

Near-colorings: non-colorable graphs and NP-completeness

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Abstract

A graph G is (d_1, \dots, d_l) -colorable if the vertex set of G can be partitioned into subsets V_1, \dots, V_l such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq l$. In this paper, we focus on complexity aspects of such colorings when $l = 2, 3$. More precisely, we prove that, for any fixed integers k, j, g with $(k, j) \neq (0, 0)$ and $g \geq 3$, either every planar graph with girth at least g is (k, j) -colorable or it is NP-complete to determine whether a planar graph with girth at least g is (k, j) -colorable. Also, for every fixed integer k , it is NP-complete to determine whether a planar graph that is either $(0, 0, 0)$ -colorable or non- $(k, k, 1)$ -colorable is $(0, 0, 0)$ -colorable. Additionally, we exhibit non- $(3, 1)$ -colorable planar graphs with girth 5 and non- $(2, 0)$ -colorable planar graphs with girth 7.

1 Introduction

A graph G is (d_1, \dots, d_k) -colorable if the vertex set of G can be partitioned into subsets V_1, \dots, V_k such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of proper k -coloring (when $d_1 = \dots = d_k = 0$) and d -improper k -coloring (when $d_1 = \dots = d_k = d \geq 1$).

Planar graphs are known to be $(0, 0, 0, 0)$ -colorable (Appel and Haken [1, 2]) and $(2, 2, 2)$ -colorable (Cowen, Cowen, and Woodall [13]). The $(2, 2, 2)$ -colorability is optimal (for any integer k , there exist non- $(k, k, 1)$ -colorable planar graphs) and holds in the choosability case (Eaton and Hull [15] or Skrekovski [23]). Improper colorings have then been considered for planar graphs with large girth or graphs with low maximum average degree. We recall that the girth of a graph G , denoted by $g(G)$, is the length of a shortest cycle in G , and the maximum average degree of a graph G , denoted by

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$\text{mad}(G)$, is the maximum of the average degrees of all subgraphs of G , i.e. $\text{mad}(G) = \max \{2|E(H)|/|V(H)|, H \subseteq G\}$.

(1, 0)-coloring.

Glebov and Zambalaeva [20] proved that every planar graph with girth at least 16 is (1, 0)-colorable. This was then strengthened by Borodin and Ivanova [3] who proved that every graph G with $\text{mad}(G) < \frac{7}{3}$ is (1, 0)-colorable. This implies that every planar graph G with girth at least 14 is (1, 0)-colorable. Borodin and Kostochka [7] then proved that every graph G with $\text{mad}(G) \leq \frac{12}{5}$ is (1, 0)-colorable. In particular, it follows that every planar graph with girth at least 12 is (1, 0)-colorable. On the other hand, they constructed graphs G with $\text{mad}(G)$ arbitrarily close (from above) to $\frac{12}{5}$ that are not (1, 0)-colorable; hence their upper bound on the maximum average degree is best possible. The last result was strengthened for triangle-free graphs: Kim, Kostochka, and Zhu [22] proved that triangle-free graphs G satisfying $11|V(G)| - 9|E(G)| \geq -4$ are (1, 0)-colorable. This implies that planar graphs with girth at least 11 are (1, 0)-colorable. On the other hand, Esperet, Montassier, Ochem, and Pinlou [16] proved that determining whether a planar graph with girth 9 is (1, 0)-colorable is NP-complete. To our knowledge, the question whether all planar graphs with girth at least 10 are (1, 0)-colorable is still open.

(k, 0)-coloring with $k \geq 2$.

Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] proved that every graph G with $\text{mad}(G) < \frac{3k+4}{k+2}$ is $(k, 0)$ -colorable. The proof in [4] extends the one in [3] but does not work for $k = 1$. Moreover, they exhibited a non- $(k, 0)$ -colorable planar graph with girth 6. Finally, Borodin and Kostochka [8] proved that for $k \geq 2$, every graph G with $\text{mad}(G) \leq \frac{3k+2}{k+1}$ is $(k, 0)$ -colorable. This result is tight in terms of maximum average degree.

