric as follows [WZ]:

$$(2) \quad \text{Ric}(X, Y) = -\frac{1}{4}B_{\bar{\mathfrak{g}}}(X, Y) - \frac{1}{2}Q(C_{\chi, Q|_{\bar{\mathfrak{h}}}}(X), Y), \quad X, Y \in \mathfrak{p},$$

where  $B_{\bar{\mathfrak{g}}}$  is the Killing form of  $\bar{\mathfrak{g}}$ ,  $\chi$  the isotropy representation of the Lie algebra  $\bar{\mathfrak{h}}$  on  $\mathfrak{p}$ , and  $C_{\phi, g}$  is the Casimir operator of the orthogonal representation  $\phi$  of  $\mathfrak{h}$  w.r.t. a nondegenerate, symmetric, bilinear form  $g$  on  $\mathfrak{h}$  invariant under  $\text{Ad } H$ , i.e.  $C_{\phi, g} = \sum_i \phi(X_i)\phi(Y_i)$  with  $g(X_i, Y_j) = \delta_{ij}$ . Note that  $Q$  need not be positive definite on  $\bar{\mathfrak{g}}$  and hence  $Q|_{\bar{\mathfrak{h}}}$  need not be positive definite.

A subalgebra  $\mathfrak{k} \subset \mathfrak{g}$  is called compactly embedded if  $\text{Ad}_G(K)$  is compact in  $\text{Ad}_G(G)$ . If  $G/H$  is a Riemannian homogeneous space, then  $\mathfrak{h}$  is compactly embedded in  $\mathfrak{g}$ . But  $H$  is only compact if  $G$  is closed in the full isometry group of  $\langle \cdot, \cdot \rangle$ . A compactly embedded subalgebra is the direct sum of its center and semisimple ideal.

(3) LEMMA. *Suppose  $G/H$  is naturally reductive with respect to the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ . Let  $\mathfrak{u}$  be a compactly embedded subalgebra of  $\mathfrak{g}$  with  $\mathfrak{h} \subset \mathfrak{u} \subset \mathfrak{g}$ . Then, unless  $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$  for some ideal  $\mathfrak{t}$  in the center of  $\mathfrak{g}$ , there exists  $X \in \mathfrak{p} \cap \mathfrak{u}$  with  $\text{Ric}(X, X) > 0$ .*

PROOF. Using the notation in (1) let  $\bar{\mathfrak{u}} = \mathfrak{u} \cap \bar{\mathfrak{g}}$ . Then  $\bar{\mathfrak{h}} \subset \bar{\mathfrak{u}} \subset \bar{\mathfrak{g}}$  and  $\bar{\mathfrak{u}}$  is compactly embedded in  $\bar{\mathfrak{g}}$ . Hence if  $X \in \bar{\mathfrak{u}} \cap \mathfrak{p}$ ,  $B_{\bar{\mathfrak{g}}}(X, X) \leq 0$  with equality iff  $X$  is in the center of  $\bar{\mathfrak{g}}$ . Hence the first term in (2) for  $\text{Ric}(X, X)$  is nonnegative, but the second term can have either sign.

Let  $\bar{U}$  be a compact Lie group with Lie algebra  $\bar{\mathfrak{u}}$ , and  $\bar{H} \subset \bar{U}$  the subgroup corresponding to  $\bar{\mathfrak{h}}$ . Then  $Q|_{\bar{\mathfrak{u}}}$  induces a naturally reductive metric on  $\bar{U}/\bar{H}$ . If  $\text{Ric}(X, X) \leq 0$  for all  $X \in \mathfrak{p} \cap \mathfrak{u}$ , then this metric on  $\bar{U}/\bar{H}$  also has nonpositive Ricci curvature since the second term in (2) is the same for both naturally reductive metrics and the first term is related by  $B_{\bar{\mathfrak{g}}}(X, X) \leq B_{\bar{\mathfrak{u}}}(X, X)$  (since  $\bar{\mathfrak{u}}$  is compactly embedded in  $\bar{\mathfrak{g}}$ ).

$\bar{U}/\bar{H}$  might not be effective, but by dividing by a common normal subgroup we obtain an effective compact homogeneous space  $\bar{U}'/\bar{H}'$  with a naturally reductive metric with  $\text{Ric} \leq 0$ . To this metric we apply Bochner's theorem [K, p. 57], which states that every Killing vector field on a compact manifold with  $\text{Ric} \leq 0$  is parallel. Hence  $\bar{H}'$  must be finite since a parallel Killing vector field cannot vanish anywhere. Therefore  $\bar{\mathfrak{h}}$  is an ideal in  $\bar{\mathfrak{u}}$  and the isotropy action is trivial on  $\bar{\mathfrak{u}} \cap \mathfrak{p}$ . Applying (2) to  $X \in \bar{\mathfrak{u}} \cap \mathfrak{p}$  we see that  $\text{Ric}(X, X) = -\frac{1}{4}B_{\bar{\mathfrak{g}}}(X, X) > 0$  unless  $X$  is in the center of  $\bar{\mathfrak{g}}$ . Since  $\bar{\mathfrak{g}}$  is an ideal in  $\mathfrak{g}$  and  $\bar{\mathfrak{h}}$  an ideal in  $\mathfrak{h}$ , this implies the lemma.

