Kyushu J. Math.
Vol. **57**, 2003, pp. 303–324

NAVIER–STOKES EQUATIONS IN THE BESOV SPACE NEAR L^∞ AND *BMO*

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(Received 26 September 2002)

Dedicated to Professor Reimund Rautmann on the occasion of his 70th birthday

Abstract. We prove a local existence theorem for the Navier–Stokes equations with the initial data in $B^0_{\infty,\infty}$ containing functions which do not decay at infinity. Then we establish an extension criterion on our local solutions in terms of the vorticity in the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$.

0. Introduction

Consider the Navier–Stokes equations in $\mathbb{R}^n$, $n \geq 2$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } x \in \mathbb{R}^n, t \in (0, T), \\ u|_{t=0} = a \end{cases} \tag{N–S}$$

where $u = (u^1(x, t), u^2(x, t), \dots, u^n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x, t) \in \mathbb{R}^n \times (0, T)$, respectively, while $a = (a^1(x), a^2(x), \dots, a^n(x))$ is the given initial velocity vector.

In this paper we first construct a solution u of (N–S) in a finite time-interval $(0, T)$ for a which does not necessarily decay at infinity. Then we give a criterion on extension beyond T of our local solution. The theory on existence of local strong solutions to (N–S) in the usual L^r-spaces had been developed by many authors [**9, 22, 25, 28, 29**], and Kato [**16**] and Giga and Miyakawa [**13**] succeeded in constructing the solution for $a \in L^n$. The L^n-strong solution is more important than any other L^r-one from a viewpoint of scaling invariance. It can be easily seen that if $\{u, p\}$ solves (N–S), then so does $\{u_\lambda, p_\lambda\}$ for all $\lambda > 0$, where $u_\lambda(x, t) \equiv \lambda u(\lambda x, \lambda^2 t)$ and $p_\lambda(x, t) \equiv \lambda^2 p(\lambda x, \lambda^2 t)$. Scaling invariance means that $\|u_\lambda(\cdot, 0)\|_{L^r} (= \lambda^{1-n/r}\|u(\cdot, 0)\|_{L^r}) = \|u(\cdot, 0)\|_{L^r}$ holds for all $\lambda > 0$, and this is valid if and only if $r = n$. Since the pioneering work of Kato and Giga and Miyakawa, much great effort has been made to prove the existence of the strong solution in spaces larger than L^r such as the Morrey space M_r, the Lorentz space $L^{r,\infty}$ and the Besov space $B^s_{p,q}$ (Giga *et al* [**15**], Giga and Miyakawa [**14**], Kozono and Yamazaki [**19, 20**], Sohr [**24**], Cannone [**5**], Cannone and Meyer [**6**], Amann [**1**]). Most of these papers treated the spaces where the norms are invariant under the scaling transform u_λ as mentioned above.

In the present paper, we deal with the space of initial data which do not decay at infinity. Giga *et al* [**11**] proved the existence of a strong solution for $a \in BUC$ (BUC; bounded and uniformly continuous functions in $\mathbb{R}^n$). In particular, we consider here the Besov space $B^0_{\infty,\infty}$ which is slightly larger than L^∞. The space L^∞ is useful enough to the nonlinear partial differential equations since it is an algebra by pointwise multiplication. On the other hand, the disadvantage of choosing L^∞ is that the Riesz transforms are not bounded. To overcome this difficulty, we make use of the auxiliary space of BMO and the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ in which the singular integral operators are bounded. Here BMO is the space of functions of bounded mean oscillation. For the detailed definition of BMO, see e.g. Stein [**26**]. Furthermore, we establish a certain Hölder-type inequality in $\dot{B}^s_{p,q}$ which plays a substitutive role for the estimate $\|f \cdot g\|_{L^r} \le \|f\|_{L^\infty}\|g\|_{L^r}$ (see Lemma 2.3 below). As a result, we can construct a solution u of (N–S) on a finite interval $(0, T)$ for $a \in B^0_{\infty,\infty}$. Our approach is motivated by the recent work of Sawada [**23**]. Indeed, Sawada [**23**] obtained a sharper Hölder-type estimate in the Besov spaces with various differential orders and proved a local existence theorem with the initial data in $B^{-s}_{p,\infty}$ for $0 \le s < 1 - n/p$ with $n < p \le \infty$. Koch and Tataru [**17**] also gave the solution for $a = \nabla b$ with $b \in BMO$. The space of initial data obtained by Koch and Tataru is closely related to the Besov spaces which contain L^∞. Although our first result on existence of the local solution is not altogether new, our approach is different, and we give a more simplified proof than Sawada [**23**] and Koch and Tataru [**17**] so far as the initial data is in $B^0_{\infty,\infty}$. Furthermore, as a by-product of our approach, we establish an extension criterion of local solutions in the scaling invariant class.

To be precise, we show that our Hölder-type inequality in $\dot{B}^s_{p,q}$ is applicable to the question whether the solution $u(t)$ on $(0, T)$ extends beyond $t > T$. In the class of the usual Sobolev space H^s, Beale *et al* [**2**] proved that the solution $u(t)$ can be continued on $(0, T')$ for some $T' > T$ provided $\int_0^T \|\mathrm{rot}\, u(t)\|_{L^\infty}\, dt < \infty$ (see also Majda [**21**]). This result was generalized by the authors [**18**] from L^∞ to $\dot{B}^0_{\infty,\infty}$. In the space BUC, Giga *et al* [**12**] gave a similar criterion under the more restrictive hypothesis that $\sup_{0<t<T} \|\mathrm{rot}\, u(t)\|_{L^\infty} < \infty$. It should be noted that one cannot control behavior at infinity of solutions in BUC so that the logarithmic type of the

Sobolev inequality for $\|\mathrm{rot}\, u\|_{L^\infty}$ does not hold as in the case of Beale *et al*. Giga *et al* established another estimate of a logarithmic type for the nonlinear evolution $\int_0^t \|e^{(t-\tau)\Delta} P(u \cdot \nabla u)(\tau)\|_{L^\infty}\, d\tau$. Since their estimate causes a singularity at $\tau = t$, they need to generalize the well-known Gronwall inequality. On the other hand, our estimate in $\dot{B}^s_{p,q}$ gives us so much freedom to choose various differential orders s that we may cancel the singularity at $\tau = t$ of the nonlinear evolution. Hence we can obtain the same criterion as Beale *et al* even in the class of solutions which do not necessarily decay at infinity. As for the Euler equations of the inviscid fluid, it turns out that the techniques of Besov spaces such as the Littlewood–Paley decomposition and Bony's paraproduct formula are useful enough to analyze singularities of the solution. This was fully developed by Chemin [**7**] and the series of papers by Vishik [**30–32**]. Our approach to (N–S) in the Besov space makes very full use of the variant of the paraproduct formula, so in this sense the method of the present paper is relatively related to those of [**7, 30–32**].

In Section 1, we state our main theorems. Section 2 is devoted to the preparation of some $L^r - L^q$-estimates for $e^{t\Delta}$ in Besov spaces. Making use of Bony's paraproduct formula, we establish our Hölder type estimate in $\dot{B}^s_{p,q}$. Finally in Section 3, we prove the main theorems.

