Navier-Stokes equations interacting with a nonlinear elastic shell

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Contents

Abstract

This dissertation is written under the supervision of Prof. Steve Shkoller. We study a moving boundary value problem consisting of a viscous incompressible fluid moving and interacting with a nonlinear elastic shell. The fluid motion is governed by the Navier-Stokes equations, while the shell is modeled by the nonlinear St. Venant-Kirchhoff constitutive law, and they are coupled together by continuity of displacements and tractions (stresses) along the moving material interface. We introduce new variables so that in order to formulate the problem more easily, and prove existence and uniqueness of solutions in Sobolev spaces.

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This dissertation includes details of my work on the interaction of fluids and nonlinear elastic shells. A more concise version of the research result has been submitted to the SIAM Journal of Mathematical Analysis.

1. Introduction

The free boundary problem in continuum mechanics is one of the most beautiful and important problems in nature. It appears when considering the motion of single fluid with free boundary, or the motion of two different materials in liquid crystals, elastic solids, porous media, and etc. Some typical examples in fluid dynamics are: an oil drop moving inside water (known as the surface tension problem), an elastic ball moving inside fluids (known as the fluid-solid interaction problem), and water balloons or veins (also the fluid-solid interaction problem, while in this case the solid is assumed to be thin enough so that it does not occupy any space). In most of the cases, we have to couple the fluid equations such as the Euler or the Navier-Stokes equations with other equations in order to study different physical phenomena.

A classical problem studied in this area is the Navier-Stokes equations without surface tension. In mathematics, this is formulated by the Navier-Stokes equations with a Neumann type of boundary condition, which is described as that the normal stress of the fluid vanishes on the moving boundary. Even for this simple boundary value problem, the existence of a global in time (weak) solution was not known until 1994. A level set method with a set-valued fixed point theorem, so-called the Kakutani fixed point theorem, is used in [17] and [18] to establish the existence of global weak solutions to the imcompressible Stokes and Navier-Stokes equations. Standard energy methods are applied in [9] for the study on the global in time weak solution in two-dimensional multiphase viscous fluids.

A more difficult problem is to consider the surface tension. In mathematical terminology, the boundary condition becomes that the normal stress of the fluid is proportional to the mean curvature vector on the moving boundary (this boundary condition comes from the fact that the surface tension tries to minimize the surface area of the fluid domain and the variation of surface area is the mean curvature vector). Since in general the fluid domain might undergo topological changes such as the separation or merging so that the mean curvature vector is no longer defined, the existence of global in time weak solutions is not yet known and is still an open problem. Because of the potential topological change of the fluid domain, level set methods are widely used to study the problem, and numerous numerical results based on the level set formulation are achieved (but without rigorous mathematical proof). Theoretically, several approaches of establishing the existence of the short time solution were successful. A slight powerful topological fixed point theorem, known as the Tychonoff fixed point theorem, was used together with the standard energy methods in [5] to show the existence and uniqueness of the short time strong solution to the single phase problem.

An even more difficult problem in this area is invicid fluids (the Euler equations) with or without surface tension. The lack of viscosity was overcome by an additional assumption that the curl of the fluid velocity vanishes everywhere inside the fluid, and almost all of the well-posedness results focused on irrotational fluids. Without this additional irrotational constraint, a Nash-Moser iteration is used to study the problem without surface tension in [16], and the well-posedness of the Euler equations with surface tension is established in [8]. In theory, the lack of viscosity is overcome by a div-curl type of elliptic estimates which depends crucially on the transport structure of the vorticity which only appears in the invicid fluids, while in numerical context, the lack of viscosity causes catastrophe, and artificial viscosity has to be introduced into the problems and good numerical schemes for simulating this phenomena are still not available.

Many physical phenomena also involve fluids interacting with an elastic or rigid structure. Fluid-solid interaction problems involving moving material interfaces have been the focus of active research since the nineties. The first problem solved in this area was for the case of a rigid body moving in a viscous fluid (see [10], [20] and also the early works of [24] and [23] for a rigid body moving in a Stokes flow in the full space). The case of an elastic body moving in a viscous fluid was considerably more challenging because of some apparent regularity incompatibilities between the two phases: fluid and solid. The first existence results in this area were for regularized elasticity laws, such as in [11] for a *finite* number of elastic modes, or in [2] and [3] for hyperviscous elasticity laws, or in [22] for phase-field regularization which "fattens" the sharp interface via a diffuse-interface model.

The treatment of classical elasticity laws for the solid phase, without any regularizing term, was only considered recently in [6] for the three-dimensional linear St. Venant-Kirchhoff constitutive law and in [7] for quasilinear elastodynamics coupled to the Navier-Stokes equations. Some of the basic new ideas introduced in those works concerned a functional framework that scales in a hyperbolic fashion (and is therefore driven by the solid phase), the introduction of approximate problems either penalized with respect to the divergence-free constraint in the moving fluid domain, or smoothed by an appropriate parabolic artificial viscosity in the solid phase (chosen in an asymptotically convergent and consistent fashion), and the tracking of the motion of the interface by difference quotients techniques.

The complimentary fluid-solid interaction problem, studied herein, consists of the motion of a viscous incompressible fluid enclosed by a moving thin nonlinear elastic shell (for example, an often used mathematical model of the cardio-vascular system). This is a moving boundary problem that models the motion of a viscous incompressible Newtonian fluid inside of a deformable elastic structure. The main mathematical differences with respect to the previous problem of a three-dimensional solid body moving inside of the fluid is the two-dimensional nature of the shell and in the appearance of "elliptic" operators that are degenerate in the (a priori unknown) tangential directions. The only cases considered until now consisted of regularized problem wherein the elliptic degeneracy occurs along a *fixed direction*, such as in [13] or [3].

In this dissertation, we are concerned here with establishing the existence and uniqueness of solutions to the time-dependent incompressible Navier-Stokes equations interacting with a quasilinear elastic shell of Koiter type, which comes directly from an asymptotic expansion in the nonlinear three-dimensional St. Venant-Kirchhoff equations as the thickness of the shell converges to zero, and is a function of the first and second fundamental forms of the moving shell (boundary of the fluid domain).

Let $\Omega_0 \subset \mathbb{R}^3$ denote an open bounded domain with boundary $\Gamma_0 := \partial\Omega_0$. For each $t \in (0, T]$, we wish to find the domain $\Omega(t)$, a divergence-free velocity field $u(t, \cdot)$, a pressure function $p(t, \cdot)$ on $\Omega(t)$, and a volume-preserving transformation $\eta(t, \cdot) : \Omega_0 \to \mathbb{R}^3$ such that

$$\Omega(t) = \eta(t, \Omega_0)\,, \tag{1.1a}$$

$$\eta_t(t, x) = u(t, \eta(t, x))\,, \tag{1.1b}$$

$$u_t + \nabla_u u - \nu\Delta u = -\nabla p + f \qquad \text{in } \Omega(t)\,, \tag{1.1c}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega(t)\,, \tag{1.1d}$$

$$(\nu \operatorname{Def} u - p\mathrm{Id})n = \mathfrak{t}_{shell} \qquad \text{on } \Gamma(t)\,, \tag{1.1e}$$

$$u(0, x) = u_0(x) \qquad \forall x \in \Omega_0\,, \tag{1.1f}$$

$$\eta(0, x) = x \qquad \forall x \in \Omega_0\,, \tag{1.1g}$$

where ν is the kinematic viscosity, $n(t, \cdot)$ is the outward pointing unit normal to $\Gamma(t)$, $\Gamma(t) := \partial\Omega(t)$ denotes the boundary of $\Omega(t)$, $\operatorname{Def} u$ is twice the rate of deformation tensor of u, given in coordinates by $u^i_{,j} + u^j_{,i}$, and $\mathfrak{t}_{shell}$ is the traction imparted onto the fluid by the elastic shell, which we describe next.

We shall consider a thin elastic shell modeled by the nonlinear Saint Venant-Kirchhoff constitutive law. With ϵ denoting the thickness of the shell, the hyperelastic stored energy function has the asymptotic expansion

$$E_{shell} = \epsilon E_{mem} + \epsilon^3 E_{ben} + \mathcal{O}(\epsilon^4).$$

The membrane energy satisfies

$$E_{mem} = \int_{\Gamma(t)} \Big[\frac{\mu}{4} \sum_{\alpha,\beta=1}^{2} (g_{\alpha\beta} - g_{0\alpha\beta})^2 + \frac{\mu\lambda}{4(2\mu+\lambda)}\Big(\sum_{\alpha=1}^{2}(g_{\alpha\alpha} - g_{0\alpha\alpha})\Big)^2\Big] dS \tag{1.2}$$

while the bending energy E_{ben} is given by

$$E_{ben} = \int_{\Gamma(t)} \Big[(4\mu + 2\lambda)H^2 - 2\mu K\Big] dS, \tag{1.3}$$

where g denotes the induced metric on the surface $\Gamma(t)$, and H, K denote the mean and Gauss curvatures on $\Gamma(t)$, respectively, and $\lambda/2$ and $\mu/2$ are the Lamé constants (see, for example, [14]).

The traction vector

$$\mathfrak{t}_{shell} = \epsilon \mathfrak{t}_{mem} + \epsilon^3 \mathfrak{t}_{ben} + \mathcal{O}(\epsilon^4)$$

is computed from the first variation of the energy function E_{shell}; the traction vector associated to the membrane energy will be defined later when we introduce a coordinate system, while the traction associated to the bending energy has a simple intrinsic characterization given by

$$\mathfrak{t}_{ben} = \sigma(\Delta_g H - 2HK + 2H^3)n\,, \tag{1.4}$$

where σ is a function of the Lamé constants and Δ_g denotes the Laplacian with respect to the induced metric g on $\Gamma(t)$:

$$\Delta_g f = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^\alpha}\Big(\sqrt{\det(g)} g^{\alpha\beta} \frac{\partial f}{\partial x^\beta}\Big)\,.$$

1.1. **Outline of this dissertation.** In Section 2, we explain why the traction associated to the bending energy has the form (1.4). In Section 3, in addition to the use of Lagrangian variables, we introduce a new coordinate system near the boundary (shell) and three new maps, η^ν, η^τ, and h, which facilitate the computation of the membrane and bending tractions $\mathfrak{t}_{\text{mem}}$ and $\mathfrak{t}_{\text{ben}}$. A key observation is the symmetry relation (3.7) which reduces the derivative count on the tangential reparameterization map η^τ.

The space of solutions (to the problem with $\mathfrak{t}_{\text{mem}} = 0$) is introduced in Section 4, and the main theorem is stated in Section 5. Section 6 defines our notation, and Section 7 provides some useful lemmas.

In Section 8, we introduce the linearized and regularized problems. The regularization requires smoothing certain variables as well as the introduction of a certain artificial viscosity term on the boundary of the fluid domain. Weak solutions of this linear problem are established via a penalization scheme which approximates the incompressibility of the fluid.

In Section 9, we establish a regularity theory for our weak solution using energy estimates for the mollified problem with constants that depend on the mollification parameters. In Section 10, we improve these estimates so that the constants are independent of the artificial viscosity as well as other regularization parameters. This requires an elliptic estimate, arising from the boundary condition (1.1e), which provides additional regularity for the shape of the boundary.

In Section 11, the Tychonoff fixed-point theorem is used to prove the existence of solutions to the original nonlinear problem (1.1) Uniqueness, following required compatibility conditions, is established in Sections 5 and Section 11. In Section 12, we consider the inclusion of the lower-order membrane traction $\mathfrak{t}_{\text{mem}}$ into the problem formulation.

2. The variation of the bending energy

In this section, we compute the variation of the bending energy and find the expression of $\mathfrak{t}_{\text{ben}}$.

2.1. **Some useful formula.** By the definition of the metric and the identity $g^{\alpha\sigma}g_{\sigma\tau} = \delta^\alpha_\tau$, it follows that

$$\delta g_{\alpha\beta} = (\delta\eta^j)_{,\alpha}\eta^j_{,\beta} + \eta^j_{,\alpha}(\delta\eta^j)_{,\beta}\,, \tag{2.1}$$

$$\delta g^{\alpha\beta} = -g^{\alpha\sigma}g^{\beta\tau}\Big[(\delta\eta^j)_{,\sigma}\eta^j_{,\tau} + \eta^j_{,\sigma}(\delta\eta^j)_{,\tau}\Big]\,. \tag{2.2}$$

Since $\Gamma^\kappa_{\alpha\beta} = g^{\kappa\varrho}\eta^j_{,\alpha\beta}\eta^j_{,\varrho}$, by (2.1) and (2.2), the variation of the Christoffel symbol is

$$\delta\Gamma^\kappa_{\alpha\beta} = -g^{\kappa\sigma}\Gamma^\tau_{\alpha\beta}\Big[(\delta\eta^j)_{,\sigma}\eta^j_{,\tau} + \eta^j_{,\sigma}(\delta\eta^j)_{,\tau}\Big] + g^{\kappa\varrho}(\delta\eta^j)_{,\alpha\beta}\eta^j_{,\varrho} + g^{\kappa\varrho}\eta^j_{,\alpha\beta}(\delta\eta^j)_{,\varrho}\,. \tag{2.3}$$

Let g be the determinant of $g_{\alpha\beta}$ (Jacobian), then

$$\frac{\partial}{\partial x_\tau}\sqrt{g} = \Gamma^\sigma_{\sigma\tau}\sqrt{g} \quad \text{or} \quad \frac{\partial}{\partial x_\tau} g = 2\Gamma^\sigma_{\sigma\tau} g\,, \tag{2.4}$$

$$\delta g = 2gg^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)_{,\tau} \quad \text{or} \quad \delta\sqrt{g} = \sqrt{g}g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)_{,\tau}\,. \tag{2.5}$$

By (2.4), we have the following identity:

$$\Big[\sqrt{g}g^{\alpha\beta}\Delta_g\eta^j\Big]_{,\alpha\beta} + \Big[\sqrt{g}g^{\alpha\beta}\Gamma^\kappa_{\alpha\beta}\Delta_g\eta^j\Big]_{,\kappa} = \sqrt{g}\Delta^2_g\eta^j\,. \tag{2.6}$$

Let $C_{\alpha\beta}$ be the covariant components of the curvature tensor which is defined through the equation

$$\eta_{,\alpha\beta} = \Gamma^\kappa_{\alpha\beta}\eta_{,\kappa} + C_{\alpha\beta}n.$$

The following identities hold true for $C_{\alpha\beta}$:

$$C_{\alpha\beta,\tau} - C_{\tau\alpha,\beta} = \Gamma^\kappa_{\alpha\tau}C_{\kappa\beta} - \Gamma^\kappa_{\alpha\beta}C_{\tau\kappa} \qquad \forall\ \alpha,\beta,\tau\,; \tag{2.7}$$

$$-\Gamma^\kappa_{\kappa\tau}C_{\sigma\sigma} + \Gamma^\kappa_{\kappa\sigma}C_{\tau\sigma} + \Gamma^\kappa_{\sigma\tau}C_{\kappa\sigma} - \Gamma^\kappa_{\sigma\sigma}C_{\tau\kappa} + \Gamma^\tau_{\kappa\kappa}C_{\sigma\sigma} - \Gamma^\tau_{\sigma\kappa}C_{\sigma\kappa} = 0 \quad \forall\ \tau\,. \tag{2.8}$$

2.2. **The variation of the bending energy.** Since the mean curvature vector

$$Hn^j = \frac{1}{2}\Delta_g\eta^j = \frac{1}{2}g^{\alpha\beta}(\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa}),$$

by (2.1), (2.2) and (2.3) we find that

$$\begin{aligned}
\delta(Hn^j) &= \frac{1}{2}\,\delta\Big[g^{\alpha\beta}(\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa})\Big] \\
&= \frac{1}{2}(\delta g^{\alpha\beta})(\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa}) + \frac{1}{2}g^{\alpha\beta}\Big[(\delta\eta^j)_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}(\delta\eta^j)_{,\kappa} - (\delta\Gamma^\kappa_{\alpha\beta})\eta^j_{,\kappa}\Big] \\
&= -\frac{1}{2}g^{\alpha\sigma}g^{\beta\tau}\Big[(\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau}\Big](\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa}) \\
&\quad + \frac{1}{2}g^{\alpha\beta}\Big[(\delta\eta^j)_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}(\delta\eta^j)_{,\kappa} + g^{\kappa\sigma}\Gamma^\tau_{\alpha\beta}\Big((\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau}\Big)\eta^j_{,\kappa} \\
&\qquad - g^{\kappa\varrho}(\delta\eta^p)_{,\alpha\beta}\eta^p_{,\varrho}\eta^j_{,\kappa} - g^{\kappa\varrho}\eta^p_{,\alpha\beta}(\delta\eta^p)_{,\varrho}\eta^j_{,\kappa}\Big]\,.
\end{aligned}$$

Therefore, the variation of H^2 can be computed as

$$\begin{aligned}
\delta H^2 &= \delta(Hn \cdot Hn) = 2Hn \cdot \delta(Hn) \\
&= \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} (\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota}) \times \Big[(\delta\eta^j)_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}(\delta\eta^j)_{,\kappa} - g^{\kappa\varrho}(\delta\eta^p)_{,\alpha\beta}\eta^p_{,\varrho}\eta^j_{,\kappa} \\
&\qquad - g^{\kappa\varrho}\eta^p_{,\alpha\beta}(\delta\eta^p)_{,\varrho}\eta^j_{,\kappa} + g^{\kappa\sigma}\Gamma^\tau_{\alpha\beta}\Big((\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau} \Big)\eta^j_{,\kappa} \Big] \\
&\quad - \frac{1}{2} g^{\gamma\delta} g^{\alpha\sigma} g^{\beta\tau} \Big[(\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau} \Big] (\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa})(\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota}) \\
&= \frac{1}{2} \Big\{ g^{\alpha\beta} g^{\gamma\delta} \Big[\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota} - g^{\kappa\varrho}(\eta^p_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^p_{,\iota})\eta^p_{,\kappa}\eta^j_{,\varrho} \Big] (\delta\eta^j)_{,\alpha\beta} \\
&\quad + g^{\alpha\beta} g^{\gamma\delta} \Big[-\Gamma^\kappa_{\alpha\beta}\eta^j_{,\gamma\delta} + \Gamma^\kappa_{\alpha\beta}\Gamma^\iota_{\gamma\delta}\eta^j_{,\iota} - g^{\kappa\varrho}(\eta^p_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^p_{,\iota})\eta^p_{,\varrho}\eta^j_{,\alpha\beta} \Big] (\delta\eta^j)_{,\kappa} \\
&\quad + g^{\alpha\beta} g^{\gamma\delta} \Big[g^{\kappa\varrho}\Gamma^\tau_{\alpha\beta}(\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota})\eta^j_{,\kappa} \Big((\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau} \Big) \Big] \\
&\quad - g^{\gamma\delta} g^{\alpha\sigma} g^{\beta\tau} \Big[(\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau} \Big] (\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa})(\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota}) \Big\} .
\end{aligned}$$

By the fact that $g^{\kappa\varrho}(\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota})\eta^j_{,\kappa} = 0$, further simplification can be done, and finally we find that

$$\begin{aligned}
\delta H^2 &= \frac{1}{2} \Big\{ g^{\alpha\beta} g^{\gamma\delta} \Big[\delta^j_p - g^{\kappa\varrho}\eta^p_{,\kappa}\eta^j_{,\varrho} \Big] \eta^p_{,\gamma\delta}(\delta\eta^j)_{,\alpha\beta} - g^{\alpha\beta} g^{\gamma\delta} \Big[\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota} \Big] \Gamma^\kappa_{\alpha\beta}(\delta\eta^j)_{,\kappa} \\
&\quad - g^{\gamma\delta} g^{\alpha\sigma} g^{\beta\tau} \Big[(\delta\eta^p)_{,\sigma}\eta^p_{,\tau} + \eta^p_{,\sigma}(\delta\eta^p)_{,\tau} \Big] (\eta^j_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^j_{,\kappa})(\eta^j_{,\gamma\delta} - \Gamma^\iota_{\gamma\delta}\eta^j_{,\iota}) \Big\} \\
&= \frac{1}{2} \Big[g^{\alpha\beta}\Delta_g\eta^j(\delta\eta^j)_{,\alpha\beta} - g^{\alpha\beta}\Gamma^\kappa_{\alpha\beta}\Delta_g\eta^j(\delta\eta^j)_{,\kappa} \Big] \quad \Big(\equiv \mathrm{I}\Big) \\
&\qquad\qquad - g^{\alpha\sigma} g^{\beta\tau}(\eta^p_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^p_{,\kappa})\eta^j_{,\sigma}\Delta_g\eta^p(\delta\eta^j)_{,\tau} . \quad \Big(\equiv \mathrm{II}\Big)
\end{aligned}$$

Intergrating by parts, it follows that

$$\int_{\Gamma_0} \mathrm{I}\sqrt{g}dx = \frac{1}{2}\int_{\Gamma_t} \Delta_g^2\eta \cdot (\delta\eta) dS_t. \tag{2.9}$$

For II, we have

$$\begin{aligned}
\mathrm{II} &= -\, g^{\alpha\sigma} g^{\beta\tau}(\eta^p_{,\alpha\beta} - \Gamma^\kappa_{\alpha\beta}\eta^p_{,\kappa})\eta^j_{,\sigma}\Delta_g\eta^p(\delta\eta^j)_{,\tau} \\
&= 2(g^{\alpha\beta}g^{\sigma\tau} - g^{\alpha\sigma}g^{\beta\tau})C_{\alpha\beta}H\eta^j_{,\sigma}(\delta\eta^j)_{,\tau} - 2g^{\alpha\beta}g^{\sigma\tau}C_{\alpha\beta}H\eta^j_{,\sigma}(\delta\eta^j)_{,\tau} \\
&= \frac{2}{g}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})C_{\alpha\beta}H\eta^j_{,\sigma}(\delta\eta^j)_{,\tau} - 4H^2 g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)_{,\tau} \\
&= \qquad\qquad\qquad \mathrm{III} \qquad\qquad\qquad + \qquad\qquad \mathrm{IV} \qquad .
\end{aligned}$$

By (2.4), we find that

$$
\begin{aligned}
-\frac{1}{2}\int_{\Gamma_0} \mathrm{III}\sqrt{g}dx &= \int_{\Gamma_0} \Big[\frac{1}{\sqrt{g}}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})C_{\alpha\beta}H\eta^j_{,\sigma}\Big]_{,\tau}(\delta\eta^j)dx \\
&= \int_{\Gamma_0} \frac{1}{\sqrt{g}}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})\Big[-\Gamma^\kappa_{\kappa\tau}C_{\alpha\beta}H\eta^j_{,\sigma} + C_{\alpha\beta,\tau}H\eta^j_{,\sigma} \\
&\qquad + C_{\alpha\beta}H_{,\tau}\eta^j_{,\sigma} + C_{\alpha\beta}H\eta^j_{,\sigma\tau}\Big](\delta\eta^j)dx \\
&= \int_{\Gamma_0} \frac{1}{\sqrt{g}}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})\Big[-\Gamma^\kappa_{\kappa\tau}C_{\alpha\beta}H\eta^j_{,\sigma} + C_{\alpha\beta,\tau}H\eta^j_{,\sigma} \\
&\qquad + C_{\alpha\beta}H_{,\tau}\eta^j_{,\sigma} + C_{\alpha\beta}H\Gamma^\kappa_{\sigma\tau}\eta^j_{,\kappa} + C_{\alpha\beta}C_{\sigma\tau}Hn^j\Big](\delta\eta^j)dx\,.
\end{aligned}
$$

Noticing that

$$
(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})C_{\alpha\beta}C_{\sigma\tau} = 2\det(C)
$$

and for K being the Gaussian curvature of the surface,

$$
K = \frac{1}{g}(C_{11}C_{22} - C_{12}^2) = \frac{1}{g}\det(C),
$$

using identities (2.7) and (2.8), we find that

$$
\begin{aligned}
-\frac{1}{2}\int_{\Gamma_0} \mathrm{III}\sqrt{g}dx &= \int_{\Gamma_0} \frac{1}{\sqrt{g}}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta^j_{,\sigma}(\delta\eta^j)dx \\
&\quad + 2\int_{\Gamma_t} HKn^j(\delta\eta^j)dS_t\,.
\end{aligned} \tag{2.10}
$$

Combining (2.9), (2.10) and IV, we have

$$
\begin{aligned}
\int_{\Gamma_0}(\delta H^2)\sqrt{g}dx &= \frac{1}{2}\int_{\Gamma_t} \Delta_g^2\eta\cdot(\delta\eta)dS_t - 2\int_{\Gamma_t} K\Delta_g\eta\cdot(\delta\eta)dS_t \\
&\quad - 2\int_{\Gamma_0} \frac{1}{\sqrt{g}}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta^j_{,\sigma}(\delta\eta^j)dx \\
&\quad - 4\int_{\Gamma_0} H^2 g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)_{,\tau}\sqrt{g}dx\,.
\end{aligned}
$$

By (2.5), it follows that

$$
\int_{\Gamma_0} H^2(\delta\sqrt{g})dx = \int_{\Gamma_0} H^2 g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)_{,\tau}\sqrt{g}dx.
$$

Noting that $(\sqrt{g}g^{\sigma\tau}\eta_{,\sigma})_{,\tau} = \sqrt{g}\Delta_g\eta$, integrating by parts leads to

$$\begin{aligned}
&\int_{\Gamma_0} H^2 g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)_{,\tau}\sqrt{g}dx \\
&= -\int_{\Gamma_t} H^2\Delta_g\eta\cdot(\delta\eta)dS_t - \int_{\Gamma_0}(H^2)_{,\tau}g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)\sqrt{g}dx\,,
\end{aligned}$$

and hence finally we obtain

$$\begin{aligned}
\delta\int_{\Gamma_t} H^2 dS_t = &\frac{1}{2}\int_{\Gamma_t}\Delta_g^2\eta^j(\delta\eta^j)dS_t - 2\int_{\Gamma_t}K\Delta_g\eta^j(\delta\eta^j)dS_t \\
&+ 3\int_{\Gamma_t}H^2\Delta_g\eta^j(\delta\eta^j)dS_t + 6\int_{\Gamma_0}HH_{,\tau}g^{\sigma\tau}\eta^j_{,\sigma}(\delta\eta^j)\sqrt{g}dx \\
&- 2\int_{\Gamma_0}\frac{1}{\sqrt{g}}(\delta^{\alpha\beta}\delta^{\sigma\tau} - \delta^{\alpha\sigma}\delta^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta^j_{,\sigma}(\delta\eta^j)dx\,.
\end{aligned}$$

Further simplication gives us

$$\begin{aligned}
\delta\int_{\Gamma_t} H^2 dS_t = &\frac{1}{2}\int_{\Gamma_t}\Delta_g^2\eta^j(\delta\eta^j)dS_t - 2\int_{\Gamma_t}K\Delta_g\eta^j(\delta\eta^j)dS_t + 3\int_{\Gamma_t}H^2\Delta_g\eta^j(\delta\eta^j)dS_t \\
&+ \int_{\Gamma_0}(g^{\alpha\beta}g^{\sigma\tau} + 2g^{\alpha\sigma}g^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta^j_{,\sigma}(\delta\eta^j)\sqrt{g}dx\,.
\end{aligned}$$

Next, we claim that the vector

$$\mathfrak{t}_{\text{ben}} \equiv \frac{1}{2}\Delta_g^2\eta - 2K\Delta_g\eta + 3H^2\Delta_g\eta + (g^{\alpha\beta}g^{\sigma\tau} + 2g^{\alpha\sigma}g^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta_{,\sigma} \tag{2.11}$$

is actually normal to Γ_t. It suffices to show that

$$\frac{1}{2}\Delta_g^2\eta + (g^{\alpha\beta}g^{\sigma\tau} + 2g^{\alpha\sigma}g^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta_{,\sigma}$$

is normal to Γ_t since $\Delta_g\eta = 2Hn$. By computing the tangential component of $\Delta_g^2\eta$,

$$\begin{aligned}
\Delta_g(Hn)\cdot\eta_{,\gamma} =&\Big[(\Delta_g H)n + 2g^{\alpha\beta}H_{,\alpha}n_{,\beta} + H(\Delta_g n)\Big]\cdot\eta_{,\gamma} \\
=& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} + H(\Delta_g n)\cdot\eta_{,\gamma} \\
=& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} + Hg^{\alpha\beta}(\Gamma^\kappa_{\alpha\beta}C_{\kappa\gamma} + \Gamma^\kappa_{\beta\gamma}C_{\alpha\kappa} - C_{\alpha\gamma,\beta}) \\
=& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} + Hg^{\alpha\beta}(\Gamma^\kappa_{\alpha\gamma}C_{\beta\kappa} + \Gamma^\kappa_{\beta\gamma}C_{\alpha\kappa} - C_{\alpha\beta,\gamma})\,,
\end{aligned}$$

where we obtain the last equality by (2.8). Furthermore, by the fact that

$$g^{\alpha\beta}_{,\gamma} = -(g^{\alpha\kappa}\Gamma^\beta_{\kappa\gamma} + g^{\beta\kappa}\Gamma^\alpha_{\kappa\gamma}),$$

we find that

$$\begin{aligned}\Delta_g(Hn)\cdot\eta_{,\gamma} =& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} + Hg^{\alpha\beta}(\Gamma^\kappa_{\alpha\gamma}C_{\beta\kappa} + \Gamma^\kappa_{\beta\gamma}C_{\alpha\kappa} - C_{\alpha\beta,\gamma})\\ =& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} - H(g^{\alpha\beta}C_{\alpha\beta})_{,\gamma}\\ =& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} - 2HH_{,\gamma}\,.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\Big[\frac{1}{2}\Delta_g^2\eta + (g^{\alpha\beta}g^{\sigma\tau} + 2g^{\alpha\sigma}g^{\beta\tau})C_{\alpha\beta}H_{,\tau}\eta_{,\sigma}\Big]\cdot\eta_{,\gamma}\\ =& -2g^{\alpha\beta}H_{,\alpha}C_{\beta\gamma} - 2HH_{,\gamma} + 2HH_{,\gamma} + 2g^{\beta\tau}C_{\gamma\beta}H_{,\tau}\end{aligned}$$

which is zero. The claim is proved.

Similarly, the normal component of $\Delta_g n$ is $-g^{\alpha\beta}g^{\sigma\tau}C_{\alpha\sigma}C_{\beta\tau}$. This implies

$$\frac{1}{2}\Delta_g^2\eta = (\Delta_g H - Hg^{\alpha\beta}g^{\sigma\tau}C_{\alpha\sigma}C_{\beta\tau})n$$

and hence we have

$$\begin{aligned}\delta\int_{\Gamma_t}H^2dS_t =& \int_{\Gamma_t}(\Delta_g H)n^j(\delta^j) + \int_{\Gamma_0}\sqrt{g}H(g^{\alpha\beta}g^{\sigma\tau} - g^{\alpha\sigma}g^{\beta\tau})C_{\alpha\beta}C_{\sigma\tau}n^j(\delta\eta^j)dS_0\\ &- 4\int_{\Gamma_t}HKn^j(\delta\eta^j)dS_t + 2\int_{\Gamma_t}H^3n^j(\delta\eta^j)dS_t\\ =& \int_{\Gamma_t}\Big[\Delta_g H - 2HK + 2H^3\Big]n^j(\delta^j)dS_t\,.\end{aligned}$$

This equation shows we can also define $\mathcal{L}$ in the following way:

$$\mathfrak{t}_{\text{ben}} = (\Delta_g H - 2HK + 2H^3)n\,. \tag{2.12}$$

3. Lagrangian formulation

3.1. **A new coordinate system near the shell.** Consider the isometric immersion $\eta_0 : (\Gamma_0, g_0) \to (\mathbb{R}^3, \mathrm{Id})$. Let $\mathcal{B} = \Gamma_0 \times (-\epsilon, \epsilon)$ where ϵ is chosen sufficiently small so that the map

$$B : \mathcal{B} \to \mathbb{R}^3 : (y, z) \mapsto y + zN(y)$$

is itself an immersion, defining a tubular neighborhood of Γ_0 in $\mathbb{R}^3$. We can choose a coordinate system $\frac{\partial}{\partial y^\alpha}$, $\alpha = 1, 2$ and $\frac{\partial}{\partial z}$ on $\mathcal{B}$ where $\frac{\partial}{\partial y^\alpha}$ denotes the tangential derivative and $\frac{\partial}{\partial z}$ denotes the normal derivative.

Let $G = B^*(\mathrm{Id})$ denote the induced metric on $\mathcal{B}$ from $\mathbb{R}^3$ so that

$$G(y, z) = G_z(y) + dz \otimes dz,$$

where G_z is the metric on the surface $\Gamma_0 \times \{z\}$; note that $G_0 = g_0$.

Remark 1. *By assumption, $g_{0\alpha\beta} = \frac{\partial}{\partial y^\alpha} \cdot \frac{\partial}{\partial y^\beta}$, where $\cdot$ denotes the usual Cartesian inner-product on $\mathbb{R}^n$. Let $C_{\alpha\beta}$ denote the covariant components of the second fundamental form of the base manifold Γ_0, so that $C_{\alpha\beta} = -N_{,\alpha} \cdot \frac{\partial}{\partial y^\beta}$. Then, G_z is given by*

$$(G_z)_{\alpha\beta} = (g_0)_{\alpha\beta} - 2zC_{\alpha\beta} + z^2 g_0^{\gamma\delta} C_{\alpha\gamma} C_{\beta\delta}.$$

Let $h : \Gamma_0 \to (-\epsilon, \epsilon)$ be a smooth height function and consider the graph of h in $\mathcal{B}$, parameterized by $\phi : \Gamma_0 \to \mathcal{B} : y \mapsto (y, h(y))$. The tangent space to graph(h), considered as a submanifold of $\mathcal{B}$, is spanned at a point $\phi(x)$ by the vectors

$$\phi_*(\frac{\partial}{\partial y^\alpha}) = \frac{\partial \phi}{\partial y^\alpha} = \frac{\partial}{\partial y^\alpha} + \frac{\partial h}{\partial y^\alpha} \frac{\partial}{\partial z},$$

and the normal to graph(h) is given by

$$n(y) = J_h^{-1}(y)\Big(- G^{\alpha\beta}_{h(y)} \frac{\partial h}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z}\Big) \tag{3.1}$$

where $J_h = (1 + h_{,\alpha} G^{\alpha\beta}_{h(y)} h_{,\beta})^{1/2}$. The mean curvature H of graph(h) is defined to be the trace of ∇n where

$$(\nabla n)_{ij} = G(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial w^i}} n, \frac{\partial}{\partial w^j}) \qquad \text{for } i, j = 1, 2, 3$$

where $\frac{\partial}{\partial w^\alpha} = \frac{\partial}{\partial y^\alpha}$ for $\alpha = 1, 2$ and $\frac{\partial}{\partial w^3} = \frac{\partial}{\partial z}$, and $\nabla^{\mathcal{B}}$ denotes the covariant derivative. Using (3.1),

$$\begin{aligned}(\nabla n)_{\alpha\beta} &= G\Big(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial y^\alpha}}\Big[-J_h^{-1}G_h^{\gamma\delta}h_{,\gamma}\frac{\partial}{\partial y^\delta} + J_h^{-1}\frac{\partial}{\partial z}\Big], \frac{\partial}{\partial y^\beta}\Big)\\ &= -(G_h)_{\delta\beta}\Big[(J_h^{-1}G_h^{\gamma\delta}h_{,\gamma})_{,\alpha} + J_h^{-1}(-G_h^{\gamma\sigma}h_{,\gamma}\Gamma^\delta_{\alpha\sigma} + \Gamma^\delta_{\alpha 3})\Big];\\ (\nabla n)_{33} &= G\Big(\nabla^{\mathcal{B}}_{\frac{\partial}{\partial z}}\Big[-J_h^{-1}G_h^{\gamma\delta}h_{,\gamma}\frac{\partial}{\partial y^\delta} + J_h^{-1}\frac{\partial}{\partial z}\Big], \frac{\partial}{\partial z}\Big),\\ &= J_h^{-1}(-G_h^{\gamma\delta}h_{,\gamma}\Gamma^3_{3\delta} + \Gamma^3_{33})\end{aligned}$$

where Γ^k_{ij} denotes the Christoffel symbols with respect to the metric G. It follows that the curvature of graph(h) (in the divergence form) is

$$H = -(J_h^{-1}G_h^{\gamma\delta}h_{,\gamma})_{,\delta} + J_h^{-1}(-G_h^{\gamma\delta}h_{,\gamma}\Gamma^j_{j\delta} + \Gamma^j_{j3}), \tag{3.2}$$

or (in the quasilinear form)

$$H = -J_h^{-1}G_h^{\alpha\beta}\Big[\delta_{\beta\gamma} - J_h^{-2}G_h^{\gamma\delta}h_{,\beta}h_{,\delta}\Big]h_{,\alpha\gamma} + G_h^{\alpha\beta}F_{\alpha\beta}(y, h, \nabla h), \tag{3.3}$$

where $F_{\alpha\beta}$ denotes a smooth generic function of y, h and ∇h.

Remark 2. *Note that G_h denotes the metric $G_{z=h(y)}$, and not the metric on the submanifold graph(h).*

Remark 3. *If the initial height function is zero, i.e., $h(0) = 0$, then $H(0) = \Gamma^j_{j3}(0)$, which is the mean curvature of the base manifold Γ_0 as required.*

Example 1. *In the case that the initial surface Γ_0 is flat, so that $\Gamma_0 \subset \mathbb{R}^2 \times \{0\}$, then $G_{\alpha\beta} = \delta_{\alpha\beta}$, and the mean curvature is given by (3.3),*

$$H = -\frac{1}{\sqrt{1+|\nabla h|^2}}\Big[\delta_{\alpha\beta} - \frac{h_{,\alpha}h_{,\beta}}{1+|\nabla h|^2}\Big]h_{,\alpha\beta}.$$

Example 2. *In the case that the initial surface $\Gamma_0 = S^2$, with local coordinates $(y_1, y_2) = (\theta, \phi)$, then the metric*

$$G = (1+z)^2 \sin^2\phi\, d\theta^2 + (1+z)^2 d\phi^2 + dz^2,$$

and the mean curvature is given by the formula

$$H = -\Big[\Big(J_h^{-1}\frac{h_\theta}{(1+h)^2\sin^2\phi}\Big)_{,\theta} + \Big(J_h^{-1}\frac{h_\phi}{(1+h)^2}\Big)_{,\phi}\Big] + J_h^{-1}\Big[G_h^{\gamma 2} h_{,\gamma}\cot\phi + \frac{2}{1+h}\Big].$$

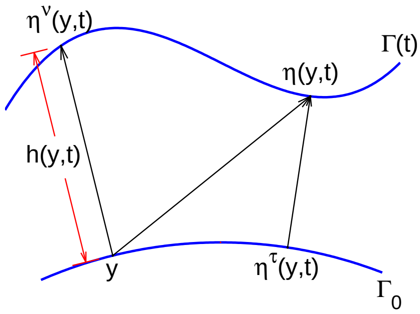
3.2. **Tangential reparameterization symmetry.** Let $\mathcal{N}$ denote the normal bundle to Γ_0, so that for each $y \in \Gamma_0$, we have the Whitney sum $\mathbb{R}^3 = T_y\Gamma_0 \oplus \mathcal{N}_y$.

Given a signed height function $h : \Gamma_0 \times [0,T) \to \mathbb{R}$, for each $t \in [0,T)$, define the *normal* map

$$\eta^\nu : \Gamma_0 \times [0,T) \to \Gamma(t), \quad (y,t) \mapsto y + h(y,t)N(y), \quad N(y) \in \mathcal{N}_y\,.$$

Then, there exists a unique *tangential* map $\eta^\tau : \Gamma_0 \times [0,T) \to \Gamma_0$ (a diffeomorphism as long as h remains a graph) such that the diffeomorphism $\eta(t)$ has the decomposition

$$\eta(\cdot,t) = \eta^\nu(\cdot,t)\circ\eta^\tau(\cdot,t), \quad \eta(y,t) = \eta^\tau(y,t) + h(\eta^\tau(y,t),t)N(\eta^\tau(y,t))\,.$$



The tangent vector $\eta_{,\alpha}$ to $\Gamma(t)$ can be decomposed with respect to the Whitney sum as $\eta_{,\alpha}(y,t) = \eta^\kappa_{,\alpha}(y,t)\frac{\partial}{\partial y^\kappa} + h_{,\kappa}(\eta^\tau(y,t),t)\eta^\kappa_{,\alpha}\frac{\partial}{\partial z}$ and hence the induced metric $g_{\alpha\beta} = \eta_{,\alpha}\cdot\eta_{,\beta}$ may be expressed as

$$g_{\alpha\beta} = \Big[\Big((G_h)_{\kappa\sigma} + h_{,\kappa}h_{,\sigma}\Big)\circ\eta^\tau\Big]\eta^\kappa_{,\alpha}\eta^\sigma_{,\beta} := \Big[\mathcal{G}_{\kappa\sigma}\circ\eta^\tau\Big]\eta^\kappa_{,\alpha}\eta^\sigma_{,\beta}. \tag{3.4}$$

Note that $\mathcal{G}_{\kappa\sigma}$ is the induced metric with respect to the *normal* map η^ν. Furthermore, we have the following useful relationship between the determinant of the two induced metrics:

$$\det(g) = \det(\nabla_0\eta^\tau)^2\Big[\det(G_h)J_h^2\Big]\circ\eta^\tau = \det(\nabla_0\eta^\tau)^2\Big[\det(\mathcal{G})\Big]\circ\eta^\tau \tag{3.5}$$

where ∇_0 denotes the surface gradient.

REMARK 4. *The identity (3.4) can also be read as* $(\eta^\tau)^* g = \mathcal{G}$.

Let y and $\tilde{y} = \varphi(y)$ denote two different coordinate systems on Γ_0 with associated metrics

$$g_{\alpha\beta} = \frac{\partial \eta^i}{\partial y^\alpha}\frac{\partial \eta^i}{\partial y^\beta}\,, \qquad \tilde{g}_{\alpha\beta} = \frac{\partial \eta^i}{\partial \tilde{y}^\alpha}\frac{\partial \eta^i}{\partial \tilde{y}^\beta}.$$

It follows that $\varphi^* \tilde{g} = g$. Let H, $\tilde{H}$, K, $\tilde{K}$, n and $\tilde{n}$ denote the mean curvature, Gauss curvature, and the unit normal vector computed with respect to y and $\tilde{y}$, respectively. Since H, K, and n depend only on the shape of $\Gamma(t)$, these geometric quantities are invariant to tangential reparameterization; thus, the identity

$$\tilde{H} = H \circ \varphi\,, \qquad \tilde{K} = K \circ \varphi\,, \qquad \tilde{n} = n \circ \varphi. \tag{3.6}$$

Similarly, computing the first variation of $\displaystyle\int_{\Gamma(t)} H^2 dS$ in our two coordinate systems yields

$$\Big[\Big(\Delta_g H + H(H^2 - K)\Big)n\Big](y) = \Big[\Big(\Delta_{\tilde{g}} \tilde{H} + \tilde{H}(\tilde{H}^2 - \tilde{K})\Big)\tilde{n}\Big](\tilde{y}) \qquad \forall\ \tilde{y} = \varphi(y).$$

By (3.6), we have the following important identity

$$\Big[\Delta_{\varphi^*\tilde{g}} H\Big](y) = \Big[\Delta_{\tilde{g}}(H \circ \varphi)\Big](\tilde{y}) \qquad \forall\ \tilde{y} = \varphi(y) \tag{3.7}$$

and hence

$$[\Delta_{\mathcal{G}}(H \circ \eta^{-\tau})] \circ \eta^\tau = \Delta_g H \tag{3.8}$$

where by (3.3),

$$H \circ \eta^{-\tau} = -J_h^{-1} G_h^{\alpha\beta}\Big[\delta_{\beta\gamma} - J_h^{-2} G_h^{\gamma\delta} h_{,\beta} h_{,\delta}\Big] h_{,\alpha\gamma} + G_h^{\alpha\beta} F_{\alpha\beta}(y, h, \nabla h).$$

3.3. **Bounds on η^τ.** Let u^τ denote the tangential velocity defined by $\eta_t^\tau = u^\tau \circ \eta^\tau$. Time-differentiating the relation $\eta = \eta^\nu \circ \eta^\tau$ and using the definition of η^ν, we find that

$$u^\tau = (\nabla_0 \eta^\nu)^{-1}\Big[u \circ \eta^\nu - h_t \frac{\partial}{\partial z}\Big]. \tag{3.9}$$

From the trace theorem, it follows that

$$\|u^\tau\|_{H^{2.5}(\Gamma_0)} \le C\mathcal{P}(\|h\|_{H^{3.5}(\Gamma_0)}, \|\eta\|_{H^3(\Omega_0)})\Big[\|v\|_{H^3(\Omega_0)} + \|h_t\|_{H^{2.5}(\Gamma_0)}\Big] \tag{3.10}$$

for some polynomial $\mathcal{P}$. Since, $\eta^\tau(y,t) = y + \int_0^t (u^\tau \circ \eta^\tau)(y,s)ds$, it follows that

$$\|\nabla_0 \eta^\tau(y,t)\|_{H^{1.5}(\Gamma_0)} \le C\Big[1 + \int_0^t \|u^\tau\|_{H^{2.5}(\Gamma_0)}\Big(1 + \|\nabla_0\eta^\tau\|_{H^{1.5}(\Gamma_0)}\Big)^4 ds\Big]$$

and hence by Gronwall's inequality,

$$\|\nabla_0 \eta^\tau(y,t)\|_{H^{1.5}(\Gamma_0)} \le C\Big[1 + \int_0^t \|u^\tau\|_{H^{2.5}(\Gamma_0)} ds\Big] \tag{3.11}$$

for $t \in [0,T]$ sufficiently small. Furthermore, we also have

$$\|\eta_t^\tau(y,t)\|_{H^{2.5}(\Gamma_0)} \le C\|u^\tau\|_{H^{2.5}(\Gamma_0)}\Big[1 + \|\nabla_0\eta^\tau\|_{H^{1.5}(\Gamma_0)}\Big]^4. \tag{3.12}$$

3.4. **An expression for $\mathbf{t}_{\mathbf{ben}}$ and $\mathbf{t}_{\mathbf{mem}}$ in terms of h and η^τ.** Now we can compute $\mathbf{t}_{ben}$ in terms of h and η^τ: the highest order term of $\Delta_g H$ is

$$\Big\{\frac{1}{\sqrt{\det(\mathcal{G})}}\frac{\partial}{\partial y^\gamma}\Big[\sqrt{\det(\mathcal{G})}\mathcal{G}^{\gamma\delta}\frac{\partial}{\partial y^\delta}\Big(J_h^{-1}(G_h^{\alpha\beta} - J_h^{-2}G_h^{\alpha\kappa}G_h^{\beta\sigma}h_{,\kappa}h_{,\sigma})h_{,\alpha\beta}\Big)\Big]\Big\} \circ \eta^\tau.$$

Since $\mathcal{G}_{\alpha\beta} = (G_h)_{\alpha\beta} + h_{,\alpha}h_{,\beta}$, the inverse of $\mathcal{G}_{\gamma\delta}$ is

$$\frac{1}{\det(\mathcal{G})}\begin{bmatrix} (G_h)_{22} + h_{,2}^2 & -(G_h)_{12} - h_{,1}h_{,2} \\ -(G_h)_{12} - h_{,1}h_{,2} & (G_h)_{11} + h_{,1}^2 \end{bmatrix}$$

which can also be written as

$$\mathcal{G}^{\alpha\beta} = J_h^{-2}\Big[G_h^{\alpha\beta} - (-1)^{\kappa+\sigma}\det(G_h)^{-1}(1-\delta_{\alpha\kappa})(1-\delta_{\beta\sigma})h_{,\kappa}h_{,\sigma}\Big].$$

Therefore, the highest order term of $\Delta_g H$ can be written as

$$\frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}A^{\alpha\beta\gamma\delta}h_{,\alpha\beta}\Big]_{,\gamma\delta} \circ \eta^\tau$$

where

$$A^{\alpha\beta\gamma\delta} = J_h^{-3}\Big[G_h^{\alpha\gamma} - (-1)^{\kappa+\sigma}\det(G_h)^{-1}(1-\delta_{\alpha\kappa})(1-\delta_{\gamma\sigma})h_{,\kappa}h_{,\sigma}\Big] \times (G_h^{\beta\delta} - J_h^{-2}G_h^{\beta\kappa}G_h^{\delta\sigma}h_{,\kappa}h_{,\sigma}) \tag{3.13}$$

is a fourth-rank tensor.