(k, 1)-coloring.

Recently, Borodin, Kostochka, and Yancey [9] proved that every graph with $\text{mad}(G) \leq \frac{14}{5}$ is (1, 1)-colorable, and the restriction on $\text{mad}(G)$ is sharp. In [5], it is proven that every graph G with $\text{mad}(G) < \frac{10k+22}{3k+9}$ is $(k, 1)$ -colorable for $k \geq 2$.

(k, j)-coloring.

A first step was made by Havet and Sereni [21] who showed that, for every $k \geq 0$, every graph G with $\text{mad}(G) < \frac{4k+4}{k+2}$ is (k, k) -colorable (in fact (k, k) -choosable). More generally, they studied k -improper l -choosability and proved that every graph G with $\text{mad}(G) < l + \frac{lk}{l+k}$ ($l \geq 2, k \geq 0$) is k -improper l -choosable; this implies that such graphs are (k, \dots, k) -colorable (where the number of partite sets is l). Borodin, Ivanova, Montassier, and Raspaud [6] gave some sufficient conditions of (k, j) -colorability depending on the density of the graphs using linear programming. Finally, Borodin and Kostochka [8]

solved the problem for a wide range of j and k : let $j \geq 0$ and $k \geq 2j + 2$; every graph G with $\text{mad}(G) \leq 2(2 - \frac{k+2}{(j+2)(k+1)})$ is (k, j) -colorable. This result is tight in terms of the maximum average degree and improves some results in [4, 5, 6].

Using the fact that every planar graph G with girth $g(G)$ has $\text{mad}(G) < 2g(G)/(g(G) - 2)$, the previous results give results for planar graphs. They are summarized in Table 1, which also shows the recent results that planar graphs with girth 5 are $(5, 3)$ -colorable (Choi and Raspaud [12]) and $(10, 1)$ -colorable (Choi, Choi, Jeong, and Suh [11]).

girth	$(k, 0)$	$(k, 1)$	$(k, 2)$	$(k, 3)$	$(k, 4)$
3,4	×	×	×	×	×
5	×	$(10, 1)$ [11]	$(6, 2)$ [8]	$(5, 3)$ [12]	$(4, 4)$ [21]
6	×	$(4, 1)$ [8]	$(2, 2)$ [21]		
7	$(4, 0)$ [8]	$(1, 1)$ [9]			
8	$(2, 0)$ [8]				
11	$(1, 0)$ [22]				

Table 1: The girth and the (k, j) -colorability of planar graphs. The symbol “ \times ” means that there exist non- (k, j) -colorable planar graphs for every k .

From the previous discussion, the following questions are natural:

Question 1.

1. Are planar graphs with girth 10 $(1, 0)$ -colorable?
2. Are planar graphs with girth 7 $(3, 0)$ -colorable?
3. Are planar graphs with girth 6 $(1, 1)$ -colorable?
4. Are planar graphs with girth 5 $(4, 1)$ -colorable?
5. Are planar graphs with girth 5 $(2, 2)$ -colorable?

(d_1, \dots, d_k) -coloring.

Finally we would like to mention two studies. Chang, Havet, Montassier, and Raspaud [10] gave some approximation results to Steinberg’s Conjecture using (k, j, i) -colorings. Dorbec, Kaiser, Montassier, and Raspaud [14] studied the particular case of (d_1, \dots, d_k) -coloring where the value of each d_i ($1 \leq i \leq k$) is either 0 or some value d , making the link between $(d, 0)$ -coloring [8] and (d, \dots, d) -coloring [21].

The aim of this paper is to provide complexity results on the subject and to obtain non-colorable planar graphs showing that some above-mentioned results are optimal.

Claim 2. *There exist 2-degenerate planar graphs that are:*

1. *non- (k, k) -colorable with girth 4, for every $k \geq 0$,*
2. *non- $(3, 1)$ -colorable with girth 5,*
3. *non- $(k, 0)$ -colorable with girth 6,*
4. *non- $(2, 0)$ -colorable with girth 7.*

Claim 2.4 shows that the $(2, 0)$ -colorability of planar graphs with girth at least 8 [8] is a tight result. Claim 2.3 has been obtained in [?] and the corresponding graph is depicted in Figure 1.