1. Results

Before stating our results, let us recall definitions of the Besov space $B^s_{p,q}$ and the *homogeneous* Besov space $\dot{B}^s_{p,q}$. For details, see e.g. Bergh and Löfström [**3**]. We first introduce the Littlewood–Paley decomposition by means of $\{\varphi_j\}_{j=-\infty}^{\infty}$. Take a function $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}\,\phi = \{\xi \in \mathbb{R}^n; 1/2 \le |\xi| \le 2\}$ such that $\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1$ for all $\xi \neq 0$. The functions φ_j $(j = 0, \pm 1, \dots)$ and ψ are defined by

$$\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi), \quad \mathcal{F}\psi(\xi) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j}\xi),$$

where $\mathcal{F}$ denotes the Fourier transform. Then, for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we write

$$\|f\|_{B^s_{p,q}} \equiv \|\psi * f\|_{L^p} + \left(\sum_{j=1}^{\infty} (2^{sj}\|\varphi_j * f\|_{L^p})^q \right)^{1/q}, \quad 1 \le q < \infty,$$

$$\|f\|_{B^s_{p,\infty}} \equiv \|\psi * f\|_{L^p} + \sup_{1 \le j < \infty} (2^{sj}\|\varphi_j * f\|_{L^p}), \quad q = \infty,$$

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \left(\sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * f\|_{L^p})^q \right)^{1/q}, \quad 1 \le q < \infty,$$

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \sup_{-\infty<j<\infty} (2^{sj} \|\varphi_j * f\|_{L^p}), \quad q = \infty,$$

where L^p denotes the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p}$. The Besov space $B^s_{p,q}$ and the homogeneous Besov space $\dot{B}^s_{p,q}$ are defined by

$$B^s_{p,q} \equiv \{f \in \mathcal{S}'; \|f\|_{B^s_{p,q}} < \infty\}, \quad \dot{B}^s_{p,q} \equiv \{f \in \mathcal{S}'; \|f\|_{\dot{B}^s_{p,q}} < \infty\}.$$

It is well known that the solution of (N–S) can be reduced to finding a solution u of the following integral equation:

$$u(t) = e^{t\Delta}a - \int_0^t e^{(t-\tau)\Delta} P(u \cdot \nabla u)(\tau)\, d\tau, \quad 0 < t < T, \tag{I.E.}$$

where $P = \{P_{jk} = \delta_{jk} + R_j R_k\}_{1 \le j,k \le n}$ ($R_j = \mathcal{F}^{-1}(\sqrt{-1}\xi_j/|\xi|)\mathcal{F}$; the Riesz transforms) denotes the Weyl–Helmholtz projection.

Our first result on the local existence and uniqueness of solutions to (I.E.) now reads as follows.

THEOREM 1. *For every* $a \in B^0_{\infty,\infty}$ *with* $\operatorname{div} a = 0$, *there exist* $T_* = T_*(\|a\|_{B^0_{\infty,\infty}})$ *and a solution u of (I.E.) on* $[0, T_*)$ *such that*

$$u \in C_{\mathrm{w}}([0, T_*); B^0_{\infty,\infty}), \quad \{\log(e + t^{-1})\}^{-1} u \in L^\infty(0, T_*; L^\infty), \tag{1.1}$$

$$t^{1/2}\nabla u \in L^\infty(0, T_*; \dot{B}^0_{\infty,1}), \tag{1.2}$$

where C_{w} *denotes weakly-$*$ continuous functions. As for uniqueness, u is the only solution of (I.E.) in the class of (1.1).*

Remarks.

(i) The time-interval $(0, T_*)$ of the existence of the solution u in Theorem 1 is characterized by

$$T_* = C_\varepsilon / \|a\|^{2/(1-\varepsilon)}_{B^0_{\infty,\infty}} \quad \text{for all sufficiently small } \varepsilon > 0, \tag{1.3}$$

where C_ε is the constant depending only on ε, but not on a.

(ii) The solution u in Theorem 1 is in fact smooth on $\mathbb{R}^n \times (0, T_*)$ and satisfies (N–S) in the usual sense; Giga *et al* [**11**] proved that once the solution $u(t)$ of (I.E.) belongs to L^∞ at some definite time $t = t_*$, $u(t)$ becomes necessarily smooth for $t > t_*$.

(iii) Koch and Tataru [**17**] showed the existence of local and global strong solutions with the initial data in BMO^{-1}. The relation between BMO^{-1} and $B^0_{\infty,\infty}$ is connected through Lemma 2.2 below. Our proof is different from that of Koch and Tataru and seems to be more straightforward along the definition of Besov spaces by the Littlewood–Paley decomposition. Sawada [**23**] treated lager Besov spaces $B^{-s}_{p,\infty}$ for $0 \le s < 1 - n/p$ with $n < p \le \infty$ as the initial data. For his proof, a skillful technique for domain decomposition in the Fourier variable of the paraproduct formula is essential. On the other hand, our proof is based on Lemma 2.2 and seems to be simpler.

Next, we are interested in the problem of whether the solution $u(t)$ can be extended beyond $t > T_*$.

THEOREM 2. *Let $a \in B^0_{\infty,\infty}$ and let u be the solution of (I.E.) in the classes of (1.1) and (1.2) for all $T_* < T$. If*

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{\dot{B}^0_{\infty,\infty}}\, dt < \infty, \tag{1.4}$$

then u can be extended to the solution of (I.E.) in the classes of (1.1) and (1.2) on $(0, \tilde{T})$ for some $\tilde{T} > T$.

Remark. For the Euler equations, Beale *et al* [**2**] treated the solution in the usual Sobolev space H^s, and proved the above extension criterion under the stronger hypothesis that

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{L^\infty}\, dt < \infty.$$

Their criterion was generalized by [**18**] from L^∞ to $\dot{B}^0_{\infty,\infty}$ like (1.4). It is easy to see that their results hold for (N–S), too. Their proof is based on the L^∞-estimate of functions by means of logarithmic growth of the H^s-norm for $s > n/2$. Therefore, it is essential for their proof that the solution decays at infinity. As for (N–S) in the class of solutions in *BUC* where the solution does not decay at infinity, Giga *et al* [**12**] established the same criterion under the stronger hypothesis that

$$\sup_{0<t<T} \|\mathrm{rot}\, u(t)\|_{L^\infty} < \infty.$$

Theorem 2 states that, even in the class of solutions which do not decay at infinity, the extension criterion does hold under the hypothesis of the norm which is invariant with respect to the scaling transform $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$. Notice that

$$\|\mathrm{rot}\, u_\lambda\|_{L^1(\mathbb{R};\dot{B}^0_{\infty,\infty})} = \|\mathrm{rot}\, u\|_{L^1(\mathbb{R};\dot{B}^0_{\infty,\infty})} \quad \text{for all } \lambda > 0.$$

2. Preliminaries

In what follows, we denote by C various constants. In particular, we denote by $C = C(*, *, \dots, *)$ the constants which depend only on the quantities appearing in parenthesis.

We first show some elementary interpolation inequalities in the Besov spaces.

LEMMA 2.1. (i) *For $s_0 < s_1$ there is a constant $C = C(s_0, s_1)$ such that*

$$\|f\|_{B^{s_0}_{p,1}} \le C(N\|f\|_{B^{s_0}_{p,\infty}} + 2^{-(s_1-s_0)N}\|f\|_{B^{s_1}_{p,\infty}}) \tag{2.1}$$

holds for all $f \in B^{s_1}_{p,\infty}$ with $1 \le p \le \infty$ and all positive integers N.

(ii) *For $s_0 < s < s_1$ there is a constant $C = C(s_0, s, s_1)$ such that*

$$\|f\|_{\dot{B}^{s}_{p,1}} \le C(2^{-(s-s_0)N}\|f\|_{\dot{B}^{s_0}_{p,\infty}} + N\|f\|_{\dot{B}^{s}_{p,\infty}} + 2^{-(s_1-s)N}\|f\|_{\dot{B}^{s_1}_{p,\infty}}) \tag{2.2}$$

holds for all $f \in \dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_0}_{p,\infty}$ with $1 \le p \le \infty$ and all positive integers N.

Proof. (i) Let $f \in B^{s_1}_{p,\infty}$. By definition we have

$$\begin{aligned}
\|f\|_{B^{s_0}_{p,1}} &= \|\psi * f\|_{L^p} + \sum_{j=1}^{N} 2^{s_0 j}\|\varphi_j * f\|_{L^p} + \sum_{j=N+1}^{\infty} 2^{s_0 j}\|\varphi_j * f\|_{L^p} \\
&\le \|\psi * f\|_{L^p} + N \sup_{j\ge 1} 2^{s_0 j}\|\varphi_j * f\|_{L^p} \\
&\quad + \sum_{j=N+1}^{\infty} 2^{-(s_1-s_0)j} \sup_{j\ge 1} 2^{s_1 j}\|\varphi_j * f\|_{L^p} \\
&\le N\|f\|_{B^{s_0}_{p,\infty}} + (2^{s_1-s_0}-1)^{-1} 2^{-(s_1-s_0)N}\|f\|_{B^{s_1}_{p,\infty}} \\
&\le C(N\|f\|_{B^{s_0}_{p,\infty}} + 2^{-(s_1-s_0)N}\|f\|_{B^{s_1}_{p,\infty}})
\end{aligned}$$

for all positive integers N with $C = \max\{1, (2^{s_1-s_0}-1)^{-1}\}$.