We also set

$$\eta^z(y,t) = h(\eta^\tau(y,t),t)\,, \qquad \forall\ y\in\Gamma_0\,. \tag{3.14}$$

Using these variables, $\mathfrak{t}_{\text{mem}}$ can be represented as

$$\mathfrak{t}_{\text{mem}} = -\Big[\mu M^\kappa + \frac{\mu\lambda}{2\mu+\lambda}N^\kappa\Big]\frac{\partial}{\partial y^\kappa} - \Big[\mu M^z + \frac{\mu\lambda}{2\mu+\lambda}N^z\Big]\frac{\partial}{\partial z} \tag{3.15}$$

where with $\bar\Gamma^\gamma_{\alpha\beta}$ denoting the Christoffel symbols with respect to the metric g_0,

$$\begin{aligned}
M^\kappa = {} & \Big[(g_{\alpha\beta}-g_{0\alpha\beta})\eta^\kappa_{,\alpha\beta} + (G_{\eta^z})_{\gamma\delta}\eta^\kappa_{,\alpha}\Big(\eta^\gamma_{,\alpha\beta}\eta^\delta_{,\beta} + \eta^\gamma_{,\alpha}\eta^\delta_{,\beta\beta}\Big) + \eta^\kappa_{,\alpha}\Big(\eta^z_{,\alpha\beta}\eta^z_{,\beta} + \eta^z_{,\alpha}\eta^z_{,\beta\beta}\Big)\Big] \\
& + \Big[\bar\Gamma^\sigma_{\sigma\beta}(g_{\alpha\beta}-g_{0\alpha\beta})\eta^\kappa_{,\alpha} + (g_{\alpha\beta}-g_{0\alpha\beta})(G_{\eta^z})^{\kappa\iota}(G_{\eta^z})_{\sigma\iota,\beta}\eta^\sigma_{,\alpha} \\
& \quad + (G_{\eta^z})_{\gamma\delta,\beta}\eta^\gamma_{,\alpha}\eta^\delta_{,\beta}\eta^\kappa_{,\alpha}\Big]\,, \\
M^z = {} & \Big[(G_{\eta^z})_{ij}\eta^z_{,\alpha}\Big(\eta^i_{,\alpha}\eta^j_{,\beta\beta} + \eta^i_{,\beta}\eta^j_{,\alpha\beta}\Big) + (g_{\alpha\beta}-g_{0\alpha\beta})\eta^z_{,\alpha\beta}\Big] \\
& + \Big[(g_{\alpha\beta}-g_{0\alpha\beta})(C_{\kappa\iota} - \eta^z\mathcal{G}^{\gamma\delta}C_{\kappa\gamma}C_{\iota\delta})\eta^\kappa_{,\alpha}\eta^\tau_{,\beta} + \bar\Gamma^\sigma_{\sigma\beta}(g_{\alpha\beta}-g_{0\alpha\beta})\eta^z_{,\alpha} \\
& \quad + (G_{\eta^z})_{\kappa\iota,\beta}\eta^\kappa_{,\alpha}\eta^\iota_{,\beta}\eta^z_{,\alpha}\Big]\,, \\
N^\kappa = {} & \Big[(g_{\alpha\alpha}-g_{0\alpha\alpha})\eta^\kappa_{,\beta\beta} + 2(G_{\eta^z})_{\gamma\delta}\eta^\kappa_{,\beta}\eta^\gamma_{,\alpha\beta}\eta^\delta_{,\alpha} + 2\eta^\kappa_{,\beta}\eta^z_{,\alpha}\eta^z_{,\alpha\beta}\Big] \\
& + \Big[\bar\Gamma^\sigma_{\sigma\beta}(g_{\alpha\alpha}-g_{0\alpha\alpha})\eta^\kappa_{,\alpha} + (g_{\alpha\alpha}-g_{0\alpha\alpha})(G_{\eta^z})^{\kappa\iota}(G_{\eta^z})_{\sigma\iota,\beta}\eta^\sigma_{,\beta} \\
& \quad + (G_{\eta^z})_{\gamma\delta,\beta}\eta^\gamma_{,\alpha}\eta^\delta_{,\alpha}\eta^\kappa_{,\beta}\Big]\,,
\end{aligned}$$

and

$$\begin{aligned}
N^z = {} & \Big[2(G_{\eta^z})_{ij}\eta^z_{,\beta}\eta^i_{,\alpha\beta}\eta^j_{,\alpha} + (g_{\alpha\alpha}-g_{0\alpha\alpha})\eta^z_{,\beta\beta}\Big] + \Big[\bar\Gamma^\sigma_{\sigma\beta}(g_{\alpha\alpha}-g_{0\alpha\alpha})\eta^z_{,\beta} \\
& + (g_{\alpha\alpha}-g_{0\alpha\alpha})(C_{\kappa\iota} - \eta^z\mathcal{G}^{\gamma\delta}C_{\kappa\gamma}C_{\iota\delta})\eta^\kappa_{,\alpha}\eta^\iota_{,\beta} + (G_{\eta^z})_{\kappa\iota,\beta}\eta^\kappa_{,\alpha}\eta^\iota_{,\alpha}\eta^z_{,\beta}\Big].
\end{aligned}$$

3.5. **Lagrangian formulation of the problem.** Let $\eta(t,x) = x + \int_0^t u(s,x)ds$ denote the Lagrangian particle placement field, a volume-preserving embedding of Ω_0 onto $\Omega(t) \subset \mathbb{R}^3$, and denote the cofactor matrix of $\nabla\eta(x,t)$ by

$$a(x,t) = [\nabla\eta(x,t)]^{-1}\,. \tag{3.16}$$

Let $v = u \circ \eta$ denote the Lagrangian or material velocity field, $q = p \circ \eta$ the Lagrangian pressure function, and $F = f \circ \eta$ the forcing function in the material frame. The system (1.1) can be reformulated as

$$\eta_t = v \qquad \text{in } (0,T)\times\Omega_0, \tag{3.17a}$$

$$v^i_t - \nu(a^j_\ell D_\eta(v)^i_\ell)_{,j} = -(a^k_i q)_{,k} + F^i \qquad \text{in } (0,T)\times\Omega_0, \tag{3.17b}$$

$$a^k_i v^i_{,k} = 0 \qquad \text{in } (0,T)\times\Omega_0, \tag{3.17c}$$

$$\begin{aligned}(\nu D_\eta(v)^i_\ell - q\delta^i_\ell)a^j_\ell N_j &= \epsilon\mathfrak{t}_{mem} + \sigma\Theta \\ &\quad \times \Big[L(h)B_*(-G^{\alpha\beta}_h h_{,\alpha},1)\Big]\circ\eta^\tau \qquad \text{on } (0,T)\times\Gamma_0,\end{aligned} \tag{3.17d}$$

$$h_t = B_*((-G^{\alpha\beta}_h h_{,\alpha},1))\cdot(v\circ\eta^{-\tau}) \qquad \text{on } (0,T)\times\Gamma_0, \tag{3.17e}$$

$$v = u_0 \qquad \text{on } \{t=0\}\times\Omega_0, \tag{3.17f}$$

$$h = 0 \qquad \text{on } \{t=0\}\times\Gamma_0, \tag{3.17g}$$

$$\eta = \text{Id} \qquad \text{on } \{t=0\}\times\Omega_0, \tag{3.17h}$$

where $D_\eta(v)^i_\ell := (a^k_\ell v^i_{,k} + a^k_i v^\ell_{,k})$, N denotes the outward-pointing unit normal to Γ_0, Θ is defined in Remark 5, and B_* is the push-forward of B defined as

$$B_*(\gamma'(0)) = (B\circ\gamma)'(0) \qquad \forall\ \gamma(t)\subset\Gamma_0.$$

$L(h)$ is the representation of $\mathfrak{t}_{shell}\cdot n$ using the height function h. It is defined as follows

$$\begin{aligned}L(h) = \epsilon^3\Big\{&\frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}A^{\alpha\beta\gamma\delta}h_{,\alpha\beta}\Big]_{,\gamma\delta} + L_1^{\alpha\beta\gamma}(y,h,Dh,D^2h)h_{,\alpha\beta\gamma} \\ &+ L_2(y,h,Dh,D^2h)\Big\}\end{aligned}$$

where L_1 and L_2 are polynomials of their variables with $L_1(y,0)=0$, g_0 is the metric tensor on Γ_0.

Remark 5. *For a point $\eta(y,t)\in\Gamma(t)$, there are two ways of defining the unit normal n to $\Gamma(t)$:*

1. *Let $n=\sqrt{g}^{-1}a^TN$ where N is the unit normal to Γ_0.*
2. *Let $n=\Big[J_h^{-1}\Big(-G_h^{\alpha\beta}h_{,\alpha}\dfrac{\partial}{\partial y^\beta}+\dfrac{\partial}{\partial z}\Big)\Big]\circ\eta^\tau$ (denoted by $[J_h^{-1}(-\nabla_0 h,1)]\circ\eta^\tau$).*

The function Θ is defined by

$$\Theta(-\nabla_0 h\circ\eta^\tau,1)=a^TN.$$

Equating the modulus of both sides, by (3.5) we must have

$$\Theta=\sqrt{\det(g)}[(J_h^{-1})\circ\eta^\tau]=\det(\nabla_0\eta^\tau)\sqrt{\det(G_h)\circ\eta^\tau}.$$

Remark 6. *An equivalent form of (3.17e) is given by*

$$\begin{aligned}h_t&=G((-G_h^{\beta\gamma}h_{,\alpha},1),B^*(v\circ\eta^{-\tau}))\\&=-(G_h)_{\beta\gamma}G_h^{\alpha\beta}h_{,\alpha}(v\circ\eta^{-1})_\gamma+(v\circ\eta^{-1})_z\\&=-h_{,\alpha}(v\circ\eta^{-\tau})_\alpha+(v\circ\eta^{-\tau})_z\end{aligned}$$

where B^* is the pullback map from $T_{B(x)}\mathbb{R}^3$ to $T_x\mathcal{B}$. This equation states that the shape of the boundary moves with the normal velocity of the fluid.

Remark 7. *By Remark (6), we also have $\eta_t^z=v_z$ which implies that*

$$\eta(y,0,t)=(\eta^\tau(y,t),\eta^z(y,t))\qquad\forall\ y\in\Gamma_0.$$

Remark 8. *For many of the nonlinear estimates that appear later, it is important that $L(h)$ is linear in the third derivative $h_{,\alpha\beta\gamma}$.*

Remark 9. *Without using the symmetry (3.8), we can still compute $\Delta_g H$ in terms of h and η^τ by using (3.4) and (3.5); however, L_1 would then depend on $\nabla_0^2\eta^\tau$ and thus lose one derivative of regularity, preventing the closure of our energy estimate (see Remark 33 in Appendix B for details).*

The study of the problem with $\mathfrak{t}_{mem} = 0$ requires that the lower order terms of L depend at most on first derivatives of η^τ.

4. Notation and conventions

As we described above, the basic difficulties in the analysis of the fluid-shell interaction problem is the directional degeneracy of the fourth-order elliptic operator arising from the bending energy. Thus, we begin our study by consider $\mathfrak{t}_{\text{shell}} = \mathfrak{t}_{\text{ben}}$. In Section 12, we will then add the membrane traction $\mathfrak{t}_{\text{mem}}$ to the analysis.
For $T > 0$, we set

$$V^1(T) = \Big\{ v \in L^2(0,T; H^1(\Omega_0)) \;\Big|\; v_t \in L^2(0,T; H^1(\Omega_0)') \Big\};$$
$$V^2(T) = \Big\{ v \in L^2(0,T; H^2(\Omega_0)) \;\Big|\; v_t \in L^2(0,T; L^2(\Omega_0)) \Big\};$$
$$V^k(T) = \Big\{ v \in L^2(0,T; H^k(\Omega_0)) \;\Big|\; v_t \in L^2(0,T; H^{k-2}(\Omega_0)) \Big\} \quad \text{for } k \geq 3 \, ;$$
$$H(T) = \Big\{ h \in L^2(0,T; H^{5.5}(\Gamma_0)) \;\Big|\; h_t \in L^2(0,T; H^{2.5}(\Gamma_0)), h_{tt} \in L^2(0,T; H^{0.5}(\Gamma_0)) \Big\}$$

with norms

$$\|v\|^2_{V^1(T)} = \|v\|^2_{L^2(0,T;H^1(\Omega_0))} + \|v_t\|^2_{L^2(0,T;H^1(\Omega_0)')};$$
$$\|v\|^2_{V^2(T)} = \|v\|^2_{L^2(0,T;H^2(\Omega_0))} + \|v_t\|^2_{L^2(0,T;L^2(\Omega_0))};$$
$$\|v\|^2_{V^k(T)} = \|v\|^2_{L^2(0,T;H^k(\Omega_0))} + \|v_t\|^2_{L^2(0,T;H^{k-2}(\Omega_0))} \quad \text{for } k \geq 3 \, ;$$
$$\|h\|^2_{H(T)} = \|h\|^2_{L^2(0,T;H^{5.5}(\Gamma_0))} + \|h_t\|^2_{L^2(0,T;H^{2.5}(\Gamma_0))} + \|h_{tt}\|^2_{L^2(0,T;H^{0.5}(\Gamma_0))}.$$

We then introduce the space (of "divergence free" vector fields)

$$\mathcal{V}_v = \Big\{ v \in H^1(\Omega_0) \;\Big|\; a^j_i(t) v^i_{,j} = 0 \; \forall \; t \in [0,T] \Big\}$$

and

$$\mathcal{V}_v(T) = \Big\{ v \in L^2(0,T; H^1(\Omega_0)) \;\Big|\; a^j_i(t) v^i_{,j} = 0 \; \forall \; t \in [0,T] \Big\}.$$

We use X_T to denote the space $V^3(T) \times H(T)$ with norm

$$\|(v,h)\|^2_{X_T} = \|v\|^2_{V^3(T)} + \|h\|^2_{H(T)}$$

and use Y_T, a subspace of X_T, to denote the space

$$Y_T = \Big\{(v,h) \in V^3(T) \times H(T) \;\Big|\; h_t \in L^\infty(0,T;H^2(\Gamma_0))\Big\}$$

with norm

$$\begin{aligned}\|(v,h)\|_{Y_T}^2 = \|(v,h)\|_{X_T}^2 + \|v\|^2_{L^\infty(0,T;H^2(\Omega_0))} + \|h\|^2_{L^\infty(0,T;H^4(\Gamma_0))} \\ + \|h_t\|^2_{L^\infty(0,T;H^2(\Gamma_0))}.\end{aligned}$$

Remark 10. *By the Sobolev embedding theorem, $u \in L^\infty(0,T;H^2(\Omega_0))$ and $h \in L^\infty(0,T;H^4(\Gamma_0))$ if $(u,h) \in X(T)$, with the following estimates:*

$$\sup_{0\le t\le T} \|u\|^2_{H^2(\Omega_0)} \le \|u_0\|^2_{H^2(\Omega_0)} + \|u\|^2_{V^3(T)},$$

$$\sup_{0\le t\le T} \|h\|^2_{H^4(\Omega_0)} \le \|h_0\|^2_{H^4(\Omega_0)} + \|h\|^2_{H(T)},$$

where u_0 and h_0 are the restriction of u and h to $\{t=0\}$, respectively.

We will solve (3.17) by a fixed-point method in an appropriate subset of Y_T.

5. The main theorem

Before stating the main theorem, we define the following quantities. Let q_0 be defined by

(5.1a) $$\Delta q_0 = -\nabla u_0 : (\nabla u_0)^T + \nu [a^k_\ell D_\eta(u_0)^i_\ell]_{,ki}(0) + \operatorname{div} F(0) \quad \text{in} \quad \Omega_0,$$

(5.1b) $$q_0 = \nu(\operatorname{Def} u_0 \cdot N)\cdot N - \sigma L(0) \quad \text{on} \quad \Gamma_0$$

and

(5.2) $$u_1 = \nu \Delta u_0 - \nabla q_0 + F(0).$$

We also define the projection operator $P_{ij}(x) : \mathbb{R}^3 \to T_{\eta(x,t)}\Gamma(t)$ by

$$\mathcal{P}_{ij}(x) = [\delta_{ij} - (J_h^{-2}\circ \eta^\tau) a^k_i a^\ell_j N_k(x) N_\ell(x)] = \Big[\delta_{ij} - \frac{a^k_i N_k(x)}{|a^k_i N_k(x)|}\frac{a^\ell_j N_\ell(x)}{|a^\ell_j N_\ell(x)|}\Big].$$

THEOREM 5.1. *Let $\nu > 0$, $\sigma > 0$ be given, and*

$$F \in L^2(0,T;H^2(\Omega_0)), F_t \in L^2(0,T;L^2(\Omega_0)), F(0) \in H^1(\Omega_0).$$

Suppose that the shell traction is given by

$$\mathsf{t}_{shell} = \mathsf{t}_{ben}.$$

Assume that the initial data satisfies

$$u_0 \in H^{2.5}(\Omega_0) \cap H^{4.5}(\Gamma_0),$$

as well as the compatibility condition

$$[\mathrm{Def}\, u_0 \cdot N]_{tan} = 0. \tag{5.3}$$

There exists $T > 0$ depending on u_0 and F such that there exists a solution $(v,h) \in Y_T$ of problem (3.17). Moreover, if $u_0 \in H^{5.5}(\Omega_0) \cap H^{7.5}(\Gamma_0)$ and the associated u_1, q_0 also satisfy the compatibility condition

$$\begin{aligned} CP := &\Big[g_0^{ki} u^j_{0,k} N_j N_\ell + g_0^{k\ell} u^j_{0,k} N_j N_i\Big]\Big[\nu(\mathrm{Def}\, u_0)^j_i - q_0 \delta^j_i\Big] N_j \\ &+ \nu(\delta_{i\ell} - N_i N_\ell)\Big[(\mathrm{Def}\, u_1)^j_i - \Big((\nabla u_0 \nabla u_0) + (\nabla u_0 \nabla u_0)^T\Big)^j_i\Big] N_j \\ &- (\delta_{i\ell} - N_i N_\ell)\Big[\nu(\mathrm{Def}\, u_0)^j_i - q_0 \delta^j_i\Big] u^k_{0,j} N_k = 0 \end{aligned} \tag{5.4}$$

then the solution $(v,h) \in Y_T$ is unique.

REMARK 11. *In (5.4), $q_t(0)$ is not needed because the projection operator $\mathcal{P}_{ij}(0) = (\delta_{ij} - N_i N_j)$ projects $q_t(0)N$ to 0. Therefore, only u_0, $u_1(= u_t(0))$ and q_0 are required in the compatibility conditions (5.4).*

6. A BOUNDED CONVEX CLOSED SET OF Y_T

DEFINITION 6.1. *Given $M > 0$. Let $C_T(M)$ denote the subset of Y_T consisting of elements of (v,h) in Y_T such that*

$$\|(v,h)\|^2_{Y_T} \le M \tag{6.1}$$

and such that $v(0) = u_0$, $h(0) = 0$ and $h_t(0) = (B_0)_*((0,1)) \cdot u_0$.

Remark 12. *For $(v,h) \in C_T(M)$, define u^τ by (3.9) and let η^τ be the associated flow map. Also define v^τ as $u^\tau \circ \eta^\tau$. By (3.11) and (3.12), we have*

$$\sup_{t\in[0,T]} \|\nabla_0 \eta^\tau(t)\|_{H^{1.5}(\Gamma_0)} + \|v^\tau\|^2_{L^2(0,T;H^{2.5}(\Gamma_0))} \le C(M) \tag{6.2}$$

for some constant $C(M)$.

We will make use of the following lemmas (proved in [6]):

Lemma 6.1. *There exists $T_0 \in (0,T)$ such that for all $T \in (0,T_0)$ and for all $v \in C_T(M)$, the matrix a is well-defined (by (3.16)) with the estimate (independent of $v \in C_T(M)$)*

$$\begin{aligned} &\|a\|_{L^\infty(0,T;H^2(\Omega_0))} + \|a_t\|_{L^\infty(0,T;H^1(\Omega_0))} + \|a_t\|_{L^2(0,T;H^2(\Omega_0))} \\ &+ \|a_{tt}\|_{L^\infty(0,T;L^2(\Omega_0))} + \|a_{tt}\|_{L^2(0,T;H^1(\Omega_0))} \le C(M). \end{aligned} \tag{6.3}$$

Lemma 6.2. *There exists $T_1 \in (0,T)$ and a constant C (independent of M) such that for all $T \in (0,T_1)$ and $v \in C_T(M)$, for all $\phi \in H^1(\Omega_0)$ and $t \in [0,T]$*

$$C\|\phi\|^2_{H^1(\Omega_0)} \le \int_{\Omega_0} \Big[|v|^2 + |D_\eta(v)|^2\Big] dx \tag{6.4}$$

where

$$|D_\eta(v)|^2 := D_\eta(v)^i_j D_\eta(v)^i_j = (a^k_j v^i_{,k} + a^k_j v^i_{,k})(a^\ell_j v^i_{,\ell} + a^\ell_i v^j_{,\ell}).$$

In the remainder of the paper, we will assume that

$$0 < T < \min\{T_0, T_1, \bar{T}\}$$

for some fixed $\bar{T}$ where the forcing F is defined on the time interval $[0,\bar{T}]$.

7. Preliminary results

7.1. **Pressure as a Lagrange multiplier.** In the following, we use $H^{1;2}(\Omega_0;\Gamma_0)$ to denote the Hilbert space $H^1(\Omega_0) \cap H^2(\Gamma_0)$ with norm

$$\|u\|^2_{H^{1;2}(\Omega_0;\Gamma_0)} = \|u\|^2_{H^1(\Omega_0)} + \|u\|^2_{H^2(\Gamma_0)}$$

and use $\bar{\mathcal{V}}_{\bar{v}}$ ($\bar{\mathcal{V}}_{\bar{v}}(T)$) to denote the space

$$\Big\{v \in \mathcal{V}_{\bar{v}} \;\Big|\; v \in H^2(\Gamma_0)\Big\}\Big(\Big\{v \in \mathcal{V}_{\bar{v}}(T) \;\Big|\; v \in L^2(0,T;H^2(\Gamma_0))\Big\}\Big).$$

LEMMA 7.1. *For all* $p \in L^2(\Omega_0)$, $t \in [0,T]$, *there exists a constant* $C > 0$ *and* $\phi \in H^{1;2}(\Omega_0;\Gamma_0)$ *such that* $a_i^j(t)\phi^i_{,j} = p$ *and*

$$\|\phi\|_{H^{1;2}(\Omega_0;\Gamma_0)} \le C\|p\|_{L^2(\Omega_0)}. \tag{7.1}$$

Proof. We solve the following problem on the time-dependent domain $\Omega(t)$:

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} \qquad \text{in} \quad \eta(t,\Omega_0) := \Omega(t).$$

The solution to this problem can be written as the sum of the solutions to the following two problems

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} - \bar{p}(t) \qquad \text{in} \quad \eta(t,\Omega_0), \tag{7.2}$$

$$\operatorname{div}(\phi \circ \eta(t)^{-1}) = \bar{p}(t) \qquad \text{in} \quad \eta(t,\Omega_0), \tag{7.3}$$

where $\bar{p}(t) = \dfrac{1}{|\Omega_0|}\displaystyle\int_{\Omega_0} p(t,x)dx$. The existence of the solution to problem (7.2) with zero boundary condition is standard (see, for example, [15] Chapter 3), and the solution to problem (7.3) can be chosen as a linear function (linear in x) , for example, $\bar{p}(t)x_1$. The estimate (7.1) follows from the estimates of the solutions to (7.2). □

Define a linear functional on $H^{1;2}(\Omega_0;\Gamma_0)$ by $(p, a_i^j(t)\varphi^i_{,j})_{L^2(\Omega_0)}$ for $\varphi \in H^{1;2}(\Omega_0;\Gamma_0)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t) : L^2(\Omega_0) \to H^{1;2}(\Omega_0;\Gamma_0)$ such that for all $\varphi \in H^{1;2}(\Omega_0;\Gamma_0)$,

$$(p, a_i^j(t)\varphi^i_{,j})_{L^2(\Omega_0)} = (Q(t)p,\varphi)_{H^{1;2}(\Omega_0;\Gamma_0)} := (Q(t)p,\varphi)_{H^1(\Omega_0)} + (Q(t)p,\varphi)_{H^2(\Gamma_0)}.$$

Letting $\varphi = Q(t)p$ shows that

$$\|Q(t)p\|_{H^{1;2}(\Omega_0;\Gamma_0)} \le C\|p\|_{L^2(\Omega_0)}$$

for some constant $C > 0$. By Lemma 7.1,

$$\|p\|^2_{L^2(\Omega_0)} \le \|Q(t)p\|_{H^{1;2}(\Omega_0;\Gamma_0)}\|\varphi\|_{H^{1;2}(\Omega_0;\Gamma_0)} \le C\|Q(t)p\|_{H^{1;2}(\Omega_0;\Gamma_0)}\|p\|_{L^2(\Omega_0)}$$

which shows that $R(Q(t))$ is closed in $H^{1;2}(\Omega_0;\Gamma_0)$. Since $\bar{\mathcal{V}}_v(t) \subset R(Q(t))^\perp$ and $R(Q(t))^\perp \subset \bar{\mathcal{V}}_v(t)$, it follows that

$$H^{1;2}(\Omega_0;\Gamma_0)(t) = R(Q(t)) \oplus_{H^{1;2}(\Omega_0;\Gamma_0)} \bar{\mathcal{V}}_v(t). \tag{7.4}$$

We can now introduce our Lagrange multiplier.

LEMMA 7.2. *Let $\mathcal{L}(t) \in H^{1;2}(\Omega_0;\Gamma_0)'$ be such that $\mathcal{L}(t)\varphi = 0$ for any $\varphi \in \bar{\mathcal{V}}_v(t)$. Then there exist a unique $q(t) \in L^2(\Omega_0)$, which is termed the pressure function, satisfying*

$$\forall\ \varphi \in H^{1;2}(\Omega_0;\Gamma_0), \quad \mathcal{L}(t)(\varphi) = (q(t), a_i^j(t)\varphi^i_{,j})_{L^2(\Omega_0)}.$$

Moreover, there is a $C > 0$ (which does not depend on $t \in [0,T]$ and ϵ and on the choice of $v \in C_T(M)$) such that

$$\|q(t)\|_{L^2(\Omega_0)} \le C\|\mathcal{L}(t)\|_{H^{1;2}(\Omega_0;\Gamma_0)'}.$$

Proof. By the decomposition (7.4), for given $\tilde{a}$, let $\varphi = v_1 + v_2$, where $v_1 \in \mathcal{V}_v(t)$ and $v_2 \in R(Q(t)$. It follows that

$$\mathcal{L}(t)(\varphi) = \mathcal{L}(t)(v_2) = (\psi(t), v_2)_{H^{1;2}(\Omega_0;\Gamma_0)} = (\psi(t), \varphi)_{H^{1;2}(\Omega_0;\Gamma_0)}$$

for a unique $\psi(t) \in R(Q(t))$.

From the definition of $Q(t)$ we then get the existence of a unique $q(t) \in L^2(\Omega_0)$ such that

$$\forall\ \varphi \in H^{1;2}(\Omega_0;\Gamma_0), \quad \mathcal{L}(t)(\varphi) = (q(t), a_i^j(t)\varphi^i_{,j})_{L^2(\Omega_0)}.$$

The estimate stated in the lemma is then a simple consequence of (7.1). □

7.2. **Standard inequalities.**

LEMMA 7.3. *Suppose $f \in H^{1.5}(\Gamma_0)$ if $n = 3$ (while $f \in H^1(\Gamma_0)$ if $n = 2$) and $g \in H^{0.5}(\Gamma_0)$, then $fg \in H^{0.5}(\Gamma_0)$ and satisfies*

$$\|fg\|_{H^{0.5}(\Gamma_0)} \le C\|f\|_{L^\infty(\Gamma_0)}\|g\|_{H^{0.5}(\Gamma_0)} \tag{7.5}$$

for some constant C depending on the geometry of Γ_0.

Lemma 7.4. *For space dimension $n = 3$,*

(7.6) $$\|f\|_{L^4(\Omega)} \le C\|f\|^{3/4}_{H^1(\Omega)}\|f\|^{1/4}_{L^2(\Omega)} \ ,$$

(7.7) $$\|f\|_{L^\infty(\Omega)} \le C\|f\|^{3/4}_{H^2(\Omega)}\|f\|^{1/4}_{L^2(\Omega)} \ ,$$

(7.8) $$\|f\|_{L^2(\Gamma_0)} \le C\|f\|^{1/2}_{H^1(\Omega_0)}\|f\|^{1/2}_{L^2(\Omega_0)} \ ,$$

while for space dimension $n = 2$,

(7.9) $$\|f\|_{L^4(\Omega)} \le C\|f\|^{1/2}_{H^1(\Omega)}\|f\|^{1/2}_{L^2(\Omega)} \ ,$$

(7.10) $$\|f\|_{L^\infty(\Omega)} \le C\|f\|^{1/2}_{H^2(\Omega)}\|f\|^{1/2}_{L^2(\Omega)} \ ,$$

(7.11) $$\|f\|_{H^{0.5}(\Omega)} \le C\|f\|^{1/2}_{H^1(\Omega)}\|f\|^{1/2}_{L^2(\Omega)} \ ,$$

and for either $n = 2$ or $n = 3$,

(7.12) $$\|v\|_{H^1(\Omega)} \le C\|v\|^{1/2}_{H^2(\Omega)}\|v\|^{1/2}_{L^2(\Omega)}$$

where C depends on Ω.

7.3. **Estimates for a and h.** We make use of near-identity transformations. The following lemmas can be found in [5] and [6].

Lemma 7.5. *There exists $K > 0$, $T_0 > 0$ such that if $0 < t \le T_0$, then, for any $(\tilde v, \tilde h) \in C_{T_0}(M)$,*

(7.13a) $$\|\tilde a^T - Id\|_{L^\infty(0,T;\mathcal{C}^0(\overline{\Omega}_0))} \le K\sqrt{t} \ ;$$

(7.13b) $$\|\tilde a - Id\|_{L^\infty(0,T;H^2(\Omega_0))} \le K\sqrt{t} \ ;$$

(7.13c) $$\|\tilde a_t - \tilde a_t(0)\|_{L^\infty(0,T;H^1(\Omega_0))} \le C(M)t \ ;$$

(7.13d) $$\|\tilde a_t\|_{L^\infty(0,T;H^1(\Omega_0))} \le K.$$

We also need the following

Lemma 7.6. *For any* $(\tilde{v}, \tilde{h}) \in C_{T_0}(M)$*,*

$$\|\tilde{h}\|_{H^{3.5}(\Gamma_0)} \le CMt^{1/4} \tag{7.14}$$

for all $0 < t \le T_0$*.*

Proof. For $(\tilde{v}, \tilde{h}) \in C_T(M)$, $\|\tilde{h}\|^2_{H^4(\Gamma_0)} + \|\tilde{h}_t\|^2_{H^2(\Gamma_0)} \le M$. By $\tilde{h}(0) = 0$,

$$\|\tilde{h}(t)\|_{H^2(\Gamma_0)} \le \int_0^t \|\tilde{h}_t\|_{H^2(\Gamma_0)} ds \le \sqrt{M}t.$$

Finally, the interpolation inequality

$$\|\nabla_0^2 f(t)\|_{H^{1.5}(\Gamma_0)} \le C\|\nabla_0^4 f\|^{3/4}_{L^2(\Gamma_0)} \|\nabla_0^2 f\|^{1/4}_{L^2(\Gamma_0)}, \tag{7.15}$$

implies

$$\|\tilde{h}\|_{H^{3.5}(\Gamma_0)} \le C\|\tilde{h}\|^{3/4}_{H^4(\Gamma_0)} \|\tilde{h}\|^{1/4}_{H^2(\Gamma_0)} \le CMt^{1/4}.$$

□

Corollary 7.1. $\|L_1(t)\|_{H^{1.5}(\Gamma_0)}$ *and* $\|L_2(t)\|_{H^{1.5}(\Gamma_0)}$ *converge to zero as* $t \to 0$*, uniformly in* $(v, h) \in C_{T_0}(M)$*. Furthermore, for* $t \le 1$*,*

$$\|L_1(t)\|_{H^{1.5}(\Gamma_0)} + \|L_2(t)\|_{H^{1.5}(\Gamma_0)} \le C(M)t^{1/4}.$$

By the fact that $\|\tilde{h}_t\|^2_{H^2(\Gamma_0)} \le M$ and $\|\tilde{h}_{tt}\|^2_{L^2(0,T;H^{0.5}(\Gamma_0))} \le M$ if $(\tilde{v}, \tilde{h}) \in C_T(M)$, similar computations lead to the following lemma.

Lemma 7.7. *For all* $(\tilde{v}, \tilde{h}) \in C_T(M)$*,*

$$\|\tilde{h}_t(t)\|_{H^{1.5}(\Gamma_0)} \le CMt^{1/8} \tag{7.16}$$

for all $0 < t \le T$*.*

8. The linearized problem

Suppose that $(\tilde v, \tilde h) \in C_T(M)$ is given. Let $\tilde\eta(t) = \mathrm{Id} + \int_0^t \tilde v(s)ds$ and $\tilde a = (\nabla\tilde\eta)^{-1}$. We are concerned with the following time-dependent linear problem, whose fixed-point $v = \tilde v$ provides a solution to (3.17):

$$v^i_t - \nu[\tilde a^k_\ell D_{\tilde\eta}(v)^i_\ell]_{,k} = -(\tilde a^k_i q)_{,k} + F^i \qquad \text{in } (0,T)\times\Omega_0\,, \tag{8.1a}$$

$$\tilde a^j_i v^i_{,j} = 0 \qquad \text{in } (0,T)\times\Omega_0\,, \tag{8.1b}$$

$$\begin{aligned}[\nu D_{\tilde\eta}(v)^j_i - q\delta^j_i]\tilde a^\ell_j N_\ell = \sigma\tilde\Theta\Big[\mathcal{L}_{\tilde h}(h)(-\nabla_0\tilde h,1)\Big]\circ\tilde\eta^\tau \qquad &\text{on } (0,T)\times\Gamma_0\,,\\ +\,\sigma\tilde\Theta\Big[[\mathcal{G}(\tilde h)(-\nabla_0\tilde h,1)]\circ\tilde\eta^\tau\Big]&\end{aligned} \tag{8.1c}$$

$$h_t\circ\tilde\eta^\tau = [\tilde h_{,\alpha}\circ\tilde\eta^\tau]v_\alpha - v_z \qquad \text{on } (0,T)\times\Gamma_0\,, \tag{8.1d}$$

$$v = u_0 \qquad \text{on } \{t=0\}\times\Omega_0\,, \tag{8.1e}$$

$$h = 0 \qquad \text{on } \{t=0\}\times\Gamma_0\,. \tag{8.1f}$$

where $D_{\tilde\eta}(v)^j_i = \tilde a^k_i v^j_{,k} + \tilde a^k_j v^i_{,k}$, $\tilde\Theta = \det(\nabla_0\tilde\eta^\tau)$, and

$$\mathcal{L}_{\tilde h}(h) = \frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\tilde A^{\alpha\beta\gamma\delta}h_{,\alpha\beta}\Big]_{,\gamma\delta}$$

with

$$\begin{aligned}\tilde A^{\alpha\beta\gamma\delta} &= J^{-3}_{\tilde h}\sqrt{\det(G_{\tilde h})}\Big[G^{\alpha\gamma}_{\tilde h} - (-1)^{\kappa+\sigma}\det(G_{\tilde h})^{-1}(1-\delta_{\alpha\kappa})(1-\delta_{\gamma\sigma})\tilde h_{,\kappa}\tilde h_{,\sigma}\Big]\\ &\quad\times(G^{\beta\delta}_{\tilde h} - J^{-2}_{\tilde h}G^{\beta\mu}_{\tilde h}G^{\delta\nu}_{\tilde h}\tilde h_{,\mu}\tilde h_{,\nu})\end{aligned}$$

and

$$\mathcal{M}(\tilde h) = \sqrt{\det(G_{\tilde h})}\Big[L^{\alpha\beta\gamma}_1(y,\tilde h, D\tilde h, D^2\tilde h)\tilde h_{,\alpha\beta\gamma} + L_2(y,\tilde h, D\tilde h, D^2\tilde h)\Big].$$

Here the thickness ϵ is assumed to be 1.

We will also use $L_{\tilde h}(h)$ to denote $\mathcal{L}_{\tilde h}(h) + \mathcal{M}(\tilde h)$.

Remark 13. *$\mathcal{L}_{\tilde h}$ is a coercive fourth order operator for small $\tilde h \le \delta$. Actually, it is easy to see that $\mathcal{L}_{\tilde h}$ is coercive at time $t=0$, and the coercivity of $\mathcal{L}_{\tilde h}$ for $t>0$ (but sufficiently small) follows from the continuity of $\tilde h$ in time into the space $H^2(\Gamma_0)$. Moreover, by Lemma 7.6, we have the following corollary.*

Corollary 8.1. *There exists a $\nu_1 > 0$ and $0 < T \leq T_0$ such that for all $0 < t \leq T$,*

$$\nu_1 \|\nabla_0^2 f(t)\|^2_{L^2(\Gamma_0)} \leq \int_{\Gamma_0} \tilde{A}^{\alpha\beta\gamma\delta} f_{,\alpha\beta}(t) f_{,\gamma\delta}(t) dS.$$

for all $0 < t \leq T$. Later on we will denote the right-hand side quantity of this inequality by $E_{\bar{h}}(f)$, where the subscript $\bar{h}$ indicates that $\bar{A}$ is a function of $\bar{h}$.

Remark 14. *Given $(\tilde{v}, \tilde{h}) \in V^3(T) \times H(T)$, for the corresponding $\tilde{\eta}^\tau$, we have*

$$\|\tilde{\eta}^\tau\|^2_{L^\infty(0,T;H^{2.5}(\Omega_0))} + \|\tilde{\eta}^\tau_t\|^2_{L^2(0,T;H^{2.5}(\Gamma_0))} \leq C(M)$$

where (3.12) and (3.11) are used to obtain this estimate.

The solution of (8.1) is found as a weak limit of a sequence of regularized problems.

Definition 8.1. **(Mollifiers on Γ_0)** *For $\epsilon > 0$, let*

$$K^p_\epsilon := (1 - \epsilon\Delta_0)^{-\frac{p}{2}} : H^s(\Gamma_0) \to H^{s+p}(\Gamma_0)$$

denote the usual self-adjoint Frederich's mollifier on the compact manifold Γ_0, where Δ_0 is the surface Laplacian defined on Γ_0, given by

$$\Delta_0 f = \frac{1}{\sqrt{\det(g_0)}} \frac{\partial}{\partial y^\alpha} \Big(\sqrt{\det(g_0)} g^{\alpha\beta} \frac{\partial f}{\partial y^\beta} \Big).$$

By the Sobolev extension theorem, there exist bounded extension operators

$$E_s : H^s(\Omega_0) \to H^s(\mathbb{R}^n), \quad s \geq 1\,.$$

For fixed (but small) ϵ and $\epsilon_1 > 0$, let ρ_ϵ be a (positive) smooth mollifier on $\mathbb{R}^n$. Set $\bar{v} = \rho_\epsilon * E_1(\tilde{v})$, $\tilde{F} = \rho_\epsilon * E_2(F)$, $\tilde{u}_0 = \rho_\epsilon * E_3(u_0)$, where $*$ denotes the convolution in space, and $\bar{h} = K^m_\epsilon(\tilde{h})$ for large enough m. Define $\bar{\eta}$ and $\bar{a}$ in the same fashion as $\tilde{\eta}$ and $\tilde{a}$. Note that $\bar{v} \to \tilde{v} \in V(T)$, $\tilde{F} \to F$ in $V^2(T)$, $\tilde{u_0} \to u_0$ in $H^{2.5}(\Omega_0)$ and $\bar{h} \to \tilde{h}$ in $H(T)$ as $\epsilon \to 0$.

The regularized problem takes the form

$$v_t^i - \nu[\bar a_\ell^k D_{\bar\eta}(v)_\ell^i]_{,k} = -(\bar a_i^k q)_{,k} + \tilde F^i \qquad \text{in } (0,T)\times\Omega_0\,, \tag{8.2a}$$

$$\bar a_i^j v^i_{,j} = 0 \qquad \text{in } (0,T)\times\Omega_0\,, \tag{8.2b}$$

$$\begin{aligned}[\nu D_{\bar\eta}(v)_i^j - q\delta_i^j]\bar a_j^\ell N_\ell &= \sigma \mathcal{L}_{\bar h}^{\epsilon_1}(h^{\epsilon_1})(-\nabla_0\bar h\circ\bar\eta^\tau,1) \qquad \text{on } (0,T)\times\Gamma_0\,,\\ &\quad + \sigma\mathcal{M}_{\bar h}^{\epsilon_1}(-\nabla_0\bar h\circ\bar\eta^\tau,1) + \kappa\Delta_0^2 v\end{aligned} \tag{8.2c}$$

$$h_t\circ\bar\eta^\tau = [(\bar h_{,\alpha})\circ\bar\eta^\tau]v_\alpha - v_z \qquad \text{on } (0,T)\times\Gamma_0\,, \tag{8.2d}$$

$$v = \tilde u_0 \qquad \text{on } \{t=0\}\times\Omega_0\,, \tag{8.2e}$$

$$h = 0 \qquad \text{on } \{t=0\}\times\Gamma_0\,, \tag{8.2f}$$

where

$$\begin{aligned}\bar{\mathcal{L}}_{\bar h}^{\epsilon_1}(f) &= \frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\Big(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} f_{,\alpha\beta}\Big)_{,\gamma\delta}\Big]^{\epsilon_1}\circ\bar\eta^\tau\,,\\ \bar{\mathcal{M}}_{\bar h}^{\epsilon_1} &= \bar\Theta\Big[\Big(L_1^{\alpha\beta\gamma}(\cdot,\bar h, D\bar h, D^2\bar h)\bar h_{,\alpha\beta\gamma} + L_2(\cdot,\bar h, D\bar h)\Big)^{\epsilon_1}\Big]^{\epsilon_1}\circ\bar\eta^\tau(y,t).\end{aligned}$$

Note that

$$\bar{\mathcal{L}}_{\bar h}^{\epsilon_1}(f) + \bar{\mathcal{M}}_{\bar h}^{\epsilon_1} = \bar\Theta\Big[L_{\bar h}(f)\Big]^{\epsilon_1}\circ\bar\eta^\tau.$$

8.1. **Weak solutions.**

Definition 8.2. *A vector $v\in\bar{\mathcal{V}}_{\bar v}(T)$ with $v_t\in\bar{\mathcal{V}}_{\bar v}(T)'$ for almost all $t\in(0,T)$ is a weak solution of (8.2) provided that*

$$\begin{aligned}\text{(i) } \langle v_t,\varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\bar\eta}v : D_{\bar\eta}\varphi dx + \sigma\int_{\Gamma_0}\bar A^{\alpha\beta\gamma\delta}h^{\epsilon_1}_{,\alpha\beta}\Big[-\bar h_{,\sigma}(\varphi^\sigma\circ\bar\eta^{-\tau})\\ + (\varphi^z\circ\bar\eta^{-\tau})\Big]^{\epsilon_1}_{,\gamma\delta} dS + \kappa\int_{\Gamma_0}\Delta_0 v\cdot\Delta_0\varphi dS = \langle\tilde F,\varphi\rangle - \sigma\langle G_{\bar h}^{\epsilon_1},\varphi\rangle_{\Gamma_0}\end{aligned} \tag{8.3a}$$

$$\text{(ii) } v(0,\cdot) = \tilde u_0 \tag{8.3b}$$

for almost all $t\in[0,T]$, where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\bar{\mathcal{V}}_v(t)$ and its dual $\bar{\mathcal{V}}_v(t)'$, and h is given by the evolution equation (8.2d) and the initial condition

(8.2f):

$$h(y,t) = \int_0^t \Big[- \bar{h}_{,\alpha}(y,s) v^\alpha(\bar{\eta}^{-\tau}(y,s),0,s) + v^z(\bar{\eta}^{-\tau}(y,s),0,s)\Big] ds \tag{8.4}$$

8.2. **Penalized problems.** Letting $\theta > 0$ denote the penalized parameter, we define w_θ (with also ϵ and ϵ_1 dependence in mind) to be the "unique" solution of the problem (whose existence can be obtained via a modified Galerkin method which will be presented in the following sections):

$$\begin{aligned} &\text{(i) } \langle w_{\theta t}, \varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\bar\eta} w_\theta : D_{\bar\eta}\varphi dx + \sigma \int_{\Gamma_0} \bar{A}^{\alpha\beta\gamma\delta} h^{\epsilon_1}_{,\alpha\beta} \Big[- \bar{h}_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) \\ &\quad + (\varphi^z \circ \bar\eta^{-\tau})\Big]^{\epsilon_1}_{,\gamma\delta} dS + \kappa \int_{\Gamma_0} \Delta_0 v \cdot \Delta_0 \varphi dS + (\frac{1}{\theta} \bar{a}^j_i v^i_{,j}, \bar{a}^\ell_k \varphi^k_{,\ell})_{L^2(\Omega_0)} \\ &\quad = \langle \tilde{F}, \varphi\rangle - \sigma \langle \bar{\mathcal{M}}^{\epsilon_1}_{\bar h}(-\nabla_0 \bar h \circ \bar\eta^\tau, 1), \varphi\rangle_{\Gamma_0} \end{aligned} \tag{8.5a}$$

$$\text{(ii) } v(0,\cdot) = \tilde{u}_0 \tag{8.5b}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $H^1(\Omega_0)$ and its dual, and h in (8.5a) satisfies (8.4) with v replaced by w_θ.