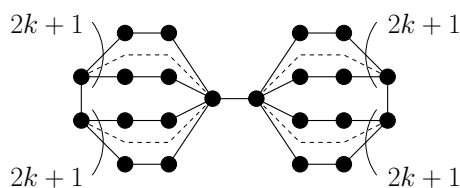


Figure 1: A non- $(k, 0)$ -colorable planar graph with girth 6 [4].

Theorem 3. *Let k, j , and g be fixed integers such that $(k, j) \neq (0, 0)$ and $g \geq 3$. Either every planar graph with girth at least g is (k, j) -colorable or it is NP-complete to determine whether a planar graph with girth at least g is (k, j) -colorable.*

Theorem 4. *Let k be a fixed integer. It is NP-complete to determine whether a 3-degenerate planar graph that is either $(0, 0, 0)$ -colorable or non- $(k, k, 1)$ -colorable is $(0, 0, 0)$ -colorable.*

We construct a non- (k, k) -colorable planar graph with girth 4 in Section 2, a non- $(3, 1)$ -colorable planar graph with girth 5 in Section 3, and a non- $(2, 0)$ -colorable planar graph with girth 7 in Section 4. We prove Theorem 3 in Section 5 and we prove Theorem 4 in Section 6.

Notation.

In the following, when we consider a (d_1, \dots, d_k) -coloring of a graph G , we color the vertices of V_i with color d_i for $1 \leq i \leq k$: for example in a $(3, 0)$ -coloring, we will use color 3 to color the vertices of V_1 inducing a subgraph with maximum degree 3 and use color 0 to color the vertices of V_2 inducing a stable set. A vertex is said to be *colored i^j* if it is colored i and has j neighbors colored i , that is, it has degree j in the subgraph induced by its color. A vertex is *saturated* if it is colored i^i , that is, if it has maximum degree in the subgraph induced by its color. A cycle (resp. face) of length k is called a k -cycle (resp. k -face). A k -vertex (resp. k^- -vertex, k^+ -vertex) is a vertex of degree k (resp. at most k , at least k). The minimum degree of a graph G is denoted by $\delta(G)$.

2 A non- (k, k) -colorable planar graph with girth 4

For every $k \geq 0$, we construct a non- (k, k) -colorable planar graph J_4 with girth 4. Let $H_{x,y}$ be a copy of $K_{2,2k+1}$, as depicted in Figure 2. In any (k, k) -coloring of $H_{x,y}$, the vertices x and y must receive the same color. We obtain J_4 from a vertex u and a star S with center v_0 and $k + 1$ leaves v_1, \dots, v_{k+1} by linking u to every vertex v_i with a copy H_{u,v_i} of $H_{x,y}$. The graph J_4 is not (k, k) -colorable: by the property of $H_{x,y}$, every vertex v_i should get the same color as u . This gives a monochromatic S , which is forbidden. Notice that J_4 is a planar graph with girth 4 and is 2-degenerate.

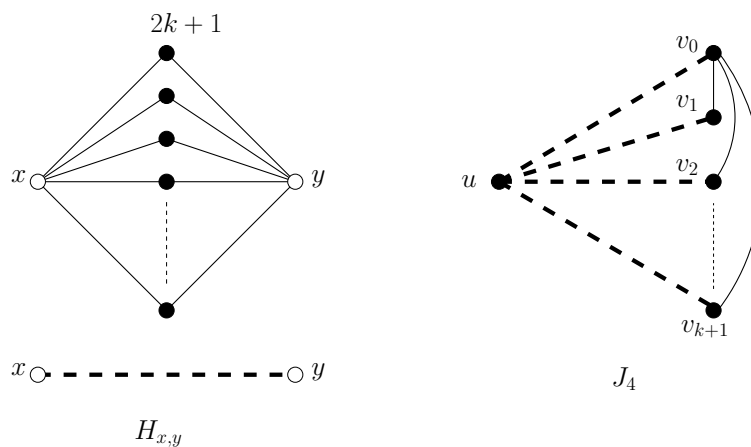


Figure 2: A non- (k, k) -colorable planar graph with girth 4.