REMARK. This lemma becomes false without the assumption that  $G/H$  is naturally reductive. For example  $\text{SL}(n, \mathbf{R})$  admits a left-invariant metric with negative Ricci curvature [LM].

Recall that a connected Lie group  $G$  admits a Levi decomposition  $G = G_1 \cdot G_2$ , where  $G_1$  is a maximal connected semisimple subgroup, unique up to conjugacy, and  $G_2$  is the solvable radical of  $G$ . The semisimple group  $G_1$  admits an Iwasawa decomposition  $G_1 = K \cdot S$ , again unique up to conjugacy, where  $S$  is solvable, the Lie algebra  $\mathfrak{k}$  of  $K$  is compactly embedded in  $\mathfrak{g}_1$ , and  $K \cap S = \{e\}$ .  $K$  is compact iff the center of  $G_1$  is finite. Under any left-invariant Riemannian metric,  $G_1/K$  is a symmetric space of nonpositive curvature on which  $S$  acts simply transitively by isometries. The reader is referred to Helgason [H] for further details.

(4) LEMMA. *Let  $M = G/H$  be naturally reductive with respect to  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  and suppose  $M$  has nonpositive Ricci curvature. Shrink  $G$  if necessary so that the group  $\bar{G}$  in (1) is equal to  $G$ . Then there exists a semisimple Levi factor  $G_1$  of  $G$  and an Iwasawa decomposition  $G_1 = K \cdot S$  such that  $K \subset H$ .*

PROOF. We first show that for suitable choices of decompositions  $G = G_1 \cdot G_2$ ,  $G_1 = K \cdot S$ , there exists a compactly embedded subalgebra  $\mathfrak{u}$  of  $\mathfrak{g}$  containing both  $\mathfrak{h}$  and  $\mathfrak{k}$ .

Let  $G'$  be the full isometry group of  $M$  and  $H'$  its (compact) isotropy group. Since  $G \subset G'$  we can choose Levi factors  $G_1$  and  $G'_1$  of  $G$  and  $G'$  and Iwasawa decompositions  $G_1 = K \cdot S$  and  $G'_1 = K' \cdot S'$  satisfying  $G_1 \subset G'_1$ ,  $K \subset K'$ ,  $S \subset S'$ . Note that  $H$  is compact iff  $G$  is closed in  $G'$  and  $K$  is compact iff  $G_1$  has finite center. Since the claim involves only the Lie algebras, we may assume, after modding out a discrete central subgroup, if necessary, that  $G'_1$  has finite center. Hence  $K'$  is compact and lies in a maximal compact subgroup  $U'$  of  $G'$ . Since all maximal compact subgroups are conjugate, there exists  $x \in G'$  with  $xH'x^{-1} \subset U'$ . Since  $G' = G \cdot H'$ , we may choose  $x$  to lie in  $G$ . Letting  $U = U' \cap G$ ,  $U$  contains both  $K$  and  $xHx^{-1}$  and has Lie algebra  $\mathfrak{u}$  compactly embedded in  $\mathfrak{g}$ . Replacing each of  $G_1$ ,  $K$ ,  $S$  and  $U$  by their conjugates under  $x^{-1}$ , we have  $H, K \subset U$  as desired.

By (3)  $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$  for some ideal  $\mathfrak{t}$  in the center of  $\mathfrak{g}$ . Since the adjoint representation of  $\mathfrak{h}$  acts trivially on  $\mathfrak{p} \cap \mathfrak{u}$ , we have  $\text{Ric}(X, X) = -\frac{1}{4}B_{\mathfrak{g}}(X, X)$  for  $X \in \mathfrak{p} \cap \mathfrak{u}$  by (2) and, hence,  $\mathfrak{p} \cap \mathfrak{u} \subset \mathfrak{z}(\mathfrak{g})$ , which implies  $\mathfrak{p} \cap \mathfrak{u} = \mathfrak{t}$ . Hence  $\mathfrak{t}$  and  $\mathfrak{h}$  are orthogonal with respect to  $Q$ . Since  $Q$  is  $\text{Ad}(G)$ -invariant and  $\mathfrak{g}_1$  is semisimple,  $\mathfrak{g}_1$ , and hence  $\mathfrak{k}$ , must also be orthogonal to  $\mathfrak{t}$  with respect to  $Q$ . But this implies  $\mathfrak{k} \subset \mathfrak{h}$ .