8.3. **Weak solutions for the penalized problem.** The goal of this section is to establish the existence of v to the problem (8.2) (or the weak formulation (8.3)), as well as the energy inequality satisfied by v and v_t. Before proceeding, we introduce variable $\tilde{q}_0$ and $\tilde{w}_1$ as follows: let $\tilde{q}_0$ be the solution of the following Laplace equation

$$\Delta \tilde{q}_0 = \nabla \tilde{u}_0 : (\nabla \tilde{u}_0)^t - \operatorname{div} \tilde{F}(0) \qquad \text{in } \Omega_0\,, \tag{8.6a}$$

$$\tilde{q}_0 = \nu (\operatorname{Def} \tilde{u}_0)^j_i N_i N_j - \sigma \mathcal{M}^{\epsilon_1}_0(0) + \kappa \Delta_0^2 \tilde{u}_0 \cdot N \qquad \text{on } \Gamma_0\,, \tag{8.6b}$$

and $\tilde{w}_1$ be defined by

$$\tilde{w}_1 = \nu \Delta \tilde{u}_0 - \nabla \tilde{q}_0 + \tilde{F}(0). \tag{8.7}$$

By elliptic regularity,

$$\begin{aligned} \|\tilde{q}_0\|^2_{H^1(\Omega_0)} &\le C\Big[\|\tilde{u}_0\|^2_{H^2(\Omega_0)} + \|\tilde{F}(0)\|^2_{L^2(\Omega_0)} + \|\mathcal{M}^{\epsilon_1}_0(0)\|^2_{H^{0.5}(\Gamma_0)} + \|\Delta_0^2 \tilde{u}_0\|^2_{H^{0.5}(\Gamma_0)}\Big] \\ &\le C(M)\Big[\|\tilde{u}_0\|^2_{H^2(\Omega_0)} + \|\tilde{u}_0\|^2_{H^{4.5}(\Gamma_0)} + \|\tilde{F}(0)\|^2_{L^2(\Omega_0)} + 1\Big]\,, \end{aligned}$$

and hence

$$\|\tilde{w}_1\|^2_{L^2(\Omega_0)} \le C(M)\Big[\|\tilde{u}_0\|^2_{H^2(\Omega_0)} + \|\tilde{u}_0\|^2_{H^{4.5}(\Gamma_0)} + \|\tilde{F}(0)\|^2_{L^2(\Omega_0)} + 1\Big].$$

REMARK 15. *By (7.14), the constant $C(M)$ in the estimates above can also be refined as a constant independent of M if T is chosen small enough.*

By introducing a (smooth) basis $(e_\ell)_{\ell=1}^\infty$ of $H^{1;2}(\Omega_0;\Gamma_0)$, and taking the approximation at rank $m \ge 2$ under the form $w_\ell(t,x) = \sum_{k=1}^{\ell} d_k(t)e_k(x)$ with

$$h_\ell(y,t) = \int_0^t \Big[-\bar{h}_{,\alpha}(y,s) w_\ell^\alpha(\bar{\eta}^{-\tau}(y,s),0,s) + w_\ell^z(\bar{\eta}^{-\tau}(y,s),0,s)\Big] ds, \tag{8.8}$$

and satisfying on $[0,T]$,

(i) $(w_{\ell tt},\varphi)_{L^2(\Omega_0)} + \nu(\bar{a}_i^j w_{\ell t,j}, \bar{a}_i^k \varphi_{,k})_{L^2(\Omega_0)} + \nu((\bar{a}_i^j\bar{a}_i^k)_t w_\ell, \varphi_{,k})_{L^2(\Omega_0)}$

$$+\nu\int_{\Omega_0} \bar{a}_r^j \bar{a}_i^k w^i_{\ell t,j}\varphi^r_{,k} dx + \nu\int_{\Omega_0}(\bar{a}_r^j\bar{a}_i^k)_t w^i_{\ell,j}\varphi^r_{,k}dx$$

$$+\sigma\int_{\Gamma_0}\bar{A}^{\alpha\beta\gamma\delta}[-\bar{h}_{,\sigma}(w_\ell^\sigma\circ\bar{\eta}^{-\tau}) + w_\ell^z\circ\bar{\eta}^{-\tau}]^{\epsilon_1}_{,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_1}_{,\gamma\delta}dS$$

$$+\sigma\int_{\Gamma_0}(\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell,\alpha\beta}[-\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \varphi^z\circ\bar{\eta}^{-\tau}]^{\epsilon_1}_{,\gamma\delta}dS$$

$$+\sigma\int_{\Gamma_0}\bar{A}^{\alpha\beta\gamma\delta}h^{\epsilon_1}_{\ell,\alpha\beta}[-\bar{h}_{t,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) + \bar{h}_{,\sigma}\bar{v}^\kappa(\varphi^\sigma_{,\kappa}\circ\bar{\eta}^{-\tau}) + \bar{v}^\kappa(\varphi^z_{,\kappa}\circ\bar{\eta}^{-\tau})]^{\epsilon_1}_{,\gamma\delta}dS$$

$$+\kappa\int_{\Gamma_0}\Delta_0 w_{\ell t}\cdot\Delta_0\varphi dS - ((\bar{a}_i^j q_\ell)_t, \varphi^i_{,j})_{L^2(\Omega_0)}$$

$$= \langle\tilde{F}_t,\varphi\rangle - \sigma\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_1}_t\Big[\bar{h}_{,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) - \varphi^z\circ\bar{\eta}^{-\tau}\Big]^{\epsilon_1}dS$$

$$-\sigma\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_1}\Big[\bar{h}_{t,\sigma}(\varphi^\sigma\circ\bar{\eta}^{-\tau}) - \bar{h}_{,\sigma}\bar{v}^\kappa(\varphi^\sigma_{,\kappa}\circ\bar{\eta}^{-\tau}) - \bar{v}^\kappa(\varphi^z_{,\kappa}\circ\bar{\eta}^{-\tau})\Big]^{\epsilon_1}dS$$

$\forall\ \varphi \in \text{span}(e_1,\cdots,e_\ell)$,

(ii) $w_{\ell t}(0) = (w_1)_\ell, w_\ell(0) = (u_0)_\ell$ in Ω_0,

where $q_\ell = \tilde{q}_0 - \frac{1}{\theta}\bar{a}_i^j w^i_{\ell,j}$, and $(\tilde{u}_0)_\ell$ denote the respective $H^{1;2}(\Omega_0;\Gamma_0)$ projections of u_0 on $\text{span}(e_1,e_2,\cdots,e_\ell)$.

Remark 16. *The existence of w_k follows from the solution of*

$$d_k''(t) + d_\ell'(t)A_{k\ell}(t) + d_\ell(t)B_{k\ell}(t) + \int_0^t d_\ell(s)C_{k\ell}(s,t)ds = F(t)$$

for functions A, B, C and F; however, the existence of the solution d_k does not immediately follow from the fundamental theorem of ODE due to the presence of the time-integral. A straightforward fix-point argument can be implemented, whose details we leave to interested reader.

The use of the test function $\varphi = w_{\ell t}$ in this system of ODE gives us in turn the energy law

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_{\ell t}\|^2_{L^2(\Omega_0)} + \frac{\nu}{2}\|D_{\bar\eta}(w_{\ell t})\|^2_{L^2(\Omega_0)} + \frac{\sigma}{2}\frac{d}{dt}E_{\bar h}(h^{\epsilon_1}_{\ell t,\alpha\beta}) + \theta\|q_{\ell t}\|^2_{L^2(\Omega_0)}\\
&+ \nu((\bar a^j_i \bar a^k_i)_t w_{\ell,j}, w_{\ell t,k})_{L^2(\Omega_0)} + \nu\int_{\Omega_0}(\bar a^j_r \bar a^k_i)_t w^i_{\ell,j} w^r_{\ell t,k} dx + \kappa\|\Delta_0 w_{\ell t}\|^2_{L^2(\Gamma_0)}\\
&+ (q_{\ell t}, \bar a^j_{it} w^i_{\ell,j})_{L^2(\Omega_0)} - (q_\ell, \bar a^j_{it} w^i_{\ell t,j})_{L^2(\Omega_0)} - \frac{\sigma}{2}\int_{\Gamma_0}(\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell t,\alpha\beta} h^{\epsilon_1}_{\ell t,\gamma\delta} dS\\
&- \sigma\int_{\Gamma_0}(\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell,\alpha\beta}\Big[h_{\ell tt} + \bar h_{t,\sigma}(w^\sigma_{\ell t}\circ\bar\eta^{-\tau})\Big]^{\epsilon_1}_{,\gamma\delta} dS + \sigma\int_{\Gamma_0}\bar A^{\alpha\beta\gamma\delta} h^{\epsilon_1}_{\ell,\alpha\beta}\times \qquad (8.9)\\
&\times\Big[-\bar h_{t,\sigma}(w^\sigma_{\ell t}\circ\bar\eta^{-\tau}) + \bar h_{,\sigma}\bar v^\kappa(w^\sigma_{\ell t,\kappa}\circ\bar\eta^{-\tau}) + \bar v^\kappa(w^z_{\ell t,\kappa}\circ\bar\eta^{-\tau})\Big]^{\epsilon_1}_{,\gamma\delta} dS\\
&= \langle\tilde F_t, w_{\ell t}\rangle - \sigma\int_{\Gamma_0}\Big[(L_1^{\alpha\beta\gamma}\bar h_{,\alpha\beta\gamma} + L_2)(-\nabla_0\bar h, 1)\Big]_t\cdot(w_{\ell t}\circ\bar\eta^{-\tau})dS\\
&- \sigma\int_{\Gamma_0}(L_1^{\alpha\beta\gamma}\bar h_{,\alpha\beta\gamma} + L_2)\bar v^\kappa\Big[-\bar h_{,\sigma}(w^\sigma_{\ell t,\kappa}\circ\bar\eta^{-\tau}) + (w^z_{\ell t,\kappa}\circ\bar\eta^{-\tau})\Big]dS.
\end{aligned}$$

For the tenth term, we have

$$\Big|\int_{\Gamma_0}(\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell t,\alpha\beta} h^{\epsilon_1}_{\ell t,\gamma\delta} dS\Big| \le C(M)\|\bar h_t\|_{H^{2.5}(\Gamma_0)}\|\nabla_0^2 h_{\ell t}\|^2_{L^2(\Gamma_0)}.$$

By ϵ_1-regularization and the identity

$$\begin{aligned}
\int_{\Gamma_0}(\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell,\alpha\beta} h^{\epsilon_1}_{\ell tt,\gamma\delta} dS &= \int_{\Gamma_0}\frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar A^{\alpha\beta\gamma\delta})_t\Big]_{,\gamma\delta} h^{\epsilon_1}_{\ell,\alpha\beta} h^{\epsilon_1}_{\ell tt} dS\\
&+ \int_{\Gamma_0}\frac{2}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar A^{\alpha\beta\gamma\delta})_t\Big]_{,\gamma} h^{\epsilon_1}_{\ell,\alpha\beta\delta} h^{\epsilon_1}_{\ell tt} dS\\
&+ \int_{\Gamma_0}(\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell,\alpha\beta\gamma\delta} h^{\epsilon_1}_{\ell tt} dS,
\end{aligned}$$

we find that

$$
\begin{aligned}
&\Big|\int_{\Gamma_0} (\bar{A}^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\ell,\alpha\beta} h^{\epsilon_1}_{\ell tt,\gamma\delta} dS\Big| \\
&\le C(\epsilon_1)\Big[1+\|\bar{h}_t\|_{H^{2.5}(\Gamma_0)}\Big]\|\nabla_0^2 h_\ell\|_{L^2(\Gamma_0)}\Big[\|w_\ell\|_{H^1(\Omega_0)}+\|w_{\ell t}\|_{H^1(\Omega_0)}\Big].
\end{aligned}
$$

Similarly, the the last two terms of the left-hand side can be bounded by

$$
C(\epsilon_1)\|\bar{h}_t\|_{H^{2.5}(\Gamma_0)}\|\nabla_0^2 h_\ell\|_{L^2(\Gamma_0)}\|w_{\ell t}\|_{H^1(\Omega_0)}
$$

where we also use the ϵ_1-regularization to control $\nabla_0^3 w_{\ell t}$. It also follows that the last two terms on the right-hand side can be bounded by

$$
C(M)\Big[1+\|\bar{h}_t\|_{H^{2.5}(\Gamma_0)}\Big]\|w_{\ell t}\|_{H^1(\Omega_0)}.
$$

With positive θ, the fourth term of the left-hand side involving the square of $q_{\ell t}$ acts as a viscous energy term. Integrating (8.9) in time from 0 to t, we then get

$$
\begin{aligned}
&\|w_{\ell t}(t)\|^2_{L^2(\Omega_0)}+\|\nabla_0^2 h_{\ell t}(t)\|^2_{L^2(\Gamma_0)} \\
&+\int_0^t \Big[\|\nabla w_{\ell t}\|^2_{L^2(\Omega_0)}+\kappa\|w_{\ell t}\|^2_{H^2(\Gamma_0)}+\theta\|q_{\ell t}\|^2_{L^2(\Omega_0)}\Big]ds \\
&\le C(M)\Big[\|w_{\ell t}(0)\|^2_{L^2(\Omega_0)}+\|w_\ell(0)\|^2_{H^1(\Omega_0)}+\|q_\ell(0)\|^2_{H^{0.5}(\Omega_0)}\Big] \\
&+C(\epsilon_1)\int_0^t \Big[1+\|\bar{h}_t(s)\|^2_{H^{2.5}(\Gamma_0)}\Big]\|\nabla_0^2 h_{\ell t}(s)\|^2_{L^2(\Gamma_0)}ds \\
&+C(\theta)\int_0^t \|\bar{v}(t')\|^2_{H^3(\Omega_0)}\int_0^{t'}\Big[\|\nabla w_{\ell t}(s)\|^2_{L^2(\Omega_0)}+\|q_{\ell t}(s)\|^2_{L^2(\Omega_0)}\Big]ds dt',
\end{aligned}
\tag{8.10}
$$

where $C(\epsilon_1), C(\theta)\to\infty$ as $\epsilon_1,\theta\to 0$, and we use

$$
\|f(t)\|_X \le \|f(0)\|_X+\int_0^t \|f_t(s)\|_X ds \le \|f(0)\|_X+\sqrt{t}\int_0^t \|f_t(s)\|^2_X ds
$$

for $f=w_\ell$, $f=h_\ell$ and $f=g_\ell$ to obtain (8.10).

Remark 17. *The θ-dependence follows from estimating the terms $(q_{\ell t}, \bar{a}^j_{it} w^i_{\ell,j})_{L^2(\Omega_0)}$:*

$$
\begin{aligned}
&\Big|(q_{\ell t}, \bar{a}^j_{it} w^i_{\ell,j})_{L^2(\Omega_0)}\Big| \le \frac{\theta}{2}\|q_{\ell t}\|^2_{L^2(\Omega_0)}+\frac{1}{2\theta}\|\bar{a}^j_{it}\|^2_{L^\infty(\Omega_0)}\|w^i_{\ell,j}\|^2_{L^2(\Omega_0)} \\
&\le \frac{\theta}{2}\|q_{\ell t}\|^2_{L^2(\Omega_0)}+\frac{C(M)}{\theta}\Big[\|\nabla w_\ell(0)\|^2_{L^2(\Omega_0)}+t\int_0^t \|\nabla w_{\ell t}\|^2_{L^2(\Omega_0)}(s)ds\Big].
\end{aligned}
$$

By the Gronwall inequality, for $0 \le t \le T$,

$$
\begin{aligned}
&\|w_{\ell t}(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\ell t}(t)\|^2_{L^2(\Gamma_0)} \\
&+ \int_0^t \Big[\|\nabla w_{\ell t}\|^2_{L^2(\Omega_0)} + \kappa \|w_{\ell t}\|^2_{H^2(\Gamma_0)} + \theta \|q_{\ell t}\|^2_{L^2(\Omega_0)}\Big] ds \\
\le\, & C(\epsilon_1, \theta) N_0(u_0, F)
\end{aligned}
\tag{8.11}
$$

where

$$
N_0(u_0, F) := \|u_0\|^2_{H^{2.5}(\Omega_0)} + \|u_0\|^2_{H^{4.5}(\Gamma_0)} + \|F_t\|^2_{L^2(0,T;H^1(\Omega_0)')} + \|F(0)\|^2_{H^{0.5}(\Omega_0)} + 1.
$$

We can then infer that w_ℓ defined on $[0, T]$, and that there is a subsequence, still denoted with the subscript ℓ, satisfying

$$
\begin{aligned}
&(8.12a) && w_\ell \rightharpoonup w_\theta && \text{in } L^2(0, T; H^{1;2}(\Omega_0; \Gamma_0)) \\
&(8.12b) && w_{\ell t} \rightharpoonup w_{\theta t} && \text{in } L^2(0, T; H^{1;2}(\Omega_0; \Gamma_0)) \\
&(8.12c) && \nabla_0^2 h_\ell \rightharpoonup \nabla_0^2 h_\theta && \text{in } L^2(0, T; L^2(\Gamma_0)) \\
&(8.12d) && \nabla_0^2 h_{\ell t} \rightharpoonup \nabla_0^2 h_{\theta t} && \text{in } L^2(0, T; L^2(\Gamma_0)) \\
&(8.12e) && q_{\ell t} \rightharpoonup q_{\theta t} && \text{in } L^2(0, T; L^2(\Omega_0))
\end{aligned}
$$

where

$$
q_\theta = \tilde{q}_0 - \frac{1}{\theta} \bar{a}_i^j w^i_{\theta,j}.
$$

From the standard procedure for weak solutions, we can now infer from these weak convergences and the definition of w_ℓ that $w_{\ell tt} \in L^2(0, T; H^1(\Omega_0)')$. In turn, $w_{\ell t} \in \mathcal{C}^0([0, T]; H^1(\Omega_0)')$, $w_\ell \in \mathcal{C}^0([0, T]; L^2(\Omega_0))$ with $w_\theta(0) = u_0$, $w_{\theta t}(0) = w_1$.

Moreover, (8.12) implies that w_θ satisfies, for all $\varphi \in L^2(0,T;H^{1;2}(\Omega_0;\Gamma_0))$,

$$
\begin{aligned}
&\text{(i)} \int_0^T \Big[(w_{\theta tt},\varphi)_{L^2(\Omega_0)} + \nu(\bar a_i^j w_{\theta t,j}, \bar a_i^k \varphi_{,k})_{L^2(\Omega_0)} + \nu((\bar a_i^j \bar a_i^k)_t w_\theta, \varphi_{,k})_{L^2(\Omega_0)}\Big] dt \\
&+ \nu \int_0^T \Big[\int_{\Omega_0} \bar a_r^j \bar a_i^k w^i_{\theta t,j} \varphi^r_{,k} dx + \nu \int_{\Omega_0} (\bar a_r^j \bar a_i^k)_t w^i_{\theta,j} \varphi^r_{,k} dx\Big] dt \\
&+ \sigma \int_0^T \int_{\Gamma_0} \bar A^{\alpha\beta\gamma\delta} [-\bar h_{,\sigma}(w_\theta^\sigma \circ \bar\eta^{-\tau}) + w_\theta^z \circ \bar\eta^{-\tau}]^{\epsilon_1}_{,\alpha\beta} \times \\
&\qquad \times [-\bar h_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) + \varphi^z \circ \bar\eta^{-\tau}]^{\epsilon_1}_{,\gamma\delta} dS dt \\
&+ \sigma \int_0^T \int_{\Gamma_0} (\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\theta,\alpha\beta} [-\bar h_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) + \varphi^z \circ \bar\eta^{-\tau}]^{\epsilon_1}_{,\gamma\delta} dS dt \\
&+ \sigma \int_0^T \int_{\Gamma_0} \bar A^{\alpha\beta\gamma\delta} h^{\epsilon_1}_{\theta,\alpha\beta} [-\bar h_{t,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) + \bar h_{,\sigma} \bar v^\kappa (\varphi^\sigma_{,\kappa} \circ \bar\eta^{-\tau}) \\
&\qquad + \bar v^\kappa (\varphi^z_{,\kappa} \circ \bar\eta^{-\tau})]^{\epsilon_1}_{,\gamma\delta} dS dt \\
&+ \kappa \int_0^T \int_{\Gamma_0} \Delta_0 w_{\theta t} \cdot \Delta_0 \varphi dS dt - \int_0^T ((\bar a_i^j q_\theta)_t, \varphi^i_{,j})_{L^2(\Omega_0)} dt \\
&= \int_0^T \Big\{ \langle \tilde F_t, \varphi\rangle ds - \sigma \int_{\Gamma_0} \Big[L_1^{\alpha\beta\gamma} \bar h_{,\alpha\beta\gamma} + L_2\Big]_t^{\epsilon_1} \Big[\bar h_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) - \varphi^z \circ \bar\eta^{-\tau}\Big]^{\epsilon_1} dS \\
&\qquad - \sigma \int_{\Gamma_0} \Big[L_1^{\alpha\beta\gamma} \bar h_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_1} \Big[\bar h_{t,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) - \bar h_{,\sigma} \bar v^\kappa (\varphi^\sigma_{,\kappa} \circ \bar\eta^{-\tau}) \\
&\qquad\qquad - \bar v^\kappa (\varphi^z_{,\kappa} \circ \bar\eta^{-\tau})\Big]^{\epsilon_1} dS \Big\} dt
\end{aligned}
\tag{8.13a}
$$

$$\text{(ii)} \quad w_{\theta t}(0) = \tilde w_1, w_\theta(0) = \tilde u_0 \text{ in } \Omega_0 . \tag{8.13b}$$

Choosing φ to be independent of time, we find that for all $t \in [0,T]$,

$$
\begin{aligned}
&(w_{\theta t},\varphi)_{L^2(\Omega_0)} + \frac{\nu}{2}\int_{\Omega_0} D_{\bar\eta}(w_\theta) : D_{\bar\eta}(\varphi) dx + \kappa \int_{\Gamma_0} \Delta_0 w_\theta \cdot \Delta_0 \varphi dS \\
&+ \sigma \int_{\Gamma_0} \bar A^{\alpha\beta\gamma\delta} h^{\epsilon_1}_{\theta,\alpha\beta} [-\bar h_{,\sigma}(\varphi^\sigma \circ \bar\eta^{-\tau}) + \varphi^z \circ \bar\eta^{-\tau}]^{\epsilon_1}_{,\gamma\delta} dS - (\bar a_i^j q_\theta, \varphi^i_{,j})_{L^2(\Omega_0)} \\
&= \langle \tilde F, \varphi\rangle + \sigma \int_{\Gamma_0} \Big[L_1^{\alpha\beta\gamma\delta} \bar h_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_1} \Big[- \bar h_{,\sigma} \varphi^\sigma \circ \bar\eta^{-\tau} + \varphi^z \circ \bar\eta^{-\tau}\Big]^{\epsilon_1} dS + c(\varphi)
\end{aligned}
$$

for all $\varphi \in H^{1;2}(\Omega_0;\Gamma_0)$, where $c(\varphi) \in \mathbb{R}$ is given by

$$
\begin{aligned}
c(\varphi) &= (\tilde w_1, \varphi)_{L^2(\Omega_0)} + \frac{\nu}{2}\int_{\Omega_0} \mathrm{Def}(\tilde u_0) : \mathrm{Def}\, \varphi dx - (\tilde q_0 - \frac{1}{\theta}\operatorname{div} \tilde u_0, \operatorname{div}\varphi)_{L^2(\Omega_0)} \\
&\quad - (\tilde F(0), \varphi)_{L^2(\Omega_0)} - \sigma(\bar{\mathcal M}_0^{\epsilon_1}(0)(0,1), \varphi)_{L^2(\Gamma_0)} + \kappa(\Delta_0 \tilde u_0, \Delta_0 \varphi)_{L^2(\Gamma_0)} .
\end{aligned}
$$

By compatibility conditions (8.6) and (8.7), $c(\varphi)=0$. Therefore, the weak limit (w_θ, h_θ) satisfies, for all $t\in[0,T]$,

$$
\begin{aligned}
&(w_{\theta t},\varphi)_{L^2(\Omega_0)}+\frac{\nu}{2}\int_{\Omega_0} D_{\bar\eta}(w_\theta):D_{\bar\eta}(\varphi)dx+\kappa\int_{\Gamma_0}\Delta_0 w_\theta\cdot\Delta_0\varphi dS\\
&\quad-(\bar a_i^j q_\theta,\varphi^i_{,j})_{L^2(\Omega_0)}+\sigma\int_{\Gamma_0}\bar A^{\alpha\beta\gamma\delta}h^{\epsilon_1}_{\theta,\alpha\beta}[-\bar h_{,\sigma}(\varphi^\sigma\circ\bar\eta^{-\tau})+\varphi^z\circ\bar\eta^{-\tau}]^{\epsilon_1}_{,\gamma\delta}dS\\
&\quad=\langle\tilde F,\varphi\rangle-\sigma\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma\delta}\bar h_{,\alpha\beta\gamma}+L_2\Big]^{\epsilon_1}\Big[-\bar h_{,\sigma}\varphi^\sigma\circ\bar\eta^{-\tau}+\varphi^z\circ\bar\eta^{-\tau}\Big]^{\epsilon_1}dS\,,
\end{aligned}
\tag{8.14}
$$

for all $\varphi\in H^{1;2}(\Omega_0;\Gamma_0)$.

Since $w_\theta\in L^2(0,T;H^{1;2}(\Omega_0;\Gamma_0))$, we can use it as a test function in (8.14) and obtain (after time integration)

$$
\begin{aligned}
&\frac12\|w_\theta\|^2_{L^2(\Omega_0)}+\frac{\sigma}{2}E_{\bar h}(h_\theta^{\epsilon_1})+\int_0^t\Big[\frac{\nu}{2}\|D_{\bar\eta}w_\theta\|^2_{L^2(\Omega_0)}+\kappa\|\Delta_0 w_\theta\|^2_{L^2(\Gamma_0)}\\
&\quad+\theta\|q_\theta\|^2_{L^2(\Omega_0)}\Big]ds-\theta\int_0^t(q_\theta,\tilde q_0)dt-\frac{\sigma}{2}\int_0^t\int_{\Gamma_0}(\bar A^{\alpha\beta\gamma\delta})_t h^{\epsilon_1}_{\theta,\alpha\beta}h^{\epsilon_1}_{\theta,\gamma\delta}dSds\\
&\quad=\frac12\|\tilde u_0\|^2_{L^2(\Omega_0)}+\int_0^t\langle\tilde F,\varphi\rangle+\sigma\langle\bar{\mathcal M}^{\epsilon_1}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau,1),\varphi\rangle_{\Gamma_0}dt.
\end{aligned}
\tag{8.15}
$$

Consequently,

$$
\begin{aligned}
&\Big[\|w_\theta(t)\|^2_{L^2(\Omega_0)}+\|\nabla_0^2 h_\theta^{\epsilon_1}(t)\|^2_{L^2(\Gamma_0)}\Big]+\int_0^t\|\nabla w_\theta\|^2_{L^2(\Omega_0)}ds+\kappa\int_0^t\|w_\theta\|^2_{H^2(\Gamma_0)}ds\\
&\quad+\theta\int_0^t\|q_\theta\|^2_{L^2(\Omega_0)}ds\\
&\le C(M)\Big[\|\tilde u_0\|^2_{L^2(\Omega_0)}+\theta\|\tilde q_0\|^2_{L^2(\Omega_0)}+\|\tilde F\|^2_{H^1(\Omega_0)'}+\|\bar{\mathcal M}^{\epsilon_1}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau,1)\|^2_{L^2(\Gamma_0)}\Big]\\
&\quad+C(M)\int_0^t\|\bar h_t\|_{H^{2.5}(\Gamma_0)}\|\nabla_0^2 h_\theta^{\epsilon_1}\|^2_{L^2(\Gamma_0)}ds\\
&\le C(M)\Big[N_1(u_0,F)+\int_0^t\|\bar h_t\|_{H^{2.5}(\Gamma_0)}\|\nabla_0^2 h_\theta^{\epsilon_1}\|^2_{L^2(\Gamma_0)}ds\Big]
\end{aligned}
$$

where

$$
N_1(u_0,F)=N_0(u_0,F)+\|F\|^2_{L^2(0,T;H^1(\Omega_0)')}+\|F(0)\|^2_{H^1(\Omega_0)}\,.
$$

By the Gronwall inequality,

$$\sup_{0\le t\le T}\Big[\|w_\theta(t)\|^2_{L^2(\Omega_0)}+\|\nabla_0^2 h_\theta^{\epsilon_1}(t)\|^2_{L^2(\Gamma_0)}\Big] + \int_0^T\Big[\|\nabla w_\theta\|^2_{L^2(\Omega_0)}+\theta\|q_\theta\|^2_{L^2(\Omega_0)}\Big]ds \le C(M)N_1(u_0,F). \tag{8.16}$$

8.4. **Improved pressure estimates.** By ϵ_1-regularization, we can rewrite (8.14) as, for a.a. $t\in[0,T]$,

$$(w_{\theta t},\varphi)_{L^2(\Omega_0)}+\frac{\nu}{2}\int_{\Omega_0}D_{\bar\eta}(w_\theta):D_{\bar\eta}(\varphi)dx+\kappa(\Delta_0 w_\theta,\Delta_0\varphi)_{L^2(\Gamma_0)}-(\bar a_i^j q_\theta,\varphi^i_{,j})_{L^2(\Omega_0)}$$
$$+\sigma\int_{\Gamma_0}\bar{\mathcal{L}}^{\epsilon_1}_{\bar h}(h_\theta^{\epsilon_1})\Big[-\bar h_{,\sigma}\circ\bar\eta^\tau\varphi^\sigma+\varphi^z\Big]dS=\langle\tilde F,\varphi\rangle+\sigma\langle\bar{\mathcal{M}}^{\epsilon_1}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau,1),\varphi\rangle_{\Gamma_0}.$$

Therefore, by the Lagrange Multiplier Lemma, we conclude that

$$\|q_\theta\|^2_{L^2(\Omega_0)}\le C(M)\Big[\|w_{\theta t}\|^2_{H^1(\Omega_0)'}+\|\nabla w_\theta\|^2_{L^2(\Omega_0)}+\|\tilde F\|^2_{H^1(\Omega_0)'}+\kappa\|\Delta_0^2 w_\theta\|^2_{H^{-2}(\Gamma_0)}$$
$$+\|[\bar{\mathcal{L}}^{\epsilon_1}_{\bar h}(h_\theta^{\epsilon_1})+\bar{\mathcal{M}}^{\epsilon_1}_{\bar h}](-\nabla_0\bar h\circ\bar\eta^\tau,1)\|^2_{H^{-2}(\Gamma_0)}\Big]$$

and hence

$$\|q_\theta\|^2_{L^2(\Omega_0)}\le C(M)\Big[\|w_{\theta t}\|^2_{L^2(\Omega_0)}+\|\nabla w_\theta\|^2_{L^2(\Omega_0)}+\kappa\|w_\theta\|^2_{H^2(\Gamma_0)}+\|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma_0)} + \|F\|^2_{H^1(\Omega_0)'}+1\Big]. \tag{8.17}$$

8.5. **Weak limits as $\theta\to 0$.** Since $w_{\theta t}\in L^2(0,T;H^{1;2}(\Omega_0;\Gamma_0))$, we can use it as a test function in (8.13). Similar to the way we obtain (8.10), we find that

$$\frac{1}{2}\|w_{\theta t}\|^2_{L^2(\Omega_0)}+\frac{\nu}{2}\int_0^t|D_{\bar\eta}w_{\theta t}\|^2_{L^2(\Omega_0)}ds+\frac{\sigma}{2}E_{\bar h}(h_{\theta t}^{\epsilon_1})+\kappa\int_0^t\|\Delta_0^2 w_{\theta t}\|^2_{L^2(\Gamma_0)}ds$$
$$+\theta\int_0^t\|q_{\theta t}\|^2_{L^2(\Omega_0)}ds+\int_0^t(q_{\theta t},\bar a_{it}^j w^i_{\theta,j})_{L^2(\Omega_0)}ds-\int_0^t(q_\theta,\bar a_i^j w^i_{\theta t,j})ds$$
$$\le C(M)N_0(u_0,F)+C(M)\int_0^t\|\bar v(t')\|^2_{H^3(\Omega_0)}\int_0^{t'}\|\nabla w_{\theta t}(s)\|^2_{L^2(\Omega_0)}dsdt'$$
$$+C(\epsilon_1)\int_0^t\Big[1+\|\bar h_t\|_{H^{2.5}(\Gamma_0)}\Big]\|\nabla_0^2 h_{\theta t}^{\epsilon_1}\|^2_{L^2(\Gamma_0)}ds.$$

By (8.17),

$$\Big|\int_0^t (q_\theta, \bar{a}_i^j w_{\theta t,j}^i) ds\Big| \leq C(M,\delta)\int_0^t \|q_\theta\|^2_{L^2(\Omega_0)} ds + \delta \int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} ds$$

$$\leq C(M)\int_0^t \Big[\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla w_\theta\|^2_{L^2(\Omega_0)} + \kappa \|w_\theta\|^2_{H^2(\Gamma_0)} + \|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma_0)}\Big] ds$$

$$+ \delta \int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} ds + C(M) N_0(u_0, F)$$

$$\leq C(M)\Big[N_1(u_0,F) + \int_0^t \Big(\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa\|w_\theta\|^2_{H^2(\Gamma_0)} + \|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma_0)}\Big) ds\Big]$$

$$+ \delta \int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} ds \tag{8.18}$$

where (8.16) is used to bound $\|\nabla w_\theta\|^2_{L^2(0,T;L^2(\Omega_0))}$. Integrating by parts in time,

$$\int_0^t (q_{\theta t}, \bar{a}_{it}^j w_{\theta,j}^i)_{L^2(\Omega_0)} ds = (q_\theta, \bar{a}_{it}^j w_{\theta,j}^i)_{L^2(\Omega_0)}(t) + (\tilde{q}_0, \tilde{u}_{0,i}^j \tilde{u}_{0,j}^i)_{L^2(\Omega_0)}$$

$$- \int_0^t (q_\theta, \bar{a}_{itt}^j w_{\theta,j}^i)_{L^2(\Omega_0)} ds - \int_0^t (q_\theta, \bar{a}_{it}^j w_{\theta t,j}^i)_{L^2(\Omega_0)} ds.$$

By ϵ-regularization, the last two term can be bounded by

$$C(M)\int_0^t \|q_\theta\|_{L^2(\Omega_0)} \Big[C(\epsilon)\|\nabla w_\theta\|_{L^2(\Omega_0)} + \|\nabla w_{\theta t}\|_{L^2(\Omega_0)}\Big] ds$$

and hence

$$\Big|\int_0^t (q_\theta, \bar{a}_{itt}^j w_{\theta,j}^i)_{L^2(\Omega_0)} ds\Big| + \Big|\int_0^t (q_\theta, \bar{a}_{it}^j w_{\theta t,j}^i)_{L^2(\Omega_0)} ds\Big|$$

$$\leq C(M,\delta)\int_0^t \|q_\theta\|^2_{L^2(\Omega_0)} ds + C(\epsilon)\int_0^t \|\nabla w_\theta\|^2_{L^2(\Omega_0)} ds + \delta\int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} ds$$

$$\leq C(\epsilon,\delta) N_1(u_0,F) + C(M,\delta)\int_0^t \|w_{\theta t}\|^2_{L^2(\Omega_0)} ds + C(\epsilon_1)\int_0^t \|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma_0)} ds$$

$$+ \delta \int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} ds. \tag{8.19}$$

For $(q_\theta, \bar{a}_{it}^j w_{\theta,j}^i)_{L^2(\Omega_0)}(t)$, it follows that

$$\Big|(q_\theta, \bar{a}_{it}^j w_{\theta,j}^i)_{L^2(\Omega_0)}(t)\Big| \leq \delta_1 \|w_{\theta t}\|^2_{L^2(\Omega_0)} + C(\epsilon,\delta_1)\|\nabla w_\theta\|^2_{L^2(\Omega_0)}$$

$$\leq C(\epsilon,\delta_1)\|\nabla w_\theta\|^2_{L^2(\Omega_0)} + \delta_1 C(\epsilon_1)\|\nabla_0^2 h_\theta\|^2_{L^2(\Gamma_0)} + \delta_1\Big[\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|F\|_{L^2(\Omega_0)} + 1\Big]$$

while for $(\tilde{q}_0, \tilde{u}^j_{0,i}\tilde{u}^i_{0,j})_{L^2(\Omega_0)}$, it is bounded by $C(M)N_1(u_0, F)$. Combining (8.18), (8.19) and the estimates above, by choosing $\delta > 0$ and $\delta_1 > 0$ small enough,

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma_0)} + \theta\|q_{\theta t}\|^2_{L^2(\Omega_0)}\Big] ds \\
&\le C(\epsilon_1, \epsilon)\Big[N_2(u_0, F) + \int_0^t \Big(\|w_{\theta t}\|^2_{L^2(\Omega_0)} + (1 + \|\bar{h}_t\|_{H^{2.5}(\Gamma_0)})\|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} \\
&\quad + \|\bar{v}\|^2_{H^3(\Omega_0)} \int_0^s \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} dt'\Big) ds\Big] + C_1(\epsilon_1, \epsilon)\|\nabla w_\theta\|^2_{L^2(\Omega_0)}
\end{aligned}
$$

where $N_2(u_0, F) = N_1(u_0, F) + \|F\|^2_{L^\infty(0,T;L^2(\Omega_0))}$. By the Gronwall inequality,

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma_0)}\Big] ds \\
&\le C(\epsilon_1, \epsilon)N_2(u_0, F) + C_1(\epsilon_1, \epsilon)\|\nabla w_\theta\|^2_{L^2(\Omega_0)}.
\end{aligned} \tag{8.20}
$$

By using $w_\theta(t) = \tilde{u}_0 + \int_0^t w_{\theta t} ds$, we find that

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma_0)}\Big] ds \\
&\le C(\epsilon_1, \epsilon)N_2(u_0, F) + C_1(\epsilon_1, \epsilon)t \int_0^t \|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} ds.
\end{aligned}
$$

Therefore, for any $0 \le t \le t_1 = \min\Big\{T, \dfrac{1}{2C_1}\Big\}$, we have

$$
\begin{aligned}
&\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \frac{1}{2}\int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa\|w_{\theta t}\|^2_{H^2(\Gamma_0)}\Big] ds \\
&\le C(\epsilon_1, \epsilon)N_2(u_0, F).
\end{aligned}
$$

By $w_\theta(t_1) = \tilde{u}_0 + \int_0^{t_1} w_{\theta t} ds$, we also have

$$
\|\nabla w_\theta(t_1)\|^2_{L^2(\Omega_0)} \le C(\epsilon_1, \epsilon)N_2(u_0, F). \tag{8.21}
$$

For $t \geq t_1$, since $w_\theta(t) = w_\theta(t_1) + \int_{t_1}^t w_{\theta t} ds$, we have from (8.20) and (8.21) that

$$\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa \|w_{\theta t}\|^2_{H^2(\Gamma_0)}\Big] ds$$

$$\leq C(\epsilon_1, \epsilon) N_2(u_0, F) + C_1(\epsilon_1, \epsilon)\Big[\|w_\theta(t_1)\|^2_{L^2(\Omega_0)} + (t - t_1) \int_{t_1}^t \|\nabla_0 w_{\theta t}\|^2_{L^2(\Omega_0)} ds\Big]$$

$$\leq C(\epsilon_1, \epsilon) N_2(u_0, F) + C_1(\epsilon_1, \epsilon)(t - t_1) \int_{t_1}^t \|\nabla_0 w_{\theta t}\|^2_{L^2(\Omega_0)} ds\Big].$$

Therefore, for any $t_1 \leq t \leq 2t_1$, we also have

$$\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \frac{1}{2} \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa \|w_{\theta t}\|^2_{H^2(\Gamma_0)}\Big] ds$$

$$\leq C(\epsilon_1, \epsilon) N_2(u_0, F)$$

which with $w_\theta(2t_1) = \tilde{u}_0 + \int_0^{2t_1} w_{\theta t} ds$ gives

$$\|\nabla w_\theta(2t_1)\|^2_{L^2(\Omega_0)} \leq C(\epsilon_1, \epsilon) N_2(u_0, F).$$

By induction, for any $t \in [0, T]$,

$$\|w_{\theta t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\theta t}\|^2_{L^2(\Gamma_0)} + \frac{1}{2} \int_0^t \Big[\|\nabla w_{\theta t}\|^2_{L^2(\Omega_0)} + \kappa \|w_{\theta t}\|^2_{H^2(\Gamma_0)}\Big] ds$$

$$\leq C(\epsilon_1, \epsilon) N_2(u_0, F). \tag{8.22}$$

We also get a θ-independent bound for $\|q_\theta\|^2_{L^2(0,T;L^2(\Omega_0))}$ by (8.17):

$$\|q_\theta\|^2_{L^2(0,T;L^2(\Omega_0))} \leq C(\epsilon_1, \epsilon) N_2(u_0, F). \tag{8.23}$$

Let $\theta = \frac{1}{m}$. Energy inequalities (8.16), (8.22) and (8.23) show that there exists a subsequence $w_{\frac{1}{m_\ell}}$ such that

$$w_{\frac{1}{m_\ell}} \rightharpoonup \mathfrak{v} \quad \text{in} \quad L^2(0, T; H^{1;2}(\Omega_0; \Gamma_0)) \tag{8.24a}$$

$$w_{\frac{1}{m_\ell}t} \rightharpoonup \mathfrak{v}_t \quad \text{in} \quad L^2(0, T; H^{1;2}(\Omega_0; \Gamma_0)) \tag{8.24b}$$

$$\nabla_0^2 h_{\frac{1}{m_\ell}} \rightharpoonup \nabla_0^2 \mathfrak{h} \quad \text{in} \quad L^2(0, T; L^2(\Omega_0)) \tag{8.24c}$$

$$\nabla_0^2 h_{\frac{1}{m_\ell}t} \rightharpoonup \nabla_0^2 \mathfrak{h}_t \quad \text{in} \quad L^2(0, T; L^2(\Omega_0)) \tag{8.24d}$$

$$q_{\frac{1}{m_\ell}} \rightharpoonup \mathfrak{q} \quad \text{in} \quad L^2(0, T; L^2(\Omega_0))\,. \tag{8.24e}$$

Moreover, (8.16) also shows that $\|\bar a_i^j w^i_{\frac{1}{m},j}\|_{L^2(0,T;L^2(\Omega_0))} \to 0$ as $m \to \infty$. Therefore the weak limit $\mathfrak{v}$ satisfies the "divergence-free" condition (8.2b), i.e.,

$$\mathfrak{v} \in \mathcal{V}_{\bar v}(T). \tag{8.25}$$

Since (8.16) is independent of θ and ϵ_1, by the property of lower-semicontinuity of norms,

$$\sup_{0\le t\le T}\Big[\|\mathfrak{v}(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2\mathfrak{h}(t)\|^2_{L^2(\Gamma_0)}\Big] + \|\nabla\mathfrak{v}\|^2_{L^2(0,T;L^2(\Omega_0))} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma_0)}$$
$$\le C(M)N_1(u_0,F). \tag{8.26}$$

By (8.24) and ϵ_1-regularization, the weak limit $(\mathfrak{v},\mathfrak{h},\mathfrak{q})$ satisfies

$$\int_0^T(\mathfrak{v}_t,\varphi)_{L^2(\Omega_0)}dt + \frac{\nu}{2}\int_0^T\int_{\Omega_0} D_{\bar\eta}(\mathfrak{v}) : D_{\bar\eta}(\varphi)dxdt + \kappa\int_0^T\int_{\Gamma_0}\Delta_0\mathfrak{v}\cdot\Delta_0\varphi dSdt$$
$$-\int_0^T(\bar a_i^j\mathfrak{q},\varphi^i_{,j})_{L^2(\Omega_0)}dt + \sigma\int_0^T\int_{\Gamma_0}\bar A^{\alpha\beta\gamma\delta}\mathfrak{h}^{\epsilon_1}_{,\alpha\beta}[-\bar h_{,\sigma}(\varphi^\sigma\circ\bar\eta^{-\tau}) + \varphi^z\circ\bar\eta^{-\tau}]^{\epsilon_1}_{,\gamma\delta}dSdt$$
$$= \int_0^T\Big\{\langle\tilde F,\varphi\rangle - \sigma\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma\delta}\bar h_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_1}\Big[-\bar h_{,\sigma}\varphi^\sigma\circ\bar\eta^{-\tau} + \varphi^z\circ\bar\eta^{-\tau}\Big]^{\epsilon_1}dS\Big\}dt$$

for all $\varphi \in L^2(0,T;H^{1;2}(\Omega_0;\Gamma_0))$. By the density argument, we find that for a.a. $t\in[0,T]$, $\varphi\in H^{1;2}(\Omega_0;\Gamma_0)$,

$$(\mathfrak{v}_t,\varphi)_{L^2(\Omega_0)} + \frac{\nu}{2}\int_{\Omega_0}D_{\bar\eta}(\mathfrak{v}) : D_{\bar\eta}(\varphi)dx + \kappa\int_{\Gamma_0}\Delta_0\mathfrak{v}\cdot\Delta_0\varphi dS - (\bar a_i^j\mathfrak{q},\varphi^i_{,j})_{L^2(\Omega_0)}$$
$$+\sigma\int_{\Gamma_0}\bar A^{\alpha\beta\gamma\delta}\mathfrak{h}^{\epsilon_1}_{,\alpha\beta}[-\bar h_{,\sigma}(\varphi^\sigma\circ\bar\eta^{-\tau}) + \varphi^z\circ\bar\eta^{-\tau}]^{\epsilon_1}_{,\gamma\delta}dS \tag{8.27}$$
$$= \langle\tilde F,\varphi\rangle - \sigma\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma\delta}\bar h_{,\alpha\beta\gamma} + L_2\Big]^{\epsilon_1}\Big[-\bar h_{,\sigma}\varphi^\sigma\circ\bar\eta^{-\tau} + \varphi^z\circ\bar\eta^{-\tau}\Big]^{\epsilon_1}dS\,,$$

or after a change of variable $y' = \bar\eta^\tau(y,t)$,

$$(\mathfrak{v}_t,\varphi)_{L^2(\Omega_0)} + \frac{\nu}{2}(D_{\bar\eta}\mathfrak{v},D_{\bar\eta}\varphi)_{L^2(\Omega_0)} + \kappa\int_{\Gamma_0}\Delta_0\mathfrak{v}\cdot\Delta_0\varphi dS - (\bar a_i^j\mathfrak{q},\varphi^i_{,j})_{L^2(\Omega_0)}$$
$$+\sigma\int_{\Gamma_0}\mathcal{L}^{\epsilon_1}_{\bar h}(\mathfrak{h})(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\varphi dS \tag{8.28}$$
$$= \langle\tilde F,\varphi\rangle - \sigma\int_{\Gamma_0}\bar{\mathcal{M}}^{\epsilon_1}_{\bar h}(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\varphi dS\,.$$

Furthermore, if $\varphi \in \mathcal{V}_{\bar{v}}$, then

$$(\mathfrak{v}_t, \varphi)_{L^2(\Omega_0)} + \frac{\nu}{2}(D_{\bar{\eta}}\mathfrak{v}, D_{\bar{\eta}}\varphi)_{L^2(\Omega_0)} + \kappa \int_{\Gamma_0} \Delta_0 \mathfrak{v} \cdot \Delta_0 \varphi dS$$
$$+ \sigma \int_{\Gamma_0} \mathcal{L}^{\epsilon_1}_{\bar{h}}(\mathfrak{h})(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \varphi dS = \langle \tilde{F}, \varphi \rangle - \sigma \int_{\Gamma_0} \bar{\mathcal{M}}^{\epsilon_1}_{\bar{h}}(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \varphi^{\epsilon_1} dS$$

for a.a. $t \in [0,T]$. In other words, $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ is a weak solution of (8.2).