3 A non- $(3, 1)$ -colorable planar graph with girth 5

We construct a non- $(3, 1)$ -colorable planar graph J_5 with girth 5. Consider the graph $H_{x,y}$ depicted in Figure 3. If x and y are colored 3 but have no neighbor colored 3, then it is not possible to extend this partial coloring to $H_{x,y}$. Now, we construct the graph S_z as follows. Let z be a vertex and $t_1 t_2 t_3$ be a path on three vertices. Take 21 copies H_{x_i, y_j} of $H_{x,y}$ with $1 \leq i \leq 7$ and $1 \leq j \leq 3$. Identify every x_i with z and identify every y_i with t_i . Finally, we obtain J_5 from three copies S_{z_1} , S_{z_2} , and S_{z_3} of S_z by adding the edges $z_1 z_2$ and $z_2 z_3$ (Figure 3). Notice that J_5 is planar with girth 5 and is 2-degenerate. Let us show that J_5 is not $(3, 1)$ -colorable. In every $(3, 1)$ -coloring of J_5 , the path $z_1 z_2 z_3$ contains a vertex z colored 3. In the copy of S_z corresponding to z , the path $t_1 t_2 t_3$ contains a vertex t colored 3. Since z (resp. t) has at most 3 neighbors colored 3, one of the seven copies of $H_{x,y}$ between z and t , does not contain a neighbor of z or t colored 3. This copy of $H_{x,y}$ is not $(3, 1)$ -colorable, and thus J_5 is not $(3, 1)$ -colorable.

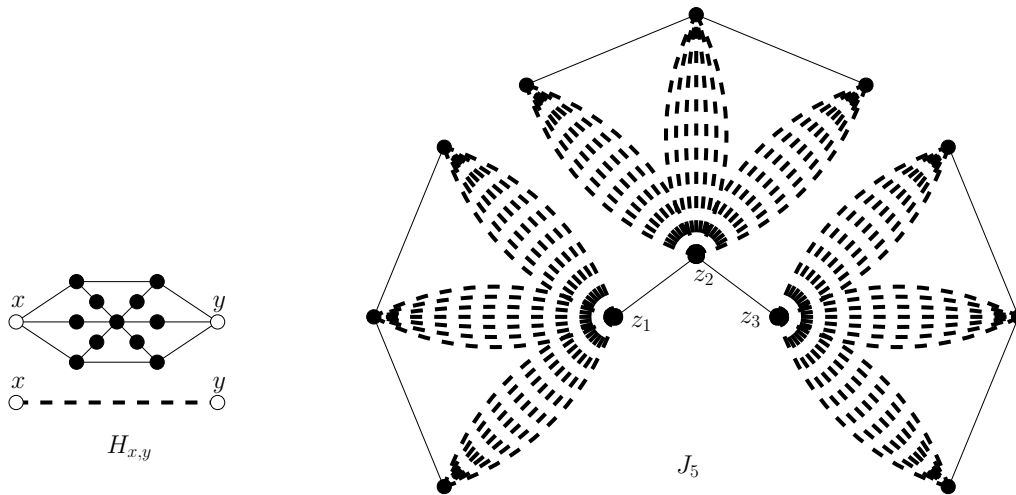


Figure 3: A non-(3,1)-colorable planar graph with girth 5.