(ii) Similarly we have

$$\begin{aligned}
\|f\|_{\dot{B}^{s}_{p,1}} &= \left(\sum_{j=-\infty}^{-N-1} + \sum_{j=-N}^{N} + \sum_{j=N+1}^{\infty}\right) 2^{sj}\|\varphi_j * f\|_{L^p} \\
&\le \sum_{j=-\infty}^{-N-1} 2^{(s-s_0)j}(\sup_{j\in\mathbb{Z}} 2^{s_0 j}\|\varphi_j * f\|_{L^p}) + \sum_{j=-N}^{N} \cdot \sup_{j\in\mathbb{Z}} 2^{sj}\|\varphi_j * f\|_{L^p} \\
&\quad + \sum_{j=N+1}^{\infty} 2^{-(s_1-s)j}(\sup_{j\in\mathbb{Z}} 2^{s_1 j}\|\varphi_j * f\|_{L^p})
\end{aligned}$$

$$\le (2^{s-s_0}-1)^{-1}2^{-(s-s_0)N}\|f\|_{\dot B^{s_0}_{p,\infty}} + (2N+1)\|f\|_{\dot B^{s}_{p,\infty}} + (2^{s_1-s}-1)^{-1}2^{-(s_1-s)N}\|f\|_{\dot B^{s_1}_{p,\infty}}$$
$$\le C(2^{-(s-s_0)N}\|f\|_{\dot B^{s_0}_{p,\infty}} + N\|f\|_{\dot B^{s}_{p,\infty}} + 2^{-(s_1-s)N}\|f\|_{\dot B^{s_1}_{p,\infty}})$$

for all $f \in \dot B^{s_1}_{p,\infty} \cap \dot B^{s_0}_{p,\infty}$ and all positive integers N, where $C = \max\{(2^{s-s_0}-1)^{-1}, 3, (2^{s_1-s}-1)^{-1}\}$. □

We next investigate the behavior of the heat semigroup $e^{t\Delta}$ in the Besov spaces.

LEMMA 2.2. (i) *Let $s_0 \le s_1$, $1 \le p, q \le \infty$. Then there holds*

$$\|e^{t\Delta}a\|_{B^{s_1}_{p,q}} \le C(1 + t^{-\frac{1}{2}(s_1-s_0)})\|a\|_{B^{s_0}_{p,q}}, \tag{2.3}$$

$$\|e^{t\Delta}a\|_{\dot B^{s_1}_{p,q}} \le Ct^{-\frac{1}{2}(s_1-s_0)}\|a\|_{\dot B^{s_0}_{p,q}}, \tag{2.4}$$

$$\|e^{t\Delta}a\|_{B^{s_1}_{p,1}} \le C(1 + t^{-\frac{1}{2}(s_1-s_0)})\log\left(e + \frac{1}{t}\right)\|a\|_{B^{s_0}_{p,\infty}}, \tag{2.5}$$

for all $t > 0$, where $C = C(s_0, s_1)$ is independent of p, q and a.

(ii) *Let $s_0 < s_1$, $1 \le p \le \infty$. Then there holds*

$$\|e^{t\Delta}a\|_{\dot B^{s_1}_{p,1}} \le Ct^{-\frac{1}{2}(s_1-s_0)}\|a\|_{\dot B^{s_0}_{p,\infty}} \tag{2.6}$$

for all $t > 0$ with C independent of a. In particular, there holds

$$\|\nabla e^{t\Delta}a\|_{\dot B^{0}_{\infty,1}} \le Ct^{-\frac{1}{2}}\|a\|_{\dot B^{0}_{\infty,\infty}} \tag{2.7}$$

for all $t > 0$ with C independent of a.

Proof. (i) Let $G(x) = (4\pi)^{-n/2}e^{-|x|^2/4}$. Then we see

$$e^{t\Delta}a = G_{\sqrt{t}} * a,$$

where $G_\varepsilon(x) \equiv \varepsilon^{-n}G(x/\varepsilon)$ for $\varepsilon > 0$. To prove (2.3) and (2.4), it suffices to show that

$$2^{s_1 j}\|\varphi_j * G_{\sqrt{t}} * a\|_{L^p} \le Ct^{-\frac{1}{2}(s_1-s_0)}2^{s_0 j}\|\varphi_j * a\|_{L^p} \tag{2.8}$$

with a constant $C = C(s_0, s_1)$ independent of $t > 0$, $j \in \mathbb{Z}$, $1 \le p \le \infty$. Since $\operatorname{supp}\hat\varphi_j = \{\xi \in \mathbb{R}^n\,;\, 2^{j-1} \le |\xi| \le 2^{j+1}\}$, we have

$$\varphi_j * G_{\sqrt{t}} * a = \tilde\varphi_j * G_{\sqrt{t}} * \varphi_j * a \quad \text{for all } j \in \mathbb{Z},$$

where $\tilde{\varphi}_j \equiv \varphi_{j-1} + \varphi_j + \varphi_{j+1}$, and hence by the Young inequality, there holds

$$\begin{aligned} 2^{s_1 j} \|\varphi_j * G_{\sqrt{t}} * a\|_{L^p} &= 2^{\alpha j} 2^{s_0 j} \|(-\Delta)^{-\alpha/2} \tilde{\varphi}_j * (-\Delta)^{\alpha/2} G_{\sqrt{t}} * \varphi_j * a\|_{L^p} \\ &\le 2^{\alpha j} \|(-\Delta)^{-\alpha/2} \tilde{\varphi}_j\|_{L^1} \|(-\Delta)^{\alpha/2} G_{\sqrt{t}}\|_{L^1} (2^{s_0 j} \|\varphi_j * a\|_{L^p}), \end{aligned}$$

where $\alpha \equiv s_1 - s_0$. It is easy to see that

$$\|(-\Delta)^{-\alpha/2} \tilde{\varphi}_j\|_{L^1} = C 2^{-\alpha j}, \quad \|(-\Delta)^{\alpha/2} G_{\sqrt{t}}\|_{L^1} = C t^{-\alpha/2},$$

for all $t > 0$ and all $j \in \mathbb{Z}$ with $C = C(s_1, s_0)$. From this and the above estimate, we obtain (2.8).

By (2.3) we have

$$\|e^{t\Delta} a\|_{B^{s_1}_{p,1}} = \|e^{(t/2)\Delta}(e^{(t/2)\Delta} a)\|_{B^{s_1}_{p,1}} \le C(1 + t^{-\alpha/2}) \|e^{(t/2)\Delta} a\|_{B^{s_0}_{p,1}}. \tag{2.9}$$

It follows from (2.1) and (2.3) that

$$\begin{aligned} \|e^{(t/2)\Delta} a\|_{B^{s_0}_{p,1}} &\le C(N \|e^{(t/2)\Delta} a\|_{B^{s_0}_{p,\infty}} + 2^{-\varepsilon N} \|e^{(t/2)\Delta} a\|_{B^{s_0+\varepsilon}_{p,\infty}}) \\ &\le C\{N + 2^{-\varepsilon N}(1 + t^{-\varepsilon/2})\} \|a\|_{B^{s_0}_{p,\infty}}, \quad \varepsilon > 0, \end{aligned}$$

for all positive integers N, where $C = C(s_0, \varepsilon)$. In the case $t \ge 1$, we take $N = 1$ and obtain

$$\|e^{(t/2)\Delta} a\|_{B^{s_0}_{p,1}} \le C \|a\|_{B^{s_0}_{p,\infty}}, \quad t \ge 1.$$

In the case $0 < t < 1$, we take N so large that $2^{-\varepsilon N} t^{-\varepsilon/2} \le 1$, i.e.

$$N \ge \frac{1}{2 \log 2} \log\left(\frac{1}{t}\right)$$

and obtain

$$\|e^{(t/2)\Delta} a\|_{B^{s_0}_{p,1}} \le C\left(1 + \log \frac{1}{t}\right) \|a\|_{B^{s_0}_{p,\infty}}, \quad 0 < t < 1.$$

In both cases, we have

$$\|e^{(t/2)\Delta} a\|_{B^{s_0}_{p,1}} \le C \log\left(e + \frac{1}{t}\right) \|a\|_{B^{s_0}_{p,\infty}} \quad \text{for all } t > 0. \tag{2.10}$$

Now, the desired estimate (2.5) is a consequence of (2.9) and (2.10).

