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Nature of the Singularity in Some Brans-Dicke Universes

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In the background of somewhat conflicting results about the possibility of nonsingular cosmological solutions of the Brans-Dicke equations, this article investigates some explicit solutions of these equations. It is found that, in the solutions presented, the material energy density never becomes infinite—either the spatial volume collapses or the gravitational 'constant' shows singular behaviour according as one adopts one or the other of the two units discussed by Dicke. While such situations are obtained with the equation of state $p=\epsilon\rho$ ($\epsilon=$ a constant $\neq 1/3$), the constant ω in Brans-Dicke equations has to be given a negative value contrary to observational requirement.

§ 1. Introduction

In course of investigation of the approach to singularity in Brans-Dicke cosmology, Nariai[1] claimed that for $p=\rho$ there exist solutions (belonging to the Bianchi type I group) in which there is no singularity and the material energy density ρ remains finite at all epochs. Such singularity-free solutions occurred for $0<(2\omega+3)\leq 1/3$, where ω is the coupling constant in the Brans-Dicke Lagrangian. As the corresponding values of ω would give perihelion motion in significant contradiction to observational data, Nariai considered the solutions unacceptable. Nevertheless, Nariai's result seems surprising for the following reasons: It is well known that in the Brans-Dicke theory considered in terms of Dicke's revised units,[2] the Einstein equations remain valid with a constant gravitational 'constant' and only the energy momentum tensor is augmented by that due to the scalar field. The energy momentum tensor $\Lambda_{\mu\nu}$ due to the massless Brans-Dicke scalar field is give by

$$\Lambda_{\mu\nu}=\frac{2\omega+3}{16\pi G\lambda^2}\left[\lambda_{,\mu}\lambda_{,\nu}-\frac{1}{2}g_{\mu\nu}\lambda_{,\alpha}\lambda_{,}{}^{\alpha}\right] \tag{1·1}$$

and obviously satisfies the condition

$$\left[\Lambda_{\mu\nu}v^\mu v^\nu-\frac{1}{2}g_{\alpha\beta}\Lambda^{\alpha\beta}v_\mu v^\mu\right]\geq 0$$

for arbitrary time-like vector v^μ if $(2\omega+3)\geq 0$. Thus one would expect that the homogeneous space-time considered by Nariai should be singular.[3]

The purpose of the present article is to throw light on the nature of the singularity in the cases considered by Nariai by building up explicit solutions for the Bianchi type I universes. It is found that in Dicke's revised units there is a spatial collapse although the material energy density remains finite. In the usual units (i.e., in units in which the gravitational 'constant' G varies but elementary particle masses remain constant) the spatial volume does not vanish and the matter energy density also remains finite but the scalar field and G have singular behaviour.

§ 2. Field equations and their integration

In Dicke's revised units, the field equations are, with $G=c=1$,

$$R_{ij}-\frac{1}{2}g_{ij}R=-8\pi(T_{ij}+\Lambda_{ij}), \tag{2·1}$$

$$\Box \ln\lambda = g^{ik}(\ln\lambda)_{;ik}=\frac{8\pi T}{(2\omega+3)} . \tag{2·2}$$

With λ being a function of t alone, the only nonvanishing components of $\Lambda_{\mu\nu}$ are

$$\Lambda_0{}^0=-\Lambda_1{}^1=-\Lambda_2{}^2=-\Lambda_3{}^3=\frac{2\omega+3}{32\pi}\left(\frac{\dot\lambda}{\lambda}\right)^2, \tag{2·3}$$

so that the scalar field energy momentum tensor is similar to that of a perfect fluid with

$$p_\lambda=\rho_\lambda=\frac{2\omega+3}{32\pi}\left(\frac{\dot\lambda}{\lambda}\right)^2. \tag{2·4}$$

With the Bianchi type I line element, we obtain

$$ds^2=dt^2-e^{2\phi}dx^2-e^{2\theta}dy^2-e^{2\psi}dz^2, \tag{2·5}$$

where ϕ, θ and ψ are functions of t alone, the field equation (2·1) is written out explicitly as follows:

$$\begin{aligned}8\pi(p+\rho_\lambda) &= -(\ddot\psi+\ddot\theta)+\frac{3\dot R}{2R}(\dot\phi-\dot\psi-\dot\theta)-\frac{1}{2}(\dot\varphi^2+\dot\psi^2+\dot\theta^2)\\ &= -(\ddot\theta+\ddot\varphi)+\frac{3}{2}\frac{\dot R}{R}(\dot\psi-\dot\varphi-\dot\theta)-\frac{1}{2}(\dot\varphi^2+\dot\psi^2+\dot\theta^2)\\ &= -(\ddot\psi+\ddot\varphi)+\frac{3}{2}\frac{\dot R}{R}(\dot\theta-\dot\varphi-\dot\psi)-\frac{1}{2}(\dot\varphi^2+\dot\psi^2+\dot\theta^2),\end{aligned} \tag{2·6}$$

$$8\pi(\rho+\rho_\lambda)=\frac{9}{2}\frac{\dot R^2}{R^2}-\frac{1}{2}(\dot\theta^2+\dot\psi^2+\dot\phi^2), \tag{2·7}$$

where $R^3\equiv\exp(\theta+\phi+\psi)$. With $p=\rho$, Eqs. (2·6) and (2·7) yield

$$\frac{8\pi}{3}(\rho+\rho_\lambda)R^6=a\,, \tag{2·8}$$

where a is a constant. Taking the divergence of Eq. (2·1) and then eliminating $\Box \ln \lambda$ with the help of Eq. (2·2), we get