4 A non-(2,0)-colorable planar graph with girth 7

We construct a non-(2,0)-colorable planar graph J_7 with girth 7. Consider the graphs $T_{x,y,z}$ and S in Figure 4. If the vertices x , y , and z of $T_{x,y,z}$ are colored 2 and have no neighbor colored 2, then w is colored 2^2 . Suppose that the vertices a, b, c, d, e, f, g of S are respectively colored 2, 0, 2, 2, 2, 2, 0, and that a has no neighbor colored 2. Using successively the property of $T_{x,y,z}$, we have that w_1, w_2 , and w_3 must be colored 2^2 . It follows that w_4 is colored 0, w_5 is colored 2, and so w_6 is colored 2^2 . Again, by the property of $T_{x,y,z}$, w_7 must be colored 2^2 . Finally, w_8 must be colored 0 and there is no choice of color for w_9 . Hence, such a coloring of the outer 7-cycle $abcdefg$ cannot be extended.

The graph H_z depicted on the left of Figure 5 is obtained as follows. We link a vertex z to every vertex of a 7-cycle $v_1 \dots v_7$ with a path of three edges. Then we embed the graph S in every 7-face F_i ($1 \leq i \leq 7$) incident to z by identifying the outer 7-cycle of S with the 7-cycle of F_i such that a is identified to z . Finally, the graph J_7 depicted on the right of Figure 5 is obtained from two adjacent vertices u and v and six copies H_{z_1}, \dots, H_{z_6} of H_z by identifying z_1, z_2, z_3 with u and z_4, z_5, z_6 with v . Notice that J_7 is planar with has girth 7. Let us prove that J_7 is not (2,0)-colorable.

- We assume that u is colored 2 since u and v cannot be both colored 0.
- In one of the three copies of H_z attached to u , say H_{z_1} , u has no neighbor colored 2.
- Since a 7-cycle is not 2-colorable, the 7-cycle $v_1 \dots v_7$ of H_{z_1} contains a monochromatic edge colored 2, say $v_1 v_2$.
- The coloring of the face F_2 cannot be extended to the copy of S embedded in F_2 .

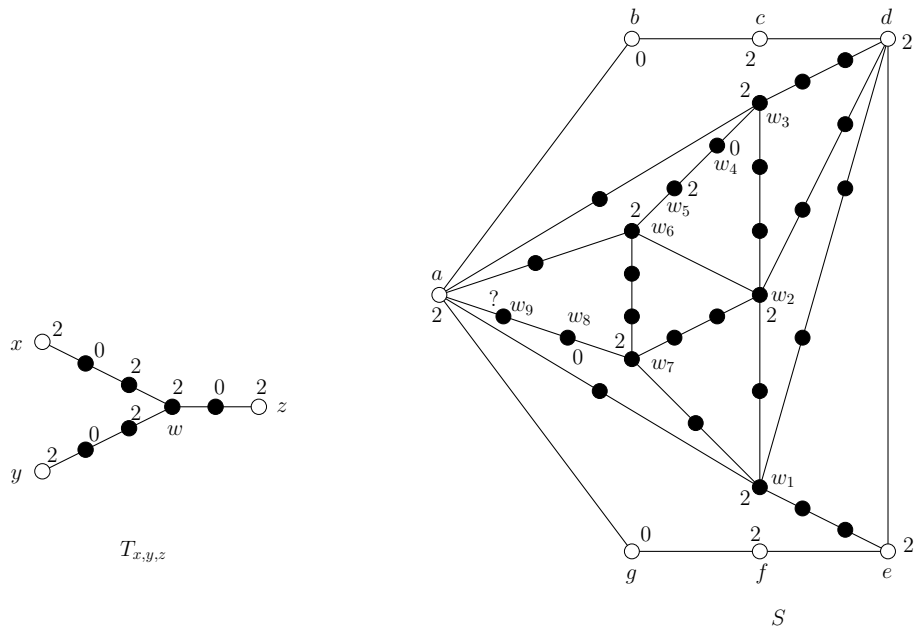


Figure 4: The graphs $T_{x,y,z}$ and S .