(ii) Since (2.4) yields $\|G * a\|_{\dot{B}^{s'}_{p,\infty}} = \|e^{\Delta}a\|_{\dot{B}^{s'}_{p,\infty}} \le C(s_0, s', p)\|a\|_{\dot{B}^{s_0}_{p,\infty}}$ for all $s' \ge s_0$, we easily see that

$$\begin{aligned}\|G * a\|_{\dot{B}^{s_1}_{p,1}} &= \sum_{j<0} 2^{(s_1-s_0)j}2^{s_0 j}\|\varphi_j * G * a\|_{L^p} + \sum_{j\ge 0} 2^{-j}2^{(s_1+1)j}\|\varphi_j * G * a\|_{L^p} \\ &\le C(\|G * a\|_{\dot{B}^{s_0}_{p,\infty}} + \|G * a\|_{\dot{B}^{s_1+1}_{p,\infty}}) \\ &\le C\|a\|_{\dot{B}^{s_0}_{p,\infty}}. \end{aligned} \tag{2.11}$$

For a non-negative integer m with $m > s/2$, Triebel [**27**] showed that

$$\|f\|_{\dot{B}^s_{p,q}} \cong \left\{ \int_0^\infty (\tau^{m-s/2}\|(-\Delta)^m e^{\tau\Delta} f\|_{L^p})^q \frac{d\tau}{\tau} \right\}^{1/q}, \quad 1 \le q < \infty,$$
$$\|f\|_{\dot{B}^s_{p,\infty}} \cong \sup_{0<\tau<\infty} \tau^{m-s/2}\|(-\Delta)^m e^{\tau\Delta} f\|_{L^p}.$$

Then we observe that

$$\|f_{\sqrt{t}}\|_{\dot{B}^{s_1}_{p,1}} \le Ct^{\frac{1}{2}(-s_1-n+n/p)}\|f\|_{\dot{B}^{s_1}_{p,1}} \quad \text{and} \quad \|f_{1/\sqrt{t}}\|_{\dot{B}^{s_0}_{p,\infty}} \le Ct^{\frac{1}{2}(s_0+n-n/p)}\|f\|_{\dot{B}^{s_0}_{p,\infty}}.$$

Therefore the desired estimate (2.6) is a consequence of (2.11), since $(e^{t\Delta}a) = (G * (a_{1/\sqrt{t}}))_{\sqrt{t}}$.

Finally, from (2.6) we easily obtain (2.7), since $\|\nabla e^{t\Delta}a\|_{\dot{B}^0_{\infty,1}} \le C\|e^{t\Delta}a\|_{\dot{B}^1_{\infty,1}}$. □

Remark. Giga *et al* [**11**, Lemma 3] showed the estimate

$$\|(-\Delta)^\alpha e^{t\Delta}a\|_{L^\infty} \le Ct^{-\alpha}\|a\|_{BMO}, \quad \alpha > 0 \tag{2.12}$$

for all $a \in BMO$. We note that (2.6) is a slightly sharper estimate than (2.12). Their proof of (2.12) is based on the estimate of the maximal function of $(-\Delta)^\alpha e^{t\Delta}a$. On the other hand, we may give another proof. Indeed, there holds

$$R_k(-\Delta)^\alpha G \in BUC \quad \text{with} \quad |R_k(-\Delta)^\alpha G(x)| \le C_{n,\alpha}|x|^{-n-2\alpha} \quad \text{for } |x| \ge 1 \tag{2.13}$$

with

$$C_{n,\alpha} = (4\pi)^{-n/2}\frac{\Gamma(n/2+\alpha)}{2\Gamma(n/2+1)},$$

which yields

$$R_k(-\Delta)^\alpha G \in L^1, \quad k = 1, \ldots, n.$$

Hence we have $(-\Delta)^\alpha G \in \mathcal{H}^1$. Here $\mathcal{H}^1$ is the Hardy space. Since $(-\Delta)^\alpha(G_{\sqrt{t}}) = t^{-\alpha}((-\Delta)^\alpha G)_{\sqrt{t}}$, it follows from the duality result $(\mathcal{H}^1)^* = BMO$ due to Fefferman–Stein [**10**] that

$$\begin{aligned}\sup_{t>0} t^\alpha \|(-\Delta)^\alpha e^{t\Delta} a\|_{L^\infty} &= \sup_{t>0} t^\alpha \|(-\Delta)^\alpha (G_{\sqrt{t}} * a)\|_{L^\infty} \\ &= \sup_{t>0} \|((-\Delta)^\alpha G)_{\sqrt{t}} * a\|_{L^\infty} \\ &\le \|(-\Delta)^\alpha G\|_{\mathcal{H}^1} \|a\|_{BMO}.\end{aligned}$$

This implies (2.12). We prove (2.13) in the appendix.

Next we establish a bilinear estimate in the homogeneous Besov space.

LEMMA 2.3. *Let* $1 \le p \le \infty$, $1 \le q \le \infty$ *and* $s > 0$. *Then we have*

$$\|f \cdot g\|_{\dot{B}^s_{p,q}} \le C(\|f\|_{\dot{B}^s_{\infty,q}} \|g\|_{L^p} + \|f\|_{L^p} \|g\|_{\dot{B}^s_{\infty,q}}) \tag{2.14}$$

for all $f, g \in \dot{B}^s_{\infty,q} \cap L^p$, *where* $C = C(p, q, s)$.

Proof. For the proof, we make use of the following paraproduct formula due to Bony [**4**]. Our method here is related to that of Christ and Weinstein [**8**, Proposition 3.3]:

$$\begin{aligned} f \cdot g &= \sum_{k=-\infty}^{\infty} (\varphi_k * f)(P_k g) + \sum_{k=-\infty}^{\infty} (P_k f)(\varphi_k * g) + \sum_{k=-\infty}^{\infty} \sum_{|l-k|\le 2} (\varphi_k * f)(\varphi_l * g) \\ &\equiv h_1 + h_2 + h_3, \end{aligned} \tag{2.15}$$

where $P_k g = \sum_{l=-\infty}^{k-3} \varphi_l * g = \psi_{2^{-(k-3)}} * g$ with $\psi_\varepsilon = \varepsilon^{-n}\psi(x/\varepsilon)$, $\varepsilon > 0$. We first consider the case when $1 \le q < \infty$. Since

$$\begin{aligned} \operatorname{supp} \mathcal{F}((\varphi_k * f)(P_k g)) &\subset \{\xi \in \mathbb{R}^n\,;\ 2^{k-2} \le |\xi| \le 2^{k+2}\}, \\ \operatorname{supp} \mathcal{F}\varphi_j &= \{\xi \in \mathbb{R}^n\,;\ 2^{j-1} \le |\xi| \le 2^{j+1}\}, \end{aligned}$$

we have by the Young inequality that

$$\begin{aligned} \|h_1\|_{\dot{B}^s_{p,q}} &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{|k-j|\le 2} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^p} \right)^q \right\}^{1/q} \\ &\le \left\{ \sum_{j=-\infty}^{\infty} \left(\sum_{|l|\le 2} 2^{sj} \|\varphi_j * ((\varphi_{j+l} * f)(P_{j+l} g))\|_{L^p} \right)^q \right\}^{1/q} \end{aligned}$$

$$\leq \left\{ \sum_{j=-\infty}^{\infty} \left(\sum_{|l|\leq 2} 2^{sj} \|\varphi_j\|_{L^1} \|\varphi_{j+l} * f\|_{L^\infty} \|P_{j+l} g\|_{L^p} \right)^q \right\}^{1/q}$$
$$\leq C\|g\|_{L^p} \left\{ \sum_{j=-\infty}^{\infty} \left(\sum_{|l|\leq 2} 2^{sj} \|\varphi_{j+l} * f\|_{L^\infty} \right)^q \right\}^{1/q},$$

where $C = \|\mathcal{F}^{-1}\phi\|_{L^1} \|\psi\|_{L^1}$. Notice that

$$\|\varphi_j\|_{L^1} = \|(\mathcal{F}^{-1}\phi)_{2^{-j}}\|_{L^1} = \|\mathcal{F}^{-1}\phi\|_{L^1} \quad \text{for all } j \in \mathbb{Z},$$
$$\sup_{k\in\mathbb{Z}} \|P_k g\|_{L^p} = \sup_{k\in\mathbb{Z}} \|\psi_{2^{-(k-3)}} * g\|_{L^p} \leq \|\psi\|_{L^1} \|g\|_{L^p}.$$