9. Estimates independent of ϵ_1

9.1. **Partition of unity.** Since Ω_0 is compact, by partition of unity, we can choose two non-negative smooth functions ζ_0 and ζ_1 so that

$$\zeta_0 + \zeta_1 = 1 \quad \text{in} \quad \Omega_0 \ ;$$
$$\operatorname{supp}(\zeta_0) \subset\subset \Omega_0 \ ;$$
$$\operatorname{supp}(\zeta_1) \subset\subset \Gamma_0 \times (-\epsilon, \epsilon) := \Omega_1.$$

We will assume that $\zeta_1 = 1$ inside the region $\Omega_1' \subset \Omega_1$ and $\zeta_0 = 1$ inside the region $\Omega_0' \subset \Omega_0$. Note that then $\zeta_1 = 1$ while $\zeta_0 = 0$ on Γ_0.

9.2. **Higher regularity.**

9.2.1. *ϵ_1-independent bounds for $\mathfrak{q}$.* Similar to (8.17), we have

$$\|\mathfrak{q}\|^2_{L^2(\Omega_0)} \leq C(M)\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\nabla \mathfrak{v}\|^2_{L^2(\Omega_0)} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma_0)} + \|\nabla_0^2 \mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)}$$
$$+ \|F\|^2_{L^2(\Omega_0)} + 1\Big]. \tag{9.1}$$

9.2.2. *Interior regularity.* Converting the fluid equation (8.2) into Eulerian variables by composing with $\bar{\eta}^{-1}$, we obtain a Stokes problem in the domain $\bar{\eta}(\Omega_0)$:

$$-\nu \Delta \mathfrak{u} + \nabla \mathfrak{p} = \tilde{F} \circ \bar{\eta}^{-1} - \mathfrak{v}_t \circ \bar{\eta}^{-1} + \nu \bar{a}^j_{\ell,j} \circ \bar{\eta}^{-1} \mathfrak{u}_{,\ell} - \mathfrak{p} \bar{a}^j_{i,j} \circ \bar{\eta}^{-1}\,, \tag{9.2a}$$

$$\operatorname{div} \mathfrak{u} = 0\,, \tag{9.2b}$$

where $\mathfrak{u} = \mathfrak{v} \circ \bar{\eta}^{-1}$ and $\mathfrak{p} = \mathfrak{q} \circ \bar{\eta}^{-1}$. By the regularity results for the Stokes problem,

$$\begin{aligned} &\|\mathfrak{u}\|^2_{H^2(\bar{\eta}(\Omega_0))} + \|\mathfrak{p}\|^2_{H^1(\bar{\eta}(\Omega_0))} \\ &\le C\Big[\|\tilde{F} \circ \bar{\eta}^{-1}\|^2_{L^2(\bar{\eta}(\Omega_0))} + \|\mathfrak{v}_t \circ \bar{\eta}^{-1}\|^2_{L^2(\bar{\eta}(\Omega_0))} + \|\nabla \mathfrak{u}\|^2_{L^2(\bar{\eta}(\Omega_0))} + \|\mathfrak{p}\|^2_{L^2(\bar{\eta}(\Omega_0))} \\ &\quad + \|\mathfrak{u}\|^2_{H^{1.5}(\Gamma_0)}\Big] \end{aligned}$$

or

$$\begin{aligned} \|\mathfrak{v}\|^2_{H^2(\Omega_0)} + \|\mathfrak{q}\|^2_{H^1(\Omega_0)} \le C\Big[\|F\|^2_{L^2(\Omega_0)} + \|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\mathfrak{v}\|^2_{H^{1.5}(\Gamma_0)}\Big] \\ + C(M)\Big[\|\nabla \mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\mathfrak{q}\|^2_{L^2(\Omega_0)}\Big] \end{aligned}$$

for some constant C independent of M, ϵ. By (9.1),

$$\begin{aligned} \|\mathfrak{v}\|^2_{H^2(\Omega_0)} + \|\mathfrak{q}\|^2_{H^1(\Omega_0)} \le C(M)\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\nabla \mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\mathfrak{v}\|^2_{H^2(\Gamma_0)} \\ + \|\nabla_0^2 \mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} + \|F\|^2_{L^2(\Omega_0)} + 1\Big] \end{aligned} \tag{9.3}$$

Similarly,

$$\begin{aligned} \|\mathfrak{v}\|^2_{H^3(\Omega_0)} + \|\mathfrak{q}\|^2_{H^2(\Omega_0)} \le C\Big[\|F\|^2_{H^1(\Omega_0)} + \|\mathfrak{v}_t\|^2_{H^1(\Omega_0)} + \|\mathfrak{v}\|^2_{H^{2.5}(\Gamma_0)}\Big] \\ + C(M)\Big[\|\nabla \mathfrak{v}\|^2_{H^1(\Omega_0)} + \|\mathfrak{q}\|^2_{H^1(\Omega_0)}\Big] \end{aligned}$$

and therefore by (9.1) and (9.3),

$$\begin{aligned} \|\mathfrak{v}\|^2_{H^3(\Omega_0)} + \|\mathfrak{q}\|^2_{H^2(\Omega_0)} \le C(M)\Big[\|\mathfrak{v}\|^2_{H^1(\Omega_0)} + \|\nabla \mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 \mathfrak{v}\|^2_{H^1(\Omega_1)} \\ + \|\nabla_0^2 \mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} + \|F\|^2_{H^1(\Omega_0)} + 1\Big]. \end{aligned} \tag{9.4}$$

For the regularized problem, because the ϵ-regularization ensures that the forcing and the initial data are smooth, while the ϵ_1-regularization ensures that the right-hand side of (8.2c) is smooth, by standard difference quotient technique, it is also easy to see that

$$\nabla_0^k \mathfrak{v} \in L^2(0,T;H^1(\Omega_1) \cap H^2(\Gamma_0)) \quad \text{for } k = 1,2,3,4 \tag{9.5}$$

Since (8.24b) implies that $\mathfrak{v}_t \in L^2(0,T;H^1(\Omega_0))$, by ϵ_1-regularization and (9.4) we conclude that

$$\mathfrak{v} \in L^2(0,T;H^3(\Omega_0)), \quad \mathfrak{q} \in L^2(0,T;H^2(\Omega_0)). \tag{9.6}$$

9.3. **Estimates for $\mathfrak{v}_t(0)$ and $\mathfrak{q}(0)$.** By (9.6) and ϵ_1-regularization, $(\mathfrak{v},\mathfrak{h},\mathfrak{q})$ satisfies the strong from (8.2). Taking the "divergence" of (8.2a) and then making use of condition (8.2b), we find that

$$-\bar{a}^k_{it}\mathfrak{v}^i_{,k} - \nu\bar{a}^k_i[\bar{a}^j_\ell D_{\bar{\eta}}(\mathfrak{v})^i_\ell]_{,jk} = -\bar{a}^k_i(\bar{a}^j_i\mathfrak{q})_{,jk} + \bar{a}^k_i\tilde{F}^i_{,k}. \tag{9.7}$$

Let $t=0$, by the identity $\bar{a}^\ell_{kt} = -\bar{a}^i_k\bar{v}^j_{,i}\bar{a}^\ell_j$,

$$\Delta\mathfrak{q}(0) = \nabla\tilde{u}_0 : (\nabla\tilde{u}_0)^T - \operatorname{div}(\tilde{F}(0)) \qquad \text{in} \quad \Omega_0$$

with

$$\mathfrak{q}(0) = \nu(\operatorname{Def}\tilde{u}_0)^j_i N_i N_j - \sigma\mathcal{M}^{\epsilon_1}_0(0) + \kappa\Delta^2_0\tilde{u}_0 \qquad \text{on} \quad \Gamma_0$$

while (8.2a) gives us

$$\mathfrak{v}_t(0) = \nu\Delta\tilde{u}_0 - \nabla\mathfrak{q}(0) + \tilde{F}(0) \qquad \text{in} \quad \Omega_0.$$

By standard elliptic regularity result,

$$\|\mathfrak{v}_t(0)\|^2_{L^2(\Omega_0)} + \|\mathfrak{q}(0)\|^2_{H^1(\Omega_0)} \le CN_0(u_0,F) \tag{9.8}$$

for some constant independent of M, ϵ and ϵ_1.

9.4. **$L^2_tL^2_x$-estimates for $\mathfrak{v}_t$.** Since $\mathfrak{v}_t \in L^2(0,T;H^1(\Omega_0))$, we can use it as a test function in (8.28). By (8.25), we find that

$$\begin{aligned}
&\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \frac{\nu}{4}\frac{d}{dt}\int_{\Omega_0}|D_{\bar{\eta}}\mathfrak{v}|^2dx - \frac{\nu}{2}\int_{\Omega_0}(D_{\bar{\eta}}\mathfrak{v})^j_i\bar{a}^k_{jt}\mathfrak{v}^i_{,k}dx + \kappa\int_{\Gamma_0}\Delta_0\mathfrak{v}\cdot\Delta_0\varphi dS \\
&+ \int_{\Omega_0}\mathfrak{q}\bar{a}^\ell_{kt}\mathfrak{v}^k_{,\ell}dx + \sigma\int_{\Gamma_0}\mathcal{L}^{\epsilon_1}_{\bar{h}}(\mathfrak{h})(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\cdot\mathfrak{v}_t dS \\
&= \langle\tilde{F},\mathfrak{v}_t\rangle - \sigma\int_{\Gamma_0}\mathcal{M}^{\epsilon_1}_{\bar{h}}(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\cdot\mathfrak{v}_t dS\,.
\end{aligned}$$

By (6.3),

$$\int_{\Omega_0} (D_{\bar\eta}\mathfrak{v})^j_i \bar a^k_{jt} \mathfrak{v}^i_{,k} dx \le C(M)C(\delta)\|\nabla \mathfrak{v}\|^2_{L^2(\Omega_0)} + \delta \|\mathfrak{v}\|^2_{H^2(\Omega_0)}$$

and

$$\Big|\int_{\Omega_0} \mathfrak{q}\bar a^\ell_{kt}\mathfrak{v}^k_{,\ell} dx\Big| \le C(M)C(\delta)\Big[\|\nabla\mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 \mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} + \|F\|^2_{L^2(\Omega_0)} + 1\Big] \\ + \delta\|\mathfrak{v}\|^2_{H^2(\Omega_0)} + \frac{1}{2}\|\mathfrak{v}_t\|^2_{L^2(\Omega)}$$

for some $C(\delta)$, where we use (9.1) and the interpolation inequality (7.6) (for $n = 3$) and (7.9) (for $n = 2$). These integrals on the boundary (with σ in front) are bounded by

$$C(M)\Big[\|\nabla_0^4\mathfrak{h}^{\epsilon_1}\|_{L^2(\Gamma_0)} + 1\Big]\|\mathfrak{v}_t\|_{H^1(\Omega_0)} \\ \le C(M)C(\delta_1)\Big[\|\nabla_0^4\mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} + 1\Big] + \delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega_0)}.$$

Combining all the estimates above,

$$\frac{1}{2}\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \frac{\nu}{4}\frac{d}{dt}\int_{\Omega_0}|D_{\bar\eta}\mathfrak{v}|^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int_{\Gamma_0}|\Delta_0\mathfrak{v}|^2 dS \\ \le C\Big[\|\nabla\mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\nabla_0^4\mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} + \|F\|^2_{L^2(\Omega_0)} + 1\Big] + \delta\|\mathfrak{v}\|^2_{H^2(\Omega_0)} + \delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega_0)}$$

for some constant C depending on M, δ and δ_1. Therefore by (8.26),

$$\text{(9.9)} \quad \int_0^t \|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} ds + \|\nabla\mathfrak{v}(t)\|^2_{L^2(\Omega_0)} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma_0)} \\ \le C\Big[N_2(u_0, F) + \int_0^t \|\nabla_0^4\mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} ds\Big] + \delta\int_0^t \|\mathfrak{v}\|^2_{H^2(\Omega_0)} ds + \delta_1\int_0^t \|\mathfrak{v}_t\|^2_{H^1(\Omega_0)} ds.$$

REMARK 18. *To obtain (9.9), we avoid using $v_t \in H^2(\Gamma_0)$ for the integrals on the boundary because this kind of terms can only be controlled by the artificial viscosity and will produce κ-dependent estimates. We will also avoid this kind of estimates in the following discussion in order to get κ-independent estimates.*

9.5. **Energy estimates for $\nabla_0^2 v$ near the boundary.** Because of (9.5), $\nabla_0^2(\zeta_1^2\nabla_0^2\mathfrak{v})$ in (8.27) can be used as a test function in (8.28). It follows that

$$\begin{aligned}&\Big|\int_{\Gamma_0}\Big[\bar{\mathcal{L}}_{\bar h}^{\epsilon_1}(\mathfrak{h}^{\epsilon_1})+\bar{\mathcal{M}}_{\bar h}^{\epsilon_1}\Big](-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^4\mathfrak{v}dS\Big|\\&\le C(M)\Big[\|\nabla_0^2\mathfrak{h}^{\epsilon_1}\|_{H^2(\Gamma_0)}+1\Big]\|\mathfrak{v}\|_{H^4(\Gamma_0)}\\&\le C(M,\delta_3)\Big[1+\|\mathfrak{h}\|^2_{H^4(\Gamma_0)}\Big]+\delta_3\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}.\end{aligned}$$

By (8.4), we find that

$$\|\mathfrak{h}\|^2_{H^4(\Gamma_0)}\le C(\epsilon)\Big[\int_0^t\|\bar h\|_{H^5(\Gamma_0)}\|\mathfrak{v}\|_{H^4(\Gamma_0)}ds\Big]^2\le C(\epsilon)\int_0^t\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}ds$$

and hence

$$\begin{aligned}&\Big|\int_{\Gamma_0}\Big[\bar{\mathcal{L}}_{\bar h}^{\epsilon_1}(\mathfrak{h}^{\epsilon_1})+\bar{\mathcal{M}}_{\bar h}^{\epsilon_1}\Big](-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot\nabla_0^4\mathfrak{v}dS\Big|\\&\le\bar C\Big[1+\int_0^t\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}\Big]+\delta_3\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}.\end{aligned}$$

for some constant $\bar C$ depending on M, ϵ and δ_3. Since

$$\Delta_0 f=\frac{1}{\sqrt{\det(g_0)}}\frac{\partial}{\partial y^\alpha}\Big[\sqrt{\det(g_0)}g_0^{\alpha\beta}\frac{\partial}{\partial y^\beta}f\Big],$$

by the regularity on Γ_0 (and hence on g_0),

$$\begin{aligned}\int_{\Gamma_0}|\Delta_0\nabla_0^2\mathfrak{v}|^2dS&\le\int_{\Gamma_0}\Delta_0^2\mathfrak{v}\cdot(\nabla_0^4 v)dS+C\|\mathfrak{v}\|_{H^3(\Gamma_0)}\|\mathfrak{v}\|_{H^4(\Gamma_0)}\\&\le\int_{\Gamma_0}\Delta_0^2 v\cdot(\nabla_0^4 v)dS+C(\delta)\|\mathfrak{v}\|^2_{H^1(\Omega_0)}+\delta\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}\end{aligned}$$

which implies, by choosing $\delta>0$ small enough, that

$$\nu_2\|v\|^2_{H^4(\Gamma_0)}\le\int_{\Gamma_0}\Delta_0^2 v\cdot(\nabla_0^4 v)dS+C\|v\|^2_{H^1(\Omega_0)}.$$

By the identity

$$
\begin{aligned}
&(\mathfrak{q}, \bar{a}_k^\ell(\nabla_0^2(\zeta_1^2\nabla_0^2\mathfrak{v}^k)_{,\ell})\\
&= (\mathfrak{q}, \nabla_0^2\bar{a}_k^\ell(\zeta_1^2\nabla_0\mathfrak{v}^k)_{,\ell}) + 4(\zeta_1\nabla_0\mathfrak{q}, \nabla_0'\bar{a}_k^\ell\zeta_{1,\ell}\nabla_0^2\mathfrak{v}^k) + 2(\mathfrak{q}, \zeta_1^2\nabla_0\bar{a}_k^\ell\nabla_0^2\mathfrak{v}^k_{,\ell})\\
&\quad - 2(\zeta_1\nabla_0\mathfrak{q}, \nabla_0(\bar{a}_k^\ell\zeta_{1,\ell}\nabla_0^2\mathfrak{v}^k) + 2(\mathfrak{q}, \nabla_0(\bar{a}_k^\ell\zeta_{1,\ell}\nabla_0\zeta_1\nabla_0^2\mathfrak{v}^k))\\
&\quad + (\nabla_0\mathfrak{q}, \nabla_0(\zeta_1^2\nabla_0\bar{a}_k^\ell\nabla_0\mathfrak{v}^k_{,\ell})),
\end{aligned}
\tag{9.10}
$$

(6.3) and (9.3) imply that

$$
\begin{aligned}
&(\mathfrak{q}, \bar{a}_k^\ell(\nabla_0'^2(\zeta_1^2\nabla_0^2\mathfrak{v}^k)_{,\ell}) \le C(M)\|\mathfrak{q}\|_{H^1(\Omega_0)}\|\mathfrak{v}\|_{H^3(\Omega_0)}\\
&\le C(M)C(\delta)\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\nabla\mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\nabla\nabla_0\mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma_0)}\\
&\qquad + \|\nabla_0^2\mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)} + \|F\|^2_{L^2(\Omega_0)} + 1\Big] + \delta\|\mathfrak{v}\|^2_{H^3(\Omega_0)}.
\end{aligned}
$$

For the viscosity term,

$$
\begin{aligned}
&\int_{\Omega_0} D_{\bar\eta}\mathfrak{v} : D_{\bar\eta}(\nabla_0^2(\zeta_1^2\nabla_0^2\mathfrak{v}))dx\\
&= \|\zeta_1 D_{\bar\eta}\nabla_0^2\mathfrak{v}\|^2_{L^2(\Omega_0)} + \frac{1}{2}\int_{\Omega_0}\Big[\nabla_0^2(\bar{a}_i^k\bar{a}_i^\ell)\mathfrak{v}^j_{,\ell} + \nabla_0^2(\bar{a}_i^k\bar{a}_j^\ell)\mathfrak{v}^i_{,\ell}\Big](\zeta_1^2\nabla_0^2\mathfrak{v}^j)_{,k}dx\\
&\quad + \int_{\Omega_0}\Big[\nabla_0(\bar{a}_i^k\bar{a}_i^\ell)\nabla_0\mathfrak{v}^j_{,\ell} + \nabla_0(\bar{a}_i^k\bar{a}_j^\ell)\nabla_0\mathfrak{v}^i_{,\ell}\Big](\zeta_1^2\nabla_0^2\mathfrak{v}^j)_{,k}dx\\
&\quad + \int_{\Omega_0} D_{\bar\eta}(\nabla_0^2\mathfrak{v})_i^j\bar{a}_i^k\zeta_1\zeta_{1,k}\nabla_0^2\mathfrak{v}^j dx
\end{aligned}
$$

and hence by (7.6) (or (7.9) if $n = 2$),

$$
\begin{aligned}
\frac{1}{2}\|\zeta_1 D_{\bar\eta}\nabla_0'^2\mathfrak{v}\|^2_{L^2(\Omega_0)} &\le \int_{\Omega_0} D_{\bar\eta}\mathfrak{v} : D_{\bar\eta}(\nabla_0^2(\zeta_1^2\nabla_0^2\mathfrak{v}))dx\\
&\quad + C(M)C(\delta)\Big[\|\nabla\mathfrak{v}\|^2_{L^2(\Omega_0)} + \|\nabla\nabla_0\mathfrak{v}\|^2_{L^2(\Omega_1')}\Big] + \delta\|\mathfrak{v}\|^2_{H^3(\Omega_0)}.
\end{aligned}
$$

Summing all the estimates, by letting $\delta_3 = \frac{\nu_2\kappa}{2}$, we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\zeta_1\nabla_0^2\mathfrak{v}\|^2_{L^2(\Omega_0)} + \frac{\nu}{4}\|\zeta_1 D_{\bar\eta}\nabla_0^2\mathfrak{v}\|^2_{L^2(\Omega_0)} + \frac{\nu_2\kappa}{2}\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}\\
&\le \bar{C}\Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\mathfrak{v}\|^2_{H^1(\Omega_0)} + \|\nabla\nabla_0\mathfrak{v}\|^2_{L^2(\Omega_1')} + \|\mathfrak{v}\|^2_{H^2(\Gamma_0)} + \|\nabla_0^2\mathfrak{h}^{\epsilon_1}\|^2_{L^2(\Gamma_0)}\\
&\qquad + \|F\|^2_{H^1(\Omega_0)} + 1\Big] + \bar{C}\int_0^t\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}ds + \delta\|\mathfrak{v}\|^2_{H^3(\Omega_0)}
\end{aligned}
$$

for some constant $\bar{C}$ depending on M, κ, ϵ and δ. Integrating the inequality above in time from 0 to t, by (8.26) we find that

$$\begin{aligned} &\|\nabla_0^2\mathfrak{v}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \Big[\|\nabla\nabla_0^2\mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}\Big]ds \\ &\le \bar{C}N_2(u_0,F) + \bar{C}\int_0^t \Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\nabla\nabla_0\mathfrak{v}\|^2_{L^2(\Omega_1')} + \|\mathfrak{v}\|^2_{H^2(\Gamma_0)}\Big]ds \\ &+ \bar{C}\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)}dr + \delta\int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega_0)}ds. \end{aligned} \tag{9.11}$$

By using $\nabla_0(\zeta_1^2\nabla_0\mathfrak{v})$ as a testing function in (8.28), similar computations leads to

$$\begin{aligned} &\|\nabla_0\mathfrak{v}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \Big[\|\nabla\nabla_0\mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa\|\mathfrak{v}\|^2_{H^3(\Gamma_0)}\Big]ds \\ &\le C(M)N_2(u_0,F) + C(M,\delta)\int_0^t \Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \kappa\|\mathfrak{v}\|^2_{H^2(\Gamma_0)}\Big]ds \\ &+ C(M)\int_0^t\int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)}drds + \delta\int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega_0)}ds. \end{aligned} \tag{9.12}$$

9.6. **Energy estimates for v_t - $L^2_tH^1_x$-estimates.** In this section, we time differentiate (8.28) and then use $\mathfrak{v}_t$ as a test function to obtain

$$\begin{aligned} &\langle\mathfrak{v}_{tt},\mathfrak{v}_t\rangle + \nu\int_{\Omega_0}\Big[\bar{a}_\ell^k(D_{\bar{\eta}}\mathfrak{v})^i_{\ell,k}\Big]_t\mathfrak{v}^i_t dx + \sigma\int_{\Gamma_0}\Big[\bar{\mathcal{L}}^{\epsilon_1}_{\bar{h}}(\mathfrak{h}^{\epsilon_1})(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS \\ &+\kappa\int_{\Gamma_0}|\Delta_0\mathfrak{v}_t|^2dS - \int_{\Omega_0}(\bar{a}^\ell_k\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dx = \langle F_t,\mathfrak{v}_t\rangle - \sigma\int_{\Gamma_0}\Big[\bar{\mathcal{M}}^{\epsilon_1}_{\bar{h}}(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS. \end{aligned}$$

By the chain rule,

$$\begin{aligned} &\int_{\Gamma_0}\Big[(\bar{\mathcal{L}}^{\epsilon_1}_{\bar{h}}(\mathfrak{h}^{\epsilon_1}) + \bar{\mathcal{M}}^{\epsilon_1}_{\bar{h}})(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS \\ &= \int_{\Gamma_0}\bar{\Theta}_t\Big[L_{\bar{h}}(\mathfrak{h}^{\epsilon_1})\Big]^{\epsilon_1}\circ\bar{\eta}^\tau(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\cdot\mathfrak{v}_t dS \\ &+\int_{\Gamma_0}\bar{\Theta}\bar{v}^\tau\cdot\Big[\nabla_0[L_{\bar{h}}(\mathfrak{h}^{\epsilon_1})]^{\epsilon_1}(-\nabla_0\bar{h},1)\Big]\circ\bar{\eta}^\tau\cdot\mathfrak{v}_t dS \\ &+\int_{\Gamma_0}\bar{\Theta}\Big[[L_{\bar{h}}(\mathfrak{h}^{\epsilon_1})]^{\epsilon_1}(\nabla_0\bar{h},-1)\Big]_t\circ\bar{\eta}^\tau\cdot\mathfrak{v}_t dS. \end{aligned}$$

By using $H^2(\Gamma_0)$-$H^{-2}(\Gamma_0)$ duality pairing with ϵ-regularization on $\bar\Theta$ and $\bar v$, it follows that

$$
\begin{aligned}
&\Big|\int_{\Gamma_0}\Big[(\bar{\mathcal{L}}_{\bar h}^{\epsilon_1}(\mathfrak{h}^{\epsilon_1})+\bar{\mathcal{M}}_{\bar h}^{\epsilon_1})(-\nabla_0\bar h\circ\bar\eta^\tau,1)\Big]_t\cdot\mathfrak{v}_t dS\Big|\\
&\le C(\epsilon)\Big[\|\nabla_0^3\mathfrak{h}\|_{L^2(\Gamma_0)}+\|\nabla_0^2\mathfrak{h}_t\|_{L^2(\Gamma_0)}+1\Big]\|\mathfrak{v}_t\|_{H^2(\Gamma_0)}\\
&\le C(\epsilon,\delta_3)\Big[\int_0^t\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}ds+\|\mathfrak{v}\|^2_{H^2(\Gamma_0)}+1\Big]+\delta_3\|\mathfrak{v}_t\|^2_{H^2(\Gamma_0)}\\
&\le \bar C\Big[\int_0^t\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}ds+\|\mathfrak{v}\|^2_{H^1(\Omega_0)}+1\Big]+\delta\|\mathfrak{v}\|^2_{H^3(\Omega_0)}+\delta_3\|\mathfrak{v}_t\|^2_{H^2(\Gamma_0)}
\end{aligned}
$$

for some constant $\bar C$ depending on M, ϵ, δ and δ_3, where we use the interpolation inequality (7.8) to estimate $\|\mathfrak{v}\|^2_{H^2(\Gamma_0)}$.

By (7.6) (or (7.9) is $n=2$),

$$
\begin{aligned}
\int_{\Omega_0}|D_{\bar\eta}\mathfrak{v}_t|^2dx &= 2\int_{\Omega_0}\Big[\bar a_i^k D_{\bar\eta}(\mathfrak{v})_i^j\Big]_t\mathfrak{v}^j_{t,k}dx-2\int_{\Omega_0}\Big[(\bar a_i^k\bar a_i^\ell)_t\mathfrak{v}^j_{,\ell}+(\bar a_i^k\bar a_j^\ell)_t\mathfrak{v}^i_{,\ell}\Big]\mathfrak{v}^j_{t,k}dx\\
&\le 2\int_{\Omega_0}\Big[\bar a_i^k D_{\bar\eta}(\mathfrak{v})_i^j\Big]_t\mathfrak{v}^j_{t,k}dx+C(M)C(\delta,\delta_1)\|\nabla\mathfrak{v}\|^2_{L^2(\Omega_0)}\\
&\qquad-\int_{\Omega_0}(\bar a_k^\ell\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dx+\delta\|\mathfrak{v}\|^2_{H^2(\Omega_0)}+\delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega_0)}.
\end{aligned}
$$

Note that

$$
\langle F_t,\mathfrak{v}_t\rangle\le C\|F_t\|_{H^1(\Omega_0)'}\|\mathfrak{v}_t\|_{H^1(\Omega_0)}\le C(\delta_1)\|F_t\|^2_{H^1(\Omega_0)'}+\delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega_0)}.
$$

Summing all the estimates above,

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)}+\frac{\nu}{4}\|\nabla\mathfrak{v}_t\|^2_{L^2(\Omega_0)}+\kappa\|\Delta_0\mathfrak{v}_t\|^2_{L^2(\Gamma_0)}\\
&\le\bar C\Big[\int_0^t\|\mathfrak{v}\|^2_{H^4(\Gamma_0)}ds+\|\mathfrak{v}\|^2_{H^1(\Omega_0)}+1\Big]+C(\delta_1)\|F_t\|^2_{H^1(\Omega_0)'}\\
&\quad+\delta\|\mathfrak{v}\|^2_{H^3(\Omega_0)}+\delta_1\|\mathfrak{v}_t\|^2_{H^1(\Omega_0)}+\delta_3\|\mathfrak{v}_t\|^2_{H^2(\Gamma_0)}+\int_{\Omega_0}(\bar a_k^\ell\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dx
\end{aligned}
\tag{9.13}
$$

for some constant $\bar C$ depending on M, κ, δ and δ_1. By Appendix C.2,

$$
\int_0^t\int_{\Omega_0}(\bar a_k^\ell\mathfrak{q})_t\mathfrak{v}^k_{t,\ell}dxds\le C(M)C(\delta,\delta_1)N_3(u_0,F)+\delta\int_0^t\|\mathfrak{v}\|^2_{H^3(\Omega_0)}ds+\delta_1\int_0^t\|\mathfrak{v}_t\|^2_{H^1(\Omega_0)}ds
$$

where

$$N_3(u_0,F) := \|u_0\|^2_{H^{2.5}(\Omega_0)} + \|u_0\|^2_{H^{4.5}(\Gamma_0)} + \|F\|^2_{L^2(0,T;H^1(\Omega_0))}$$
$$+ \|F_t\|^2_{L^2(0,T;H^1(\Omega_0)')} + \|F(0)\|^2_{H^1(\Omega_0)} + 1.$$

Integrating (9.13) in time from 0 to t and choosing $\delta_1, \delta_3 > 0$ small enough, (8.26) and (9.9) imply that, for all $t \in [0,T]$,

$$\|\mathfrak{v}_t(t)\|^2_{L^2(\Omega_0)} + \int_0^t \Big[\|\nabla \mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \kappa \|\mathfrak{v}_t\|^2_{H^2(\Gamma_0)}\Big] ds$$
$$\le \bar{C} N_3(u_0,F) + \bar{C} \int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)} dr ds + \delta \int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega_0)} ds \tag{9.14}$$

for some constant $\bar{C}$ depending on M, κ, δ and δ_2. In (9.14), (9.8) is used to bound $\|v_t(0)\|^2_{L^2(\Omega_0)}$.

9.7. **ϵ_1-independent estimates.** Integrating (9.3) in time from 0 to t, (8.26), (9.9) and (9.12) imply that

$$\int_0^t \Big[\|\mathfrak{v}\|^2_{H^2(\Omega_0)} + \|\mathfrak{q}\|^2_{H^1(\Omega_0)}\Big] ds$$
$$\le C(M) N_1(u_0,F) + \int_0^t \Big[\|\mathfrak{v}_t\|^2_{L^2(\Omega_0)} + \|\mathfrak{v}\|^2_{H^2(\Gamma_0)}\Big] ds$$
$$\le \bar{C} N_3(u_0,F) + \bar{C} \int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)} dr ds + \delta \int_0^t \|\mathfrak{v}\|^2_{H^3(\Omega_0)} ds \tag{9.15}$$

for some constant $\bar{C}$ depending on M, κ and δ. Integrating (9.4) in time from 0 to t, making use of (9.11), (9.12), (9.14), (9.15), and then choosing $\delta > 0$ small enough and T even smaller, we find that

$$\int_0^t \Big[\|\mathfrak{v}\|^2_{H^3(\Omega_0)} + \|\mathfrak{q}\|^2_{H^2(\Omega_0)}\Big] ds \le \bar{C} N_3(u_0,F) + \bar{C} \int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)} dr ds \tag{9.16}$$

for some constant $\bar{C}$ depending on M, κ and ϵ.

Having (9.16), by choosing $\delta_2 > 0$ small enough, the estimates (9.11) can be rewritten as

$$\|\nabla_0^2 \mathfrak{v}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \Big[\|\nabla \nabla_0^2 \mathfrak{v}\|^2_{L^2(\Omega_1)} + \kappa \|\mathfrak{v}\|^2_{H^4(\Gamma_0)}\Big] ds$$
$$\le \bar{C} N_3(u_0,F) + \bar{C} \int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)} dr ds \tag{9.17}$$

for some constant $\bar{C}$ depending on M, κ and ϵ. Therefore,

$$X(t) \le \bar{C}\Big[\int_0^t X(s)ds + N_3(u_0, F)\Big]$$

where

$$X(t) = \int_0^t \|\mathfrak{v}\|^2_{H^4(\Gamma_0)} ds.$$

By the Gronwall inequality,

$$\int_0^t \int_0^s \|\mathfrak{v}(r)\|^2_{H^4(\Gamma_0)} dr ds \le \bar{C} N_3(u_0, F) \tag{9.18}$$

for all $t \in [0, T]$ for some constant $\bar{C}$ depending on M, κ, and ϵ. Having (9.18), estimates (9.9), (9.14), (9.16) and (9.17) along with the standard embedding theorem lead to

$$\begin{aligned} &\sup_{0\le t\le T}\Big[\|\mathfrak{v}(t)\|^2_{H^2(\Omega_0)} + \|\mathfrak{v}_t(t)\|^2_{L^2(\Omega_0)}\Big] + \|\mathfrak{v}\|^2_{V^3(T)} + \|\mathfrak{q}\|^2_{L^2(0,T;H^2(\Omega_0))} \\ &+ \kappa\|\mathfrak{v}\|^2_{L^2(0,T;H^4(\Gamma_0))} \le \bar{C} N_3(u_0, F) \end{aligned} \tag{9.19}$$

for some constant $\bar{C}$ depending on M, κ and ϵ.

9.8. **Weak limits as $\epsilon_1 \to 0$.** Since the estimate (9.19) is independent of ϵ_1, the weak limit as $\epsilon_1 \to 0$ of the sequence $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ exists. We will denote the weak limit of $(\mathfrak{v}, \mathfrak{h}, \mathfrak{q})$ by $(v_\kappa, h_\kappa, q_\kappa)$. By lower semi-continuity, (9.8) and thus (9.19) hold for the weak limit $(v_\kappa, h_\kappa, q_\kappa)$. Furthermore,

$$\begin{aligned} &\langle v_{\kappa t}, \varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\bar{\eta}} v_\kappa : D_{\bar{\eta}}\varphi dx + \sigma\int_{\Gamma_0} \bar{\Theta}\Big[[\mathcal{L}_{\bar{h}}(h_\kappa)(-\nabla_0\bar{h}, 1)] \circ \bar{\eta}^\tau\Big] \cdot \varphi dS \\ &+ \kappa\int_{\Gamma_0} \Delta_0 v_\kappa \cdot \Delta_0\varphi dS - (q_\kappa, \bar{a}^\ell_k \varphi^k_{,\ell})_{L^2(\Omega_0)} \\ &= \langle F, \varphi\rangle - \sigma\int_{\Gamma_0} \bar{\Theta}\Big[[\mathcal{M}(\bar{h})(-\nabla_0\bar{h}, 1)] \circ \bar{\eta}^\tau\Big] \cdot \varphi dS \end{aligned} \tag{9.20}$$

for all $\varphi \in H^{1;2}(\Omega_0; \Gamma_0)$ and a.a. $t \in [0, T]$.

10. Estimates independent of κ and ϵ

10.1. **Energy estimates which are independent of κ.** Although (9.19) doesn't imply that $h_\kappa \in H^4(\Gamma_0)$, h_κ is indeed in $H^4(\Gamma_0)$ by (8.4). Therefore, we know that $(v_\kappa, h_\kappa, q_\kappa)$ satisfies

$$v_{\kappa t}^{\ i} - \nu[\bar{a}_\ell^k D_{\bar{\eta}}(v_\kappa)_\ell^i]_{,k} = -(\bar{a}_i^k q_\kappa)_{,k} + \tilde{F}^i \quad \text{in } (0,T)\times\Omega_0\,, \tag{10.1a}$$

$$\bar{a}_i^j v_{\kappa}{}_{,j}^{i} = 0 \quad \text{in } (0,T)\times\Omega_0\,, \tag{10.1b}$$

$$\begin{aligned}[\nu D_{\bar{\eta}}(v_\kappa)_i^j - q_\kappa \delta_i^j]\bar{a}_j^\ell N_\ell &= \sigma\bar{\Theta}[\mathcal{L}_{\bar{h}}(h_\kappa)(-\nabla_0\bar{h},1)]\circ\bar{\eta}^\tau \\ &\quad + \sigma\bar{\Theta}[\mathcal{M}_{\bar{h}}(-\nabla_0\bar{h},1)]\circ\bar{\eta}^\tau + \kappa\Delta_0^2 v_\kappa \quad \text{on } (0,T)\times\Gamma_0\,,\end{aligned} \tag{10.1c}$$

$$h_t\circ\bar{\eta}^\tau = [(\bar{h}_{,\alpha})\circ\bar{\eta}^\tau]v_\alpha - v_z \quad \text{on } (0,T)\times\Gamma_0\,, \tag{10.1d}$$

$$v = \tilde{u}_0 \quad \text{on } \{t=0\}\times\Omega_0, \tag{10.1e}$$

$$h = 0 \quad \text{on } \{t=0\}\times\Gamma_0\,. \tag{10.1f}$$

Having (10.1c), (A.10) in Appendix A implies that h_κ is in $H^5(\Gamma_0)$ for a.a. $t\in[0,T]$ with estimate

$$\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} ds \le C(\epsilon)\int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|v_\kappa\|^2_{H^3(\Omega_0)} + \|q_\kappa\|^2_{H^2(\Omega_0)} + 1\Big] ds,$$

where the forcing f in (A.10) is given by

$$[\nu D_{\bar{\eta}}(v_\kappa)_i^j - q_\kappa\delta_i^j]\bar{a}_j^\ell N_\ell - \sigma\bar{\Theta}[\mathcal{M}_{\bar{h}}(-\nabla_0\bar{h},1)]\circ\bar{\eta}^\tau.$$

By the same argument, (8.17) holds with all θ replaced by κ. Therefore, by (9.4) (which follows from (8.17)),

$$\begin{aligned}\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} ds &\le C(\epsilon)\int_0^t \Big[\|v_{\kappa t}\|^2_{H^1(\Omega_0)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)}\Big] ds \\ &\quad + C(\epsilon)N_2(u_0,F).\end{aligned} \tag{10.2}$$

With this extra regularity of h_κ, the energy estimate (9.19) can be made independent of κ. In Appendix B.2, we prove that

$$\frac{\nu_1}{2}\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \le \int_0^t\int_{\Gamma_0}\bar\Theta\Big[[L_{\bar h}(h_\kappa)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa)dSds$$
$$+C'\int_0^t\Big[1+\|\tilde v\|^2_{H^3(\Omega_0)}+\|\tilde h_t\|^2_{H^{2.5}(\Gamma_0)}+\|\tilde h\|^2_{H^5(\Gamma_0)}\Big]\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds$$
$$+C'\int_0^t\Big[\|\tilde h\|^2_{H^5(\Gamma_0)}+1\Big]ds+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega_0)}ds+\delta_1\int_0^t\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)}ds$$

for some constant C' depending on M, ϵ, δ and δ_1. By (10.2),

$$\frac{\nu_1}{2}\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \le \int_0^t\int_{\Gamma_0}\bar\Theta\Big[[L_{\bar h}(h_\kappa)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa)dSds$$
$$+C'N_2(u_0,F)+C'\int_0^t\Big[\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)}+K(s)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}\Big]ds \tag{10.3}$$
$$+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega_0)}ds+\delta_1\int_0^t\|v_{\kappa t}\|^2_{H^1(\Omega_0)}ds$$

where

$$K(s):=1+\|\tilde v\|^2_{H^3(\Omega_0)}+\|\tilde h_t\|^2_{H^{2.5}(\Gamma_0)}+\|\tilde h\|^2_{H^5(\Gamma_0)}.$$

With (10.3), (9.11) now is replaced by

$$\Big[\|\nabla_0^2 v_\kappa(t)\|^2_{L^2(\Omega_1)}+\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)}\Big]+\int_0^t\Big[\|\nabla\nabla_0^2 v_\kappa\|^2_{L^2(\Omega_1)}+\kappa\|v_\kappa\|^2_{H^4(\Gamma_0)}\Big]ds$$
$$\le C'N_2(u_0,F)+C'\int_0^t\Big[\|v_{\kappa t}\|^2_{L^2(\Omega_0)}+\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)}+K(s)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}\Big]ds$$
$$+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega_0)}ds+\delta_1\int_0^t\|v_{\kappa t}\|^2_{H^1(\Omega_0)}ds \tag{10.4}$$

for some C' depending on M, ϵ, δ and δ_1, where (A.8) is applied to bound $\kappa\|v_\kappa\|^2_{H^3(\Gamma_0)}$ (this is where $\|v_{\kappa t}\|^2_{L^2(\Omega_0)}$ comes from). Similar computations leads to

$$\Big[\|\nabla_0 v_\kappa(t)\|^2_{L^2(\Omega_1)}+\|\nabla_0^3 h_\kappa(t)\|^2_{L^2(\Gamma_0)}\Big]+\int_0^t\Big[\|\nabla\nabla_0 v_\kappa\|^2_{L^2(\Omega_1)}+\kappa\|v_\kappa\|^2_{H^3(\Gamma_0)}\Big]ds$$
$$\le CN_2(u_0,F)+C\int_0^t\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds+\delta\int_0^t\|v_\kappa\|^2_{H^3(\Omega_0)}ds \tag{10.5}$$

for some constant C depending on M and δ.