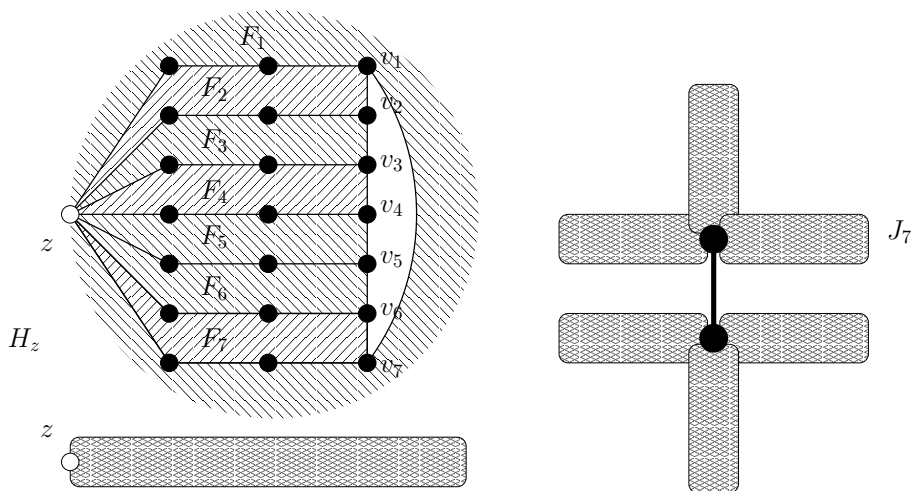


Figure 5: The graphs H_z and J_7 .

5 NP-completeness of (k, j) -colorings

Let $g_{k,j}$ be the largest integer g such that there exists a planar graph with girth g that is not (k, j) -colorable. Because of large odd cycles, $g_{0,0}$ is not defined. For $(k, j) \neq (0, 0)$, we have $4 \leq g_{k,j} \leq 10$ by the example in Figure 2 and the result that planar graphs with girth at least 11 are $(0, 1)$ -colorable [22]. We prove this equivalent form of Theorem 3:

Theorem 5. *Let k and j be fixed integers such that $(k, j) \neq (0, 0)$. It is NP-complete to determine whether a planar graph with girth $g_{k,j}$ is (k, j) -colorable.*

Let us define the partial order \preceq . Let $n_3(G)$ be the number of 3^+ -vertices in G . For any two graphs G_1 and G_2 , we have $G_1 \prec G_2$ if and only if at least one of the following conditions holds:

- $|V(G_1)| < |V(G_2)|$ and $n_3(G_1) \leq n_3(G_2)$.
- $n_3(G_1) < n_3(G_2)$.

Note that the partial order \preceq is well-defined and is a partial linear extension of the subgraph poset. The following lemma is useful.

Lemma 6. *Let k and j be fixed integers such that $(k, j) \neq (0, 0)$. There exists a planar graph $G_{k,j}$ with girth $g_{k,j}$, minimally non- (k, j) -colorable for the subgraph order, such that $\delta(G_{k,j}) = 2$.*

Proof. We have $\delta(G_{k,j}) \geq 2$, since a non- (k, j) -colorable graph that is minimal for the subgraph order does not contain a 1^- -vertex. Suppose that for some pair (k, j) , we construct a 2-degenerate non- (k, j) -colorable planar graph with girth $g_{k,j}$. Then this graph contains a (not necessarily proper) minimally non- (k, j) -colorable subgraph with minimum degree 2. Thus, we can prove the lemma for the following pairs (k, j) by using Claim 2.

- Pairs (k, j) such that $g_{k,j} \leq 4$: We actually have $g_{k,j} = 4$ by Claim 2.1.
- Pairs (k, j) such that $g_{k,j} \geq 6$: Indeed, a planar graph with girth at least 6 is 2-degenerate. In particular, Claim 2.3 shows that $g_{k,0} \geq 6$, so the lemma is proved for all pairs $(k, 0)$.
- Pairs $(k, 1)$ such that $1 \leq k \leq 3$: If $g_{k,j} \geq 6$, then we are in a previous case. Otherwise, we have $g_{k,j} = 5$ by Claim 2.2.

The remaining pairs satisfy $g_{k,j} = 5$. There are two types of remaining pairs (k, j) :

- Type 1: $k \geq 4$ and $j = 1$.
- Type 2: $2 \leq j \leq k$.