Then the Minkowski inequality yields

$$\|h_1\|_{\dot{B}^s_{p,q}} \leq C\|g\|_{L^p} \sum_{|l|\leq 2} \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_{j+l} * f\|_{L^\infty})^q \right\}^{1/q}$$
$$= C\|g\|_{L^p} \sum_{|l|\leq 2} 2^{-sl} \left\{ \sum_{j=-\infty}^{\infty} (2^{s(j+l)} \|\varphi_{j+l} * f\|_{L^\infty})^q \right\}^{1/q}$$
$$= C\|g\|_{L^p} \|f\|_{\dot{B}^s_{\infty,q}}.$$

For $q = \infty$, we have similarly that

$$\|h_1\|_{\dot{B}^s_{p,\infty}} \leq \sup_{j\in\mathbb{Z}} \left(2^{sj} \sum_{|k-j|\leq 2} \|\varphi_j * ((\varphi_k * f)(P_k g))\|_{L^p} \right)$$
$$\leq C\|g\|_{L^p} \sum_{|l|\leq 2} 2^{-sl} \sup_{j\in\mathbb{Z}} (2^{s(j+l)} \|\varphi_{j+l} * f\|_{L^\infty})$$
$$= C\|g\|_{L^p} \|f\|_{\dot{B}^s_{\infty,\infty}}.$$

Hence, in both cases, there holds

$$\|h_1\|_{\dot{B}^s_{p,q}} \leq C\|g\|_{L^p} \|f\|_{\dot{B}^s_{\infty,q}}, \quad 1 \leq q \leq \infty. \tag{2.16}$$

By replacing the role of f and g with that of g and f, respectively, we see that the second term can be handled in the exactly same way as above. Hence there holds

$$\|h_2\|_{\dot{B}^s_{p,q}} \leq C\|f\|_{L^p} \|g\|_{\dot{B}^s_{\infty,q}}, \quad 1 \leq q \leq \infty. \tag{2.17}$$

To deal with the third term, we should note that

$$\operatorname{supp} \mathcal{F}(\varphi_k * f \cdot \varphi_l * g) \subset \{\xi \in \mathbb{R}^n \,;\, |\xi| \leq 2^{\max\{k,l\}+2}\},$$

so there holds

$$\varphi_j * ((\varphi_k * f)(\varphi_l * g)) = 0 \quad \text{for } \max\{k, l\} \leq j - 3.$$

Hence if $1 \leq q < \infty$, by the Minkowski inequality we have

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,q}} &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{j-2 \leq \max\{k,l\}} \sum_{|k-l| \leq 2} \varphi_j * ((\varphi_k * f)(\varphi_l * g)) \right\|_{L^p} \right)^q \right\}^{1/q} \\
&\leq \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{\alpha \geq -4} \sum_{|m| \leq 2} \|\varphi_j\|_{L^1} \|\varphi_{j+\alpha} * f\|_{L^\infty} \|\varphi_{j+\alpha+m} * g\|_{L^p} \right)^q \right\}^{1/q} \\
&\leq C\|g\|_{L^p} \left\{ \sum_{j=-\infty}^{\infty} \left(\sum_{\alpha \geq -4} 2^{-s\alpha} 2^{s(j+\alpha)} \|\varphi_{j+\alpha} * f\|_{L^\infty} \right)^q \right\}^{1/q} \\
&\leq C\|g\|_{L^p} \sum_{\alpha \geq -4} 2^{-s\alpha} \left\{ \sum_{j=-\infty}^{\infty} (2^{s(j+\alpha)} \|\varphi_{j+\alpha} * f\|_{L^\infty})^q \right\}^{1/q} \\
&\leq C\|g\|_{L^p} \|f\|_{\dot{B}^s_{\infty,q}},
\end{aligned}$$

where $C = \|\mathcal{F}^{-1}\phi\|^2_{L^1}$. Notice that

$$\sup_{k \in Z} \|\varphi_k * g\|_{L^p} = \sup_{k \in Z} \|\varphi_k\|_{L^1} \|g\|_{L^p} \leq \|\mathcal{F}^{-1}\phi\|_{L^1} \|g\|_{L^p}.$$

If $q = \infty$, we have similarly

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,\infty}} &\leq \sup_{j \in \mathbb{Z}} \left(2^{sj} \sum_{j-4 \leq k} \sum_{|m| \leq 2} \|\varphi_j * ((\varphi_k * f)(\varphi_{k+m} * g))\|_{L^p} \right) \\
&\leq \sup_{j \in \mathbb{Z}} \left(2^{sj} \|\varphi_j\|_{L^1} \sum_{j-4 \leq k} \sum_{|m| \leq 2} \|\varphi_k * f\|_{L^\infty} \|g\|_{L^p} \right) \\
&\leq C\|g\|_{L^p} \sup_{j \in \mathbb{Z}} \left(2^{sj} \sum_{\alpha \geq -4} \|\varphi_{j+\alpha} * f\|_{L^\infty} \right) \\
&\leq C\|g\|_{L^p} \sup_{j \in \mathbb{Z}} \left(\sum_{\alpha \geq -4} 2^{-s\alpha} 2^{s(j+\alpha)} \|\varphi_{j+\alpha} * f\|_{L^\infty} \right) \\
&\leq C\|g\|_{L^p} \|f\|_{\dot{B}^s_{\infty,\infty}}.
\end{aligned}$$

In both cases, we obtain

$$\|h_3\|_{\dot{B}^s_{p,q}} \leq C\|g\|_{L^p} \|f\|_{\dot{B}^s_{p,q}}, \quad 1 \leq q \leq \infty. \tag{2.18}$$

Now it follows from (2.16), (2.17) and (2.18) that

$$\|f \cdot g\|_{\dot{B}^s_{p,q}} \leq C(\|f\|_{\dot{B}^s_{\infty,q}} \|g\|_{L^p} + \|f\|_{L^p} \|g\|_{\dot{B}^s_{\infty,q}})$$

where $C = C(p, q, s)$. This yields the desired estimate. □

The following lemma plays an important role for our extension criterion.

LEMMA 2.4. *For* $0 < \varepsilon \le \delta \le 1$ *and* $s \ge 0$ *there is a constant* $C = C(\varepsilon, \delta, s)$ *such that*

$$\|e^{t\Delta}P(u\cdot\nabla u)\|_{\dot B^s_{p,1}} \le C(t^{-\frac12(s-\delta)-\frac12} + t^{-\frac12(s+\varepsilon)-\frac12} + \|\mathrm{rot}\, u\|_{\dot B^s_{\infty,\infty}}\|u\|_{L^p}\log(e+\|u\|_{L^{2p}}))$$

holds for all $u \in \dot B^{s+1}_{\infty,\infty} \cap L^p$, $1 < p \le \infty$, *with* $\mathrm{div}\, u = 0$ *and for all* $t > 0$.