$$\dot{\rho}+3(p+\rho)R^{-1}\dot{R}+\frac{1}{2}(\rho-3p)\lambda^{-1}\dot{\lambda}=0$$

which, on integration yields with $p=\rho$,

$$\frac{8\pi}{3}\rho R^6\lambda^{-1}=b\,, \tag{2·9}$$

where b is another constant. From Eq. (2·6) we get

$$\begin{aligned}\dot{\phi}&=\frac{\alpha}{R^3}+\frac{\dot{R}}{R}\,,\\ \dot{\psi}&=\frac{\beta}{R^3}+\frac{\dot{R}}{R}\,,\\ \dot{\theta}&=\frac{\gamma}{R^3}+\frac{\dot{R}}{R}\,,\end{aligned} \tag{2·10}$$

where α, β, γ are constants with the constraint

$$\alpha+\beta+\gamma=0\,. \tag{2·11}$$

By the use of Eqs. (2·8) and (2·10), Eq. (2·7) may be integrated to give

$$R^3=3At\,, \tag{2·12}$$

where

$$A^2=a+\frac{1}{2}(\alpha^2+\beta^2+\gamma^2), \tag{2·13}$$

and the origin of time has been chosen to coincide with the epoch $R=0$. Using Eqs. (2·4), (2·9) and (2·12) in Eq. (2·8) and integrating, we get

$$\frac{1\mp(1-b\lambda a^{-1})^{1/2}}{1\pm(1-b\lambda a^{-1})^{1/2}}=CR^{\pm(12/(2\omega+3))^{1/2}\cdot a^{1/2}/A}\,, \tag{2·14}$$

where C is an arbitrary constant of integration. Thus near the singularity $R\to 0$, it follows that

$$\lambda\sim R^{(12/(2\omega+3))^{1/2}\cdot a^{1/2}/A}\,.$$

Referring to Eq. (2·9) we see that ρ will remain finite as $R\to 0$ if

$$\left(\frac{12}{2\omega+3}\right)^{1/2}\frac{a^{1/2}}{A}=b\,.$$

Otherwise,

$$0<(2\omega+3)=\frac{A^{-2}a}{3}\leq\frac{1}{3}\,, \tag{2·15}$$

where the sign of equality corresponds to the case of vanishing shear ($\alpha=\beta=\gamma=0$). Equation (2·15) is essentially the condition of Nariai. Thus here the singularity appears, besides geodetic incompleteness, as a vanishing of the spatial volume (i.e., the collapse of the homogeneous varieties) and one has infinity in the value of the expansion and the shear (provided it exists). However in these units the material energy density ρ remains finite. One may change them over to the 'atomic' units in which the masses remain constant but G varies. One would then have (for simplicity we present only the shear-free case)

$$\begin{aligned} R &\sim T^{-1/3}(T+1), \\ \rho &\sim (T+1)^{-6}T^2, \\ G &\sim T^{-1}(T+1)^2, \end{aligned} \tag{2·16}$$

where T is the time in these units and R, ρ, G are also reckoned in these 'atomic units'. As $T\to 0$ or ∞, the linear dimensions become arbitrarily large and ρ vanishes at either limit. Therefore the geometry does not show any collapse within this time interval. The singularity now appears as an arbitrarily large value of the gravitational 'constant' G.

One may ask whether or not this novel nature of the singularity is obtained only in the rather abnormal equation of state $p=\rho$. It is easy to see that similar situations occur even for $p=0$ but not for $p=\rho/3$. The reason is that in case $p=\rho/3$, one gets the relation $\rho R^4=\text{const}$, replacing Eq. (2·9) and hence $\rho\to\infty$ as $R\to 0$. For $p=0$, there exist solutions for the Robertson-Walker metric with flat space in which[4)]

$$\begin{aligned} R &\sim t^{1/3}, \\ \lambda &\sim R^{\pm(12/2\omega+3)^{1/2}}, \end{aligned} \tag{2·17}$$

and Eq. (2·9) is replaced by

$$\rho R^3\lambda^{1/2}=\text{const}, \tag{2·18}$$

so that using the negative sign before the exponent of R and again for $(2\omega+3)=\frac{1}{3}$, one obtains a finite ρ as $R\to 0$. Indeed as may be easily shown, for $p=\epsilon\rho$ where ϵ is a constant, ρ remains finite as $R\to 0$ for flat space isotropic models if

$$(2\omega+3)=\frac{1}{3}\left(\frac{1-3\epsilon}{1+\epsilon}\right)^2>0 \tag{2·19}$$

which can be satisfied for any $\epsilon\neq\frac{1}{3}$, provided one admits negative values of ω.

§ 3. Concluding remarks

The above discussion is useful in widening our horizon about the nature of the singularity in Brans-Dicke cosmologies. Negative values of ω are apparently ruled out by observations. Nevertheless, there does not appear to exist any a priori theoretical reasons for excluding these values of ω. Indeed if one allows negative values of ω, one can also have a class of solutions in which the geometry is static and G is time dependent. Thus one has the solution for the flat space Robertson-Walker metric

$$R=\text{const}, \qquad \rho=\text{const},$$
$$p=\epsilon\rho,$$
$$\omega=-(1-\epsilon)^{-1},$$
$$G=\frac{(2-4\epsilon)}{(2-3\epsilon)(1-\epsilon)}(4\pi\rho t^2)^{-1}.$$

For a positive G, one would demand $\epsilon<\frac{1}{2}$. The geometry would not, however, be static in Dicke's revised units.

References

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