In Appendix C.1, we establish the following κ- and ϵ-independent inequality for the time-differentiated problem:

$$\int_0^t \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} ds \le \int_0^t \int_{\Gamma_0} \Big[[L_{\bar h}(h_\kappa)(\nabla_0 \bar h, -1)] \circ \bar\eta^\tau \Big]_t \cdot v_{\kappa t} dS$$
$$+ CN_3(u_0, F) + C \int_0^t K(s) \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} \Big] ds$$
$$+ (\delta + Ct^{1/2}) \int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)} ds + (\delta_1 + Ct^{1/2}) \int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)} ds + \delta_2 \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}$$

for some constant C depending on M, δ, δ_1 and δ_2. Therefore, (9.14) can be replaced by the following estimate:

$$\Big[\|v_{\kappa t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} \Big] + \int_0^t \Big[\|\nabla v_{\kappa t}\|^2_{L^2(\Omega_0)} + \kappa \|\Delta_0 v_{\kappa t}\|^2_{L^2(\Gamma_0)} \Big] ds$$
$$\le CN_3(u_0, F) + C \int_0^t K(s) \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} \Big] ds \tag{10.6}$$
$$+ (\delta + Ct^{1/2}) \int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)} ds + (\delta_1 + Ct^{1/2}) \int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)} ds$$
$$+ \delta_2 \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}.$$

10.2. **κ-independent estimates.** Just as in Section 9.7, we find that

$$\int_0^t \Big[\|v_\kappa\|^2_{H^3(\Omega_0)} + \|q_\kappa\|^2_{H^2(\Omega_0)} \Big] ds$$
$$\le C(M) N_2(u_0, F) + C(M) \int_0^t \Big[\|v_{\kappa t}\|^2_{H^1(\Omega_0)} + \|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)} \Big] ds. \tag{10.7}$$

By choosing $\delta = \delta_1 = \delta_2 = 1/8$ and $T > 0$ so that $CT^{1/2} < 1/8$ in (10.6), we find that

$$\int_0^t \Big[\|v_\kappa\|^2_{H^3(\Omega_0)} + \|q_\kappa\|^2_{H^2(\Omega_0)} \Big] ds \le CN_3(u_0, F) + \frac{1}{8} \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}$$
$$+ C(M) \int_0^t \Big[\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1)} + K(s) \Big(\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} \Big) \Big] ds. \tag{10.8}$$

Combining the estimates (8.26), (9.9), (10.4) and (10.5) with (10.6),

$$\Big[\|v_\kappa\|^2_{H^1(\Omega_0)} + \|\nabla_0^2 v_\kappa\|^2_{L^2(\Omega_1)} + \|\nabla_0^2 h_\kappa\|^2_{H^2(\Gamma_0)} + \|v_{\kappa t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)}\Big](t)$$
$$+ \int_0^t \Big[\|\nabla v_\kappa\|^2_{L^2(\Omega_0)} + \|\nabla\nabla_0 v_\kappa\|^2_{L^2(\Omega_1)} + \|\nabla\nabla_0^2 v_\kappa\|^2_{L^2(\Omega_1)} + \|v_{\kappa t}\|^2_{H^1(\Omega_0)}\Big] ds$$
$$\leq C' N_3(u_0, F) + C' \int_0^t \Big[\|v_{\kappa t}\|^2_{L^2(\Omega_0)} + K(s)\Big(\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)}\Big)\Big] ds$$

for some constant C' depending on M and ϵ. By the Gronwall inequality and (9.4),

$$\sup_{0\leq t\leq T} \Big[\|v_\kappa(t)\|^2_{H^2(\Omega_0)} + \|v_{\kappa t}(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\kappa t}(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)}$$
$$+ \|q_\kappa(t)\|^2_{H^1(\Omega_0)}\Big] + \|v_\kappa\|^2_{V^3(T)} + \|q_\kappa\|^2_{L^2(0,T;H^2(\Omega_0))} \leq C(\epsilon) N_3(u_0, F)\,.$$

10.3. **Weak limits as $\kappa \to 0$.** Just as in Section 9.8, the weak limit $(v_\epsilon, h_\epsilon, q_\epsilon)$ of $(v_\kappa, h_\kappa, q_\kappa)$ as $\kappa \to 0$ exists in $V(T)\times L^2(0,T;H^4(\Gamma_0))\times L^2(0,T;H^2(\Omega_0))$ with estimate

$$\sup_{0\leq t\leq T} \Big[\|v_\epsilon(t)\|^2_{H^2(\Omega_0)} + \|v_{\epsilon t}(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\epsilon t}(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^4 h_\epsilon(t)\|^2_{L^2(\Gamma_0)}$$
$$+ \|q_\epsilon(t)\|^2_{H^1(\Omega_0)}\Big] + \|v_\kappa\|^2_{V^3(T)} + \|q_\epsilon\|^2_{L^2(0,T;H^2(\Omega_0))} \leq C(\epsilon) N_3(u_0, F)\,. \tag{10.9}$$

(10.9) implies that for a.a. $t \in [0,T]$,

$$\|v_\kappa(t)\|_{H^{2.5}(\Gamma_0)} \leq \bar{C}(t)$$

for some $\bar{C}(t)$ independent of κ, and therefore for a.a. $t \in [0,T]$,

$$\kappa \int_{\Gamma_0} \Delta_0 v_\kappa \cdot \Delta_0 \varphi dS \to 0$$

as $\kappa \to 0$. This observation with (9.20) shows that $(v_\epsilon, h_\epsilon, q_\epsilon)$ satisfies, for a.a. $t \in [0,T]$ and for all $\varphi \in H^{1;2}(\Omega_0;\Gamma_0)$,

$$(v_{\kappa t}, \varphi)_{L^2(\Omega_0)} + \frac{\nu}{2}\int_{\Omega_0} D_{\bar\eta} v_\kappa : D_{\bar\eta}(\varphi) dx + \sigma \int_{\Gamma_0} \bar\Theta \mathcal{L}_{\bar h}(h_\kappa)(-\nabla_0 \bar h \circ \bar\eta^\tau, 1)\cdot \varphi dS$$
$$- (\bar a_i^j q_\kappa, \varphi^i_{,j})_{L^2(\Omega_0)} = \langle \tilde F, \varphi\rangle + \sigma \langle \bar\Theta \mathcal{M}_{\bar h}(-\nabla_0 \bar h \circ \bar\eta^\tau, 1), \varphi\rangle_{\Gamma_0}\,. \tag{10.10}$$

Since (10.10) also defines a linear functional on $H^1(\Omega_0)$, by the density argument, we know that (10.10) holds for all $\varphi \in H^1(\Omega_0)$. As $(v_\epsilon, h_\epsilon, q_\epsilon)$ are smooth enough, we can integrate by parts and find that $(v_\epsilon, h_\epsilon, q_\epsilon)$ satisfies (8.2) with (8.2c) replaced by

$$[\nu D_{\bar\eta}(v_\epsilon)^j_i - q_\epsilon \delta^j_i]\bar a^\ell_j N_\ell = \sigma\Big[\bar\Theta[(\mathcal{L}_{\bar h}(h_\epsilon) + \mathcal{M}(\bar h))(\nabla_0 \bar h, -1)]\circ \bar\eta^\tau\Big] \text{ on } (0,T)\times\Gamma_0 . \tag{10.11}$$

10.4. $H^{5.5}$**-regularity of** h_κ**.** By (10.11), we have

Lemma 10.1. *For a.a.* $t \in [0,T]$, $h_\epsilon(t) \in H^{5.5}(\Gamma_0)$ *with*

$$\begin{aligned} \|h_\epsilon\|^2_{H^{5.5}(\Gamma_0)} \le C(M)\Big[&\|v_{\epsilon t}\|^2_{H^1(\Omega_0)} + \|\nabla v_\epsilon\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 v_\epsilon\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma_0)} \\ &+ \|F\|^2_{H^1(\Omega_0)} + 1\Big] , \end{aligned} \tag{10.12}$$

and hence

$$\|h_\epsilon\|^2_{L^2(0,T;H^{5.5}(\Gamma_0))} \le C(M) e^{C(M)+T} N_3(u_0, F). \tag{10.13}$$

Proof. We write the boundary condition (10.11) as

$$\mathcal{L}_{\bar h}(h_\epsilon) = \frac{1}{\sigma} J^{-2}_{\bar h}(-\nabla_0 \bar h, 1)\cdot \Big\{\bar\Theta^{-1}\Big[[\nu D_{\bar\eta}(v_\epsilon)^j_i - q_\epsilon\delta^j_i]\bar a^\ell_j N_\ell\Big]\Big\}\circ\bar\eta^{-\tau} - \mathcal{M}(\bar h). \tag{10.14}$$

By Corollary 8.1, $\mathcal{L}_{\bar h}$ is uniformly elliptic with the elliptic constant ν_1 which is independent of M which defines our convex subset $C_T(M)$. Since $\bar h \in H(T)$, $\mathcal{M}(\bar h) \in L^2(0,T;H^{2.5}(\Gamma_0)) \cap L^\infty(0,T;H^1(\Gamma_0))$, and hence by (9.19), the right-hand side of (10.14) is bounded in $H^{1.5}(\Gamma_0)$. The important point is that these bounds are independent of ϵ. Thus, elliptic regularity of $\mathcal{L}_{\bar h}$ proves the estimate

$$\|h_\epsilon\|^2_{H^{5.5}(\Gamma_0)} \le C(M)\Big[\|D_{\bar\eta}(v_\epsilon)\|^2_{H^{1.5}(\Gamma_0)} + \|q_\epsilon\|^2_{H^{1.5}(\Gamma_0)} + 1\Big] ,$$

so that with (9.4), (10.12) is proved. □

10.5. **Energy estimates which are independent of** ϵ**.** Having estimate (10.12), one can follow exactly the same procedure as in Section 10.2 to show that the constant C' appearing in (10.9) is independent of ϵ, provided that we have an ϵ-independent version of (10.4). By Appendix B.2, we indeed have such an estimate:

$$
\begin{aligned}
\frac{\nu_1}{2}\|\nabla_0^4 h_\epsilon(t)\|^2_{L^2(\Gamma_0)} \le & \int_0^t \int_{\Gamma_0} \bar\Theta\Big[[L_{\bar h}(h_\epsilon)(-\nabla_0 \bar h, 1)]\circ \bar\eta^\tau\Big]\cdot \nabla_0^2(\zeta_1^2 \nabla_0^2 v_\epsilon) dS ds \\
& + CN_2(u_0,F) + C\int_0^t K(s)\|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma_0)} ds + (\delta + Ct^{1/2})\int_0^t \|v_\epsilon\|^2_{H^3(\Omega_0)} ds \\
& + (\delta_1 + Ct^{1/2})\int_0^t \|v_{\epsilon t}\|^2_{H^1(\Omega_0)} ds
\end{aligned}
$$

for some constant C depending on M, δ and δ_1. Therefore, we can conclude that

$$
\begin{aligned}
& \sup_{0\le t\le T}\Big[\|v_\epsilon\|^2_{H^2(\Omega_0)} + \|v_{\epsilon t}\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\epsilon t}\|^2_{L^2(\Gamma_0)} + \|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma_0)} \\
& \qquad + \|q_\epsilon\|^2_{H^1(\Omega_0)}\Big](t) + \|v_\epsilon\|^2_{V^3(T)} + \|q_\epsilon\|^2_{L^2(0,T;H^2(\Omega_0))} \\
& \le C(M) e^{C(M)+T} N_3(u_0,F).
\end{aligned}
\tag{10.15}
$$

Remark 19. *Literally speaking, we cannot use $\nabla_0^2(\zeta_1^2\nabla_0^2 v_\epsilon)$ as a test function in (10.10) since it is not a function in $H^1(\Omega_0)$. However, since $h_\epsilon \in H^{5.5}(\Gamma_0)$ for a.a. $t\in[0,T]$, (10.10) also holds for all $\varphi \in H^1(\Omega_0)' \cap H^{-1.5}(\Gamma_0)$ and $\nabla_0^2(\zeta_1^2\nabla_0^2 v_\epsilon)$ is a function of this kind.*

Remark 20. *Having only (10.9), the integral*

$$
\int_{\Gamma_0}\Big[\bar\Theta[L_{\bar h}(h_\epsilon)(\nabla_0\bar h, -1)]\circ\bar\eta^\tau\Big]_t v_{\epsilon t} dS
$$

does not make sense (while it makes perfect sense with κ mollification), so literally we cannot do $L_t^2 H_x^1$ estimate for $v_{\epsilon t}$ as we did in the appendix. However, since (10.8) is independent of κ and ϵ, by the property of lower semicontinuity of norm (on the left-hand side) and the strong convergence of h_κ (on the right-hand side), we also know that (10.8) holds for the weak limit v_ϵ and h_ϵ.

10.6. **Weak limits as $\epsilon \to 0$.** The same argument leads to that weak limits of $(v_\epsilon, h_\epsilon, q_\epsilon)$ (denoted by (v,h,q)) as $\epsilon\to 0$ exists and (v,h,q) satisfies (8.1).

10.7. **Uniqueness.** In this section, we show that for a given $(\tilde{v}, \tilde{h}) \in Y_T$, the solution to (8.1) is unique in Y_T. Suppose (v_1, h_1) and (v_2, h_2) are two solutions (in Y_T) to (8.3). Let $w = v_1 - v_2$ and $g = h_1 - h_2$, then w and g satisfy

$$\langle w_t, \varphi\rangle + \frac{\nu}{2}\int_\Omega D_{\tilde{\eta}} w : D_{\tilde{\eta}}\varphi dx + \sigma\int_{\Gamma_0}\tilde{\Theta}\Big[\tilde{L}_{\tilde{h}}\Big(\int_0^t(\tilde{h}_{,\alpha}w_\alpha - w_z)ds\Big)\Big]\circ\tilde{\eta}^\tau\times$$

$$\times(-\tilde{h}_{,\alpha}\circ\tilde{\eta}^\tau\varphi^\alpha + \varphi^z)dS = 0 \tag{10.16}$$

for all $\varphi \in \mathcal{V}_v(T)$ with $w(0) = 0$, where $\tilde{L}$ equals L except $L_1 = L_2 = 0$. Since w is in $\mathcal{V}_v(T)$, letting $w = \varphi$ in (10.16) leads to

$$\Big[\|v\|^2_{H^1(\Omega_0)} + \|\nabla_0^2 v\|^2_{L^2(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} + \|v_t\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_t\|^2_{L^2(\Gamma_0)}\Big](t)$$
$$+\int_0^t\Big[\|\nabla v\|^2_{L^2(\Omega_0)} + \|\nabla\nabla_0 v\|^2_{L^2(\Omega_1)} + \|\nabla\nabla_0^2 v\|^2_{L^2(\Omega_1)} + \|v_t\|^2_{H^1(\Omega_0)}\Big]ds$$
$$\le C(M)\int_0^t K(s)\Big[\|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_t\|^2_{L^2(\Gamma_0)}\Big]ds.$$

Therefore, by the Gronwall inequality and the zero initial condition ($w(0) = 0$), we know that w (and hence g) is identical to zero.

Remark 21. *The reason we have an 1 in the expression of $N_3(u_0, F)$ is that we linearized our problem in the way that we treat L_1 and L_2 as extra forcings on the boundary. When there is no such forcings, we don't need that 1 in $N_3(u_0, F)$.*

11. Fixed-Point argument

From previous sections, we establish a map Θ_T from Y_T into Y_T, i.e., given $(\tilde{v}, \tilde{h}) \in C_T(M)$, there exists a unique $\Theta_T(\tilde{v}, \tilde{h}) = (v, h)$ satisfying (8.1). Theorem 5.1 is then proved if this mapping Θ_T has a fixed point. We shall make use of the Tychonoff Fixed-Point Theorem which states as follows:

Theorem 11.1. *For a reflexive Banach space X, and $C \subset X$ a closed, convex, bounded subset, if $F : C \to C$ is weakly sequentially continuous into X, then F has at least one fixed-point.*

In order to apply the Tychonoff Fixed-Point Theorem, we need to show that $\Theta(\tilde{v},\tilde{h}) \in C_T(M)$ and this is the case if T is small enough. In the following discussion, we will always assume T is smaller than a fixed constant (for example, say $T \le 1$) so that the right-hand side of (10.15) can be written as $C(M)N_3(u_0,F)$.

Remark 22. *The space Y_T is not reflexive. We will treat $C_T(M)$ as a convex subset of X_T and applied the Tychonoff Fixed-Point Theorem on the space X_T.*

Before proceeding the fixed-point proof, we note that lemma 7.5 implies that for a short time, the constant $C(M)$ in (9.1) and (9.4) can be chosen to be independent of M. To be more precise, for almost all $0 < t \le T_1$,

$$\|q\|^2_{L^2(\Omega_0)} \le C\Big[\|v_t\|^2_{L^2(\Omega_0)} + \|\nabla v\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} + \|F\|^2_{L^2(\Omega_0)} + 1\Big], \tag{11.1}$$

$$\begin{aligned}\|v\|^2_{H^3(\Omega_0)} + \|q\|^2_{H^2(\Omega_0)} \le C\Big[&\|v_t\|^2_{H^1(\Omega_0)} + \|\nabla v\|^2_{H^1(\Omega_0)} + \|\nabla_0 v\|^2_{H^1(\Omega_1)} \\ &+ \|\nabla_0^2 v\|^2_{H^1(\Omega_0)} + \|F\|^2_{H^1(\Omega_0)} + 1\Big],\end{aligned} \tag{11.2}$$

and

$$\begin{aligned}\|h\|^2_{H^{5.5}(\Gamma_0)} \le C\Big[&\|v_t\|^2_{H^1(\Omega_0)} + \|\nabla v\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} \\ &+ \|F\|^2_{H^1(\Omega_0)} + 1\Big]\end{aligned} \tag{11.3}$$

for some constant C independent of M.

11.1. **Continuity in time of h.** By the evolution equation (8.1d) and the fact that $v \in V^3(T_1)$, $h_t \in L^2(0,T_1;H^{2.5}(\Gamma_0))$. Since $h \in L^2(0,T_1;H^{5.5}(\Gamma_0))$, we know that $h \in \mathcal{C}^0([0,T_1];H^4(\Gamma_0))$ by standard interpolation theorem. Although there is no uniform rate that h converges to zero in $H^4(\Gamma_0)$, we have the following.

Lemma 11.1. *Let $(v,h) = \Theta_{T_1}(\tilde{v},\tilde{h})$. Then $\|h(t)\|_{H^{2.5}(\Gamma_0)}$ converges to zero as $t \to 0$, uniformly for all $(\tilde{v},\tilde{h}) \in C_{T_1}(M)$.*

Proof. By the evolution equation (8.1d),

$$\|h(t)\|_{H^{2.5}(\Gamma_0)} \le \int_0^t \|\tilde{h}_{,\alpha} v_\alpha - v_z\|_{H^{2.5}(\Gamma_0)} dS \le C(M)N_3(u_0,F)^{1/2} t^{1/2}.$$

The lemma follows directly from the inequality. □

Remark 23. *We can also conclude from* $\|\nabla_0^2 h_t\|^2_{L^2(\Gamma_0)} \le C(M)N_3(u_0,F)$ *that*

$$\|\nabla_0^2 h\|_{L^2(\Gamma_0)} \le C(M)N_3(u_0,F)t$$

for all $0 < t \le T_1$.

By lemma 11.1 and the interpolation inequality, we can also conclude that

Lemma 11.2. $\|\nabla_0^2 h(t)\|_{H^{1.5}(\Gamma_0)}$ *converges to zero as* $t \to 0$, *uniformly for all* $\tilde{h} \in C_{T_1}(M)$ *with estimate*

$$\|\nabla_0^2 h(t)\|_{H^{1.5}(\Gamma_0)} \le C(M)N_3(u_0,F)t^{1/4} \tag{11.4}$$

for all $0 < t \le T_1$.

11.2. **Improved energy estimates.** In order to apply the fixed-point theorem, we have to use the fact that the forcing F is in $V^2(T)$. We also define a new constant

$$N(u_0,F) := \|u_0\|^2_{H^{2.5}(\Omega_0)} + \|F\|^2_{V^2(T_1)} + \|F\|^2_{L^\infty(0,T_1;L^2(\Omega_0))} + \|F(0)\|^2_{H^1(\Omega_0)} + 1.$$

Noting that $N_3(u_0,F) \le N(u_0,F)$.

Remark 24. *For the linearized problem (8.1), we only need* $F \in V^1(T)$ *to obtain a unique solution* $(v,h) \in Y_T$.

11.2.1. *Estimates for* $\nabla_0^2 v$ *near the boundary.* Note that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|\zeta_1\nabla_0^2 v\|^2_{L^2(\Omega_0)} + \sigma\int_{\Gamma_0}\tilde{\Theta}B\tilde{A}^{\alpha\beta\gamma\delta}\nabla_0^2 h_{,\alpha\beta}\nabla_0^2 h_{,\gamma\delta}dS\Big] + \frac{\nu}{2}\|\zeta_1 D_{\tilde{\eta}}(\nabla_0^2 v)\|^2_{L^2(\Omega_0)}\\
&= \langle F, \nabla_0^2(\zeta_1^2\nabla_0^2 v)\rangle - \frac{\nu}{4}\int_{\Omega_0}\Big[\nabla_0^2(\tilde{a}_i^k\tilde{a}_i^\ell)v^j_{,\ell} + \nabla_0^2(\tilde{a}_i^k\tilde{a}_j^\ell)v^i_{,\ell}\Big](\zeta_1^2\nabla_0^2 v^j)_{,k}dx\\
&\quad - \frac{\nu}{2}\int_{\Omega_0}\Big[\nabla_0(\tilde{a}_i^k\tilde{a}_i^\ell)\nabla_0 v^j_{,\ell} + \nabla_0(\tilde{a}_i^k\tilde{a}_j^\ell)\nabla_0 v^i_{,\ell}\Big](\zeta_1^2\nabla_0^2 v^j)_{,k}dx\\
&\quad - \frac{\nu}{2}\int_{\Omega_0}D_{\tilde{\eta}}(\nabla_0^2 v)^j_i\tilde{a}_i^k\zeta_1\zeta_{1,k}\nabla_0^2 v^j dx + \int_{\Omega_0}q\tilde{a}_k^\ell[\nabla_0^2(\zeta_1^2\nabla_0^2 v^k)]_{,\ell}dx\\
&\quad - \sigma\Big(\sum_{k=1}^3 I_k + \sum_{k=1}^8 J_k\Big)
\end{aligned}$$

where I_k's and J_k's are defined in Appendix B.1 (with $^-$ replaced by $\tilde{}$, and no ϵ and ϵ_1).

As in [6] and [7], we study the time integral of the right-hand side of the identity above in order to prove the validity of the requirement of applying Tychonoff Fixed-Point Theorem.

Step 1. Let $A_1 = \int_0^t \int_{\Omega_0} \Big[\nabla_0^2(\tilde{a}_i^k \tilde{a}_i^\ell) v^j_{,\ell} + \nabla_0^2(\tilde{a}_i^k \tilde{a}_j^\ell) v^i_{,\ell}\Big](\zeta_1^2 \nabla_0^2 v^j)_{,k} dx ds$. By (7.7) and (10.9),

$$\begin{aligned} A_1 &\le C \int_0^t \|\tilde{a}\tilde{a}\|_{H^2(\Omega_0)} \|\nabla v\|_{L^\infty(\Omega_0)} \|v\|_{H^3(\Omega_0)} ds \\ &\le C(M)C(\delta) \int_0^t \|v\|^{1/2}_{H^3(\Omega_0)} \|v\|^{1/2}_{H^1(\Omega_0)} ds + \delta \|v\|^2_{L^2(0,T;H^3(\Omega_0))} \\ &\le C(M)C(\delta) N(u_0, F)^{1/2} \int_0^t \|v\|^{1/2}_{H^3(\Omega_0)} ds + \delta C(M) N(u_0, F) \\ &\le C(M) N(u_0, F) \Big[C(\delta) t^{3/4} + \delta\Big]. \end{aligned}$$

Step 2. Let $A_2 = \int_0^t \int_{\Omega_0} \Big[\nabla_0(\tilde{a}_i^k \tilde{a}_i^\ell) \nabla_0 v^j_{,\ell} + \nabla_0(\tilde{a}_i^k \tilde{a}_j^\ell) \nabla_0 v^i_{,\ell}\Big](\zeta_1^2 \nabla_0^2 v^j)_{,k} dx ds$. Similar to Step 1, by (7.12) and (10.9),

$$\begin{aligned} A_2 &\le C \int_0^t \|\tilde{a}\tilde{a}\|_{W^{1,4}(\Omega_0)} \|\nabla \nabla_0 v\|_{L^4(\Omega_1')} \|v\|_{H^3(\Omega_0)} ds \\ &\le C(M)C(\delta) \int_0^t \|v\|_{H^3(\Omega_0)} \|v\|_{H^2(\Omega_0)} ds + \delta \|v\|^2_{L^2(0,T;H^3(\Omega_0))} \\ &\le C(M)C(\delta) \int_0^t \|v\|^2_{H^1(\Omega_0)} ds + 3\delta \|v\|^2_{L^2(0,T;H^3(\Omega_0))} \\ &\le C(M) N(u_0, F) \Big[C(\delta) t + \delta\Big]. \end{aligned}$$

Step 3. Let $A_3 = \int_0^t \int_{\Omega_0} D_{\tilde{\eta}} (\nabla_0^2 v)^j_i \tilde{a}_i^k \zeta_1 \zeta_{1,k} \nabla_0^2 v^j dx ds$. It is easy to see that

$$A_3 \le C(M) \int_0^t \|v\|_{H^3(\Omega_0)} \|\nabla_0^2 v\|_{L^2(\Omega_1')} ds \le C(M) N(u_0, F) t^{1/2}.$$

Step 4. Let $A_4 = \int_0^t \int_{\Omega_0} q\tilde{a}_k^\ell [\nabla_0^2(\zeta_1^2 \nabla_0^2 v^k)],_\ell dxds$. By identity (9.10) and interpolation inequalities ,

$$\begin{aligned}
A_4 &\le C(M) \int_0^t \Big[\|q\|_{L^\infty(\Omega_0)} + \|q\|_{W^{1,4}(\Omega_0)} + \|q\|_{H^1(\Omega_0)}\Big]\|v\|_{H^3(\Omega_0)} ds \\
&\le C(M)C(\delta)\int_0^t \|q\|^2_{H^1(\Omega_0)} ds + \delta\Big[\|v\|^2_{L^2(0,T;H^3(\Omega_0)} + \|q\|^2_{L^2(0,T;H^2(\Omega_0))}\Big] \\
&\le C(M)N(u_0,F)\Big[C(\delta)t^{1/2} + \delta\Big].
\end{aligned}$$

Step 5. Let

$$\begin{aligned}
A_5 = \int_0^t \int_{\Gamma_0} [L_{\tilde{h}}(h)\circ\tilde{\eta}^\tau]\Big[&\nabla_0^4(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau,-1)\cdot v + 4\nabla_0^3(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau,-1)\cdot\nabla_0 v \\
&+ 6\nabla_0^2(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau,-1)\cdot\nabla_0^2 v\Big] dSds.
\end{aligned}$$

By Appendix B.3 with (7.14), we find that

$$\begin{aligned}
|A_5| &\le C(M)\int_0^t \|h\|_{H^{5.5}(\Gamma_0)}\|v\|_{H^{2.5}(\Gamma_0)}\|\tilde{h}\|_{H^{3.5}(\Gamma_0)} ds \\
&\le C(M)t^{1/4}\int_0^t \Big[\|h\|^2_{H^{5.5}(\Gamma_0)} + \|v\|^2_{H^3(\Omega_0)}\Big] ds \\
&\le C(M)N(u_0,F)t^{1/4}.
\end{aligned}$$

Step 6. Let $A_6 = \int_0^t \int_{\Gamma_0} [L_{\tilde{h}}(h)\circ\tilde{\eta}^\tau](\nabla_0(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau,-1)\cdot\nabla_0^3 v)dS$.
By $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing,

$$\begin{aligned}
|A_6| &\le \int_0^t \|L_{\tilde{h}}(h)(\nabla_0(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau,-1)\|_{H^{0.5}(\Gamma_0)}\|\nabla_0^3 v\|_{H^{-0.5}(\Gamma_0)} ds \\
&\le C(M)\int_0^t \|h\|_{H^{4.5}(\Gamma_0)}\|\tilde{h}\|_{H^{3.5}(\Gamma_0)}\|v\|_{H^{2.5}(\Gamma_0)} ds \\
&\le C(M)N(u_0,F)t^{1/4}.
\end{aligned}$$

Step 7. Let

$$A_7 = \int_0^t \int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}}\nabla_0^2\Big[\sqrt{\det(g_0)}\Big(L_1^{\alpha\beta\gamma}\tilde{h}_{,\alpha\beta\gamma} + L_2\Big)\circ\tilde{\eta}^\tau\Big]\nabla_0^2(h_t\circ\tilde{\eta}^\tau)dSds.$$

By Appendix B.1,

$$\begin{aligned}|A_7| &\le C(M)\int_0^t (1+\|\tilde{h}\|_{H^5(\Gamma_0)}\|\nabla_0^2 v\|_{H^1(\Omega_0)}ds \\ &\le C(M)\int_0^t \|v\|_{H^3(\Omega_0)}ds + C(M)\int_0^t \|\tilde{h}\|_{H^5(\Gamma_0)}\|v\|_{H^3(\Omega_0)}ds \\ &\le C(M)N(u_0,F)^{1/2}t^{1/2} + C(M)C(\delta)\int_0^t \|\tilde{h}\|^2_{H^5(\Gamma_0)}ds + \delta\int_0^t \|v\|^2_{H^3(\Omega_0)}ds \\ &\le C(M)N(u_0,F)\Big[t^{1/2}+C(\delta)t^{2/3}+\delta\Big].\end{aligned}$$

Step 8. Let $A_8 = \int_0^t (J_1+J_2+J_3)ds$. Since

$$\begin{aligned}|J_1|+|J_2|+|J_3| &\le C(M)\|h\|_{H^5(\Gamma_0)}\Big[\|h_t\|_{H^2(\Gamma_0)} + \|\nabla_0 h_t\|_{L^4(\Gamma_0)}\Big] \\ &\le C(M)\|h\|_{H^5(\Gamma_0)}\|h_t\|_{H^2(\Gamma_0)},\end{aligned}$$

$$\begin{aligned}|A_8| &\le C(M)\int_0^t \|h\|_{H^5(\Gamma_0)}\|v\|_{H^2(\Gamma_0)}ds \\ &\le C(M)C(\delta)\int_0^t \|h\|_{H^5(\Gamma_0)}ds + \delta\int_0^t \|v\|^2_{H^3(\Omega_0)}ds \\ &\le C(M)N(u_0,F)\Big[C(\delta)t^{2/3}+\delta\Big].\end{aligned}$$

Step 9. Let $A_9 = \int_0^t J_4 ds = -\frac{1}{2}\int_0^t\int_{\Gamma_0} (\tilde{\Theta}B\tilde{A}^{\alpha\beta\gamma\delta})_t \nabla_0^2 h_{,\alpha\beta}\nabla_0^2 h_{,\gamma\delta}dS$. Then

$$\begin{aligned}|A_9| &\le C(M)\int_0^t (\|\tilde{v}\|_{H^3(\Omega_0)} + \|\tilde{h}_t\|_{H^{2.5}(\Gamma_0)})\|h\|^2_{H^4(\Gamma_0)}ds \\ &\le C(M)N(u_0,F)t^{1/2}.\end{aligned}$$

Step 10. Let $A_{10} = \int_0^t (J_5+J_6)ds$. By Appendix B.2,

$$|J_5|+|J_6| \le C(M)\|\tilde{h}\|_{H^{3.5}(\Gamma_0)}\|h\|_{H^{4.5}(\Gamma_0)}\|v\|_{H^{2.5}(\Gamma_0)}$$

and hence

$$|A_{10}| \le C(M)t^{1/4}\int_0^t \Big[\|h\|^2_{H^{4.5}(\Gamma_0)} + \|v\|^2_{H^3(\Omega_0)}\Big]ds \le C(M)N(u_0,F)t^{1/4}.$$

Step 11. Let $A_{11} = \int_0^t (J_7 + J_8) ds$. By interpolation and (7.14),

$$|A_{11}| \leq C(M) \int_0^t \|h\|_{H^{4.5}(\Gamma_0)} \|h_t\|_{H^{2.5}(\Gamma_0)} ds \leq C(M) N(u_0, F) t^{1/2}.$$

Step 12. Let $A_{12} = \int_0^t \langle F, \nabla_0^2(\zeta_1^2 \nabla_0^2 v)\rangle ds$. By (7.12),

$$\begin{aligned} A_{12} \leq& \int_0^t \|F\|_{H^2(\Omega_0)} \|v\|_{H^2(\Omega_0)} ds \leq N(u_0, F) + \int_0^t \|v\|^2_{H^2(\Omega_0)} ds \\ &\leq N(u_0, F) + C(M) N(u_0, F) t. \end{aligned}$$

Step 13. Summing A_1 to A_{12}, we find that

$$\begin{aligned} &\left[\|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)} + \sigma E_{\tilde{h}}(\nabla_0^2 h)\right] + \nu \int_0^t \|D_{\tilde{\eta}}(\nabla_0^2 v)\|^2_{L^2(\Omega_1)} ds \\ &\leq \|u_0\|^2_{H^2(\Omega_0)} + C N(u_0, F) + C(M) N(u_0, F)\left[C(\delta)(t^{3/4} + t^{2/3} + t^{1/2} + t) + \delta\right]. \end{aligned}$$

By Corollary 8.1,

$$\begin{aligned} &\left[\|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^4 h(t)\|^2_{L^2(\Gamma_0)}\right] + \int_0^t \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} ds \\ &\leq C N(u_0, F) + C(M) N(u_0, F)\left[C(\delta)\mathcal{O}(t) + \delta\right] \quad \text{as} \quad t \to 0 \end{aligned} \tag{11.5}$$

where C depends on ν, σ, ν_1 and the geometry of Γ_0.

By similar computations, we can also conclude (the (8.26), (9.9) and (10.5) variants) that

$$\begin{aligned} &\left[\|v(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h(t)\|^2_{L^2(\Gamma_0)}\right] + \int_0^t \|v\|^2_{H^1(\Omega_0)} ds \\ &\leq C N(u_0, F) + C(M) N(u_0, F)\mathcal{O}(t) \qquad \text{as} \quad t \to 0\ ; \end{aligned} \tag{11.6}$$

$$\begin{aligned} &\left[\|\nabla_0 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^3 h(t)\|^2_{L^2(\Gamma_0)}\right] + \int_0^t \|\nabla_0 v\|^2_{H^1(\Omega_1)} ds \\ &\leq C N(u_0, F) + C(M) N(u_0, F)\mathcal{O}(t) \qquad \text{as} \quad t \to 0\ ; \end{aligned} \tag{11.7}$$

$$\begin{aligned} &\|\nabla v(t)\|^2_{L^2(\Omega_0)} + \int_0^t \|v_t\|^2_{L^2(\Omega_0)} ds \\ &\leq C N(u_0, F) + C(M) N(u_0, F)\mathcal{O}(t) \qquad \text{as} \quad t \to 0 \end{aligned} \tag{11.8}$$

where C depends on ν, σ, ν_1 and the geometry of Γ_0.

11.2.2. *$L^2_tH^1_x$-estimate for v_t.* For the time-differentiated problem, we are not able to use estimates such as those in sections 9.6 and 11.2.1, since no ϵ-regularization is present; nevertheless, we can obtain estimates at the ϵ-regularization level and then pass ϵ to the limit once the estimate is found to be ϵ-independent. We have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_t\|^2_{L^2(\Omega_0)} + \frac{\nu}{2}\|D_{\bar\eta}v_t\|^2_{L^2(\Omega_0)} + \frac{\sigma}{2}\frac{d}{dt}\int_{\Gamma_0}\bar\Theta\bar A^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}h_{t,\gamma\delta}dS \\
&= \langle F_t, v_t\rangle - \nu\int_{\Omega_0}\Big[(\bar a^k_i\bar a^\ell_j)_t v^j_{,\ell} + (\bar a^k_i\bar a^\ell_j)_t v^i_{,\ell}\Big]v^j_{t,k}dx + \int_{\Omega_0} q_t\bar a^\ell_{kt}v^k_{,\ell}dx \\
&+ \frac{1}{2}\int_{\Gamma_0}(\bar\Theta\bar A^{\alpha\beta\gamma\delta})_t h_{t,\alpha\beta}h_{t,\gamma\delta}dS - \int_{\Gamma_0}\frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar A^{\alpha\beta\gamma\delta})_t h_{,\alpha\beta}\Big]_{,\gamma\delta}h_{tt}dS \\
&- 2\int_{\Gamma_0}\bar\Theta_{,\gamma}\bar A^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}h_{tt,\delta}dS - \int_{\Gamma_0}\bar\Theta_{,\gamma\delta}\bar A^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}h_{tt}dS \\
&- \int_{\Gamma_0}\bar\Theta\Big[L_1^{\alpha\beta\gamma}\bar h_{,\alpha\beta\gamma}\Big]_t h_{tt}dS - \int_{\Gamma_0}\bar\Theta(L_2)_t h_{tt}dS + K_1 + K_3 + K_4 + K_5 + K_6
\end{aligned}$$

where $K_i, i = 1, ..., 6$ is defined in Appendix C.1 (withoutϵ_1).

First we estimate the time integral of terms due to the viscosity, pressure and forcing terms:

Step 1. Let $B_1 = \int_0^t\int_{\Omega_0}\Big[(\tilde a^k_i\tilde a^\ell_j)_t v^j_{,\ell} + (\tilde a^k_i\tilde a^\ell_j)_t v^i_{,\ell}\Big]v^j_{t,k}dxds$. By (7.6) if $n = 3$ (or (7.9) if $n = 2$),

$$\begin{aligned}
|B_1| &\le C(M)\int_0^t\|\nabla v\|_{L^4(\Omega_0)}\|v_t\|_{H^1(\Omega_0)}ds \\
&\le C(M)C(\delta)\int_0^t\|\nabla v\|^2_{L^2(\Omega_0)}ds + \delta\Big[\|v\|^2_{L^2(0,T;H^2(\Omega_0))} + \|v_t\|^2_{L^2(0,T;H^1(\Omega_0))}\Big] \\
&\le C(M)N(u_0, F)\Big[C(\delta)t + \delta\Big].
\end{aligned}$$

Step 2. Let $B_2 = \int_0^t\int_{\Omega_0}(q\bar a^\ell_k)_t v^k_{t,\ell}dxds$. As in Appendix C.2,

$$B_2 = \int_0^t\int_{\Omega_0}\bar a^\ell_{kt}qv^k_{t,\ell}dxds - \int_0^t\int_{\Omega_0}\bar a^\ell_{kt}q_t v^k_{,\ell}dxds.$$

By (7.6) if $n=3$ (or (7.9) if $n=2$) and (9.1), it follows that

$$\begin{aligned}&\Big|\int_0^t\int_{\Omega_0}\bar a^\ell_{kt}qv^k_{t,\ell}dxds\Big|\\&\le C(M)C(\delta)\int_0^t\|q\|^2_{L^2(\Omega_0)}ds+\delta\Big[\|q\|^2_{L^2(0,T;H^1(\Omega_0))}+\|v_t\|^2_{L^2(0,T;H^1(\Omega_0))}\Big]\\&\le C(M)N(u_0,F)\Big[C(\delta)t+\delta\Big].\end{aligned}$$

By Appendix C.2,

$$\begin{aligned}\int_0^t\int_{\Omega_0}\bar a^\ell_{kt}q_tv^k_{,\ell}dxds=&\int_{\Omega_0}(\bar a^\ell_{kt}qv^k_{,\ell})(t)dx-\int_{\Omega_0}\bar a^\ell_{kt}(0)q(0)u^k_{0,\ell}dx\\&-\int_0^t\int_{\Omega_0}(\bar a^\ell_{kt}v^k_{,\ell})_tqdxds.\end{aligned}$$

By the identity $\bar a^\ell_{kt}=-\bar a^i_k\bar v^j_{,i}\bar a^\ell_j$,

$$\begin{aligned}\Big|\int_0^t\int_{\Omega_0}(\bar a^\ell_{kt}v^k_{,\ell})_tqdxds\Big|\le&\int_0^t\int_{\Omega_0}\Big|\Big[\bar a^\ell_{ktt}v^k_{,\ell}+\bar a^\ell_{kt}v^k_{t,\ell}\Big]q\Big|dxds\\\le&\, C(M)\int_0^t(1+\|\bar v_t\|_{H^1(\Omega_0)})\|\nabla v\|_{L^4(\Omega_0)}\|q\|_{L^4(\Omega_0)}ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big|\int_0^t\int_{\Omega_0}(\bar a^\ell_{kt}v^k_{,\ell})_tqdxds\Big|\\&\le C(M)C(\delta)N(u_0,F)\int_0^t\|q\|^{2\alpha}_{H^1(\Omega_0)}\|q\|^{2(1-\alpha)}_{L^2(\Omega_0)}ds+\delta\int_0^t(1+\|\bar v_t\|^2_{H^1(\Omega_0)})ds\\&\le C(M)N(u_0,F)^2\Big[C(\delta)(t+t^{\frac{1-\alpha}{2}})+\delta\Big]\end{aligned}$$

where $\alpha=\dfrac{3}{4}$ if $n=3$ and $\alpha=\dfrac{1}{2}$ if $n=2$.

The second integral equals $\int_{\Omega_0}\nabla u_0:(\nabla u_0)^Tq(0)dx$ which is bounded by $CN(u_0,F)$.

It remains to estimate the first integral. By adding and substracting $\int_{\Omega_0}\bar a^\ell_{kt}(0)qv^k_{,\ell}dx$, we find, by $\bar a_t(0)\in H^2(\Omega_0)$, that

$$\begin{aligned}\Big|\int_{\Omega_0}(\bar a^\ell_{kt}qv^k_{,\ell})(t)dx\Big|\le&\int_{\Omega_0}\Big|(\bar a^\ell_{kt}-\bar a^\ell_{kt}(0))(qv^k_{,\ell})(t)\Big|dx+\int_{\Omega_0}\Big|\bar a^\ell_{kt}(0)qv^k_{,\ell}\Big|dx\\\le&\, C\|\bar a_t(t)-\bar a_t(0)\|_{L^4(\Omega_0)}\|q\|_{L^2(\Omega_0)}\|\nabla v\|_{L^4(\Omega_0)}\\&+C(\delta_1)\|\nabla v\|^2_{L^2(\Omega_0)}+\delta_1\|q\|^2_{L^2(\Omega_0)}.\end{aligned}$$

Noting that

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega_0)} &= \|\nabla u_0 + \int_0^t \nabla v_t ds\|^2_{L^2(\Omega_0)} \\ &\le \Big[\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^t \|\nabla v_t\|_{L^2(\Omega_0)} ds\Big]^2 \\ &\le 2\Big[\|u_0\|^2_{H^1(\Omega_0)} + C(M)N(u_0,F)t\Big],\end{aligned}$$

(7.13c), (10.9) and (11.1) imply

$$\begin{aligned}\Big|\int_{\Omega_0} (\bar a^\ell_{kt} q v^k_{,\ell}(t) dx\Big| &\le C(M)N(u_0,F)t^{1/2} + C(\delta_1)N(u_0,F) \\ &\quad + \delta_1\Big[\|v_t\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)}\Big].\end{aligned}$$

Summing all the estimates above, we find that

$$\begin{aligned}|B_2| &\le C(\delta_1)N(u_0,F) + C(M)N(u_0,F)^2\Big[C(\delta)(t + t^{\frac{1-\alpha}{2}}) + \delta\Big] \\ &\quad + \delta_1\Big[\|v_t\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)}\Big].\end{aligned}$$

Remark 25. *It may be tempting to use an interpolation inequality to show that $q \in \mathcal{C}([0,T];X)$ for some Banach space X by analyzing q_t via Laplace's equation. The problem, however, is that the boundary condition for q_t has regularity $L^2(0,T;H^{-1.5}(\Gamma_0))$ (by the fact that $h_t \in L^2(0,T;H^{2.5}(\Gamma_0))$), and thus standard elliptic estimates do not provide the desired conclusion that $q_t \in L^2(0,T;H^1(\Omega_0)')$ (and hence by interpolation, $q \in \mathcal{C}([0,T];H^{0.5}(\Omega_0))$. However, suppose that $q_t \in L^2(0,T;H^1(\Omega_0)')$; then we can estimate $\int_0^t \int_{\Omega_0} \bar a^\ell_{kt} q_t v^k_{,\ell} dx ds$ by the following method:*

$$\begin{aligned}\Big|\int_0^t \int_{\Omega_0} \bar a^\ell_{kt} q_t v^k_{,\ell} dx ds\Big| &\le \int_0^t \|\bar a^i_k \bar v^j_{,i} \bar a^\ell_j v^k_{,\ell}\|_{H^1(\Omega_0)} \|q_t\|_{H^1(\Omega_0)'} ds \\ &\le C(M)N(u_0,F)\Big[t + t^{5/8}\Big].\end{aligned}$$

Step 3. Let $B_3 = \int_0^t \langle F_t, v_t\rangle ds$. By the fact that $F_t \in L^2(0,T;L^2(\Omega_0))$, it follows that

$$B_3 \le \int_0^t \|F_t\|_{L^2(\Omega_0)} \|v_t\|_{L^2(\Omega_0)} ds \le C(M)N(u_0,F)t^{1/2}.$$

Next, we estimate the time integral of terms from the boundary.