Proof. First, notice that $\dot B^{s+1}_{\infty,\infty} \cap L^p \subset L^\infty$, and hence $u \in L^{2p}$. By Lemma 2.3 we have $u \otimes u \in \dot B^{s+1}_{p,\infty}$ with $\|u\otimes u\|_{\dot B^{s+1}_{p,\infty}} \le C\|u\|_{\dot B^{s+1}_{\infty,\infty}}\|u\|_{L^p}$. Since $\mathrm{div}\, u = 0$ implies $u\cdot\nabla u = \nabla\cdot(u\otimes u)$, there holds $u\cdot\nabla u \in \dot B^s_{p,\infty}$ with

$$\|u\cdot\nabla u\|_{\dot B^s_{p,\infty}} \le C\|u\|_{\dot B^{s+1}_{\infty,\infty}}\|u\|_{L^p}.$$

By the Biot–Savart law, we see the representation $\nabla u = R(R\wedge \mathrm{rot}\, u)$ by the Riesz transforms $R = (R_1, \dots, R_n)$. Since R is a bounded operator from $\dot B^s_{\infty,\infty}$ into itself, it follows that

$$\|u\|_{\dot B^{s+1}_{\infty,\infty}} = \|\nabla u\|_{\dot B^s_{\infty,\infty}} \le C\|\mathrm{rot}\, u\|_{\dot B^s_{\infty,\infty}}$$

from which and the above estimate, we obtain

$$\|u\cdot\nabla u\|_{\dot B^s_{p,\infty}} \le C\|\mathrm{rot}\, u\|_{\dot B^s_{\infty,\infty}}\|u\|_{L^p}. \tag{2.19}$$

Now applying (2.2) with $s_0 = s-\delta$, $s_1 = s+\varepsilon$, we have by (2.4) and (2.19), together with the boundedness of P in $\dot B^\alpha_{p,\infty}$ for $\alpha = 0, s$, that

$$\begin{aligned}
\|e^{t\Delta}P(u\cdot\nabla u)\|_{\dot B^s_{p,1}} &\le C(2^{-\delta N}\|\nabla\cdot e^{t\Delta}P(u\otimes u)\|_{\dot B^{s-\delta}_{p,\infty}} + N\|e^{t\Delta}P(u\cdot\nabla u)\|_{\dot B^s_{p,\infty}} \\
&\quad + 2^{-\varepsilon N}\|\nabla\cdot e^{t\Delta}P(u\otimes u)\|_{\dot B^{s+\varepsilon}_{p,\infty}}) \\
&\le C(2^{-\delta N}\|e^{t\Delta}P(u\otimes u)\|_{\dot B^{s-\delta+1}_{p,\infty}} + N\|e^{t\Delta}P(u\cdot\nabla u)\|_{\dot B^s_{p,\infty}} \\
&\quad + 2^{-\varepsilon N}\|e^{t\Delta}P(u\otimes u)\|_{\dot B^{s+\varepsilon+1}_{p,\infty}}) \\
&\le C(2^{-\delta N}t^{-\frac12(s-\delta)-\frac12}\|P(u\otimes u)\|_{\dot B^0_{p,\infty}} + N\|P(u\cdot\nabla u)\|_{\dot B^s_{p,\infty}} \\
&\quad + 2^{-\varepsilon N}t^{-\frac12(s+\varepsilon)-\frac12}\|P(u\otimes u)\|_{\dot B^0_{p,\infty}}) \\
&\le C(2^{-\delta N}t^{-\frac12(s-\delta)-\frac12}\|u\|^2_{L^{2p}} + N\|\mathrm{rot}\, u\|_{\dot B^s_{\infty,\infty}}\|u\|_{L^p} \\
&\quad + 2^{-\varepsilon N}t^{-\frac12(s+\varepsilon)-\frac12}\|u\|^2_{L^{2p}}),
\end{aligned}$$

for all $t > 0$ with $C = C(\varepsilon, \delta, s)$.

In the case $\|u\|_{L^{2p}} \le 1$, we take $N = 1$ and obtain

$$\|e^{t\Delta}P(u\cdot\nabla u)\|_{\dot{B}^s_{p,1}} \le C(t^{-\frac{1}{2}(s-\delta)-\frac{1}{2}} + t^{-\frac{1}{2}(s+\varepsilon)-\frac{1}{2}} + \|\mathrm{rot}\, u\|_{\dot{B}^s_{\infty,\infty}}\|u\|_{L^p}).$$

In the case $\|u\|_{L^{2p}} > 1$, we take N so large that $2^{-\varepsilon N}\|u\|^2_{L^{2p}} \le 1$, i.e.

$$N \ge \frac{2}{\varepsilon \log 2}\log\|u\|_{L^{2p}}$$

and obtain

$$\begin{aligned}\|e^{t\Delta}P(u\cdot\nabla u)\|_{\dot{B}^s_{p,1}} &\le C(t^{-\frac{1}{2}(s-\delta)-\frac{1}{2}}\|u\|^{2(1-\frac{\delta}{\varepsilon})}_{L^{2p}} + t^{-\frac{1}{2}(s+\varepsilon)-\frac{1}{2}} \\ &\quad + \|\mathrm{rot}\, u\|_{\dot{B}^s_{\infty,\infty}}\|u\|_{L^p}\log\|u\|_{L^{2p}}) \\ &\le C(t^{-\frac{1}{2}(s-\delta)-\frac{1}{2}} + t^{-\frac{1}{2}(s+\varepsilon)-\frac{1}{2}} \\ &\quad + \|\mathrm{rot}\, u\|_{\dot{B}^s_{\infty,\infty}}\|u\|_{L^p}\log\|u\|_{L^{2p}}).\end{aligned}$$

In both cases, we get the desired estimate. □

Remark. Giga *et al* [**12**, Lemma 3] showed a similar estimate to Lemma 2.4 in L^∞. If we deal with time-dependent functions $u = u(t)$ for $t \in (0, T)$, a significant difference of our estimate from theirs [**12**] consists of the separation of the singularity at $t = 0$ and singularities of $u(t)$ on $(0, T)$. This is the reason why we can avoid the generalized Gronwall inequality which they used for the proof the extension criterion on local solutions.

3. Proofs

3.1. *Proof of Theorem 1*

3.1.1. Existence. We first prove the existence of the solution u of (I.E.) by the following successive approximation:

$$\begin{cases} u_0(t) = e^{t\Delta}a, \\ u_{m+1}(t) = u_0(t) - \int_0^t e^{(t-\tau)\Delta}P(u_m\cdot\nabla u_m)(\tau)\,d\tau, \quad m = 0, 1, \ldots . \end{cases} \tag{3.1}$$

We first show that

$$u_m \in C_{\mathrm{w}}([0, T_*); B^0_{\infty,\infty}) \quad \text{with} \quad \sup_{0<t<T_*}\|u_m(t)\|_{B^0_{\infty,\infty}} \le K_m, \tag{3.2}$$

$$\{\log(e+1/t)\}^{-1}u_m \in L^\infty(0,T_*;L^\infty)$$
$$\text{with} \quad \sup_{0<t<T_*}\{\log(e+1/t)\}^{-1}\|u_m(t)\|_{L^\infty} \le K'_m, \tag{3.3}$$
$$t^{\frac{1}{2}}u_m \in L^\infty(0,T_*;\dot{B}^1_{\infty,1}) \quad \text{with} \quad \sup_{0<t<T_*} t^{\frac{1}{2}}\|u_m(t)\|_{\dot{B}^1_{\infty,1}} \le K''_m. \tag{3.4}$$

Indeed, for $m=0$, we have by Lemma 2.2 that

$$\|u_0(t)\|_{B^0_{\infty,\infty}} \le \|a\|_{B^0_{\infty,\infty}},$$
$$\|u_0(t)\|_{L^\infty} \le \|u_0(t)\|_{B^0_{\infty,1}} \le C\log(e+1/t)\|a\|_{B^0_{\infty,\infty}},$$
$$\|u_0(t)\|_{\dot{B}^1_{\infty,1}} \le Ct^{-\frac{1}{2}}\|a\|_{\dot{B}^0_{\infty,\infty}} \le Ct^{-\frac{1}{2}}\Big(\sup_{j<0}\|\varphi_j * \psi * a\|_\infty + \sup_{j\ge 0}\|\varphi_j * a\|_\infty\Big)$$
$$\le Ct^{-\frac{1}{2}}\|a\|_{B^0_{\infty,\infty}},$$

for all $t>0$, so we may take $K_0 = K'_0 = K''_0 = C\|a\|_{B^0_{\infty,\infty}}$. Suppose that (3.2), (3.3) and (3.4) are true for m. Since $\operatorname{div} u_m = 0$, we have $u_m\cdot\nabla u_m = \nabla\cdot(u_m\otimes u_m)$, and it follows from Lemma 2.2(ii) that

$$\begin{aligned}
\left\|\int_0^t e^{(t-\tau)\Delta}P(u_m\cdot\nabla u_m)(\tau)\,d\tau\right\|_{L^\infty} &\le \int_0^t \|\nabla\cdot e^{(t-\tau)\Delta}P(u_m\otimes u_m)(\tau)\|_{L^\infty}\,d\tau \\
&\le \int_0^t \|\nabla\cdot e^{(t-\tau)\Delta}P(u_m\otimes u_m)(\tau)\|_{\dot{B}^0_{\infty,1}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|P(u_m\otimes u_m)(\tau)\|_{\dot{B}^0_{\infty,\infty}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|u_m\otimes u_m(\tau)\|_{\dot{B}^0_{\infty,\infty}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|u_m(\tau)\|^2_{L^\infty}\,d\tau \\
&\le C(K'_m)^2\int_0^t (t-\tau)^{-\frac{1}{2}}\{\log(e+1/\tau)\}^2\,d\tau.
\end{aligned}$$