PROOF OF THE THEOREM. If  $G_1$  is a Levi factor of  $G$  we can write  $G_1 = G_{\text{nc}}G_c$  where  $G_{\text{nc}}$  and  $G_c$ , the noncompact and compact parts of  $G_1$ , are the products of all noncompact, respectively compact, simple normal subgroups of  $G_1$ . Then  $K = (K \cap G_{\text{nc}}) \cdot G_c$  and  $S \subset G_{\text{nc}}$ . Similarly for the full isometry group  $G'$  we write  $G'_1 = G'_{\text{nc}} \cdot G'_c$ . To finish the proof, we will use the following result from [G]:

(5) Let  $G/H$  be naturally reductive. Then there exists a nilpotent normal subgroup  $N$  of  $G$  such that  $G_1 \cdot N$  acts transitively on  $M$ ,  $G_{\text{nc}}$  commutes with  $N$ , and  $G_{\text{nc}} = G'_{\text{nc}}$ .

Since, by (4),  $K \subset H$  and since the center of  $G_{\text{nc}}$  lies in  $K \cap G_{\text{nc}} \subset H$ , the effectiveness of  $G/H$  implies that  $G_{\text{nc}}$  has trivial center and, hence,  $K \cap G_{\text{nc}}$  and  $K$  are compact. Since  $G_1 \cdot N$  acts transitively, (4) also implies that  $G_{\text{nc}} \cdot N$  and  $S \cdot N$

act transitively. We first claim that  $H \cap (G_{\text{nc}}N) = (K \cap G_{\text{nc}})(H \cap N)$  and, therefore, that  $H \cap (S \cdot N) \subset N$ . Since  $N \subset G'_2$ ,  $G_{\text{nc}}G'_2$  also acts transitively, and since  $G_{\text{nc}} = G'_{\text{nc}}$ ,  $G_{\text{nc}}G'_2$  is closed in  $G'$ . Hence the isotropy subgroup  $L$  of  $G_{\text{nc}}G'_2$  is compact. Since  $G_{\text{nc}}$  has no center,  $G_{\text{nc}} \cap G'_2 = \{e\}$ . The projection of  $L$  onto  $G_{\text{nc}}$  is compact and contains the maximal compact subgroup  $K \cap G_{\text{nc}}$ . Hence the projection is  $K \cap G_{\text{nc}}$  and we obtain  $L = (K \cap G_{\text{nc}})(L \cap G'_2)$ ; and hence

$$H \cap (G_{\text{nc}}N) = L \cap (G_{\text{nc}}N) = (K \cap G_{\text{nc}})(L \cap N) = (K \cap G_{\text{nc}})(H \cap N).$$

Now  $\mathfrak{h} \cap \mathfrak{n} = \{0\}$  since  $\text{ad}_{\mathfrak{g}} \mathfrak{h}$  contains no nilpotent operators and, hence,  $\mathfrak{h} \cap (\mathfrak{s} + \mathfrak{n}) = \{0\}$ , i.e.,  $SN$  acts almost simply transitively on  $M$ . Under the identification of  $\mathfrak{s} + \mathfrak{n}$  with the tangent space  $T_p M$ , the isotropy action of  $K \cap G_{\text{nc}} \subset H$  is trivial on  $\mathfrak{n}$  and acts on  $\mathfrak{s}$  without any trivial factors. Hence  $\mathfrak{s}$  and  $\mathfrak{n}$  are orthogonal w.r.t. the Riemannian metric and  $M$  is the Riemannian direct product  $S \times N/N \cap H$ . The metric on  $S = G_{\text{nc}}/K$  is left  $G_{\text{nc}}$ -invariant and hence symmetric.  $N$  may be given a left-invariant metric of nonpositive Ricci curvature so that  $N$  is a Riemannian covering of  $N/N \cap H$ . But a left-invariant metric on a nilpotent Lie group is either flat or else has Ricci curvatures of both signs (see [M, Theorem 2.4]). Thus the metric on  $N$  is flat and hence  $M$  is symmetric.

ADDED IN PROOF. There is an error in the second paragraph of the proof of Lemma 4; the discrete center of  $G'_1$  need not be closed in  $G'$ . For a different proof of the existence of  $\mathfrak{u}$ , see [G-W, Remark 3.4].

## REFERENCES

- [DZ] J. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc., No. 215 (1979).
- [D] E. D. Deloff, *Naturally reductive metrics and metrics with volume preserving geodesic symmetries on NC algebras*, Thesis, Rutgers, 1979.
- [G] C. Gordon, *Naturally reductive Riemannian manifolds*, preprint 1984.
- [G-W] C. Gordon and E. N. Wilson, *The fine structure of transitive Riemannian isometry groups*, I, preprint 1984.
- [H] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
- [K] S. Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag, Berlin and New York, 1972.
- [LM] M. L. Leite and I. D. Miatello, *Metrics of negative Ricci curvature on  $SL(n, \mathbf{R})$ ,  $n \geq 3$* , J. Differential Geom. **17** (1982), 635–641.
- [M] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 243–329.
- [WZ] M. Wang and W. Ziller, *On normal homogeneous Einstein metrics*, preprint 1984.

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104