Step 4. Let

$$B_4 = \int_0^t \int_{\Gamma_0} \Big[\frac{1}{2}(\bar{\Theta}\bar{A}^{\alpha\beta\gamma\delta})_t h_{t,\alpha\beta} h_{t,\gamma\delta} - \frac{\bar{\Theta}}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t h_{,\alpha\beta}\Big]_{,\gamma\delta} h_{tt} - 2\bar{\Theta}_{,\gamma}\bar{A}^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt,\delta} - \bar{\Theta}_{,\gamma\delta}\bar{A}^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt}\Big] dSds$$

By (C.4), (C.5), estimates of K_8, K_9 and Remark 34, we find that

$$|B_4| \le C(M)N(u_0,F)t^{1/2}.$$

Step 5. Let $B_5 = \int_0^t \int_{\Gamma_0} \bar{\Theta}\Big[[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma}]_t h_{tt} + (L_2)_t h_{tt}\Big] dSds$. It follows that

$$\Big|\int_0^t \int_{\Gamma_0} \bar{\Theta}(L_2)_t h_{tt} dSds\Big| \le C(M)\int_0^t \Big[\|v\|_{L^\infty(\Gamma_0)} + \|v_t\|_{L^2(\Gamma)}\Big] ds \le C(M)N(u_0,F)^{1/2}(t + t^{3/4}).$$

For parts involving L_1, we have

$$\int_0^t \int_{\Gamma_0} \bar{\Theta}\Big[[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma}]_t h_{tt} dSds = \int_0^t \int_{\Gamma_0} \bar{\Theta}\Big[L_1^{\alpha\beta\gamma}\Big]_t \bar{h}_{,\alpha\beta\gamma} h_{tt} dSds \quad (\equiv B_5^1)$$
$$+ \int_0^t \int_{\Gamma_0} \bar{\Theta} L_1^{\alpha\beta\gamma}\bar{h}_{t,\alpha\beta\gamma} h_{tt} dSds. \quad (\equiv B_5^2)$$

By (7.9),

$$\begin{aligned} |B_5^1| &\le C(M)\int_0^t \|\bar{\Theta}\|_{L^\infty(\Gamma_0)}\|\tilde{h}\|_{W^{1,4}(\Gamma_0)}\|h_{tt}\|_{L^4(\Gamma_0)} dSds \\ &\le C(M)\int_0^t \Big[\|v\|_{H^2(\Omega_0)} + \|v_t\|_{H^1(\Omega_0)}\Big] ds \\ &\le C(M)N(u_0,F)^{1/2}t^{1/2} \end{aligned}$$

while by (7.14) and Corollary 7.1,

$$\begin{aligned} |B_5^2| \le& \int_0^t \|\bar{\Theta}\|_{H^{1.5}(\Gamma_0)}\|\tilde{h}_t\|_{H^{2.5}(\Gamma_0)}\|L_1^{\alpha\beta\gamma}\|_{H^{1.5}(\Gamma_0)}\|h_{tt}\|_{H^{0.5}(\Gamma_0)} ds \\ &\le C(M)\|L_1^{\alpha\beta\gamma}\|_{H^{1.5}(\Gamma_0)}\int_0^t \|\tilde{h}\|_{H^{2.5}(\Gamma_0)}\Big[\|v\|_{H^2(\Omega_0)} + \|v_t\|_{H^1(\Omega_0)}\Big] ds \\ &\le C(M)N(u_0,F)t^{1/4}. \end{aligned}$$

Therefore,

$$|B_5| \leq C(M)N(u_0,F)(t+t^{3/4}+t^{1/4}).$$

Step 6. Let $B_6 = \int_0^t K_3 ds = \int_0^t \int_{\Gamma_0} \bar{\Theta}[L_{\bar{h}}(h)]_t[(\bar{v}\circ\bar{\eta}^{-\tau})\cdot(\nabla_0 h_t)]dSds$. The L_1 and L_2 part of B_6 is bounded by

$$C(M)\int_0^t \|\bar{\Theta}\|_{H^{1.5}(\Gamma_0)}\|\bar{v}\|_{H^{1.5}(\Gamma_0)}\|\bar{h}\|_{H^{3.5}(\Gamma_0)}\|\bar{h}_t\|_{H^2(\Gamma_0)}\|h_t\|_{H^2(\Omega_0)}ds$$

and hence

$$\Big|\int_0^t \bar{\Theta}\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma}+L_2\Big]_t[(\bar{v}\circ\bar{\eta}^{-\tau})\cdot(\nabla_0 h_t)]dSds\Big| \leq C(M)N(u_0,F)t^{1/4}.$$

Integrating by parts, the highest order part of B_6 can be expressed as

$$\begin{aligned}
&\int_0^t\int_{\Gamma_0}\frac{\bar{\Theta}(\bar{v}\circ\bar{\eta}^{-\tau})}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t h_{,\alpha\beta}\Big]_{,\gamma\delta}\nabla_0 h_t dSds \quad (\equiv B_6^1)\\
&+\int_0^t\int_{\Gamma_0}\bar{\Theta}(\bar{v}\circ\bar{\eta}^{-\tau})\bar{A}^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}\nabla_0 h_{t,\gamma\delta}dSds \quad (\equiv B_6^2)\\
&+2\int_0^t\int_{\Gamma_0}[\bar{\Theta}(\bar{v}\circ\bar{\eta}^{-\tau})]_{,\gamma}\bar{A}^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}\nabla_0 h_{t,\delta}dSds \quad (\equiv B_6^3)\\
&+\int_0^t\int_{\Gamma_0}[\bar{\Theta}(\bar{v}\circ\bar{\eta}^{-\tau})]_{,\gamma\delta}\bar{A}^{\alpha\beta\gamma\delta}h_{t,\alpha\beta}\nabla_0 h_t dSds. \quad (\equiv B_6^4)
\end{aligned}$$

It follows that

$$\begin{aligned}
|B_6^1| &\leq C(M)\int_0^t \|\bar{\Theta}\bar{v}\circ\bar{\eta}^{-\tau}\|_{H^{1.5}(\Gamma_0)}\|\bar{h}_t\|_{H^2(\Gamma_0)}\|h\|_{H^4(\Gamma_0)}\|h_t\|_{H^2(\Gamma_0)}dS\\
&\leq C(M)N(u_0,F)t
\end{aligned}$$

and

$$\begin{aligned}
|B_6^3| &\leq C(M)\int_0^t \|\bar{\Theta}\bar{v}\circ\bar{\eta}^{-\tau}\|_{W^{1,4}(\Gamma_0)}\|\bar{A}\|_{L^\infty(\Gamma_0)}\|h_t\|_{H^2(\Gamma_0)}\|h_t\|_{W^{2,4}(\Gamma_0)}dS\\
&\leq C(M)N(u_0,F)t^{1/2}.
\end{aligned}$$

For B_6^2, integrating by parts,

$$B_6^2 = \frac{1}{2}\int_0^t \int_{\Gamma_0} \bar{\Theta}(\bar{v}\circ\bar{\eta}^{-\tau})\bar{A}^{\alpha\beta\gamma\delta}\nabla_0\Big[h_{t,\alpha\beta}h_{t,\gamma\delta}\Big]dSds$$
$$= -\frac{1}{2}\int_0^t \int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}}\nabla_0\Big[\sqrt{\det(g_0)}\bar{\Theta}(\bar{v}\circ\bar{\eta}^{-\tau})\bar{A}^{\alpha\beta\gamma\delta}\Big]h_{t,\alpha\beta}h_{t,\gamma\delta}dSds$$

and hence

$$\begin{aligned}|B_6^2| \leq & \int_0^t \Big[\|\nabla_0\bar{\Theta}\|_{L^4(\Gamma_0)}\|\bar{v}\bar{A}\|_{L^\infty(\Gamma_0)} + \|\bar{\Theta}\|_{L^\infty(\Gamma_0)}\|\bar{v}\bar{A}\|_{W^{1,4}(\Gamma_0)}\Big] \\ & \quad \times \|h_t\|_{W^{2,4}(\Gamma_0)}\|h_t\|_{H^2(\Gamma_0)}ds \\ & \leq C(M)N(u_0,F)^{1/2}\int_0^t \|v\|_{H^3(\Omega_0)}ds \\ & \leq C(M)N(u_0,F)t^{1/2}.\end{aligned}$$

For B_6^4, noting that

$$\begin{aligned}\bar{\Theta}_{,\gamma\delta} = & \det(\nabla_0\bar{\eta}^\tau)_{,\gamma\delta}\sqrt{\det(G_{\bar{h}})\circ\bar{\eta}^\tau} + \det(\nabla_0\bar{\eta}^\tau)_{,\gamma}\sqrt{\det(G_{\bar{h}})\circ\bar{\eta}^\tau}_{,\delta} \\ & + \det(\nabla_0\bar{\eta}^\tau)_{,\delta}\sqrt{\det(G_{\bar{h}})\circ\bar{\eta}^\tau}_{,\gamma} + \det(\nabla_0\bar{\eta}^\tau)\sqrt{\det(G_{\bar{h}})\circ\bar{\eta}^\tau}_{,\gamma\delta}\end{aligned}$$

and $\|\nabla_0\det(\nabla_0\bar{\eta}^\tau)\|_{H^{0.5}(\Gamma_0)} \leq C(M)t^{1/2}$, we find that

$$\begin{aligned}|B_6^4| \leq & C(M)\int_0^t \|\nabla_0\det(\nabla_0\bar{\eta}^\tau)\|_{H^{0.5}(\Gamma_0)}\|\nabla_0^2 h_t\|_{H^{0.5}(\Gamma_0)}\|\nabla_0 h_t\|_{H^{1.5}(\Gamma_0)}ds \\ & + C(M)\int_0^t \|\det(\nabla_0\bar{\eta}^\tau)\|_{L^\infty(\Gamma_0)}\|\nabla_0\bar{\eta}^\tau\|^2_{L^\infty(\Gamma_0)}\|\nabla_0^2 h_t\|_{L^2(\Gamma_0)}\|\nabla_0 h_t\|_{L^2(\Gamma_0)}ds \\ \leq & C(M)N(u_0,F)t^{1/2} + C(M)N(u_0,F)^{3/4}\int_0^t \|v\|^{1/2}_{H^3(\Omega_0)}ds \\ \leq & C(M)N(u_0,F)(t^{1/2}+t^{3/4}).\end{aligned}$$

Combining all the estimates, we find that

$$|B_6| \leq C(M)N(u_0,F)(t+t^{1/2}+t^{3/4}).$$

Step 7. Let $B_7 = \int_0^t K_4 ds = \int_0^t \int_{\Gamma_0} \bar{\Theta}\Big[L_{\bar{h}}(h)\Big]_t [(\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})] dS ds$. Integrating by parts in time,

$$B_7 = -\int_0^t \int_{\Gamma_0} L_{\bar{h}}(h)\Big[\bar{\Theta}_t (\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau}) + \bar{\Theta}(\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})_t$$
$$+ \bar{\Theta}(\nabla_0 \bar{h}, -1)_{tt} \cdot (v \circ \bar{\eta}^{-\tau})\Big] dS ds + \int_{\Gamma_0} \bar{\Theta} L_{\tilde{h}}(h)[(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})] dS.$$

For the first integral, (7.16) implies

$$\begin{aligned}
&\Big| \int_{\Gamma_0} \bar{\Theta} L_{\tilde{h}}(h)[(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})] dS \Big| \\
&\leq \|\bar{\Theta}\|_{L^\infty(\Gamma_0)} \|L_{\tilde{h}}(h)\|_{L^2(\Gamma_0)} \|\nabla_0 \tilde{h}_t\|_{L^4(\Gamma_0)} \|v \circ \bar{\eta}^{-\tau}\|_{L^4(\Gamma_0)} \\
&\leq C(M) N(u_0, F) \|\tilde{h}_t\|_{H^{1.5}(\Gamma_0)} \\
&\leq C(M) N(u_0, F) t^{1/8}.
\end{aligned}$$

It also follows that

$$\begin{aligned}
&\Big| \int_0^t \int_{\Gamma_0} L_{\bar{h}}(h)\Big[\bar{\Theta}_t (\nabla_0 \bar{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau}) + \bar{\Theta}(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \bar{\eta}^{-\tau})_t\Big] dS ds \Big| \\
&\leq C(M) \int_0^t \Big[\|v\|_{L^\infty(\Gamma_0)} + \|v_t\|_{L^4(\Gamma_0)}\Big] \|L_{\tilde{h}}(h)\|_{L^2(\Gamma_0)} \|\nabla_0 \tilde{h}_t\|_{L^4(\Gamma_0)} ds \\
&\leq C(M) N(u_0, F)^{1/2} \int_0^t \Big[\|v\|_{H^3(\Omega_0)} + \|v_t\|_{H^1(\Omega_0)}\Big] ds \\
&\leq C(M) N(u_0, F) t^{1/2}.
\end{aligned}$$

For the remaining terms, $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing leads to

$$\begin{aligned}
&\Big| \int_0^t \bar{\Theta} \int_{\Gamma_0} L_{\tilde{h}}(h)(\nabla_0 \tilde{h}, -1)_{tt} \cdot v dS ds \Big| \\
&\leq \int_0^t \|\bar{\Theta}\|_{H^{1.5}(\Gamma_0)} \|L_{\tilde{h}}(h)\|_{H^{0.5}(\Gamma_0)} \|v\|_{H^{1.5}(\Gamma_0)} \|\tilde{h}_{tt}\|_{H^{0.5}(\Gamma_0)} ds.
\end{aligned}$$

By (7.11) and (7.12),

$$\|L_{\tilde{h}}(h)\|_{H^{0.5}(\Gamma_0)} \leq C(M)\Big[\|h\|^{1/2}_{H^{5.5}(\Gamma_0)} \|h\|^{1/2}_{H^{3.5}(\Gamma_0)} + 1\Big]$$

and hence

$$\Big|\int_0^t\int_{\Gamma_0} L_{\tilde h}(h)(\nabla_0\tilde h,-1)_{tt}\cdot(v\circ\bar\eta^{-\tau})dSds\Big|$$
$$\le C(M)N(u_0,F)\int_0^t \|\tilde h_{tt}\|_{H^{0.5}(\Gamma_0)}\Big[\|\nabla_0^5 h\|^{1/2}_{L^2(\Gamma_0)}+1\Big]ds$$
$$\le C(M)C(\delta)N(u_0,F)\int_0^t\Big[\|\nabla_0^5 h\|_{L^2(\Gamma_0)}+1\Big]ds+\delta C(M)N(u_0,F)$$
$$\le C(M)N(u_0,F)\Big[C(\delta)(t^{1/2}+t)+\delta\Big].$$

All the inequalities above give us

$$|B_7|\le C(M)N(u_0,F)\Big[C(\delta)(t^{1/2}+t)+t^{1/8}+\delta\Big].$$

Step 8. Let $B_6=\displaystyle\int_0^t K_5 ds=\int_0^t\int_{\Gamma_0}\bar\Theta[L_{\bar h}(h)]_t[(\bar v\circ\bar\eta^{-\tau})(\nabla_0^2\bar h,0)\cdot(v\circ\bar\eta^{-\tau})]dSds.$ By $H^{1.5}(\Gamma_0)$-$H^{-1.5}(\Gamma_0)$ duality pairing,

$$|B_6|\le C(M)\int_0^t\|\bar\Theta\|_{H^{1.5}(\Gamma_0)}(\|h_t\|_{H^{2.5}(\Gamma_0)}+1)\|\bar v\|_{H^{1.5}(\Gamma_0)}\|\bar h\|_{H^{3.5}(\Gamma_0)}\|v\|_{H^{1.5}(\Gamma_0)}ds$$
$$\le C(M)t^{1/4}\int_0^t(\|v\|_{H^3(\Omega_0)}+1)\|v\|_{H^2(\Omega_0)}ds$$
$$\le C(M)N(u_0,F)t^{1/4}.$$

Step 9. Let $B_9=\displaystyle\int_0^t K_6 ds=\int_0^t\bar\Theta\int_{\Gamma_0}L_{\tilde h}(h)(\nabla_0\tilde h,-1)_t\cdot(v_t\circ\bar\eta^{-\tau})dSds.$ By (7.8),

$$|B_9|\le C(M)\int_0^t\|\tilde h_t\|_{H^{2.5}(\Gamma_0)}\|v_t\|_{L^2(\Gamma_0)}\Big[\|\nabla_0^4 h\|_{L^2(\Gamma_0)}+1\Big]ds$$
$$\le C(M)C(\delta)N(u_0,F)^{1/2}\int_0^t\|v_t\|_{L^2(\Omega_0)}\|v_t\|_{H^1(\Omega_0)}ds+\delta C(M)N(u_0,F)$$
$$\le C(M)C(\delta)N(u_0,F)^2t^{1/2}+\delta C(M)N(u_0,F).$$

Step 10. Let $B_{10}=\displaystyle\int_0^t K_1 ds.$ By (C.1),

$$B_{10}\le C(M)N(u_0,F)(\delta+C(\delta)(t^{1/2}+t)).$$

Step 11. Summing B_1 to B_{10}, we find that

$$
\begin{aligned}
&\Big[\|v_t\|^2_{L^2(\Omega_0)} + \sigma \int_{\Gamma_0} \bar{\Theta}\bar{A}^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{t,\gamma\delta}|^2 dS\Big](t) + \nu \int_0^t \|D_{\tilde{\eta}} v_t\|^2_{L^2(\Omega_0)} ds \\
&\le \|v_t(0)\|^2_{L^2(\Omega_0)} + \sigma \int_{\Gamma_0} |G_0^{\alpha\beta} h_{t,\alpha\beta}(0)|^2 dS + (C + C(\delta_1))N(u_0, F) \\
&\quad + C(M)N(u_0,F)\Big[C(\delta)(t + t^{3/4} + t^{1/2} + t^{1/4} + t^{1/8} + t^{\frac{1-\alpha}{2}}) + \delta\Big] \\
&\quad + \delta_1 \Big[\|v_t\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)}\Big]
\end{aligned}
$$

and by Corollary 8.1,

$$
\begin{aligned}
&\Big[\|v_t(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma_0)}\Big] + \int_0^t \|v_t\|^2_{H^1(\Omega_0)} ds \\
&\le (C + C(\delta_1))N(u_0, F) + C(M)N(u_0,F)\Big[C(\delta)\mathcal{O}(t) + \delta\Big] \\
&\quad + \delta_1 \Big[\|v_t\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)}\Big]
\end{aligned}
\tag{11.9}
$$

where C depends on ν, σ, ν_1 and the geometry of Γ_0. Since this estimate is independent of ϵ, we pass ϵ to zero and conclude that the solution (v, h) to (8.1) also satisfies (11.9).

11.3. **Mapping from $C_T(M)$ into $C_T(M)$.** In this section, we are going to choose M so that $\Theta(\tilde{v}, \tilde{h}) \in C_T(M)$ if $(\tilde{v}, \tilde{h}) \in C_T(M)$.

Summing (11.5), (11.6), (11.7), (11.8) and (11.9), by (7.13) we find that

$$
\begin{aligned}
&\Big[\|v(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)} + \|v_t(t)\|^2_{L^2(\Omega_0)} \\
&\quad + \|\nabla_0^2 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^3 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^4 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma_0)}\Big] \\
&\quad + \int_0^t \Big[\|v\|^2_{H^1(\Omega_0)} + \|\nabla_0 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|v_t\|^2_{H^1(\Omega_0)}\Big] ds \\
&\le (C + C(\delta_1))N(u_0, F) + C(M)N(u_0,F)\Big[C(\delta)\mathcal{O}(t) + \delta\Big] \\
&\quad + \delta_1 \Big[\|v_t\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)}\Big]
\end{aligned}
$$

where C depends on ν, σ, ν_1 and the geometry of Γ_0. Choose $\delta_1 = \dfrac{1}{2}$,

$$\begin{aligned}
&\Big[\|v(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0 v(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^2 v(t)\|^2_{L^2(\Omega_1)} + \|v_t(t)\|^2_{L^2(\Omega_0)} \\
&+ \|\nabla_0^2 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^3 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^4 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma_0)}\Big] \\
&+ \int_0^t \Big[\|v\|^2_{H^1(\Omega_0)} + \|\nabla_0 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|v_t\|^2_{H^1(\Omega_0)}\Big] ds \\
&\leq C_1 N(u_0, F) + C(M) N(u_0, F)^2 \Big[C(\delta)\mathcal{O}(t) + \delta\Big]
\end{aligned}$$

where C_1 depends on ν, σ μ and the geometry of Γ_0. Similar to Section 9.7, for almost all $0 < t \leq T$,

$$\begin{aligned}
&\Big[\|v(t)\|^2_{H^2(\Omega_0)} + \|v_t(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h(t)\|^2_{H^2(\Gamma_0)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma_0)}\Big] \\
&+ \int_0^t \Big[\|v\|^2_{H^3(\Omega_0)} + \|v_t\|^2_{H^1(\Omega_0)} + \|q\|^2_{H^2(\Omega_0)}\Big] ds \qquad (11.10)\\
&\leq C_2 N(u_0, F) + C(M) N(u_0, F)^2 \Big[C(\delta)\mathcal{O}(t) + \delta\Big]
\end{aligned}$$

for some constant C_2 depending on C_1.

By (7.14), (7.16) and (8.1d),

$$\begin{aligned}
\int_0^t \|h_t\|^2_{H^{2.5}(\Gamma_0)} ds &\leq \int_0^t \Big[1 + \|\tilde{h}\|^2_{H^{3.5}(\Gamma_0)}\Big] \|v\|^2_{H^{2.5}(\Gamma_0)} ds \\
&\leq C(M) N(u_0, F) t^{1/4} \qquad (11.11)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|h_{tt}\|^2_{H^{0.5}(\Gamma_0)} ds &\leq C(M) \int_0^t \Big[\|\tilde{h}_t\|^2_{H^{1.5}(\Gamma_0)} \|v\|^2_{H^2(\Omega_0)} + \|\tilde{h}\|^2_{H^{2.5}(\Gamma_0)} \|v_t\|^2_{H^1(\Omega_0)}\Big] ds \\
&\leq C(M) N(u_0, F) \Big[t^{1/4} + t^{1/2}\Big]. \qquad (11.12)
\end{aligned}$$

Also, by (11.3) and (11.10),

$$\begin{aligned}
\int_0^t \|h\|^2_{H^{5.5}(\Gamma_0)} ds \leq C \int_0^t \Big[&\|v_t\|^2_{H^1(\Omega_0)} + \|\nabla v\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} \\
&+ \|F\|^2_{H^1(\Omega_0)} + 1\Big] ds \\
\leq\ & C_3 N(u_0, F) + C(M) N(u_0, F)^2 \Big[C(\delta)\mathcal{O}(t) + \delta\Big] \qquad (11.13)
\end{aligned}$$

for some constant C_3 depending on C_2.

Combining (11.10), (11.11), (11.12) and (11.13), we have the following inequality:

$$\begin{aligned}&\Big[\|v(t)\|^2_{H^2(\Omega_0)} + \|v_t(t)\|^2_{L^2(\Omega_0)} + \|h(t)\|^2_{H^4(\Gamma_0)} + \|h_t(t)\|^2_{H^2(\Gamma_0)}\Big]\\ &+ \int_0^t \Big[\|v\|^2_{H^3(\Omega_0)} + \|v_t\|^2_{H^1(\Omega_0)} + \|h\|^2_{H^{5.5}(\Gamma_0)} + \|h_t\|^2_{H^{2.5}(\Gamma_0)} + \|h_{tt}\|^2_{H^{0.5}(\Gamma_0)}\Big] ds\\ &\le (C_2 + C_3)N(u_0, F) + C(M)N(u_0, F)^2\Big[C(\delta)\mathcal{O}(t) + \delta\Big].\end{aligned}$$

Let $M = 2(C_2 + C_3)N(u_0, F) + 1$ (and hence corresponding T_0 and T in Lemma 7.5 and Corollary 8.1 are fixed). Choose $\delta > 0$ small enough (but fixed one such δ) so that

$$C(M)N(u_0, F)^2\delta \le \frac{1}{4}$$

and then choose $T > 0$ small enough so that

$$C(M)N(u_0, F)^2 C(\delta)T \le \frac{1}{4}.$$

Then for almost all $0 < t \le T$,

$$\begin{aligned}&\Big[\|v(t)\|^2_{H^2(\Omega_0)} + \|v_t(t)\|^2_{L^2(\Omega_0)} + \|h(t)\|^2_{H^4(\Gamma_0)} + \|h_t(t)\|^2_{H^2(\Gamma_0)}\Big]\\ &+ \int_0^t \Big[\|v\|^2_{H^3(\Omega_0)} + \|v_t\|^2_{H^1(\Omega_0)} + \|h_t\|^2_{H^{2.5}(\Gamma_0)} + \|h_{tt}\|^2_{H^{0.5}(\Gamma_0)}\Big] ds\\ &\le C_2 N(u_0, F) + \frac{1}{2}\end{aligned}$$

and therefore

$$\begin{aligned}&\sup_{0\le t\le T}\Big[\|v(t)\|^2_{H^2(\Omega_0)} + \|v_t(t)\|^2_{L^2(\Omega_0)} + \|h(t)\|^2_{H^4(\Gamma_0)} + \|h_t(t)\|^2_{H^2(\Gamma_0)}\Big]\\ &+ \|v\|^2_{V^3(T)} + \|h\|^2_{H(T)} \le 2C_2 N(u_0, F) + 1, \qquad (11.14)\end{aligned}$$

or in other words,

$$\|(v, h)\|^2_{Y(T)} \le 2C_2 N(u_0, F) + 1.$$

REMARK 26. *(11.14) implies that for* $(\tilde{v}, \tilde{h}) \in C_T(M)$ *(with* M *and* T *chosen as above), the corresponding solution to the linear problem (8.1)* $(v, h) = \Theta_T(\tilde{v}, \tilde{h})$ *is also in* $C_T(M)$.

11.4. **Weak continuity of the mapping Θ_T.**

LEMMA 11.3. *The mapping Θ_T is weakly sequentially continuous from $C_T(M)$ into $C_T(M)$ (endowed with the norm of X_T).*

Proof. Let $(v_p,h_p)_{p\in\mathbb{N}}$ be a given sequence of elements of $C_T(M)$ weakly convergent (in Y_T) toward a given element $(v,h)\in C_T(M)$ ($C_T(M)$ is sequentially weakly closed as a closed convex set) and let $(v_{\sigma(p)},h_{\sigma(p)})_{p\in\mathbb{N}}$ be any subsequence of this sequence.

Since $V^3(T)$ is compactly embedded into $L^2(0,T;H^2(\Omega))$, we deduce the following strong convergence results in $L^2(0,T;L^2(\Omega))$ as $p\to\infty$:

$$(a^j_\ell)_p(a^k_\ell)_p \to a^j_\ell a^k_\ell \quad\text{and}\quad (a^j_\ell)_p(a^\ell_k)_p \to a^j_\ell a^\ell_k, \tag{11.15a}$$

$$[(a^j_\ell)_p(a^k_\ell)_p]_{,j} \to (a^j_\ell a^k_\ell)_{,j} \quad\text{and}\quad [(a^j_\ell)_p(a^\ell_k)_p]_{,j} \to (a^j_\ell a^\ell_k)_{,j}\ , \tag{11.15b}$$

$$(a^k_i)_p \to a^k_i. \tag{11.15c}$$

Now, let $(w_p,g_p)=\Theta_T(v_p,h_p)$ and let q_p be the associated pressure, so that $(q_p)_{p\in\mathbb{N}}$ is in a bounded set of $V^2(T)$. Since X_T is a reflexive Hilbert space, let $(w_{\sigma(p)},g_{\sigma(p)},q_{\sigma(p)})_{p\in\mathbb{N}}$ be a subsequence weakly converging in $X_T\times V^2(T)$ toward an element $(w,g,q)\in X_T\times V^2(T)$. Since $C_T(M)$ is weakly closed in X_T, we also have $(w,g)\in C_T(M)$.

For each $\phi\in L^2(0,T;H^1(\Omega))$, we deduce from (8.3) (and Remark 6) that

$$\int_0^T \Big[(w_t,\phi)_{L^2(\Omega)} + \frac{\mu}{2}\int_\Omega D_\eta w : D_\eta\phi dx + \sigma\int_{\Gamma_0} L_h(g)(g_{,\alpha}\phi_\alpha - \phi_z)dS + \int_{\Omega_0} q a^j_i \phi^i_{,j} dx\Big] dt = \int_0^T \langle F,\phi\rangle dt$$

which with the fact that, from (11.15), for all $t\in[0,T]$, $w\in\mathcal{V}_v$, provides that (w,g) is a solution of (3.17) in $C_T(M)$, i.e., $(w,g)=\Theta_T(v,h)$.

Therefore, we deduce that the whole sequence $(\Theta_T(v_n,h_n))_{n\in\mathbb{N}}$ weakly converges in $C_T(M)$ toward $\Theta_T(v,h)$, which concludes the lemma. □

11.5. **Uniqueness.** For the uniqueness result, we assume that u_0, F and Γ_0 are smooth enough (e.g. $u_0\in H^{5.5}(\Omega_0)$, $F\in V^4(T)$, Γ_0 is a $H^{8.5}$ surface) so that u_0 and the associated u_1, q_0 satisfy compatibility conditions (5.4). Therefore, the solution

(v,h,q) are such that $v \in V^6(T)$, $q \in L^2(0,T;H^5(\Omega_0))$ and $h \in L^\infty(0,T;H^7(\Gamma_0)) \cap L^2(0,T;H^{8.5}(\Gamma_0))$, $h_t \in L^\infty(0,T;H^5(\Gamma_0)) \cap L^2(0,T;H^{5.5}(\Gamma_0))$, $h_{tt} \in L^\infty(0,T;H^2(\Gamma_0)) \cap L^2(0,T;H^{3.5}(\Gamma_0))$. This implies $a \in L^\infty(0,T;H^5(\Omega_0))$ and hence by studying the elliptic problem

$$
\begin{aligned}
(a_i^\ell a_i^k q_{t,k})_{,\ell} &= \Big[\nu a_i^\ell (a_p^k a_p^j v^i_{,j})_{,k\ell} + a^\ell_{it} v^i_{,\ell} + a_i^\ell F_{,\ell}\Big]_t - [(a_i^\ell a_i^k)_t q_{,k}]_{,\ell} \qquad \text{in} \quad \Omega_0, \\
q_t &= J_h^{-2}\Big[\Big(\sigma L_h(h) N_i - \nu D_\eta(v)^\ell_i a_i^j N_j\Big)_t - (a_i^j N_j)_t q\Big] a_i^\ell N_\ell \quad \text{on} \quad \Gamma_0,
\end{aligned}
$$

we find that $q_t \in L^2(0,T;H^2(\Omega_0))$ and this implies $v_{tt} \in L^2(0,T;H^1(\Omega_0))$. By the interpolation theorem, we also conclude that $v_t \in L^\infty(0,T;H^{2.5}(\Omega_0))$.

Remark 27. *It is somewhat surprising that v_{tt} loses more derivatives than that the Navier-Stokes equation might suggest. In the fixed domain case, Navier-Stokes equations scales like the heat equations which implies that one time derivative, roughly speaking, equals two space derivatives for smooth boundary data. With moving boundary, this is no longer true since the boundary condition depends on the solution itself.*

Suppose (v,h,q) and $(\tilde v,\tilde h,\tilde q)$ are two set of solutions of (1.1). Then

$$
(v-\tilde v)_t - \nu[a_\ell^k D_\eta(v-\tilde v)^i_\ell]_{,k} = -a_i^k(q-\tilde q)_{,k} + \delta F \tag{11.16a}
$$

$$
a_i^j (v-\tilde v)^i_{,j} = \delta a \tag{11.16b}
$$

$$
\begin{aligned}
\Big[\nu[D_\eta(v-\tilde v)]^\ell_i - (q-\tilde q)\delta^\ell_i\Big] a_\ell^j N_j &= \sigma\Theta\Big[L_h(h-\tilde h)(-\nabla_0 h, 1)\Big]\circ\eta^\tau \\
&\quad + \delta L_1 + \delta L_2 + \delta L_3
\end{aligned} \tag{11.16c}
$$

$$
\begin{aligned}
(h-\tilde h)_t \circ \eta^\tau &= [h_{,\alpha}\circ\eta^\tau](v_\alpha - \tilde v_\alpha) - (v_z - \tilde v_z) \\
&\quad + \delta h_1 + \delta h_2 + \delta h_3
\end{aligned} \tag{11.16d}
$$

$$
(v-\tilde v)(0) = 0 \tag{11.16e}
$$

$$
(h-\tilde h)(0) = 0 \tag{11.16f}
$$

where

$$\delta F = f\circ\eta - f\circ\tilde\eta + \nu[(a^k_\ell a^j_\ell - \tilde a^k_\ell \tilde a^j_\ell)\tilde v^i_{,j}]_{,k} + \nu[(a^k_\ell a^j_i - \tilde a^k_\ell \tilde a^j_i)\tilde v^\ell_{,j}]_{,k} - (a^k_i - \tilde a^k_i)\tilde q_{,k} \tag{11.17a}$$

$$\delta a = (a^j_i - \tilde a^j_i)\tilde v^i_{,j} \tag{11.17b}$$

$$\delta L_1 = \sigma\Theta\Big[L_h(\tilde h)(\nabla_0 h - \nabla_0\tilde h, 0)\Big]\circ\eta^\tau - \nu(a^k_i a^j_\ell - \tilde a^k_i \tilde a^j_\ell)\tilde v^\ell_{,k}N_j - \nu(a^k_\ell a^j_\ell - \tilde a^k_\ell\tilde a^j_\ell)\tilde v^i_{,k}N_j + (a^j_i - \tilde a^j_i)\tilde q N_j \tag{11.17c}$$

$$\delta L_2 = \tilde\Theta[L_{\tilde h}(\tilde h)\circ\eta^\tau](\nabla_0\tilde h\circ\eta^\tau - \nabla_0\tilde h\circ\tilde\eta^\tau, 0) + \Big[\Theta L_h(\tilde h)\circ\eta^\tau - \tilde\Theta L_h(\tilde h)\circ\tilde\eta^\tau\Big](\nabla_0\tilde h\circ\tilde\eta^\tau, -1) \tag{11.17d}$$

$$\delta L_3 = \tilde\Theta\Big[[L_h(\tilde h) - L_{\tilde h}(\tilde h)](\nabla_0\tilde h, -1)\Big]\circ\tilde\eta^\tau \tag{11.17e}$$

$$\delta h_1 = (h_{,\alpha}\circ\eta^\tau - h_{,\alpha}\circ\tilde\eta^\tau)\tilde v_\alpha \tag{11.17f}$$

$$\delta h_2 = \Big[(h_{,\alpha} - \tilde h_{,\alpha})\circ\tilde\eta^\tau\Big]\tilde v_\alpha \tag{11.17g}$$

$$\delta h_3 = -(\tilde h_t\circ\eta^\tau - \tilde h_t\circ\tilde\eta^\tau) \tag{11.17h}$$

We will also use δL and δh to denote $\sum_{k=1}^{3} L_k$ and $\sum_{k=1}^{3}\delta h_k$ respectively.

Before proceeding, we state some useful lemmas.

Lemma 11.4. *For each non-negative integer k,*

$$\|(a-\tilde a)(t)\|^2_{H^k(\Omega_0)} \le Ct\int_0^t \|v - \tilde v\|^2_{H^{k+1}(\Omega_0)}ds \tag{11.18}$$

where the constant C depends on k.

By (11.18), it follows that for $k = 0,\ 1$,

$$\|\delta F\|^2_{H^k(\Omega_0)} \le Ct\int_0^t \|v-\tilde v\|^2_{H^{k+1}(\Omega_0)}ds$$

and for $k = 0, 1, 2$,

$$\|\delta a\|^2_{H^k(\Omega_0)} \le Ct\int_0^t \|v-\tilde v\|^2_{H^{k+1}(\Omega_0)}ds.$$

By the identity $a^j_{it} = -a^k_i v^\ell_{,k} a^j_\ell$,

$$\|\delta F_t\|^2_{L^2(\Omega_0)} \le C\Big[t(1+\|q_t\|^2_{H^1(\Omega_0)})\int_0^t \|v-\tilde v\|^2_{H^2(\Omega_0)}ds + \|v-\tilde v\|^2_{H^2(\Omega_0)}\Big].$$

Similar to (11.3) in [7], we also have the following estimates.

LEMMA 11.5. *For $f \in H^2(\Omega_0)$ and $g \in H^{1.5}(\Gamma_0)$,*

$$\|f\circ\eta - f\circ\tilde\eta\|_{L^2(\Omega_0)} \le C\sqrt{t}\|f\|_{H^2(\Omega_0)}\Big[\int_0^t \|v-\tilde v\|^2_{H^1(\Omega_0)}ds\Big]^{1/2}, \tag{11.19}$$

$$\|g\circ\eta^\tau - g\circ\tilde\eta^\tau\|_{L^2(\Gamma_0)} \le C\sqrt{t}\|g\|_{H^{1.5}(\Gamma_0)}\Big[\int_0^t \|v-\tilde v\|^2_{H^1(\Omega_0)}ds\Big]^{1/2}. \tag{11.20}$$

for some constant C.

REMARK 28. *Assuming the regularity of h, h_t and h_{tt} given in the beginning of this section, we have*

$$\|\delta L_2\|_{H^2(\Gamma_0)} + \|\delta h_1 + \delta h_3\|_{H^{2.5}(\Gamma_0)} \le C\sqrt{t}\Big[\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)}ds\Big]^{1/2} \tag{11.21}$$

and

$$\begin{aligned}&\|(\delta L_2)_t\|_{L^2(\Gamma_0)} + \|(\delta h_1+\delta h_3)_t\|_{H^1(\Gamma_0)}\\ &\le C\Big[\|v-\tilde v\|_{H^1(\Omega_0)} + \sqrt{t}\Big(\int_0^t \|v-\tilde v\|^2_{H^2(\Omega_0)}ds\Big)^{1/2}\Big]\end{aligned} \tag{11.22}$$

and

$$\begin{aligned}\|\nabla_0^2(\delta h_3)_t\|_{L^2(\Gamma_0)} \le C\Big[&\|v-\tilde v\|_{H^1(\Omega_0)} + \|v-\tilde v\|_{H^3(\Omega_0)}\\ &+\sqrt{t}\|\tilde h_{tt}\|_{H^{3.5}(\Gamma_0)}\Big(\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)}ds\Big)^{1/2}\Big].\end{aligned} \tag{11.23}$$

First, we establish an inequality similar to (9.9) (without κ). We multiply (11.16a) by $(v-\tilde{v})_t$ and then integrate over Ω_0 to obtain

$$
\begin{aligned}
&\frac{1}{2}\|(v-\tilde{v})_t\|^2_{L^2(\Omega_0)} + \frac{\nu}{4}\frac{d}{dt}\int_{\Omega_0}|D_\eta(v-\tilde{v})|^2dx \\
&\le \frac{1}{2}\|\delta F\|^2_{L^2(\Omega_0)} + 2\nu\int_{\Omega_0}[D_\eta(v-\tilde{v})]^j_i a^k_{jt}(v-\tilde{v})^i_{,k}dx - \int_{\Omega_0}(q-\tilde{q})(a^k_i-\tilde{a}^k_i)\tilde{v}^i_{t,k}dx \\
&\quad - \int_{\Omega_0}(q-\tilde{q})(a^k_i-\tilde{a}^k_i)_t\tilde{v}^i_{,k}dx + \int_{\Omega_0}(q-\tilde{q})a^k_{it}(v-\tilde{v})^i_{,k}dx \\
&\quad - \sigma\int_{\Gamma_0}\Big[[L_h(h-\tilde{h})(\nabla_0 h,-1)]\circ\eta^\tau\Big](v-\tilde{v})_t dS - \int_{\Gamma_0}\delta L(v-\tilde{v})_t dS \\
&\le C\Big[t\int_0^t\|v-\tilde{v}\|^2_{H^2(\Omega_0)}ds + \|q-\tilde{q}\|_{H^1(\Omega_0)}\Big[\|a-\tilde{a}\|_{H^1(\Omega_0)} + \|v-\tilde{v}\|_{H^1(\Omega_0)}\Big] \\
&\quad + \|v-\tilde{v}\|_{H^1(\Omega_0)}\|v-\tilde{v}\|_{H^2(\Omega_0)} + \|h-\tilde{h}\|_{H^4(\Gamma_0)}\|(v-\tilde{v})_t\|_{H^1(\Omega_0)}\Big] \\
&\quad + \Big[\|\delta L_2\|_{L^2(\Gamma_0)}\Big]\|(v-\tilde{v})_t\|_{H^1(\Omega_0)}
\end{aligned}
$$

where (11.19) is used in the estimate of $\|\delta F\|_{L^2(\Omega_0)}$. By (11.21) and Young's inequality,

$$
\begin{aligned}
&\|(v-\tilde{v})_t\|^2_{L^2(\Omega_0)} + \nu\frac{d}{dt}\int_{\Omega_0}|D_\eta(v-\tilde{v})|^2dx \\
&\le C(\delta)\Big[t\int_0^t\|v-\tilde{v}\|_{H^2(\Omega_0)}ds + \|v-\tilde{v}\|^2_{H^1(\Omega_0)} + \|h-\tilde{h}\|^2_{H^4(\Gamma_0)}\Big] \\
&\quad + \delta\Big[\|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)} + \|v-\tilde{v}\|^2_{H^2(\Omega_0)} + \|q-\tilde{q}\|^2_{H^1(\Omega_0)}\Big].
\end{aligned}
$$

Integrating the inequality above in time from 0 to t, we find that

$$
\begin{aligned}
&\|\nabla(v-\tilde{v})(t)\|^2_{L^2(\Omega_0)} + \int_0^t\|(v-\tilde{v})_t\|^2_{L^2(\Omega_0)}ds \\
&\le C(\delta)\int_0^t\Big[\|v-\tilde{v}\|^2_{H^1(\Omega_0)} + \|h-\tilde{h}\|^2_{H^4(\Gamma_0)}\Big]ds \\
&\quad + (C(\delta)t^2+\delta)\int_0^t\|v-\tilde{v}\|^2_{H^2(\Omega_0)}ds \\
&\quad + \delta\int_0^t\Big[\|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)} + \|q-\tilde{q}\|^2_{H^1(\Omega_0)}\Big]ds.
\end{aligned}
\tag{11.24}
$$

For the $L_t^2 H_x^3$ estimate for $v - \tilde{v}$ and $L^\infty H^4$ estimate for $h - \tilde{h}$, we need to estimate

$$D_1 := \int_{\Omega_0} \zeta_1^2 \nabla_0^2 (q - \tilde{q}) \nabla_0^2 \delta a dx, \quad D_2 := \int_{\Gamma_0} \Theta\Big[[L_h(h - \tilde{h})] \circ \eta^\tau\Big](\nabla_0^4 \delta h) dS,$$
$$D_3 := \int_{\Gamma_0} \delta L \cdot \nabla_0^4 (v - \tilde{v}) dS,$$

while for the the $L_t^2 H_x^1$ estimates for the time derivative of $v - \tilde{v}$, we need only estimate the following terms:

$$E_1 := \int_{\Omega_0} (q - \tilde{q})_t (\delta a)_t dx, \qquad E_2 := \int_{\Gamma_0} \Big[\Theta[L_h(h - \tilde{h})] \circ \eta^\tau\Big]_t (\delta h)_t dS,$$
$$E_3 := \int_{\Gamma_0} (\delta L)_t \cdot (v - \tilde{v})_t dS.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\Big[\|\zeta_1 \nabla_0^2 (v - \tilde{v})\|^2_{L^2(\Omega_0)} + 2\sigma E_h(\nabla_0^2 (h - \tilde{h}))\Big] + \frac{\nu}{4}\|\zeta_1 D_{\bar{\eta}} \nabla_0^2 (v - \tilde{v})\|^2_{L^2(\Omega_0)}$$
$$\leq C\Big[\|\delta F\|^2_{H^1(\Omega_0)} + \|(v - \tilde{v})_t\|^2_{L^2(\Omega_0)} + \|\nabla(v - \tilde{v})\|^2_{L^2(\Omega_0)} + \|\nabla \nabla_0 (v - \tilde{v})\|^2_{L^2(\Omega_1')}$$
$$+ \|\nabla_0^4 (h - \tilde{h})\|^2_{L^2(\Gamma_0)}\Big] + \delta \|v - \tilde{v}\|^2_{H^3(\Omega_0)} + D_1 + D_2 + D_3$$

and

$$\frac{1}{2}\frac{d}{dt}\Big[\|(v - \tilde{v})_t\|^2_{L^2(\Omega_0)} + 2\sigma E_h((h - \tilde{h})_t)\Big] + \frac{\nu}{4}\|\nabla (v - \tilde{v})_t\|^2_{L^2(\Omega_0)}$$
$$\leq C\Big[(\|\nabla_0^4 (h - \tilde{h})\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 (h - \tilde{h})_t\|^2_{L^2(\Gamma_0)}) + \|\delta F_t\|^2_{H^1(\Omega_0)'}\Big] + \delta \|v - \tilde{v}\|^2_{H^3(\Omega_0)}$$
$$+ E_1 + E_2 + E_3.$$

With the estimate already established,

$$D_1 \leq C(\delta) t \int_0^t \|v - \tilde{v}\|^2_{H^3(\Omega_0)} ds + \delta \|q - \tilde{q}\|^2_{H^2(\Omega_0)}. \tag{11.25}$$

For D_2, by the elliptic estimate (from (11.16c) and (11.21))

$$\begin{aligned} \|h - \tilde{h}\|^2_{H^{5.5}(\Gamma_0)} &\leq C\Big[\|D_\eta (v - \tilde{v})\|^2_{H^{1.5}(\Gamma_0)} + \|q - \tilde{q}\|^2_{H^{1.5}(\Gamma_0)} + \|\delta L\|^2_{H^{1.5}(\Gamma_0)}\Big] \\ &\leq C\Big[\|v - \tilde{v}\|^2_{H^3(\Omega_0)} + \|q - \tilde{q}\|^2_{H^2(\Omega_0)} + \|h - \tilde{h}\|^2_{H^4(\Gamma_0)} \\ &\qquad + t \int_0^t \|v - \tilde{v}\|^2_{H^3(\Omega_0)} ds\Big], \end{aligned} \tag{11.26}$$

we find, by (11.21), that

$$
\begin{aligned}
D_2 &\le \|L_h(h-\tilde h)\|_{H^{1.5}(\Gamma_0)}\|\delta h\|_{H^{2.5}(\Gamma_0)} \\
&\le \delta\Big[\|v-\tilde v\|^2_{H^3(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)} + t\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)}ds\Big] \\
&\quad + C(\delta)\|h-\tilde h\|^2_{H^4(\Gamma_0)}
\end{aligned}
\tag{11.27}
$$

Similarly, D_3 is bounded by the same right-hand side in (11.27).

Summing all the estimates above, we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|\zeta_1\nabla_0^2(v-\tilde v)\|^2_{L^2(\Omega_0)} + 2\sigma E_h(\nabla_0^2(h-\tilde h))\Big] + \frac{\nu}{4}\|\zeta_1 D_{\bar\eta}\nabla_0^2(v-\tilde v)\|^2_{L^2(\Omega_0)} \\
&\le C(\delta)\Big[t\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)}ds + \|(v-\tilde v)_t\|^2_{L^2(\Omega_0)} + \|\nabla(v-\tilde v)\|^2_{L^2(\Omega_0)} \\
&\qquad + \|\nabla_0^4(h-\tilde h)\|^2_{L^2(\Gamma_0)}\Big] + \delta\Big[\|v-\tilde v\|^2_{H^3(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)}\Big],
\end{aligned}
$$

where we bound the term $\|\nabla\nabla_0(v-\tilde v)\|^2_{L^2(\Omega_1')}$ by the interpolation inequality

$$
\|v-\tilde v\|^2_{H^2(\Omega_0)} \le C(\delta_1)\|v-\tilde v\|^2_{H^1(\Omega_0)} + \delta_1\|v-\tilde v\|^2_{H^3(\Omega_0)}
$$

and then choose $\delta_1 > 0$ small enough so that $C(\delta)\delta_1 \le \delta$.

Integrating the inequality above in time from 0 to t, (11.24) implies that

$$
\begin{aligned}
&\Big[\|\nabla_0^2(v-\tilde v)(t)\|^2_{L^2(\Omega_1)} + \|\nabla_0^4(h-\tilde h)(t)\|^2_{L^2(\Gamma_0)}\Big] + \int_0^t \|\nabla\nabla_0^2(v-\tilde v)\|^2_{L^2(\Omega_1)}ds \\
&\le C(\delta)\int_0^t \Big[\|(v-\tilde v)_t\|^2_{L^2(\Omega_0)} + \|\nabla_0(v-\tilde v)\|^2_{L^2(\Omega_0)} + \|\nabla_0^4(h-\tilde h)\|^2_{L^4(\Gamma_0)}\Big]ds \\
&\quad + (C(\delta)t^2+\delta)\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)}ds + \delta\int_0^t \|q-\tilde q\|^2_{H^2(\Omega_0)}ds
\end{aligned}
\tag{11.28}
$$

For the estimates of $E_i's$, note that

$$
\Big[L_h(h-\tilde h)\circ\eta^\tau\Big]_t = \Big[L_h(h-\tilde h)\Big]_t\circ\eta^\tau + v^\tau\cdot\Big[[\nabla_0 L_h(h-\tilde h)]\circ\eta^\tau\Big]
$$

and hence by $H^{-2}(\Gamma_0)$-$H^2(\Gamma_0)$ duality pairing and standard Hölder inequality,

$$
\begin{aligned}
E_2 &\le C\Big[\|[L_h(h-\tilde{h})]_t\|_{H^{-2}(\Gamma_0)}\|\delta h_t\|_{H^2(\Gamma_0)} + \|L_h(h-\tilde{h})\|_{H^1(\Gamma_0)}\|\delta h_t\|_{L^2(\Gamma_0)}\Big] \\
&\le C\Big[\|\nabla_0^2(h-\tilde{h})_t\|_{L^2(\Gamma_0)}\|(\delta h)_t\|_{H^2(\Gamma_0)} + \|h-\tilde{h}\|_{H^5(\Gamma_0)}\|(\delta h)_t\|_{L^2(\Gamma_0)}\Big] \\
&\le C(\delta)\Big[\|v-\tilde{v}\|^2_{H^1(\Omega_0)} + \|h-\tilde{h}\|^2_{H^4(\Gamma_0)} + (1+\|\tilde{h}_{tt}\|^2_{H^{3.5}(\Gamma_0)})\|\nabla_0^2(h-\tilde{h})_t\|^2_{L^2(\Gamma_0)} \\
&\quad + t\int_0^t \|v-\tilde{v}\|^2_{H^3(\Omega_0)}ds\Big] + \delta\Big[\|v-\tilde{v}\|^2_{H^3(\Omega_0)} + \|q-\tilde{q}\|^2_{H^2(\Omega_0)}\Big]
\end{aligned} \tag{11.29}
$$

where (11.22), (11.23), (11.26) and Young's inequality are used in the bound for $\|\delta h_t\|_{L^2(\Gamma_0)}$ and $\|h-\tilde{h}\|_{H^5(\Gamma_0)}$.