Note that $\|\nabla f\|_\infty \le \|\nabla f\|_{\dot{B}^0_{\infty,1}}$ holds for all $f\in BMO$ with $\nabla f\in \dot{B}^0_{\infty,1}$ and that the projection P is a bounded operator from $\dot{B}^0_{\infty,\infty}$ into itself. Since

$$\log(e+1/\tau) \le 2+\varepsilon^{-1}\tau^{-\varepsilon'} \quad \text{for all } \tau>0,\ \varepsilon'>0,$$

we obtain from the above estimates with $\varepsilon' = 2\varepsilon$ that

$$\left\|\int_0^t e^{(t-\tau)\Delta}P(u_m\cdot\nabla u_m)(\tau)\,d\tau\right\|_{L^\infty} \le C_\varepsilon(K'_m)^2(t^{\frac{1}{2}}+t^{\frac{1}{2}-\varepsilon/2}) \le C(K'_m)^2 t^{\frac{1}{2}-\varepsilon/2}$$

for all $0 < t \le 1$. Since $\sup_{0<t\le 1} t^{\varepsilon/2}\log(e+1/t) \le C_\varepsilon$ and $L^\infty \subset B^0_{\infty,\infty}$, we may take K_{m+1} and K'_{m+1} so that

$$K_{m+1} = K'_{m+1} = C\|a\|_{B^0_{\infty,\infty}} + C_\varepsilon (K'_m)^2 T_*^{\frac{1}{2}-\varepsilon} \tag{3.5}$$

with sufficiently small $\varepsilon > 0$. Moreover, we have from (2.6) that

$$\begin{aligned}
&\left\| \int_0^t e^{(t-\tau)\Delta} P(u_m\cdot\nabla u_m)(\tau)\,d\tau \right\|_{\dot B^1_{\infty,1}} \\
&\quad\le \int_0^t \|e^{(t-\tau)\Delta}P(u_m\cdot\nabla u_m)(\tau)\|_{\dot B^1_{\infty,1}}\,d\tau \\
&\quad\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|P(u_m\cdot\nabla u_m)(\tau)\|_{\dot B^0_{\infty,\infty}}\,d\tau \\
&\quad\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|(u_m\cdot\nabla u_m)(\tau)\|_{L^\infty} \\
&\quad\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|u_m(\tau)\|_{\dot B^1_{\infty,1}}\|u_m(\tau)\|_{L^\infty}\,d\tau \\
&\quad\le CK''_mK'_m \int_0^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\log(e+1/\tau)\,d\tau \\
&\quad\le C_\varepsilon K''_mK'_m \int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau^{-\frac{1}{2}-\varepsilon})\,d\tau \\
&\quad\le C_\varepsilon K''_mK'_m (t^{\frac{1}{2}} + t^{-\varepsilon})
\end{aligned}$$

for all $0 < t \le 1$. Hence, in the same was as in (3.5), we may take K''_{m+1} as

$$K''_{m+1} = C\|a\|_{B^0_{\infty,\infty}} + C_\varepsilon K''_m K'_m T_*^{\frac{1}{2}-\varepsilon}. \tag{3.6}$$

Setting $L_m \equiv \max\{K_m, K'_m, K''_m\}$, we obtain from (3.5) and (3.6) that

$$L_{m+1} \le C\|a\|_{B^0_{\infty,\infty}} + C_\varepsilon T_*^{\frac{1}{2}-\varepsilon} L_m^2, \quad m = 0, 1, \ldots, \tag{3.7}$$

for sufficiently small $\varepsilon > 0$. If we choose T_* so small that

$$T_*^{\frac{1}{2}-\varepsilon} < \frac{1}{4CC_\varepsilon \|a\|_{B^0_{\infty,\infty}}}, \tag{3.8}$$

then the sequence $\{L_m\}_{m=0}^\infty$ is bounded by $1/2C_\varepsilon T_*^{\frac{1}{2}-\varepsilon}$. This implies that sequences (3.2), (3.3) and (3.4) are bounded by the same constant. Now by the standard argument, we get the limit function u of u_m as $m \to \infty$ in the spaces of (1.1) and (1.2). Letting $m \to \infty$ in (3.1), we see easily that u is the desired solution of (I.E.).

Remark. Giga *et al* [**11**, Lemma 3] gave an estimate $\|\nabla e^{t\Delta} Pf\|_{L^\infty} \le Ct^{-\frac{1}{2}}\|f\|_{L^\infty}$ and constructed a local solution for $a \in BUC$.

3.1.2. Uniqueness. We next prove the uniqueness of solutions in the class of (1.1). Let u and v be two solutions of (I.E.) in the class of (1.1). For $w \equiv u - v$, we have

$$w(t) = -\int_0^t \nabla \cdot e^{(t-\tau)\Delta} P(w \otimes u + v \otimes w)(\tau)\, d\tau,$$

and by Lemma 2.2(ii) there holds

$$\begin{aligned}
\|w(t)\|_{L^\infty} &\le \int_0^t \|\nabla \cdot e^{(t-\tau)\Delta} P(w \otimes u + v \otimes w)(\tau)\|_{L^\infty}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|P(w \otimes u + v \otimes w)(\tau)\|_{BMO}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|w \otimes u(\tau) + v \otimes w(\tau)\|_{BMO}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|w(\tau)\|_{L^\infty}(\|u(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty})\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|w(\tau)\|_{L^\infty} \log(e + 1/\tau)\, d\tau \\
&\le C_\varepsilon \sup_{0<\tau<t} \|w(\tau)\|_{L^\infty} \int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau^{-\varepsilon})\, d\tau \\
&\le C_\varepsilon \sup_{0<\tau<t} \|w(\tau)\|_{L^\infty}(t^{\frac{1}{2}} + t^{\frac{1}{2}-\varepsilon}) \\
&\le C_\varepsilon \sup_{0<\tau<t} \|w(\tau)\|_{L^\infty} t^{\frac{1}{2}-\varepsilon}
\end{aligned}$$

for all $0 < t \le 1$ with sufficiently small $\varepsilon > 0$. Since the right-hand side of the above estimate is monotone increasing in $t \in (0, 1]$, we have

$$g(t) \le C_\varepsilon g(t) t^{\frac{1}{2}-\varepsilon}, \quad 0 < t \le 1,$$

where $g(t) \equiv \sup_{0<\tau<t} \|w(\tau)\|_{L^\infty}$. Hence, if we take $T_1 < 1/C^{2/(1-2\varepsilon)}$, then there holds

$$w(t) \equiv 0 \quad \text{for all } t \in [0, T_1].$$

We next show that

$$w(t) \equiv 0 \quad \text{for all } t \in [T_1, T).$$

To this end, it suffices to show the following proposition.

PROPOSITION 3.1. *There exists $\kappa > 0$ such that if $w(t) \equiv 0$ on $[0, T_2]$, then there holds*

$$w(t) \equiv 0 \quad on\ [0, T_2 + \kappa).$$

Proof. Since $u(T_2) = v(T_2)$, we have

$$w(t) = -\int_{T_2}^{t} \nabla \cdot e^{(t-\tau)\Delta} P(w \otimes u + v \otimes w)(\tau)\, d\tau, \quad T_2 < t < T.$$

Hence in the same way as above, we see that

$$\begin{aligned}\|w(t)\|_{L^\infty} &\le C \int_{T_2}^{t} (t-\tau)^{-\frac{1}{2}} \|w(\tau)\|_{L^\infty} (\|u(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty})\, d\tau \\ &\le C(\sup_{T_1<t<T} \|u(\tau)\|_{L^\infty} + \sup_{T_1<t<T} \|v(\tau)\|_{L^\infty}) \sup_{T_2<\tau<t} \|w(\tau)\|_{L^\infty} (t - T_2)^{\frac{1}{2}},\end{aligned}$$

for all $T_2 < t < T$. If we take κ so that

$$\kappa^{\frac{1}{2}} \equiv \frac{1}{2C(\sup_{T_1<t<T} \|u(\tau)\|_{L^\infty} + \sup_{T_1<t<T} \|v(\tau)\|_{L^\infty})},$$

then there holds

$$\tilde{g}(t) \le \frac{1}{2}\tilde{g}(t) \quad \text{for } T_2 \le t < T_2 + \kappa,$$

where $\tilde{g}(t) \equiv \sup_{T_2<\tau<t} \|w(t)\|_{L^\infty}$. Then it follows that

$$w(t) \equiv 0 \quad \text{on } [T_2, T_2 + \kappa),$$

and we obtain the desired uniqueness result. □

3.2. *Proof of Theorem 2*

Since the existence time interval $[0, T_*)$ with respect to the initial data $a \in B^0_{\infty,\infty}$ is characterized as in (3.8)(see also (1.3)) and since $L^\infty \subset B^0_{\infty,\infty}$, it suffices to show an *a priori* estimate of $\|u(t)\|_{L^\infty}$ for $t \in (\eta, T)$ in terms of (1.4). Here $0 < \eta < T$ can be taken as $u(\eta) \in L^\infty$. Since $\|\nabla f\|_{L^\infty} \le \|\nabla f\|_{\dot{B}^0_{\infty,1}}$ holds for all $f \in BMO$ with

$\nabla f \in \dot{B}^0_{\infty,1}$, we have by Lemma 2.4 with $\varepsilon = 1/2$, $\delta = 1$, $s = 0$ and $p = \infty$ that

$$\begin{aligned}
\|u(t)\|_{L^\infty} &\le \|u(\eta)\|_{L^\infty} + \int_\eta^t \|\nabla \cdot e^{(t-\tau)\Delta} P(u \otimes u)(\tau)\|_{L^\infty}\, d\tau \\
&\le \|u(\eta)\|_{L^\infty} + \int_\eta^t \|\nabla \cdot e^{(t-\tau)\Delta} P(u \otimes u)(\tau)\|_{\dot{B}^0_{\infty,1}}\, d\tau \\
&\le \|u(\eta)\|_{L^\infty} + \int_\eta^t \|e^{(t-\tau)\Delta} P(u \cdot \nabla u)\|_{\dot{B}^0_{\infty,1}} d\tau \\
&\le \|u(\eta)\|_{L^\infty} + C\int_\eta^t (1 + (t-\tau)^{-\frac{3}{4}} \\
&\quad + \|\operatorname{rot} u(\tau)\|_{\dot{B}^0_{\infty,\infty}} \|u(\tau)\|_{L^\infty} \log(e + \|u(\tau)\|_{L^\infty}))\, d\tau \\
&\le \|u(\eta)\|_{L^\infty} \\
&\quad + C + C\int_\eta^t \|\operatorname{rot} u(\tau)\|_{\dot{B}^0_{\infty,\infty}} \|u(\tau)\|_{L^\infty} \log(e + \|u(\tau)\|_{L^\infty})\, d\tau
\end{aligned}$$

for all $t \in (\eta, T)$ with $C = C(T)$. Hence the Gronwall inequality yields

$$\|u(t)\|_{L^\infty} \le (\|u(\eta)\|_{L^\infty} + C) \exp\left(C\int_\eta^t \|\operatorname{rot} u(\tau)\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|u(\tau)\|_{L^\infty})\, d\tau \right),$$

$$\eta < t < T.$$

Setting $z(t) = \log(e + \|u(t)\|_{L^\infty})$, we obtain from the above estimate

$$z(t) \le z(\eta) + C + C\int_\eta^t \|\operatorname{rot} u(\tau)\|_{\dot{B}^0_{\infty,\infty}} z(\tau)\, d\tau, \qquad \eta < t < T.$$

Again by the Gronwall inequality, we have

$$z(t) \le (z(\eta) + C) \exp\left(C\int_\eta^t \|\operatorname{rot} u(\tau)\|_{\dot{B}^0_{\infty,\infty}}\, d\tau \right), \quad \eta < t < T,$$

which yields

$$\sup_{\eta < t < T} \|u(t)\|_{L^\infty} \le \{e^C (\|u(\eta)\|_{L^\infty} + e)\}^{\exp(C\int_0^T \|\operatorname{rot} u(\tau)\|_{\dot{B}^0_{\infty,\infty}} d\tau)}.$$

This proves Theorem 2.

Appendix

In this appendix, we prove (2.13). We make use of the representation

$$R_k(-\Delta)^\alpha G(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{i\xi_k}{|\xi|} |\xi|^{2\alpha} e^{-|\xi|^2}\, d\xi, \quad k = 1, \dots, n.$$

Obviously, $R_k(-\Delta)^\alpha G \in BUC$. Introducing the polar coordinate $\xi = \rho\omega$ with $\rho = |\xi|$, $\omega = (\omega_1, \dots, \omega_n) \in S^{n-1}$, we have

$$\begin{aligned} R_k(-\Delta)^\alpha G(x) &= (2\pi)^{-n} \lim_{R\to\infty} \int_{\omega\in S^{n-1}} d\omega\, \omega_k \int_0^R e^{ir\rho\cos\theta} e^{-\rho^2} \rho^{n+2\alpha-1}\, d\rho \\ &\qquad \text{(by changing the variable } \rho \to s = r\rho\cos\theta) \\ &= i(2\pi)^{-n} r^{-n-2\alpha} \lim_{R\to\infty} \int_{\omega\in S^{n-1}} d\omega\, \omega_k \\ &\quad \times \int_0^{Rr\cos\theta} e^{is} e^{-\frac{s^2}{r^2\cos^2\theta}} s^{n+2\alpha-1} \cos^{-n-2\alpha}\theta\, ds, \end{aligned} \tag{A.1}$$

where $r = |x|$ and θ is the angle between x and ω. Since

$$\begin{aligned} &\left| \int_0^{Rr\cos\theta} e^{is} e^{-s^2/r^2\cos^2\theta} s^{n+2\alpha-1} \cos^{-n-2\alpha}\theta\, ds \right| \\ &\quad \le \int_0^{Rr|\cos\theta|} e^{-s^2/\cos^2\theta} s^{n+2\alpha-1} |\cos\theta|^{-n-2\alpha}\, ds \\ &\qquad \text{(by changing the variable } s \to t = s/|\cos\theta|) \\ &\quad \le \int_0^{Rr} e^{-t^2} t^{n+2\alpha-1}\, dt \\ &\quad \le \frac{1}{2} \int_0^\infty e^{-t} t^{n/2+\alpha-1}\, dt = \frac{1}{2}\Gamma\left(\frac{n}{2} + \alpha\right) \end{aligned}$$

holds for all $R > 0$ and all $r \ge 1$, it follows from (A.1) that

$$\begin{aligned} |R_k(-\Delta)^\alpha G(x)| &\le (2\pi)^{-n} |S^{n-1}| \tfrac{1}{2} \Gamma(n/2 + \alpha) r^{-n-2\alpha} \\ &= (4\pi)^{-n/2} \frac{\Gamma(n/2+\alpha)}{2\Gamma(n/2+1)} r^{-n-2\alpha} \end{aligned}$$

for all $r \ge 1$, which yields (2.13).

Acknowledgement. The authors would like to thank Dr Sawada for their valuable discussions with him.

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