Now we turn to the estimates for E_3. By the regularity of Θ, h and $\tilde{h}$,

$$
\begin{aligned}
\|(\delta L_1)_t\|^2_{L^2(\Gamma_0)} \le C\Big[&t\int_0^t \|v-\tilde{v}\|^2_{H^3(\Omega_0)}ds + \|v-\tilde{v}\|^2_{H^2(\Omega_0)} + \|h-\tilde{h}\|^2_{H^2(\Gamma_0)} \\
&+ \|(h-\tilde{h})_t\|^2_{H^2(\Gamma_0)}\Big]
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_{\Gamma_0}(\delta L_1)_t(v-\tilde{v})_t dS \le C(\delta)\|(\delta_1)_t\|^2_{L^2(\Gamma_0)} + \delta\|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)} \\
&\le C(\delta)\Big[t\int_0^t \|v-\tilde{v}\|^2_{H^3(\Omega_0)}ds + \|v-\tilde{v}\|^2_{H^1(\Omega_0)} + \|h-\tilde{h}\|^2_{H^4(\Gamma_0)} + \|(h-\tilde{h})_t\|^2_{H^2(\Gamma_0)}\Big] \\
&\quad + \delta\Big[\|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)} + \|v-\tilde{v}\|^2_{H^3(\Omega_0)}\Big].
\end{aligned}
$$

By (11.22),

$$
\begin{aligned}
\int_{\Gamma_0}(\delta L_2)_t(v-\tilde{v})_t dS \le C(\delta)\Big[&t\int_0^t \|v-\tilde{v}\|^2_{H^2(\Omega_0)} + \|v-\tilde{v}\|^2_{H^1(\Omega_0)}\Big] \\
&+ \delta\|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)}.
\end{aligned}
$$

It remains to estimate the term $\int_{\Gamma_0}(\delta L_3)_t\cdot(v-\tilde{v})_t dS$. Note that

$$
\begin{aligned}
(\delta L_3)_t = &\Big[\tilde{\Theta}[(L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h}))(\nabla_0\tilde{h}, -1)\Big]_t \circ \tilde{\eta}^\tau \quad (\equiv (\delta L_3)^1_t) \\
&+ \tilde{v}^\tau\cdot\Big[\nabla_0[\tilde{\Theta}(L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h}))(\nabla_0\tilde{h}, -1)]\Big]\circ\eta^\tau \quad (\equiv (\delta L_3)^2_t).
\end{aligned}
$$

It is easy to see that

$$\int_{\Gamma_0} (\delta L_3)_t^2 (v-\tilde{v})_t dS \leq C\|\nabla_0[(L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h}))(\nabla_0\tilde{h}, -1)]\|_{L^2(\Gamma_0)}\|(v-\tilde{v})_t\|_{H^1(\Omega_0)}^2$$
$$\leq C(\delta)\|h-\tilde{h}\|_{H^4(\Gamma_0)}^2 + \delta\|(v-\tilde{v})_t\|_{H^1(\Omega_0)}^2.$$

By the identity

$$\begin{aligned}
&(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau, -1)\cdot(v-\tilde{v})_t \\
&= [(\nabla_0\tilde{h}\circ\tilde{\eta}^\tau, -1)\cdot(v-\tilde{v})]_t - ([\nabla_0\tilde{h}\circ\tilde{\eta}^\tau]_t, 0)\cdot(v-\tilde{v}) \\
&= (h-\tilde{h})_t\circ\tilde{\eta}^\tau + (\delta\tilde{h})_t - ([\nabla_0\tilde{h}\circ\tilde{\eta}^\tau]_t, 0)\cdot(v-\tilde{v}),
\end{aligned}$$

we find that

$$\begin{aligned}
&\int_{\Gamma_0} (\delta L_3)_t^1 \cdot (v-\tilde{v})_t dS \\
&= \int_{\Gamma_0} [\tilde{\Theta}(L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h}))]_t \Big[(h-\tilde{h})_t - ([\nabla_0\tilde{h}\circ\tilde{\eta}^\tau]_t, 0)\cdot(v-\tilde{v})\Big] dS \quad (\equiv M_1) \\
&+ \int_{\Gamma_0} [\tilde{\Theta}(L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h}))]_t (\delta\tilde{h})_t dS \quad (\equiv M_2) \\
&+ \int_{\Gamma_0} [\tilde{\Theta}(L_h(\tilde{h}) - L_{\tilde{h}}(\tilde{h}))](\nabla_0\tilde{h}_t\circ\tilde{\eta}^\tau, 0)\cdot(v-\tilde{v})_t dS. \quad (\equiv M_3)
\end{aligned}$$

Since in $[\tilde{\Theta}(L_h(h) - L_h(\tilde{h}))]_t$, the highest order term is $\nabla_0^3(h-\tilde{h})_t$, we can estimate M_1 and M_3 by $H^{-1}(\Gamma_0)$-$H^1(\Gamma_0)$ duality pairing and obtain

$$\begin{aligned}
&M_1 \leq C(\delta)\|\nabla_0^2(h-\tilde{h})_t\|_{L^2(\Gamma_0)}^2 + \delta\|v-\tilde{v}\|_{H^3(\Omega_0)}^2, \\
&M_2 \leq C\Big[\|v-\tilde{v}\|_{H^1(\Omega_0)}^2 + \|\nabla_0^2(h-\tilde{h})_t\|_{L^2(\Gamma_0)}^2\Big] + t\int_0^t \|v-\tilde{v}\|_{H^2(\Omega_0)}^2 ds
\end{aligned}$$

while for E_3, standard Hölder inequality is applied and we have

$$M_3 \leq C(\delta)\|h-\tilde{h}\|_{H^4(\Gamma_0)}^2 + \delta\|(v-\tilde{v})_t\|_{H^1(\Omega_0)}^2.$$

Therefore

$$\begin{aligned}
E_3 \leq C(\delta)\Big[t\int_0^t \|v-\tilde{v}\|_{H^3(\Omega_0)}^2 ds + \|v-\tilde{v}\|_{H^1(\Omega_0)}^2 + \|h-\tilde{h}\|_{H^4(\Gamma_0)}^2 \\
+ \|(h-\tilde{h})_t\|_{H^2(\Gamma_0)}^2\Big] + \delta\Big[\|(v-\tilde{v})_t\|_{H^1(\Omega_0)}^2 + \|v-\tilde{v}\|_{H^3(\Omega_0)}^2\Big].
\end{aligned} \tag{11.30}$$

Finally, similar to the estimates in Appendix C.2, we estimate the time integral of E_1. The estimate of E_1 requires the estimates of δa_t and δa_{tt}. By (11.18) and the identity $a^j_{it} = -a^k_i v^\ell_{,k} a^j_\ell$, we find that

$$\begin{aligned}\|\delta a_t\|^2_{L^2(\Omega_0)} &\leq C\Big[\|a-\tilde{a}\|^2_{H^1(\Omega_0)} + \|v-\tilde{v}\|^2_{H^1(\Omega_0)}\Big]\\ &\leq Ct\int_0^t \Big[\|v-\tilde{v}\|^2_{H^2(\Omega_0)} + \|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)}\Big]ds\end{aligned}$$

and

$$\begin{aligned}\|\delta a_{tt}\|^2_{L^2(\Omega_0)} \leq C\Big[&\|a-\tilde{a}\|^2_{H^2(\Omega_0)}\|\tilde{v}_{tt}\|^2_{H^1(\Omega_0)} + \|a-\tilde{a}\|^2_{H^1(\Omega_0)} + \|v-\tilde{v}\|^2_{H^2(\Omega_0)}\\ &+ \|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)}\Big]\\ \leq C\Big[&t\Big(1+\|\tilde{v}_{tt}\|^2_{H^1(\Omega_0)}\Big)\int_0^t \|v-\tilde{v}\|^2_{H^3(\Omega_0)}ds + \|v-\tilde{v}\|^2_{H^2(\Omega_0)}\\ &+ \|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)}\Big].\end{aligned}$$

Therefore, integrating by parts in time,

$$\begin{aligned}&\int_0^t\int_{\Omega_0}(q-\tilde{q})_t(\delta a)_t dxds = \int_{\Omega_0}(q-\tilde{q})(t)(\delta a)_t(t)dx - \int_0^t\int_{\Omega_0}(q-\tilde{q})\delta a_{tt}dxds\\ &\leq C(\delta)\Big[\|\delta a_t\|^2_{L^2(\Omega_0)} + \int_0^t \|q-\tilde{q}\|^2_{L^2(\Omega_0)}ds\Big] + \delta\Big[\|q-\tilde{q}\|^2_{L^2(\Omega_0)} + \int_0^t \|\delta a_{tt}\|^2_{L^2(\Omega_0)}ds\Big]\\ &\leq C(\delta)\Big[t\int_0^t\Big(\|v-\tilde{v}\|^2_{H^3(\Omega_0)} + \|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)}\Big)ds + \int_0^t \|v-\tilde{v}\|^2_{H^1(\Omega_0)}ds\Big]\\ &\quad + \delta\Big[\|q-\tilde{q}\|^2_{L^2(\Omega_0)} + \int_0^t\Big(\|v-\tilde{v}\|^2_{H^3(\Omega_0)} + \|(v-\tilde{v})_t\|^2_{H^1(\Omega_0)} + \|q-\tilde{q}\|^2_{L^2(\Omega_0)}\Big)ds\Big]\end{aligned}$$

where we also use the face that $q(0) = \tilde{q}(0)$ defined by 5.1.

Remark 29. *In order to close up all the estimates, $h_{tt} \in H^{3.5}(\Gamma_0)$ is needed when estimating E_2 and $v_{tt} \in H^1(\Omega_0)$ is needed when estimating δa_{tt}. These two estimates necessitate the regularity used above.*

Combining all the estimates and then integrating the inequality above in time from 0 to t, we find that

$$
\begin{aligned}
&\Big[\|(v-\tilde v)_t(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2(h-\tilde h)_t\|^2_{L^2(\Gamma_0)}\Big] + \int_0^t \|\nabla(v-\tilde v)_t\|^2_{L^2(\Omega_0)} ds \\
&\le C(\delta)\int_0^t \Big[\|v-\tilde v\|^2_{H^1(\Omega_0)} + \|\nabla_0^4(h-\tilde h)\|^2_{L^2(\Gamma_0)} + (1+\|\tilde h_{tt}\|^2_{H^{4.5}(\Gamma_0)}) \\
&\qquad \times \|\nabla_0^2(h-\tilde h)_t\|^2_{L^2(\Gamma_0)}\Big] ds \\
&\quad + (C(\delta)(t+t^2)+\delta)\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)} ds + \delta\|q-\tilde q\|^2_{L^2(\Omega_0)} \\
&\quad + \delta\int_0^t \Big[\|(v-\tilde v)_t\|^2_{H^1(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)}\Big] ds.
\end{aligned}
\tag{11.31}
$$

Similar to the estimates in (9.3) and (9.4) (except that the vector field is not divergence-free), (11.16) implies that

$$
\begin{aligned}
&\|v-\tilde v\|^2_{H^1(\Omega_0)} + \|q-\tilde q\|^2_{L^2(\Omega_0)} \\
&\le C\Big[\|\delta F\|^2_{H^1(\Omega_0)'} + \|(v-\tilde v)_t\|^2_{H^1(\Omega_0)'} + \|\delta a\|^2_{L^2(\Omega_0)} + \|v-\tilde v\|^2_{H^{0.5}(\Gamma_0)}\Big] \\
&\le C\Big[t\int_0^t \|v-\tilde v\|^2_{H^1(\Omega_0)} ds + \|(v-\tilde v)_t\|^2_{L^2(\Omega_0)} + \|v-\tilde v\|^2_{H^1(\Omega_1)}\Big]
\end{aligned}
\tag{11.32}
$$

and

$$
\begin{aligned}
&\|v-\tilde v\|^2_{H^3(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)} \\
&\le C\Big[t\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)} ds + \|(v-\tilde v)_t\|^2_{H^1(\Omega_0)} + \|\nabla_0^2(v-\tilde v)\|^2_{H^1(\Omega_1)}\Big].
\end{aligned}
\tag{11.33}
$$

Remark 30. *Note that (11.33) also implies that*

$$
\begin{aligned}
&\int_0^t \Big[\|v-\tilde v\|^2_{H^3(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)}\Big] ds \\
&\le Ct^2\int_0^t \|v-\tilde v\|^2_{H^3(\Omega_0)} ds + C\int_0^t \Big[\|(v-\tilde v)_t\|^2_{H^1(\Omega_0)} + \|\nabla_0^2(v-\tilde v)\|^2_{H^1(\Omega_1)}\Big] ds.
\end{aligned}
$$

For a fixed $T>0$ (depending on the constant C) small enough, for $0<t\le T$,

$$
\begin{aligned}
&\int_0^t \Big[\|v-\tilde v\|^2_{H^3(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)}\Big] ds \\
&\le C\int_0^t \Big[\|(v-\tilde v)_t\|^2_{H^1(\Omega_0)} + \|\nabla_0^2(v-\tilde v)\|^2_{H^1(\Omega_1)}\Big] ds
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_0^t \Big[\|v-\tilde v\|^2_{H^3(\Omega_0)} + \|q-\tilde q\|^2_{H^2(\Omega_0)}\Big] ds \\
&\le C(t+1)\int_0^t \Big[\|(v-\tilde v)_t\|^2_{H^1(\Omega_0)} + \|\nabla_0^2(v-\tilde v)\|^2_{H^1(\Omega_1)}\Big] ds. \qquad (11.34)
\end{aligned}
$$

Summing (11.24), (11.28) and (11.31) and then applying (11.32) and (11.34), we find that

$$
Y(t) + \int_0^t Z(s)ds \le C(\delta)\int_0^t k(s)Y(s)ds + (C(\delta)(t^2+t)+\delta)\int_0^t Z(s)ds \qquad (11.35)
$$

where

$$
\begin{aligned}
k(t) &= 1 + \|\tilde h_{tt}(t)\|^2_{H^{3.5}(\Gamma_0)} \\
Y(t) &= \Big[\|v-\tilde v(t)\|^2_{H^1(\Omega_0)} + \|\nabla_0^2(v-\tilde v)(t)\|^2_{L^2(\Omega_1)} + \|(v-\tilde v)_t(t)\|^2_{L^2(\Omega_0)} \\
&\quad + \|h-\tilde h\|^2_{H^4(\Gamma_0)} + \|(h-\tilde h)_t\|^2_{H^2(\Gamma_0)}\Big], \\
Z(t) &= \|(v-\tilde v)_t(t)\|^2_{H^1(\Omega_0)} + \|\nabla\nabla_0^2(v-\tilde v)(t)\|^2_{L^2(\Omega_1)}.
\end{aligned}
$$

By letting $\delta = 1/4$ and choosing $T_u \le T$ so that $C(\delta)(T_u^2 + T_u) \le 1/4$,

$$
Y(t) + \int_0^t Z(s)ds \le C\int_0^t k(s)Y(s)ds \qquad (11.36)
$$

for all $0 < t \le T_u$. Since $Y(0) = 0$, the uniqueness of the solution follows from that $Y(t) = 0$ for all $0 < t \le T_u$.

12. The membrane energy

In order to study the problem with the membrane traction $\mathfrak{t}_{\text{mem}}$ included in our formulation, we need a modification of the closed convex set to which apply the Tychonoff fixed-point theorem. Define

$$
C_T(M) = \Big\{(v,h)\in V^3(T)\times H(T)\Big| \Big\|\int_0^t v^\tau(s)ds\Big\|_{L^2(0,T;H^{3.5}(\Gamma_0))} \le M\Big\}
$$

where v^τ are defined as in Remark 12.

Remark 31. *If* $\int_0^t v^\tau ds \in L^2(0,T;H^{3.5}(\Gamma_0))$ *and* $v \in L^2(0,T;H^{2.5}(\Gamma_0))$, *we will have* $\int_0^t v^\tau ds \in \mathcal{C}^0([0,T];H^3(\Gamma_0))$ *with*

$$\Big\| \int_0^t v^\tau(s)ds \Big\|^2_{H^3(\Gamma_0)} \leq C\Big[\int_0^t \|v(s)\|^2_{H^3(\Omega_0)} ds + \int_0^t \Big\| \int_0^{t'} v^\tau(s)ds \Big\|^2_{H^{3.5}(\Gamma_0)} dt' \Big]$$

for some constant C *depending on the geometry of* Ω_0.

Taking $\mathfrak{t}_{\text{mem}}$ into account, we linearize problem (3.17) by replacing (8.1c) with

$$[\nu D_{\tilde\eta}(v)_i^j - q\delta_i^j]\tilde a_j^\ell N_\ell = \sigma\tilde\Theta\Big[[L_{\tilde h}(h)](\nabla_0\tilde h,-1)\Big]\circ\tilde\eta^\tau \qquad \text{on } (0,T)\times\Gamma_0\,, \tag{12.1}$$
$$+ L_{\tilde m}(\eta^\tau) + \mathcal{E}_2(\tilde\eta) + \mathcal{E}_1(\tilde\eta)$$

where

$$L_{\tilde m}(\eta^\tau) = -(G_{\tilde\eta^z})_{\gamma\delta}\Big[\mu\tilde\eta^\kappa_{,\alpha}\Big(\tilde\eta^\delta_{,\beta}\eta^\gamma_{,\beta\alpha} + \tilde\eta^\gamma_{,\alpha}\eta^\delta_{,\beta\beta}\Big) + \frac{2\mu\lambda}{2\mu+\lambda}\tilde\eta^\kappa_{,\beta}\tilde\eta^\delta_{,\alpha}\eta^\gamma_{,\alpha\beta}\Big]\frac{\partial}{\partial y^\kappa}$$

and

$$\begin{aligned}
\mathcal{E}_2(\tilde\eta) = & -\Big[\mu(\tilde g_{\alpha\beta} - g_{0\alpha\beta})\tilde\eta^\kappa_{,\alpha\beta} + \frac{\mu\lambda}{2\mu+\lambda}(\tilde g_{\alpha\alpha} - g_{0\alpha\alpha})\tilde\eta^\kappa_{,\beta\beta})\Big]\frac{\partial}{\partial y^\kappa} \\
& -\Big[\mu(\tilde g_{\alpha\beta} - g_{0\alpha\beta})\tilde\eta^z_{,\alpha\beta} + \frac{\mu\lambda}{2\mu+\lambda}(\tilde g_{\alpha\alpha} - g_{0\alpha\alpha})\tilde\eta^z_{,\beta\beta}\Big]\frac{\partial}{\partial z} \\
& -\Big[\mu\tilde\eta^\kappa_{,\alpha}\Big(\tilde\eta^z_{,\alpha\beta}\tilde\eta^z_{,\beta} + \tilde\eta^z_{,\alpha}\tilde\eta^z_{,\beta\beta}\Big) + \frac{2\mu\lambda}{2\mu+\lambda}\tilde\eta^\kappa_{,\beta}\tilde\eta^z_{,\alpha}\tilde\eta^z_{,\alpha\beta}\Big]\frac{\partial}{\partial y^\kappa} \\
& -\Big[\mu(G_{\tilde\eta^z})_{ij}\eta^z_{,\alpha}\Big(\tilde\eta^i_{,\alpha}\tilde\eta^j_{,\beta\beta} + \tilde\eta^i_{,\beta}\tilde\eta^j_{,\alpha\beta}\Big) + \frac{2\mu\lambda}{2\mu+\lambda}(G_{\tilde\eta^z})_{ij}\tilde\eta^z_{,\beta}\tilde\eta^i_{,\alpha\beta}\tilde\eta^j_{,\alpha}\Big]\frac{\partial}{\partial z},
\end{aligned}$$

and $\mathcal{E}_1(\tilde\eta)$ contains the remaining terms which we also treat as a given forcing on the boundary. Note that $\mathcal{E}_1(\tilde\eta)$ consists of only the lower order terms of $\mathfrak{t}_{\text{mem}}$, and

$$\|\mathcal{E}_1(\tilde\eta)\|_{H^{0.5}(\Gamma_0)} + \|\mathcal{E}_2(\tilde\eta)\|_{H^{0.5}(\Gamma_0)} \leq C(M)t^{1/2}\,, \tag{12.2a}$$

$$\|\mathcal{E}_1(\tilde\eta)\|_{H^{1.5}(\Gamma_0)} + \|\mathcal{E}_2(\tilde\eta)\|_{H^{1.5}(\Gamma_0)} \leq C(M)t^{1/2}\Big\| \int_0^t \tilde v^\tau(s)ds \Big\|_{H^{3.5}(\Gamma_0)} \tag{12.2b}$$

for some constant C depending on the geometry of Γ_0. The regularized problem is (8.2) with (8.2c) replaced by (12.1) and with $\tilde\eta$ replaced by $\bar\eta$.

Note that since $L_{\tilde m}(\mathrm{Id})=0$, it follows that

$$L_{\tilde m}(\eta^\tau)=L_{\tilde m}(\int_0^t v^\tau(s)ds).$$

LEMMA 12.1. *There exists a $T>0$ such that $L_{\tilde m}$ is uniformly elliptic. In other words, there exists a constant $\nu_2>0$ such that*

$$(G_{\tilde\eta^z})_{\kappa\sigma}(G_{\tilde\eta^z})_{\gamma\delta}\Big[\mu\tilde\eta^\kappa_{,\beta}\tilde\eta^\delta_{,\alpha}+\mu\tilde\eta^\kappa_{,\iota}\tilde\eta^\delta_{,\iota}\delta^\alpha_\beta+\frac{2\mu\lambda}{2\mu+\lambda}\tilde\eta^\kappa_{,\beta}\tilde\eta^\delta_{,\alpha}\Big]\xi^\gamma_\alpha\xi^\sigma_\beta\geq\nu_2|\xi|^2$$

for all $0\leq t\leq T$.

Proof. Let $\mathcal{N}_{\alpha\beta\gamma\sigma}=(G_{\tilde\eta^z})_{\kappa\sigma}(G_{\tilde\eta^z})_{\gamma\delta}\Big[\mu\tilde\eta^\kappa_{,\beta}\tilde\eta^\delta_{,\alpha}+\mu\tilde\eta^\kappa_{,\iota}\tilde\eta^\delta_{,\iota}\delta^\alpha_\beta+\frac{2\mu\lambda}{2\mu+\lambda}\tilde\eta^\gamma_{,\beta}\tilde\eta^\delta_{,\alpha}\Big]$. Then

$$\mathcal{N}_{\alpha\beta\gamma\sigma}(0)=g_{0\kappa\sigma}g_{0\gamma\delta}\Big[\mu\Big(\delta^\kappa_\beta\delta^\delta_\alpha+\delta^\delta_\kappa\delta^\beta_\alpha\Big)+\frac{2\mu\lambda}{2\mu+\lambda}\delta^\kappa_\beta\delta^\delta_\alpha\Big]$$

and hence

$$\begin{aligned}\mathcal{N}_{\alpha\beta\gamma\sigma}(0)\xi^\gamma_\alpha\xi^\sigma_\beta&=\mu\Big[g_{0\kappa\sigma}g_{0\gamma\delta}\xi^\gamma_\delta\xi^\sigma_\kappa+g_{0\kappa\sigma}g_{0\kappa\gamma}\xi^\gamma_\beta\xi^\sigma_\beta\Big]+\frac{2\mu\lambda}{2\mu+\lambda}g_{0\alpha\gamma}g_{0\beta\sigma}\xi^\gamma_\alpha\xi^\sigma_\beta\\&=\mu\sum_{\beta,\kappa=1}^{2}|g_{0\kappa\sigma}\xi^\sigma_\beta|^2+\Big(\mu+\frac{2\mu\lambda}{2\mu+\lambda}\Big)|g_{0\alpha\gamma}\xi^\gamma_\alpha|^2\\&\geq c|\xi|^2\end{aligned}$$

for some constant $c>0$. Since $\mathcal{N}\in\mathcal{C}([0,T];H^{1.5}(\Gamma_0))$, we conclude the lemma by continuity. ☐

Taking the inner-product of (12.1) with $\frac{\partial}{\partial y^\sigma}+(G_{\tilde\eta^z})_{\gamma\sigma}\tilde h_{,\gamma}\circ\tilde\eta^\tau\frac{\partial}{\partial z}$, we find that

$$\begin{aligned}(G_{\tilde\eta^z})_{\kappa\sigma}L_{\tilde m}(\eta^\tau)=&\langle[\nu D_{\tilde\eta}(v)^j_i-q\delta^j_i]\tilde a^\ell_jN_\ell,\frac{\partial}{\partial y^\sigma}+(G_{\tilde\eta^z})_{\gamma\sigma}h_{,\gamma}\circ\tilde\eta^\tau\frac{\partial}{\partial z}\rangle\\&-\langle\mathcal{E}_1(\tilde\eta)+\mathcal{E}_2(\tilde\eta),\frac{\partial}{\partial y^\sigma}+(G_{\tilde\eta^z})_{\gamma\sigma}h_{,\gamma}\circ\tilde\eta^\tau\frac{\partial}{\partial z}\rangle\end{aligned}\tag{12.3}$$

By (9.3), (9.4) and Lemma 12.1,

$$\begin{aligned}\Big\|\int_0^t v^\tau(s)ds\Big\|^2_{H^{2.5}(\Gamma_0)}\leq C(M)\Big[&\|v_t\|^2_{L^2(\Omega_0)}+\|\nabla v\|_{L^2(\Omega_0)}+\|\nabla_0 v\|^2_{H^1(\Omega_1)}\\&+\|\nabla_0^4h\|^2_{L^2(\Gamma_0)}+\|F\|^2_{L^2(\Omega_0)}+1\Big]\end{aligned}\tag{12.4}$$

and

$$\Big\| \int_0^t v^\tau(s)ds \Big\|^2_{H^{3.5}(\Gamma_0)} \le C(M)\Big[\|v_t\|^2_{H^1(\Omega_0)} + \|\nabla v\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} + \|F\|^2_{H^1(\Omega_0)} + 1 + t\Big\| \int_0^t \tilde v^\tau(s)ds \Big\|^2_{H^{3.5}(\Gamma_0)} \Big]. \tag{12.5}$$

Taking the inner-product of (12.1) with $(\nabla_0 \tilde h \circ \tilde\eta^\tau, -1)$, then

$$L_{\tilde h}(h) = \frac{1}{\sigma} J_{\tilde h}^{-2} \Big[\langle [\nu D_{\tilde\eta}(v)^j_i - q\delta^j_i] \tilde a^\ell_j N_\ell, (\nabla_0 \tilde h \circ \tilde\eta^\tau, -1)\rangle - \langle L_{\tilde m}(\eta^\tau) + \mathcal{E}_1(\tilde\eta) + \mathcal{E}_2(\tilde\eta), (\nabla_0 \tilde h \circ \tilde\eta^\tau, -1)\rangle \Big] \circ \tilde\eta^{-\tau} \tag{12.6}$$

and hence by ellipticity of $L_{\tilde h}$ and (12.5),

$$\|h\|_{H^{5.5}(\Gamma_0)} \le C(M)\Big[\|v_t\|^2_{H^1(\Omega_0)} + \|\nabla v\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} + \|\nabla_0^4 h\|^4_{L^2(\Gamma_0)} + \|F\|^2_{H^1(\Omega_0)} + 1 + t\Big\| \int_0^t \tilde v^\tau(s)ds \Big\|^2_{H^{3.5}(\Gamma_0)} \Big]. \tag{12.7}$$

Because of (3.14), we also have that

$$\Big\| \int_0^t v^z(s)ds \Big\|^2_{H^{3.5}(\Gamma_0)} \le C(M) t^{1/4} \Big[1 + \Big\| \int_0^t v^\tau(s)ds \Big\|_{H^{3.5}(\Gamma_0)} \Big]. \tag{12.8}$$

Similarly, we find that

$$\begin{aligned} &\Big[\|\nabla_0^2 v\|^2_{L^2(\Omega_1)} + \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} + \Big\| \int_0^t \nabla_0^3 v^\tau(s)ds \Big\|^2_{L^2(\Gamma_0)} \Big] + \int_0^t \|\nabla\nabla_0^2 v\|^2_{L^2(\Omega_1)} ds \\ &\le C(M)\Big[N_4(u_0,F) + \int_0^t \Big(\|v_t\|^2_{H^1(\Omega_0)} + K(s)\|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} \Big) ds \Big] \\ &\quad + (\delta + T^{1/2}) \int_0^t \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} ds \end{aligned}$$

and

$$\begin{aligned} &\Big[\|v_t(t)\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0 v^\tau(t)\|^2_{L^2(\Gamma_0)} \Big] + \int_0^t \|\nabla v_t\|^2_{L^2(\Omega_0)} ds \\ &\le C(M)\Big[N_4(u_0,F) + \int_0^t K(s)\Big(\|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_t\|^2_{L^2(\Gamma_0)} \Big) ds \Big] \\ &\quad + (\delta + C(M)t^{1/2}) \int_0^t \|\nabla_0^2 v\|^2_{H^1(\Omega_1)} ds + \delta \|\nabla_0^4 h\|^2_{L^2(\Gamma_0)} \end{aligned}$$

where $N_4(u_0,F) := N(u_0,F) + \|u_0^\tau\|^2_{H^3(\Gamma_0)}$. Finally, by (9.4) and the Gronwall inequality,

$$\sup_{0\le t\le T}\Big[\|v(t)\|^2_{H^2(\Omega_0)} + \|v_t(t)\|^2_{L^2(\Omega_0)} + \|v^\tau(t)\|^2_{H^1(\Gamma_0)} + \Big\|\int_0^t v^\tau(s)ds\Big\|^2_{H^3(\Gamma_0)}$$
$$+ \|\nabla_0^4 h(t)\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_t(t)\|^2_{L^2(\Gamma_0)}\Big] + \|v\|^2_{V^3(T)} + \|q\|^2_{L^2(0,T;H^2(\Omega_0))}$$
$$\le C(M)N_4(u_0,F)$$

and by (12.5), (12.7), as well as the evolution equation (8.2d), we also have

$$\int_0^T \Big[\|h\|^2_{H^{5.5}(\Gamma_0)} + \|h_t\|^2_{H^{2.5}(\Gamma_0)} + \|h_{tt}\|^2_{H^{0.5}(\Gamma_0)} + \Big\|\int_0^\cdot v^\tau(s)ds\Big\|^2_{H^{3.5}(\Gamma_0)}\Big](t)dt$$
$$\le C(M)N_4(u_0,F).$$

This establishes the map Θ_T from $(\tilde v,\tilde h) \in \tilde V^3(T)\times H(T)$ to $(v,h)\in \tilde V^3(T)\times H(T)$.

For the existence, with the help of (12.2b) we can also show that, with suitable choice of M and T, the mapping Θ_T maps from $C_T(M)$ into itself. Therefore, the existence follows from the Tychonoff fixed-point theorem. For the uniqueness, since t_{mem} does not involve any composition type of operations, a straightforward argument leads to the same conclusion provided that the solution is in the same space stated in Section 11.5. Therefore, we have

THEOREM 12.1. *Let $\nu > 0$, $\mu > 0$ and $\lambda > 0$ be given, and*

$$F \in L^2(0,T;H^2(\Omega_0)), F_t \in L^2(0,T;L^2(\Omega_0)), F(0)\in H^1(\Omega_0).$$

Assume that the initial data satisfies

$$u_0 \in H^{2.5}(\Omega_0)\cap H^{4.5}(\Gamma_0)$$

as well as the compatibility condition

$$\nu[\mathrm{Def}\, u_0\cdot N]_{\tan} = -\Big[\mu g_{0\kappa\sigma,\sigma} + \frac{\mu\lambda}{2\mu+\lambda} g_{0\sigma\sigma,\kappa}\Big]\frac{\partial}{\partial y^\kappa}.$$

There exists $T > 0$ depending on u_0 and F such that there exists a solution $(v,h) \in \tilde{V}^3(T) \times H(T)$ of problem (3.17). Moreover, if the initial data satisfies

$$\begin{aligned}
CP = &-\mu\Big[g_{0\gamma\sigma}u^\gamma_{0,\sigma\kappa} + g_{0\kappa\gamma}u^\gamma_{0,\sigma\sigma} + g_{0\kappa\gamma,\sigma}u^\gamma_{0,\sigma} + g_{0\gamma\sigma,\sigma}u^\kappa_{0,\gamma} + g_{0\gamma\sigma,\sigma}u^\gamma_{0,\kappa} \\
&\qquad + \bar\Gamma^\sigma_{\sigma\beta}g_{\kappa\beta t}(0) + g_{\alpha\beta t}(0)g_0^{\kappa\gamma}g_{0\alpha\gamma,\beta} - 2u^z_{0,\sigma}C_{\kappa\sigma} - 2u^z_0 C_{\kappa\sigma,\sigma}\Big]\frac{\partial}{\partial y^\kappa} \\
&- \frac{\mu\lambda}{2\mu+\lambda}\Big[2g_{0\gamma\sigma}u^\gamma_{0,\sigma\kappa} + g_{0\gamma\sigma,\kappa}u^\gamma_{0,\sigma} + g_{0\gamma\sigma,\kappa}u^\sigma_{0,\gamma} + g_{0\sigma\sigma,\beta}u^\kappa_{0,\beta} \\
&\qquad + \bar\Gamma^\sigma_{\sigma\kappa}g_{\alpha\alpha t}(0) + g_{\alpha\alpha t}(0)g_0^{\kappa\iota}g_{0\sigma\iota,\sigma} - 2u^z_{0,\kappa}C_{\sigma\sigma} - 2u^z_0 C_{\sigma\sigma,\kappa}\Big]\frac{\partial}{\partial y^\kappa} \\
&- L_2(y,0,0,0)u^z_{0,\kappa}\frac{\partial}{\partial y^\kappa}
\end{aligned}$$

then the solution (v,h) is unique, where CP is defined in (5.4), $q(0)$ and $v_t(0) = u_1$ are defined in (5.1) and (5.2).

Appendix A. Elliptic regularity

We establish a κ-independent elliptic estimate for solutions of

$$\frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\Big(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}\Big)_{,\gamma\delta}(-\nabla_0\bar h, 1)\Big]\circ\bar\eta^\tau + \kappa\Delta_0^2 v_\kappa = f \tag{A.1}$$

where h_κ and v_κ satisfy the evolution (8.4) with $h_\kappa \in H^4(\Gamma_0)$, $v_\kappa \in H^4(\Gamma_0)$, and $f \in H^{1.5}(\Gamma_0)$. Letting $w = v_\kappa \circ \bar\eta^{-\tau}$, (A.1) is equivalent to

$$\frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta}(-\nabla_0\bar h, 1) + \kappa\Delta_0^2 w = f\circ\bar\eta^\tau \tag{A.2}$$

which implies

$$\begin{aligned}
&\frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta} + \kappa J_{\bar h}^{-2}\Delta_0^2 w\cdot(-\nabla_0\bar h, 1) \\
&= J_{\bar h}^{-2} f\circ\bar\eta^\tau\cdot(-\nabla_0\bar h, 1)\,.
\end{aligned} \tag{A.3}$$

Recall that $w\cdot(-\nabla_0\bar h, 1) = h_{\kappa t}$.

We start with taking the inner-product of (A.3) with $\nabla_0^4 h_\kappa$ to obtain

$$\begin{aligned}
&\int_{\Gamma_0}\frac{\bar\Theta}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta}\nabla_0^4 h_\kappa dS + \kappa\int_{\Gamma_0}\Delta_0^2 h_{\kappa t}\nabla_0^4 h_\kappa dS \\
&= \int_{\Gamma_0}\Big[J_{\bar h}^{-2} f\circ\bar\eta^\tau\cdot(-\nabla_0\bar h, 1)\Big]\Big[\nabla_0^4 h_\kappa\Big]dS + \kappa\times\text{l.o.t.}
\end{aligned}$$

where l.o.t. can by bounded by

$$C(M)\|w\|_{H^3(\Gamma_0)}\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}.$$

Therefore, by Corollary 8.1,

$$\begin{aligned}
&\nu_1\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \frac{\kappa}{2}\frac{d}{dt}\int_{\Gamma_0}|\nabla_0^2\Delta_0 h_\kappa|^2 dS \\
&\le C(\epsilon)\Big[\|h_\kappa\|_{H^3(\Gamma_0)} + \|f\|_{L^2(\Gamma_0)} + \kappa\|w\|_{H^3(\Gamma_0)}\Big]\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\Big[\|h_\kappa\|^2_{H^3(\Gamma_0)} + \|f\|^2_{L^2(\Gamma_0)} + \kappa\|w\|^2_{H^3(\Gamma_0)}\Big] + \frac{\nu_1}{2}\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}
\end{aligned}$$

and hence, after integrating in time from 0 to t,

$$\begin{aligned}
&\nu_1\int_0^t \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} ds + \kappa\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\int_0^t \Big[\|h_\kappa\|^2_{H^3(\Gamma_0)} + \|f\|^2_{L^2(\Gamma_0)} + \kappa\|w\|^2_{H^3(\Gamma_0)}\Big] ds.
\end{aligned} \tag{A.4}$$

Similarly, taking the inner-product of (A.3) with $\nabla_0^2 h_\kappa$ or h_κ, we find that

$$\begin{aligned}
&\nu_1\int_0^t \|\nabla_0^3 h_\kappa\|^2_{L^2(\Gamma_0)} ds + \kappa\|\nabla_0^3 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\int_0^t \Big[\|h_\kappa\|^2_{H^2(\Gamma_0)} + \|f\|^2_{L^2(\Gamma_0)} + \kappa\|w\|^2_{H^2(\Gamma_0)}\Big] ds \\
&\le C(\epsilon)\int_0^t \Big[\|f\|^2_{L^2(\Gamma_0)} + \|v\|^2_{H^3(\Omega_0)}\Big] ds
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
&\nu_1\int_0^t \|\nabla_0^2 h_\kappa\|^2_{L^2(\Gamma_0)} ds + \kappa\|\nabla_0^2 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\int_0^t \Big[\|f\|^2_{L^2(\Gamma_0)} + \|v\|^2_{H^2(\Omega_0)}\Big] ds.
\end{aligned} \tag{A.6}$$

Let D_h denotes the difference quotients (w.r.t. the surface coordinate system). Taking the inner-product of (A.3) with $D_{-h}D_h\nabla_0^4 h_\kappa$,

$$\begin{aligned}
&\int_{\Gamma_0} \frac{\bar{\Theta}}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta} D_{-h}D_h\nabla_0^4 h_\kappa dS \\
&+\kappa\int_{\Gamma_0}\Delta_0^2 h_{\kappa t}D_{-h}D_h\nabla_0^4 h_\kappa dS \\
=&\int_{\Gamma_0} D_h\Big[J_{\bar{h}}^{-2} f\circ\bar{\eta}^\tau\cdot(-\nabla_0\bar{h},1)\Big]\Big[D_h\nabla_0^4 h_\kappa\Big]dS+\kappa\times\text{l.o.t.}
\end{aligned}$$

where l.o.t. can be bounded by

$$C(M)\|w\|_{H^4(\Gamma_0)}\|D_h\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}.$$

Therefore, by Corollary 8.1,

$$\begin{aligned}
&\nu_1\|D_h\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}+\frac{\kappa}{2}\frac{d}{dt}\int_{\Gamma_0}|D_h\nabla_0^2\Delta_0 h_\kappa|^2 dS \\
&\le C(\epsilon)\Big[\|h_\kappa\|_{H^4(\Gamma_0)}+\|f\|_{H^1(\Gamma_0)}+\kappa\|w\|_{H^4(\Gamma_0)}\Big]\|D_h\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\Big[\|h_\kappa\|^2_{H^4(\Gamma_0)}+\|f\|^2_{H^1(\Gamma_0)}+\kappa\|w\|^2_{H^4(\Gamma_0)}\Big]+\frac{\nu_1}{2}\|D_h\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}
\end{aligned}$$

and hence, after integrating in time from 0 to t, by (A.4) and (A.5) we find that

$$\begin{aligned}
&\nu_1\int_0^t\|D_h\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds+\kappa\|D_h\nabla_0^2\Delta_0 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\int_0^t\Big[\|h_\kappa\|^2_{H^4(\Gamma_0)}+\|f\|^2_{H^1(\Gamma_0)}+\kappa\|w\|^2_{H^4(\Gamma_0)}\Big]ds \\
&\le C(\epsilon)\int_0^t\Big[\|h_\kappa\|^2_{H^2(\Gamma_0)}+\|f\|^2_{H^1(\Gamma_0)}+\kappa\|w\|^2_{H^4(\Gamma_0)}\Big]ds.
\end{aligned}$$

Since the right-hand side is independent of the difference parameter h, it follows that $h_\kappa\in H^5(\Gamma_0)$ (as it is already a H^4-function) with the estimate

$$\begin{aligned}
&\int_0^t\|\nabla_0^5 h_\kappa\|^2_{L^2(\Gamma_0)}ds+\kappa\|\nabla_0^5 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \\
&\le C(\epsilon)\int_0^t\Big[\|h_\kappa\|^2_{H^2(\Gamma_0)}+\|f\|^2_{H^1(\Gamma_0)}+\kappa\|w\|^2_{H^4(\Gamma_0)}\Big]ds. \qquad \text{(A.7)}
\end{aligned}$$

Next, we obtain a κ-independent estimate of $\kappa\|w\|^2_{H^4(\Gamma_0)}$. By taking the inner-product of (A.2) with $\nabla_0^2 w$, we find that

$$\int_{\Gamma_0} \frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \left[\sqrt{\det(g_0)} \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}\right]_{,\gamma\delta} \nabla_0^2 h_{\kappa t} dS + \kappa \int_{\Gamma_0} \Delta_0^2 w \cdot \nabla_0^2 w dS$$
$$= \int_{\Gamma_0} f \circ \bar{\eta}^{-\tau} \cdot \nabla_0^2 w dS + \text{l.o.t.}$$

where l.o.t. can be bounded by

$$C(\epsilon)\|\nabla_0^3 h_\kappa\|_{L^2(\Gamma_0)}\|w\|_{H^2(\Gamma_0)}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma_0} \bar{\Theta}\bar{A}^{\alpha\beta\gamma\delta}\nabla_0 h_{\kappa,\alpha\beta}\nabla_0 h_{\kappa,\gamma\delta} dS + \frac{\kappa}{2}\int_{\Gamma_0} |\nabla_0\Delta_0 w|^2 dS$$
$$\leq C(\epsilon)\Big[\|\nabla_0^3 h_\kappa\|^2_{L^2(\Gamma_0)} + \|f\|^2_{L^2(\Gamma_0)} + \|w\|^2_{H^{2.5}(\Omega_0)}\Big]$$

where we use the fact that

$$\int_{\Gamma_0} |\nabla_0\Delta_0 w|^2 dS \leq \int_{\Gamma_0} \Delta_0^2 w \cdot \nabla_0^2 w dS + C(\epsilon)\|w\|_{H^3(\Gamma_0)}\|w\|_{H^2(\Gamma_0)}.$$

After integrating in time from 0 to t,

$$\|\nabla_0^3 h_\kappa(t)\|^2_{L^2(\Gamma_0)} + \kappa\int_0^t \|w\|^2_{H^3(\Gamma_0)} ds$$
$$\leq C(\epsilon)\int_0^t \Big[\|\nabla_0^3 h_\kappa\|^2_{L^2(\Gamma_0)} + \|f\|^2_{L^2(\Gamma_0)} + \|w\|^2_{H^{2.5}(\Omega_0)}\Big] ds. \tag{A.8}$$

Similarly, by taking the inner-product of (A.2) with $\nabla_0^4 w$, we find that

$$\int_{\Gamma_0} \frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \left[\sqrt{\det(g_0)} \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}\right]_{,\gamma\delta} \nabla_0^4 h_{\kappa t} dS + \kappa \int_{\Gamma_0} \Delta_0^2 w \cdot \nabla_0^2 w dS$$
$$= \int_{\Gamma_0} f \circ \bar{\eta}^{-\tau} \cdot \nabla_0^4 w dS + \text{l.o.t.}$$

where l.o.t. can be bounded by

$$C(\epsilon)\|\nabla_0^5 h_\kappa\|_{L^2(\Gamma_0)}\|w\|_{H^2(\Gamma_0)}.$$

Therefore, following the procedure of obtaining (A.8), we find that

$$\begin{aligned}
&\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)} + \kappa \int_0^t \|w\|^2_{H^4(\Gamma_0)} ds \\
&\le C(\epsilon,\delta_1) \int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|f\|^2_{H^{1.5}(\Gamma_0)} + \|w\|^2_{H^3(\Omega_0)}\Big] ds \\
&\quad + \delta_1 \int_0^t \|\nabla_0^5 h_\kappa\|^2_{L^2(\Gamma_0)} dS + C(\epsilon)\kappa \int_0^t \|w\|^2_{H^3(\Gamma_0)} ds \\
&\le C(\epsilon,\delta_1) \int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|f\|^2_{H^{1.5}(\Gamma_0)} + \|w\|^2_{H^3(\Omega_0)}\Big] ds \\
&\quad + \delta_1 \int_0^t \|\nabla_0^5 h_\kappa\|^2_{L^2(\Gamma_0)} dS
\end{aligned} \tag{A.9}$$

where we use (A.8) to estimate $\kappa \int_0^t \|w\|_{H^3(\Gamma_0)} ds$. (A.9) provides a κ-independent estimate for $\kappa\|w\|^2_{H^4(\Gamma_0)}$; hence by choosing $\delta_1 > 0$ small enough, (A.7) implies that for all $t \in [0,T]$,

$$\begin{aligned}
&\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} ds + \kappa \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} \\
&\le C(\epsilon) \int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|f\|^2_{H^{1.5}(\Gamma_0)} + \|w\|^2_{H^3(\Omega_0)}\Big] ds
\end{aligned} \tag{A.10}$$

for some constant C' depending on ϵ.

Remark 32. *With the inclusion of $\mathfrak{t}_{mem}$ in the shell traction, by treating it as an extra forcing, we find estimates similar to those in (A.7):*

$$\begin{aligned}
&\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} ds + \kappa \|\nabla_0^2 h_\kappa(t)\|^2_{H^3(\Gamma_0)} \\
&\le C(\epsilon) \int_0^t \Big[\|h_\kappa\|^2_{H^2(\Gamma_0)} + \Big\|\int_0^s v_\kappa^\tau(r) dr\Big\|^2_{H^3(\Gamma_0)} + \|f\|^2_{H^1(\Gamma_0)} + \kappa\|w\|^2_{H^4(\Gamma_0)}\Big] ds.
\end{aligned}$$

Moreover, just as in (A.9),

$$\begin{aligned}
&\Big\|\int_0^t v_\kappa^\tau ds\Big\|^2_{H^3(\Gamma_0)} + \|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)} + \kappa \int_0^t \|w\|^2_{H^4(\Gamma_0)} ds \\
&\le C(\epsilon,\delta_1) \int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \Big\|\int_0^s v_\kappa^\tau(r) dr\Big\|^2_{H^{2.5}(\Gamma_0)} + \|f\|^2_{H^{1.5}(\Gamma_0)} + \|w\|^2_{H^3(\Omega_0)}\Big] ds \\
&\quad + \delta_1 \int_0^t \|\nabla_0^5 h_\kappa\|^2_{L^2(\Gamma_0)} dS.
\end{aligned}$$

Therefore, by choosing $\delta_1 > 0$ small enough, we have κ-independent estimates

$$\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} ds + \kappa \|\nabla_0^2 h_\kappa(t)\|^2_{H^3(\Gamma_0)}$$
$$\leq C(\epsilon) \int_0^t \Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \Big\| \int_0^s v_\kappa^\tau(r) dr \Big\|^2_{H^{2.5}(\Gamma_0)} + \|f\|^2_{H^{1.5}(\Gamma_0)} + \|w\|^2_{H^3(\Omega_0)} \Big] ds.$$

Appendix B. Inequalities in the estimates for $\nabla_0^2 v$ near the boundary

B.1. **κ-independent estimates.** By $\zeta_1 \equiv 1$ on Γ_0 and

$$\begin{aligned}(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0^4 v_\kappa = \nabla_0^4((-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot v_\kappa) &- \nabla_0^4(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot v_\kappa \\ &- 4\nabla_0^3(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0 v_\kappa - 6\nabla_0^2(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0^2 v_\kappa \\ &- 4\nabla_0(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0^3 v_\kappa \,,\end{aligned}$$

we find that

$$\begin{aligned}&\int_{\Gamma_0} \bar{\Theta} \Big[L_{\bar{h}}(h_\kappa) \circ \bar{\eta}^\tau \Big] ((-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0^2(\zeta_1^2 \nabla_0^2 v_\kappa)) dS \\ &= - \int_{\Gamma_0} \bar{\Theta} \Big[L_{\bar{h}}(h_\kappa) \circ \bar{\eta}^\tau \Big] \Big[\nabla_0^4(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot v_\kappa + 4\nabla_0^3(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0 v_\kappa \\ &\qquad + 6\nabla_0^2(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0^2 v_\kappa \Big] dS \qquad (\equiv I_1) \\ &\quad - 4 \int_{\Gamma_0} \bar{\Theta} \Big[L_{\bar{h}}(h_\kappa) \circ \bar{\eta}^\tau \Big] (\nabla_0(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) \cdot \nabla_0^3 v_\kappa) dS \qquad (\equiv I_2) \\ &\quad + \int_{\Gamma_0} \frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \nabla_0^2 \Big[\sqrt{\det(g_0)} \Big(L_1^{\alpha\beta\gamma} \tilde{h}_{,\alpha\beta\gamma} + L_2 \Big) \circ \bar{\eta}^\tau \Big] \nabla_0^2(h_{\kappa t} \circ \bar{\eta}^\tau) dS \qquad (\equiv I_3) \\ &\quad + \int_{\Gamma_0} \frac{2\nabla_0 \bar{\Theta}}{\sqrt{\det(g_0)}} \nabla_0 \Big[\sqrt{\det(g_0)} \Big(L_1^{\alpha\beta\gamma} \tilde{h}_{,\alpha\beta\gamma} + L_2 \Big) \circ \bar{\eta}^\tau \Big] \nabla_0^2(h_{\kappa t} \circ \bar{\eta}^\tau) dS \qquad (\equiv I_4) \\ &\quad + \int_{\Gamma_0} (\nabla_0^2 \bar{\Theta}) \Big[\Big(L_1^{\alpha\beta\gamma} \tilde{h}_{,\alpha\beta\gamma} + L_2 \Big) \circ \bar{\eta}^\tau \Big] \nabla_0^2(h_{\kappa t} \circ \bar{\eta}^\tau) dS \qquad (\equiv I_5) \\ &\quad + \int_{\Gamma_0} \frac{\bar{\Theta}}{\sqrt{\det(g_0)}} \Big[(\sqrt{\det(g_0)} \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})_{,\gamma\delta} \circ \bar{\eta}^\tau \Big] \nabla_0^4(h_{\kappa t} \circ \bar{\eta}^\tau) dS.\end{aligned}$$

The last term of the identity above, by a change of coordinates, can be written as

$$\begin{aligned}
&\int_{\Gamma_0} \frac{\bar\Theta}{\sqrt{\det(g_0)}} \Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^4(h_{\kappa t}\circ\bar\eta^\tau)dS \\
&= \int_{\Gamma_0} \frac{B}{\sqrt{\det(g_0)}} \nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0^2 h_{\kappa t} dS + R_1 \\
&+ 2\int_{\Gamma_0} \frac{\nabla_0\bar\Theta}{\sqrt{\det(g_0)}}\nabla_0\Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv J_1)\\
&+ \int_{\Gamma_0} \frac{\nabla_0^2\bar\Theta}{\sqrt{\det(g_0)}}\Big[(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})_{,\gamma\delta}\circ\bar\eta^\tau\Big]\nabla_0^2(h_{\kappa t}\circ\bar\eta^\tau)dS \qquad (\equiv J_2)\\
&= \frac12\frac{d}{dt}\int_{\Gamma_0} B\bar A^{\alpha\beta\gamma\delta}\nabla_0^2 h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa,\gamma\delta}dS + R_1'
\end{aligned}$$

where $B = b^t\otimes b^t\otimes b^t\otimes b^t$ with $b=\nabla_0\bar\eta^\tau$, and

$$\begin{aligned}
R_1(t) = &\int_{\Gamma_0} b^t\otimes b^t\otimes(\nabla_0 b^t)\otimes(\nabla_0 b^t)\nabla_0(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0 h_{\kappa t}dS \quad (\equiv J_3)\\
&+\int_{\Gamma_0} b^t\otimes b^t\otimes b^t\otimes(\nabla_0 b^t)\nabla_0(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0^2 h_{\kappa t}dS \quad (\equiv J_4)\\
&+\int_{\Gamma_0} b^t\otimes b^t\otimes b^t\otimes(\nabla_0 b^t)\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{,\gamma\delta}\nabla_0 h_{\kappa t}dS \quad (\equiv J_5)
\end{aligned}$$

and

$$\begin{aligned}
R_1'(t) = R_1(t) + J_1(t) + J_2(t) &- \frac12\int_{\Gamma_0}(B\bar A^{\alpha\beta\gamma\delta})_t\nabla_0^2 h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa,\gamma\delta}dS \quad (\equiv J_6)\\
&+ 2\int_{\Gamma_0}\frac{B}{\sqrt{\det(g_0)}}\nabla_0(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta})\nabla_0 h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa t,\gamma\delta}dS \quad (\equiv J_7)\\
&+ \int_{\Gamma_0}\frac{B}{\sqrt{\det(g_0)}}\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta}) h_{\kappa,\alpha\beta}\nabla_0^2 h_{\kappa t,\gamma\delta}dS \quad (\equiv J_8)\\
&+ 2\int_{\Gamma_0}\frac{B_{,\gamma}}{\sqrt{\det(g_0)}}\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})\nabla_0^2 h_{\kappa t,\delta}dS \quad (\equiv J_9)\\
&+ \int_{\Gamma_0}\frac{B_{,\gamma\delta}}{\sqrt{\det(g_0)}}\nabla_0^2(\sqrt{\det(g_0)}\bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta})\nabla_0^2 h_{\kappa t}dS. \quad (\equiv J_{10})
\end{aligned}$$

It follows that

$$|I_1| \le C(\epsilon)(1+\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)})\|\nabla_0^2 v_\kappa\|_{H^1(\Omega_1')}\ ;$$

$$|I_3|+|I_4|+|I_5| \le C(M)(1+\|\tilde h\|_{H^5(\Gamma_0)})\|\nabla_0^2 v_\kappa\|_{H^1(\Omega_1)}\ ;$$

and hence that

$$|I_1|+|I_3|+|I_4|+|I_5| \leq C(\epsilon)\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\tilde{h}\|^2_{H^5(\Gamma_0)} + 1\Big] + \delta\|v_\kappa\|^2_{H^3(\Omega_0)}.$$

It follows that

$$|J_2|+|J_3|+|J_5|+|J_{10}| \leq C(\epsilon)\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\|\nabla_0^2 h_{\kappa t}\|_{L^2(\Gamma_0)}$$

$$|J_6| \leq C(M)(\|\tilde{v}\|_{H^3(\Omega_0)} + \|\tilde{h}_t\|_{H^{2.5}(\Gamma_0)})\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}.$$

We need only obtain κ-independent estimates for the terms I_2, J_1, J_4, J_7, J_8 and J_9. By $H^{-0.5}(\Gamma_0)$-$H^{0.5}(\Gamma_0)$ duality pairing and (7.5),

$$\begin{aligned}|I_2| &\leq C\|\bar{\Theta}\|_{H^{1.5}(\Gamma_0)}\|L_{\bar{h}}(h_\kappa)\|_{H^{0.5}(\Gamma_0)}\|\nabla_0(\nabla_0\bar{h}\circ\bar{\eta}^\tau)\|_{H^{1.5}(\Gamma_0)}\|\nabla_0^3 v_\kappa\|_{H^{-0.5}(\Gamma_0)} \\ &\leq C(M)\Big[\|\nabla_0^2 h_\kappa\|_{H^{2.5}(\Gamma_0)} + 1\Big]\|v_\kappa\|_{H^{2.5}(\Gamma_0)}.\end{aligned}$$

Therefore, by (7.11) and Young's inequality,

$$|I_2| \leq C\Big[\|h_\kappa\|^2_{H^4(\Gamma_0)} + 1\Big] + \delta_1\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^3(\Omega_0)} \tag{B.1}$$

for some C depending on M, δ and δ_1.

For J_1, J_4 and J_9, by $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing, we find that

$$\begin{aligned}|J_1|+|J_4|+|J_9| &\leq C(\epsilon)\|h_\kappa\|_{H^{4.5}(\Gamma_0)}\|v_\kappa\|_{H^{2.5}(\Gamma_0)} \\ &\leq C'\Big[\|\nabla_0^2 h_\kappa\|^2_{H^2(\Gamma_0)} + 1\Big] + \delta_1\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^3(\Omega_0)}\end{aligned}$$

for some constant C' depending on M, ϵ, δ and δ_1.

For J_7 and J_8, by $H^{-1.5}(\Gamma_0)$-$H^{1.5}(\Gamma_0)$ duality pairing,

$$|J_7|+|J_8| \leq C(M)\|B\|_{H^{1.5}(\Gamma_0)}\|\bar{h}\|_{H^{3.5}(\Gamma_0)}\|h_\kappa\|_{H^{4.5}(\Gamma_0)}\|v_\kappa\|_{H^{2.5}(\Gamma_0)}.$$

Similar to the estimate in (B.1), we find that

$$|J_7|+|J_8| \leq C(M)\Big[\|h_\kappa\|^2_{H^4(\Gamma_0)} + 1\Big] + \delta_1\|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^3(\Omega_0)}.$$

Summing all the estimates and then integrating in time from 0 to t, by Corollary 8.1 and the fact that B is close to 1 in the uniform norm for T small,

$$
\begin{aligned}
\frac{\nu_1}{2}\|\nabla_0^4 h_\kappa(t)\|^2_{L^2(\Gamma_0)} \le & \int_0^t \int_{\Gamma_0} \bar{\Theta}\Big[[L_{\tilde{h}}(h_\kappa)(-\nabla_0\bar{h},1)]\circ\bar{\eta}^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\kappa)dSds \\
& + C'\int_0^t K(s)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds + C'\int_0^t \Big[\|\tilde{h}\|^2_{H^5(\Gamma_0)}+1\Big]ds \\
& + \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)}ds + \delta_1\int_0^t \|\nabla_0^2 h_\kappa\|^2_{H^3(\Gamma_0)}ds
\end{aligned}
$$

for some constant C' depending on M, ϵ, δ and δ_1, where

$$
K(s) := 1 + \|\tilde{v}\|^2_{H^3(\Omega_0)} + \|\tilde{h}\|^2_{H^5(\Gamma_0)} + \|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)}.
$$

B.2. **ϵ-independent estimates.** We next obtain ϵ-independent estimates for the first two terms of I_1, as well as those for I_2, J_1, J_2, J_3, J_4, J_5, J_9 and J_{10} with h_κ replaced by h_ϵ. Let

$$
\begin{aligned}
I_1^1 = & -\int_{\Gamma_0}\bar{\Theta}\Big[L_{\tilde{h}}(h_\epsilon)\circ\bar{\eta}^\tau\Big]\Big[\nabla_0^4(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\cdot v_\epsilon\Big]dS, \\
I_1^2 = & -4\int_{\Gamma_0}\bar{\Theta}\Big[L_{\tilde{h}}(h_\epsilon)\circ\bar{\eta}^\tau\Big]\Big[\nabla_0^3(-\nabla_0\bar{h}\circ\bar{\eta}^\tau,1)\cdot\nabla_0 v_\epsilon\Big]dS
\end{aligned}
$$

By $H^{-1.5}(\Gamma_0)$-$H^{1.5}(\Gamma_0)$ duality pairing,

$$
|I_1^1|+|I_1^2| \le C(M)\|L_{\tilde{h}}(h_\epsilon)\|_{H^{1.5}(\Gamma_0)}\|v_\epsilon\|_{H^{2.5}(\Gamma_0)}\|(\nabla_0\tilde{h})\circ\bar{\eta}^\tau\|_{H^{2.5}(\Gamma_0)}.
$$

Therefore, by (7.14) and (10.12),

$$
\begin{aligned}
|I_1^1|+|I_1^2| \le & \ C(M)t^{1/4}\Big[\|h_\epsilon\|^2_{H^{5.5}(\Gamma_0)}+1\Big]\|v_\epsilon\|_{H^3(\Omega_0)} \\
\le & \ C(M,\delta)t^{1/2}\|h_\epsilon\|^5_{H^{5.5}(\Gamma_0)} + \delta\|v_\epsilon\|^2_{H^3(\Omega_0)} \\
\le & \ Ct^{1/2}\Big[\|v_{\epsilon t}\|^2_{H^1(\Omega_0)} + \|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma_0)} + \|F\|^2_{H^1(\Omega_0)} + 1\Big] \\
& + (\delta + Ct^{1/2})\|v_\epsilon\|^2_{H^3(\Omega_0)}
\end{aligned} \tag{B.2}
$$

for some constant C depending on M and δ.

REMARK 33. *Without the inclusion of $\mathfrak{t}_{mem}$ into the shell traction, η^τ only inherits the regularity of v and h_t, and hence the only way to estimate I_1 without any artificial regularization (in our case ϵ-regularization) is to have $h \in H^{5.5}(\Gamma_0)$. We obtain* *this regularity through the elliptic problem (10.11) where the $H^{1.5}$ regularity of the lower order terms L_1 and L_2 are crucially used. With the inclusion of* $\mathfrak{t}_{mem}$*, we have* $\eta^\tau \in H^{3.5}(\Gamma_0)$ *and with this regularity,* $h \in H^{4.5}(\Gamma_0)$ *is enough to estimate* I_1*.*

For J_1, we use an L^4-L^4-L^2 type of Hölder's inequality and conclude that

$$|J_1| \le C(M)t^{1/2}\|h_\epsilon\|_{H^{5.5}(\Gamma_0)}\|v_\epsilon\|_{H^{2.5}(\Gamma_0)}$$

while for the other J terms, we use $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing to obtain

$$|J_2| + |J_3| + |J_4| + |J_5| + |J_9| + |J_{10}| \le C(M)t^{1/2}\|h_\epsilon\|_{H^{5.5}(\Gamma_0)}\|v_\epsilon\|_{H^{2.5}(\Gamma_0)}.$$

and hence all the J terms are bounded by the same right-hand side of the inequality in (B.2). Therefore,

$$\begin{aligned}\frac{\nu_1}{2}\|\nabla_0^4 h_\epsilon(t)\|^2_{L^2(\Gamma_0)} \le &\int_0^t\int_{\Gamma_0}\bar\Theta\Big[[L_{\bar h}(h_\epsilon)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau\Big]\cdot\nabla_0^2(\zeta_1^2\nabla_0^2 v_\epsilon)dSds\\ &+ CN_2(u_0,F) + C\int_0^t K(s)\|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma_0)}ds + (\delta + Ct^{1/2})\int_0^t\|v_\epsilon\|^2_{H^3(\Omega_0)}ds\\ &+ (\delta_1 + Ct^{1/2})\int_0^t\|v_{\epsilon t}\|^2_{H^1(\Omega_0)}ds\end{aligned}$$

for some constant C depending on M, δ and δ_1.

Appendix C. $L^2_tH^1_x$ estimates for v_t

C.1. **Estimates for the integral over Γ_0.** By the chain rule,

$$\begin{aligned}\int_{\Gamma_0}\Big[\bar\Theta[L_{\bar h}(h_\kappa)(-\nabla_0\bar h,1)]\circ\bar\eta^\tau\Big]_t\cdot v_{\kappa t}dS = &\int_{\Gamma_0}\bar\Theta_t\Big[L_{\bar h}(h_\kappa)\Big]\circ\bar\eta^\tau(-\nabla_0\bar h\circ\bar\eta^\tau,1)\cdot v_{\kappa t}dS\\ &+\int_{\Gamma_0}\bar\Theta\bar v^\tau\cdot\Big[\nabla_0[L_{\bar h}(h_\kappa)](-\nabla_0\bar h,1)\Big]\circ\bar\eta^\tau\cdot v_{\kappa t}dS \quad (\equiv K_1)\\ &+\int_{\Gamma_0}\bar\Theta\Big[[L_{\bar h}(h_\kappa)](\nabla_0\bar h,-1)\Big]_t\circ\bar\eta^\tau\cdot v_{\kappa t}dS. \quad (\equiv K_2)\end{aligned}$$

The first term is bounded by

$$C(M)\|\bar v\|_{H^3(\Omega_0)}\Big[\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)} + 1\Big]\|v_{\kappa t}\|_{L^2(\Gamma_0)}$$

Integrating by parts, we find that

$$
\begin{aligned}
K_1 = & -\int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}} \nabla_0(\sqrt{\det(g_0)}) \Big[[L_{\bar h}(h_\kappa)](-\nabla_0 \bar h, 1) \Big] (\bar v \circ \bar\eta^{-\tau})(v_{\kappa t} \circ \bar\eta^{-\tau}) dS \\
& - \int_{\Gamma_0} \Big[[L_{\bar h}(h_\kappa)](-\nabla_0 \bar h, 1) \Big] (b^t)^{-1} (\nabla_0 \bar v) \circ \bar\eta^{-\tau} (v_{\kappa t} \circ \bar\eta^{-\tau}) dS \\
& - \int_{\Gamma_0} \Big[[L_{\bar h}(h_\kappa)](-\nabla_0 \bar h, 1) \Big] (\bar v \circ \bar\eta^{-\tau}) (b^t)^{-1} (\nabla_0 v_{\kappa t}) \circ \bar\eta^{-\tau} dS. \quad (\equiv K_1')
\end{aligned}
$$

The first two terms of the right-hand side can be bounded by

$$
C(M) \Big[\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)} + 1 \Big] \|v_{\kappa t}\|_{H^1(\Gamma_0)}.
$$

The terms containing L_1 and L_2 in K_1', by using $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing, are bounded by

$$
C(M) \|v_{\kappa t}\|_{H^1(\Omega_0)}.
$$

In order to estimate K_1, we only have to find a bound for

$$
\int_{\Gamma_0} \frac{\bar v}{\sqrt{\det(g_0)}} \Big[[\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{,\gamma\delta} (\nabla_0 \bar h, -1) \Big] \circ \bar\eta^\tau \nabla_0 v_{\kappa t} dS.
$$

Integrating from 0 to t and integrating by parts in time, we find that

$$
\begin{aligned}
& \int_0^t \int_{\Gamma_0} \frac{\bar v}{\sqrt{\det(g_0)}} \Big[[\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{,\gamma\delta} (\nabla_0 \bar h, -1) \Big] \circ \bar\eta^\tau \nabla_0 v_{\kappa t} dS ds \\
= & \Big(\int_{\Gamma_0} \frac{\bar v}{\sqrt{\det(g_0)}} \Big[[(\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{,\gamma\delta} (\nabla_0 \bar h, -1) \Big] \circ \bar\eta^\tau \nabla_0 v_\kappa dS \Big)(t) \ (\equiv K_1^1) \\
& - \int_0^t \int_{\Gamma_0} \frac{\bar v_t}{\sqrt{\det(g_0)}} \Big[[\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{,\gamma\delta} (\nabla_0 \bar h, -1) \Big] \circ \bar\eta^\tau \nabla_0 v_\kappa dS ds \ (\equiv K_1^2) \\
& - \int_0^t \int_{\Gamma_0} \frac{\bar v \otimes \bar v}{\sqrt{\det(g_0)}} \nabla_0 \Big\{ \Big[[\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{,\gamma\delta} (\nabla_0 \bar h, -1) \Big] \circ \bar\eta^\tau \Big\} \nabla_0 v_\kappa dS ds \ (\equiv K_1^3) \\
& - \int_0^t \int_{\Gamma_0} \frac{\bar v}{\sqrt{\det(g_0)}} \Big[[\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{,\gamma\delta} (\nabla_0 \bar h_t, 0) \Big] \circ \bar\eta^\tau \nabla_0 v_\kappa dS ds. \ (\equiv K_1^4) \\
& - \int_0^t \int_{\Gamma_0} \frac{\bar v}{\sqrt{\det(g_0)}} \Big[[\sqrt{\det(g_0)} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}]_{t,\gamma\delta} (\nabla_0 \bar h, -1) \Big] \circ \bar\eta^\tau \nabla_0 v_\kappa dS ds. \ (\equiv K_1^5)
\end{aligned}
$$

With Young's inequality,

$$
\begin{aligned}
|K_1^1| &\le C(M)\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\|v_\kappa\|_{H^1(\Gamma_0)} \\
&\le C(M)C(\delta,\delta_2)\|v_\kappa\|^2_{H^1(\Omega_0)} + \delta\|v_\kappa\|^2_{H^2(\Omega_0)} + \delta_2\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} \\
&\le C(M)C(\delta,\delta_2)t\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)}ds + \delta\int_0^t \Big[\|v_{\kappa t}\|^2_{H^1(\Omega_0)} + \|v_\kappa\|^2_{H^3(\Omega_0)}\Big]ds \\
&\quad + \delta_2\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}.
\end{aligned}
$$

and

$$
\begin{aligned}
|K_1^2| &\le C(M)\int_0^t \|\tilde v_t\|_{H^1(\Omega_0)}\|v_\kappa\|_{H^2(\Omega_0)}\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}ds \\
&\le C(M)C(\delta)\int_0^t \|\tilde v_t\|^2_{H^1(\Omega_0)}\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds + \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)}ds
\end{aligned}
$$

and

$$
\begin{aligned}
|K_1^4| &\le C(M)\int_0^t \|\tilde v\|_{L^\infty(\Omega_0)}\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\|\nabla_0\bar h_t\|_{L^4(\Gamma_0)}\|\nabla_0 v_\kappa\|_{L^4(\Omega_0)}ds \\
&\le C(M)C(\delta)\int_0^t \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds + \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)}ds.
\end{aligned}
$$

For K_1^3, by $H^1(\Gamma_0)$-$H^{-1}(\Gamma_0)$ pairing, we find that

$$
\begin{aligned}
|K_1^3| &\le C(M)\int_0^t \|(\tilde v\otimes\tilde v)\nabla_0 v_\kappa\|_{H^1(\Gamma_0)}\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}ds \\
&\le C(M)\int_0^t \|v_\kappa\|_{H^3(\Omega_0)}\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}ds \\
&\le C(M)C(\delta)\int_0^t \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}ds + \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)}ds.
\end{aligned}
$$

It remains to estimate K_1^5. Using that

$$
\begin{aligned}
(-\nabla_0\bar h,1)\circ\bar\eta^\tau\nabla_0 v_\kappa &= \nabla_0[h_{\kappa t}\circ\bar\eta^\tau] - \nabla_0[(-\nabla_0\bar h\circ\bar\eta^\tau,1)]\cdot v_\kappa \\
&= b^t(\nabla_0 h_{\kappa t})\circ\bar\eta^\tau + b^t(\nabla_0^2\bar h\circ\bar\eta^\tau,0)\cdot v_\kappa
\end{aligned}
$$

and integrating by parts, we find that

$$
\begin{aligned}
K_1^5 &= -\int_0^t \int_{\Gamma_0} \Big[\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta}\Big]_t \Big[\bar{\Theta}\bar{v} b^t \nabla_0 h_{\kappa t} - ((\nabla_0^2 \bar{h}) \circ \bar{\eta}^\tau, 0) \cdot v_\kappa \circ \bar{\eta}^{-\tau}\Big]_{,\gamma\delta} dS ds \\
&= -\int_0^t \int_{\Gamma_0} \bar{\Theta}\bar{v} b^t \bar{A}^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta} \nabla_0 h_{\kappa t,\gamma\delta} dS ds + R_1^5 \\
&= \frac{1}{2}\int_0^t \int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}} \nabla_0 \Big[\sqrt{\det(g_0)} \bar{\Theta}\bar{v} b^t \bar{A}^{\alpha\beta\gamma\delta}\Big] h_{\kappa t,\alpha\beta} h_{\kappa t,\gamma\delta} dS ds + R_1^5
\end{aligned}
$$

where

$$
\begin{aligned}
|R_1^5| &\le C(M) \int_0^t \|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)} \|v_\kappa\|_{H^3(\Omega_0)} ds \\
&\le C(M)C(\delta) \int_0^t \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} ds + \delta \int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)} ds.
\end{aligned}
$$

By interpolation,

$$
\begin{aligned}
|K_1^5| \le C(M) \int_0^t \Big[&\|\nabla_0(\bar{\Theta} b^t)\|_{L^4(\Gamma_0)} \|\bar{v}\bar{A}^{\alpha\beta\gamma\delta}\|_{L^\infty(\Gamma_0)} \|\nabla_0^2 h_{\kappa t}\|_{L^4(\Gamma_0)} \|\nabla_0^2 h_{\kappa t}\|_{L^2(\Gamma_0)} \\
&+ \|\bar{\Theta} b^t\|_{L^\infty(\Gamma_0)} \|\nabla_0(\bar{v}\bar{A}^{\alpha\beta\gamma\delta})\|_{L^4(\Gamma_0)} \|\nabla_0^2 h_{\kappa t}\|_{L^2(\Gamma_0)} \|\nabla_0^2 h_{\kappa t}\|_{L^4(\Gamma_0)} \Big] ds + |R_1^5| \\
\le C(M)C(\delta) &\Big[N(u_0, F) + \int_0^t \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} ds \Big] + \delta \int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)} ds \\
&+ C(M)C(\delta) t \int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)} ds
\end{aligned}
$$

Therefore, K_1 satisfies

$$
\begin{aligned}
\Big| \int_0^t K_1 ds \Big| &\le C \int_0^t \Big[K(s) \Big(\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} \Big) + 1 \Big] ds + \delta_2 \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} \\
&+ (\delta + C t^{1/2}) \int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)} ds + (\delta + C t^{1/2}) \int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)} ds \qquad \text{(C.1)}
\end{aligned}
$$

for some constant C depending on M, δ and δ_2.

For K_2, by time differentiating the evolution equation, we find that

$$
\begin{aligned}
(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, 1) v_{\kappa t} = h_{\kappa tt} \circ \bar{\eta}^\tau + \bar{v}^\tau \cdot (\nabla_0 h_{\kappa t}) \circ \bar{\eta}^\tau - \bar{v}^\tau \cdot (\nabla_0^2 \bar{h} \circ \bar{\eta}^\tau, 0) \cdot v_\kappa \\
- (\nabla_0 \bar{h}_t \circ \bar{\eta}^\tau, 0) \cdot v_\kappa \qquad \text{(C.2)}
\end{aligned}
$$

and hence (after a change of coordinates)

$$
\begin{aligned}
K_2 = &\int_{\Gamma_0} [L_{\bar h}(h_\kappa)]_t h_{\kappa tt} dS + \int_{\Gamma_0} [L_{\bar h}(h_\kappa)]_t [(\bar v^\tau \circ \bar\eta^{-\tau}) \cdot (\nabla_0 h_{\kappa t})] dS \quad (\equiv K_3) \\
&- \int_{\Gamma_0} [L_{\bar h}(h_\kappa)]_t [(\nabla_0 \bar h_t, 0) \cdot (v_\kappa \circ \bar\eta^{-\tau})] dS \quad (\equiv K_4) \\
&- \int_{\Gamma_0} [L_{\bar h}(h_\kappa)]_t [(\bar v^\tau \circ \bar\eta^{-\tau}) \cdot (\nabla_0^2 \bar h, 0)(v_\kappa \circ \bar\eta^{-\tau})] dS. \quad (\equiv K_5) \\
&+ \int_{\Gamma_0} [L_{\bar h}(h_\kappa)][(\nabla_0 \bar h_t, 0) \cdot (v_{\kappa t} \circ \bar\eta^{-\tau})] dS \quad (\equiv K_6)
\end{aligned}
$$

For the first term, we have

$$
\begin{aligned}
\int_{\Gamma_0} [L_{\bar h}(h_\kappa)]_t h_{\kappa tt} dS = &\frac{1}{2}\frac{d}{dt}\int_{\Gamma_0} \bar A^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta} h_{\kappa t,\gamma\delta} dS - \frac{1}{2}\int_{\Gamma_0} (\bar A^{\alpha\beta\gamma\delta})_t h_{\kappa t,\alpha\beta} h_{\kappa t,\gamma\delta} dS \\
&+ \int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} (\bar A^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta} h_{\kappa tt} dS \\
&+ \int_{\Gamma_0} \Big[L_1^{\alpha\beta\gamma} \bar h_{,\alpha\beta\gamma}\Big]_t h_{\kappa tt} dS + \int_{\Gamma_0} (L_2)_t h_{\kappa tt} dS,
\end{aligned} \tag{C.3}
$$

The second term on the right-hand side verifies

$$
\Big|\int_{\Gamma_0} (\bar A^{\alpha\beta\gamma\delta})_t h_{\kappa t,\alpha\beta} h_{\kappa t,\gamma\delta} dS\Big| \le C(M) \|\tilde h_t\|_{H^{2.5}(\Gamma_0)} \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)}, \tag{C.4}
$$

and the third term satisfies the equality

$$
\begin{aligned}
&\int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} (\bar A^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta} h_{\kappa tt} dS \\
=&\int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} (\bar A^{\alpha\beta\gamma\delta})_t\Big]_{,\gamma\delta} h_{\kappa,\alpha\beta} h_{\kappa tt} dS \quad (\equiv K_7) \\
&+ 2\int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}} \Big[\sqrt{\det(g_0)} (\bar A^{\alpha\beta\gamma\delta})_t\Big]_{,\delta} h_{\kappa,\alpha\beta\gamma} h_{\kappa tt} dS \quad (\equiv K_8) \\
&+ \int_{\Gamma_0} \bar A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta\gamma\delta} h_{\kappa tt} dS \quad (\equiv K_9).
\end{aligned}
$$

By (C.2),

$$
\|h_{\kappa tt}\|_{L^4(\Gamma_0)} \le C(M)\Big[\|v_\kappa\|_{H^2(\Omega_0)} + \|v_{\kappa t}\|_{H^1(\Omega_0)}\Big]
$$

and hence

$$|K_8| + |K_9| \le C(M)\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\Big[\|v_\kappa\|_{H^2(\Omega_0)} + \|v_{\kappa t}\|_{H^1(\Omega_0)}\Big]$$
$$\le C(M)C(\delta,\delta_1)\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^2(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}.$$

For the term K_7, we use the $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing and (7.5) to obtain

$$|K_7| \le C\|[\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t]_{,\gamma\delta}\|_{H^{-0.5}(\Gamma_0)}\Big\|\frac{1}{\sqrt{\det(g_0)}}h_{\kappa,\alpha\beta}h_{\kappa tt}\Big\|_{H^{0.5}(\Gamma_0)}$$
$$\le C(M)\|\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t\|_{H^{1.5}(\Gamma_0)}\|\nabla_0^2 h_\kappa\|_{H^{1.5}(\Gamma_0)}\|h_{\kappa tt}\|_{H^{0.5}(\Gamma_0)}$$
$$\le C(M)\|\tilde{h}_t\|_{H^{2.5}(\Gamma_0)}\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\Big[\|v_\kappa\|_{H^2(\Omega_0)} + \|v_{\kappa t}\|_{H^1(\Omega_0)}\Big]$$
$$\le C(M)C(\delta,\delta_1)\|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)}\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^2(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}.$$

Therefore,

$$\text{(C.5)}\quad \Big|\int_{\Gamma_0}\frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta}\Big]_{,\gamma\delta}h_{\kappa tt}dS\Big|$$
$$\le C(M)C(\delta,\delta_1)(1+\|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)})\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^2(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}.$$

Remark 34. *The bound for K_7 can be refined even further as*

$$|K_7| \le C(M)C(\delta)\|\tilde{h}_t\|^2_{H^{1.5}(\Gamma_0)}\|\nabla_0^2 h_\kappa\|^2_{H^{1.5}(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^3(\Omega_0)} + \delta\|v_{\kappa t}\|^2_{H^1(\Omega_0)};$$

it is this inequality that will be used in the proof of the fixed-point argument.

Now let us turn our attention to the other terms in (C.3). Integrating by parts, for the fourth term we have

$$\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma}\Big]_t h_{\kappa tt}dS$$
$$= \int_{\Gamma_0}(L_1^{\alpha\beta\gamma})_t\bar{h}_{,\alpha\beta\gamma}h_{tt}dS - \int_{\Gamma_0}\frac{1}{\sqrt{\det(g_0)}}\bar{h}_{t,\alpha\beta}\Big[\sqrt{\det(g_0)}h_{\kappa tt}L_1^{\alpha\beta\gamma}\Big]_{,\gamma}dS$$

and the first integral on the right-hand side is bounded by

$$C(M)\Big[\|v_\kappa\|_{H^2(\Omega_0)} + \|v_{\kappa t}\|_{H^1(\Omega_0)}\Big]$$

which is dominated by

$$C(M)C(\delta,\delta_1) + \delta\|v_\kappa\|^2_{H^2(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}.$$

The worst term in the second integral is

$$\int_{\Gamma_0} L_1^{\alpha\beta\gamma}\bar{h}_{t,\alpha\beta}h_{\kappa tt,\gamma}dS.$$

Using $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing, we find that

$$\begin{aligned}\Big|\int_{\Gamma_0} L_1^{\alpha\beta\gamma}\bar{h}_{t,\alpha\beta}h_{\kappa tt,\gamma}dS\Big| &\leq C(M)\|\bar{h}_{t,\alpha\beta}\|_{H^{0.5}(\Gamma_0)}\|h_{\kappa tt,\gamma}\|_{H^{-0.5}(\Gamma_0)}\\ &\leq C(M)C(\delta,\delta_1)\|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)} + \delta\|v_\kappa\|^2_{H^2(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}\end{aligned}$$

and hence

$$\begin{aligned}&\Big|\int_{\Gamma_0}\Big[L_1^{\alpha\beta\gamma}\bar{h}_{,\alpha\beta\gamma}\Big]_t h_{\kappa tt}dS\Big|\\ &\leq C(M)C(\delta,\delta_1)\Big[1+\|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)}\Big] + \delta\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1')} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}.\end{aligned} \tag{C.6}$$

For the last term on the right-hand side in (C.3), it follows that

$$\int_{\Gamma_0}(L_2)_t h_{\kappa tt}dS \leq C(M)C(\delta,\delta_1) + \delta\|\nabla_0^2 v_\kappa\|^2_{H^1(\Omega_1')} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}. \tag{C.7}$$

It remains to estimate K_3 to K_6. It is obvious that

$$\begin{aligned}|K_6| &\leq C(M)\Big[\|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}+1\Big]\|v_{\kappa t}\|_{H^1(\Omega_0)}\\ &\leq C(M)C(\delta)\Big[\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}+1\Big] + \delta\|v_{\kappa t}\|^2_{H^1(\Omega_0)}.\end{aligned} \tag{C.8}$$

Similar to the estimates in (C.6) and (C.7), we know that the lower order terms in K_3 to K_5, i.e, terms containing L_1 and L_2, can be bounded by

$$C(M)\|\tilde{h}_t\|_{H^{2.5}(\Gamma_0)}\|v_\kappa\|_{H^{1.5}(\Gamma_0)}. \tag{C.9}$$

For the highest order term, we note that by (7.14),

$$\begin{aligned}&\|(\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_{t,\gamma\delta}\|_{H^{-1.5}(\Gamma_0)} \leq \|\sqrt{\det(g_0)}(\bar{A}^{\alpha\beta\gamma\delta}h_{\kappa,\alpha\beta})_t\|_{H^{0.5}(\Gamma_0)}\\ &\leq C(M)\Big[t^{1/4}\|h_{\kappa t}\|_{H^{2.5}(\Gamma_0)} + \|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\Big].\end{aligned}$$

Therefore, combining with an upper bound (C.9) for the lower order terms, we find that

$$
\begin{aligned}
|K_3| + |K_5| &\leq C(M)\Big[t^{1/4}\|h_{\kappa t}\|_{H^{2.5}(\Gamma_0)} + \|\nabla_0^4 h_\kappa\|_{L^2(\Gamma_0)}\Big]\Big[\|h_{\kappa t}\|_{H^{2.5}(\Gamma_0)} + \|v_\kappa\|_{H^{1.5}(\Gamma_0)}\Big] \\
&\quad + C(M)\|\tilde{h}_t\|_{H^{2.5}(\Gamma_0)}\|v_\kappa\|_{H^{1.5}(\Gamma_0)} \\
&\leq C\Big[1 + \|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)}\Big]\Big[1 + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}\Big] + (\delta + Ct^{1/2})\|v_\kappa\|^2_{H^3(\Omega_0)}
\end{aligned}
$$

for some constant C depending on M and δ.

For K_4, most of the terms can be estimated in the same fashion except the term

$$
\int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta}\Big][(\nabla_0 \bar{h}_{t,\gamma\delta}, 0)\cdot(v_\kappa \circ \bar{\eta}^{-\tau})]dS
$$

which is identical to

$$
\begin{aligned}
&\int_{\Gamma_0} \Big\{\frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta}\Big][(\nabla_0 \bar{h}_{,\gamma\delta}, 0)\cdot(v_\kappa \circ \bar{\eta}^{-\tau})]\Big\}_t dS \quad (\equiv K_{10}) \\
&- \int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta}\Big]_t[(\nabla_0 \bar{h}_{,\gamma\delta}, 0)\cdot(v_\kappa \circ \bar{\eta}^{-\tau})]dS \quad (\equiv K_{11}) \\
&- \int_{\Gamma_0} \frac{1}{\sqrt{\det(g_0)}}\Big[\sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa t,\alpha\beta}\Big][(\nabla_0 \bar{h}_{,\gamma\delta}, 0)\cdot(v_\kappa \circ \bar{\eta}^{-\tau})_t]dS. \quad (\equiv K_{12})
\end{aligned}
$$

Time integrating K_{10} and use the inequality (7.6) (or (7.9) if $n = 2$) together with Young's inequality, we find that

$$
\begin{aligned}
\Big|\int_0^t K_{10}(s)ds\Big| &\leq C(M)\Big[\|u_0\|^2_{H^{2.5}(\Omega_0)} + \|\nabla_0^2 h_{\kappa t}\|_{L^2(\Omega_0)}\|v_\kappa\|_{L^4(\Omega_0)}\Big] \\
&\leq C(M)C(\delta_1,\delta_2)N_3(u_0,F) + \delta_2\|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} + \delta_1\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)}ds, \qquad \text{(C.10)}
\end{aligned}
$$

where

$$
\begin{aligned}
N_3(u_0,F) := &\|u_0\|^2_{H^{2.5}(\Omega_0)} + \|u_0\|^2_{H^{4.5}(\Gamma_0)} + \|F\|^2_{L^2(0,T;H^1(\Omega_0))} \\
&+ \|F_t\|^2_{L^2(0,T;H^1(\Omega_0)')} + \|F(0)\|^2_{H^1(\Omega_0)} + 1
\end{aligned}
$$

and we use $\|v_\kappa\|^2_{H^1(\Omega_0)} \leq C\Big[\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega)}ds + \|u_0\|^2_{H^1(\Omega_0)}\Big]$ to obtain (C.10). The worst term in K_{11} is

$$
\int_{\Gamma_0} \sqrt{\det(g_0)}\bar{A}^{\alpha\beta\gamma\delta} h_{\kappa tt,\alpha\beta}[(\nabla_0 \bar{h}_{,\gamma\delta}, 0)\cdot(v_\kappa \circ \bar{\eta}^{-\tau})]dS
$$

which, by $H^{-1.5}(\Gamma_0)$-$H^{1.5}(\Gamma_0)$ duality pairing, can be bounded by

$$\begin{aligned} &C(M)\|h_{\kappa tt}\|_{H^{0.5}(\Gamma_0)}\|\tilde{h}\|_{H^{4.5}(\Gamma_0)}\|v_\kappa\|_{H^{1.5}(\Gamma_0)} \\ &\leq C(M)C(\delta,\delta_1)\|\tilde{h}\|^2_{H^{5.5}(\Gamma_0)}\|v_\kappa\|^2_{L^2(\Omega_0)} + \delta\|v_\kappa\|^2_{H^3(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}. \end{aligned}$$

Therefore,

$$|K_{11}| \leq C\Big[\|\tilde{h}\|^2_{H^{5.5}(\Gamma_0)}\|v_\kappa\|^2_{L^2(\Omega_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)}\Big] + \delta\|v_\kappa\|^2_{H^3(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}$$

for some constant C depending on M, δ and δ_1. Also,

$$|K_{12}| \leq C(M)C(\delta)\|\tilde{h}\|^2_{H^{4.5}(\Gamma_0)}\|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)} + \delta\|v_{\kappa t}\|^2_{H^1(\Omega_0)}$$

and hence

$$\begin{aligned} \sum_{i=3}^{6}|K_i| &\leq C\Big[1+\|\tilde{h}\|^2_{H^{5.5}(\Gamma_0)} + \|\tilde{h}_t\|^2_{H^{2.5}(\Gamma_0)}\Big]\Big[1+\|v_\kappa\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)}\Big] \\ &\quad + (\delta + Ct^{1/2})\|v_\kappa\|^2_{H^3(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)} + K_8 \end{aligned} \tag{C.11}$$

with K_{10} satisfying inequality (C.10). Finally, combining all the estimates,

(C.12)

$$\begin{aligned} &\int_0^t \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)}ds \leq \int_0^t\int_{\Gamma_0}\Big[[L_{\bar{h}}(h_\kappa)(\nabla_0\bar{h},-1)]\circ\bar{\eta}^\tau\Big]_t\cdot v_{\kappa t}dS \\ &+ CN_3(u_0,F) + C\int_0^t K(s)\Big[\|v_\kappa\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} + \|\nabla_0^2 h_{\kappa t}\|^2_{L^2(\Gamma_0)}\Big]ds \\ &+ (\delta + Ct^{1/2})\int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)}ds + (\delta_1 + Ct^{1/2})\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)}ds + \delta_2\|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} \end{aligned}$$

for some constant C depending on M, δ, δ_1 and δ_2.

C.2. **Estimates for the terms with pressure.** By the "divergence free" condition (8.2b),

$$\int_{\Omega_0}(\bar{a}^\ell_k q_\kappa)_t v_{\kappa t,\ell}^{\ k}dx = \int_{\Omega_0}\bar{a}^\ell_{kt}q_\kappa v_{\kappa t,\ell}^{\ k}dx - \int_{\Omega_0}\bar{a}^\ell_{kt}q_{\kappa t}v_{\kappa,\ell}^{\ k}dx.$$

By (6.3), (7.6) (or (7.9) if $n = 2$), (9.1), and (9.3),

$$
\begin{aligned}
\int_{\Omega_0} \bar{a}^\ell_{kt} q_\kappa v_{\kappa t,\ell}^{\ k} dx &\le C(M)C(\delta_1)\|q_\kappa\|_{H^1(\Omega_0)}\|q_\kappa\|_{L^2(\Omega_0)} + \delta_1 \|v_{\kappa t}\|^2_{H^1(\Omega)} \\
&\le C(M)C(\delta,\delta_1)\Big[\|v_{\kappa t}\|^2_{L^2(\Omega_0)} + \|\nabla v_\kappa\|^2_{L^2(\Omega_0)} + \|\nabla_0^4 h_\kappa\|^2_{L^2(\Gamma_0)} \\
&\quad + \|F\|^2_{L^2(\Omega_0)} + 1\Big] + \delta\|v_\kappa\|^2_{H^3(\Omega_0)} + \delta_1\|v_{\kappa t}\|^2_{H^1(\Omega_0)}. \qquad \text{(C.13)}
\end{aligned}
$$

For the second integral, we time integrate it and have the identity

$$
\begin{aligned}
\int_0^t \int_{\Omega_0} \bar{a}^\ell_{kt} q_{\kappa t} v_{\kappa,\ell}^{\ k} ds = &\int_{\Omega_0} (\bar{a}^\ell_{kt} q_\kappa v_{\kappa,\ell}^{\ k})(t) dx - \int_{\Omega_0} \bar{a}^\ell_{kt}(0) q_\kappa(0) (\tilde{u}_0)^k_{,\ell} dx \\
&- \int_0^t \int_{\Omega_0} (\bar{a}^\ell_{kt} v_{\kappa,\ell}^{\ k})_t q_\kappa dx ds.
\end{aligned}
$$

The first term on the right-hand side can be bounded by

$$
\|\bar{a}_t\|_{L^4(\Omega_0)} \|\nabla v_\kappa\|_{L^4(\Omega_0)} \|q_\kappa\|_{L^2(\Omega_0)}
$$

while the second term on the right-hand side is bounded by $CN_3(u_0, F)$. Because of (C.13), it remains to estimate

$$
\int_{\Omega_0} \bar{a}^\ell_{ktt} v_{\kappa,\ell}^{\ k} q_\kappa dx.
$$

By the identity $\bar{a}^\ell_{kt} = -\bar{a}^i_k \bar{v}^j_{,i} \bar{a}^\ell_j$,

$$
\begin{aligned}
\int_{\Omega_0} \bar{a}^\ell_{ktt} v_{\kappa,\ell}^{\ k} q_\kappa dx = &- \int_{\Omega_0} \bar{a}^i_{kt} \bar{v}^j_{,i} \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa dx - \int_{\Omega_0} \bar{a}^i_k \bar{v}^j_{t,i} \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa dx \\
&- \int_{\Omega_0} \bar{a}^i_k \bar{v}^j_{,i} \bar{a}^\ell_{jt} v_{\kappa,\ell}^{\ k} q_\kappa dx.
\end{aligned}
$$

The first and the third integrals are bounded by

$$
\|\bar{a}_t\|_{L^4(\Omega_0)} \|\nabla \bar{v}\|_{L^4(\Omega_0)} \|\bar{a}\|_{L^\infty(\Omega_0)} \|\nabla v_\kappa\|_{L^\infty(\Omega_0)} \|q_\kappa\|_{L^2(\Omega_0)}.
$$

For the second integrals, we integrate by parts to obtain

$$
-\int_{\Omega_0} \bar{a}^i_k \bar{v}^j_{t,i} \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa dx = \int_{\Omega_0} (\bar{a}^i_k \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa)_{,i} \bar{v}^j_t dx - \int_{\Gamma_0} \bar{a}^i_k \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa \bar{v}^j_t n^i dS.
$$

It follows that

$$\int_{\Omega_0} (\bar{a}^i_k \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa)_{,i} \bar{v}^j_t dx$$
$$\le C(M)\Big[\|\nabla v_\kappa\|_{L^\infty(\Omega_0)}\Big(\|q_\kappa\|_{L^4(\Omega_0)} + \|\nabla q_\kappa\|_{L^2(\Omega_0)}\Big) + \|\nabla^2 v_\kappa\|_{L^4(\Omega_0)}\|q_\kappa\|_{L^4(\Omega_0)}\Big].$$

For the second term on the right-hand side (the integral over Γ_0), we use $H^{0.5}(\Gamma_0)$-$H^{-0.5}(\Gamma_0)$ duality pairing to obtain

$$\int_{\Gamma_0} \bar{a}^i_k \bar{a}^\ell_j v_{\kappa,\ell}^{\ k} q_\kappa \bar{v}^j_t n^i dS$$
$$\le C(M)\|v_\kappa\|_{H^2(\Gamma_0)}\Big[\|q_\kappa\|_{L^4(\Gamma_0)} + \|q_\kappa\|_{H^{0.5}(\Gamma_0)}\Big]$$
$$\le C(M)C(\delta,\delta_1)\Big[\|v_\kappa\|^2_{H^1(\Omega_0)} + \|q_\kappa\|^2_{L^2(\Omega_0)}\Big] + \delta\|v_\kappa\|^2_{H^3(\Omega_0)} + \delta_1\|q_\kappa\|^2_{H^2(\Omega_0)}.$$

Combining all the estimates, (9.3) and (9.9) imply

$$\int_0^t \int_{\Omega_0} (\bar{a}^\ell_k q_\kappa)_t v_{\kappa t,\ell}^{\ k} dxds \le C(M)C(\delta,\delta_1)N_3(u_0,F)$$
$$+ \delta\int_0^t \|v_\kappa\|^2_{H^3(\Omega_0)}ds + \delta_1\int_0^t \|v_{\kappa t}\|^2_{H^1(\Omega_0)}ds.$$

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