Consider a planar graph G with girth 5 that is non- (k, j) -colorable and is minimal for the order \preceq . Suppose for contradiction that G does not contain a 2-vertex. Also, suppose that G contains a 3-vertex a adjacent to three 4^- -vertices a_1, a_2 , and a_3 . For colorings of type 1, we can extend to G a coloring of $G \setminus \{a\}$ by assigning to a the color of improperly at least 4. For colorings of type 2, we consider the graph G' obtained from $G \setminus \{a\}$ by adding three 2-vertices b_1, b_2 , and b_3 adjacent to, respectively, a_2 and a_3, a_1 and a_3, a_1 and a_2 . Notice that $G' \preceq G$, so G' admits a coloring c of type 2. We can extend to

G the coloring of $G \setminus \{a\}$ induced by c as follows. If a_1, a_2 , and a_3 have the same color, then we assign to a the other color. Otherwise, we assign to a the color that appears at least twice among the colors of b_1, b_2 , and b_3 . Now, since G does not contain a 2-vertex nor a 3-vertex adjacent to three 4^- -vertices, we have $\text{mad}(G) \geq \frac{10}{3}$. This can be seen using the discharging procedure such that the initial charge of each vertex is its degree and every 5^+ -vertex gives $\frac{1}{3}$ to each adjacent 3-vertex. The final charge of a 3-vertex is at least $3 + \frac{1}{3} = \frac{10}{3}$, the final charge of a 4-vertex is $4 > \frac{10}{3}$, and the final charge of a k -vertex with $k \geq 5$ is at least $k - k \times \frac{1}{3} = \frac{2k}{3} \geq \frac{10}{3}$. Now, $\text{mad}(G) \geq \frac{10}{3}$ contradicts the fact that G is a planar graph with girth 5, and this contradiction shows that G contains a 2-vertex. \square

We are ready to prove Theorem 5. The case of $(1, 0)$ -coloring is proved in a stronger form which takes into account restrictions on both the girth and the maximum degree of the input planar graph [16].

Proof of the case $(k, 0)$, $k \geq 2$.

We consider a graph $G_{k,0}$ as described in Lemma 6, which contains a path uxv where x is a 2-vertex. By minimality, any $(k, 0)$ -coloring of $G_{k,0} \setminus \{x\}$ is such that u and v get distinct saturated colors. Let G be the graph obtained from $G_{k,0} \setminus \{x\}$ by adding three 2-vertices x_1, x_2 , and x_3 to create the path $ux_1x_2x_3v$. So any $(k, 0)$ -coloring of G is such that x_2 is colored k^1 . To prove the NP-completeness, we reduce the $(k, 0)$ -coloring problem to the $(1, 0)$ -coloring problem. Let I be a planar graph with girth $g_{1,0}$. For every vertex s of I , add $(k - 1)$ copies of G such that the vertex x_2 of each copy is adjacent to s , to obtain the graph I' . By construction, I' is $(k, 0)$ -colorable if and only if I is $(1, 0)$ -colorable. Moreover, I' is planar, and since $g_{k,0} \leq g_{1,0}$, the girth of I' is $g_{k,0}$.

Proof of the case $(1, 1)$.

By Claim 2.2 and [9], $g_{1,1}$ is either 5 or 6. There exist two independent proofs [17, 19] that $(1, 1)$ -coloring is NP-complete for triangle-free planar graphs with maximum degree 4. We use a reduction from that problem to prove that $(1, 1)$ -coloring is NP-complete for planar graphs with girth $g_{1,1}$. We consider a graph $G_{1,1}$ as described in Lemma 6, which contains a path uxv where x is a 2-vertex. By minimality, any $(1, 1)$ -coloring of $G_{1,1} \setminus \{x\}$ is such that u and v get distinct saturated colors. Let G be the graph obtained from $G_{1,1} \setminus \{x\}$ by adding a vertex u' adjacent to u and a vertex v' adjacent to v . So any $(1, 1)$ -coloring of G is such that u' and v' get distinct colors and u' (resp. v') has a color distinct from the color of its (unique) neighbor. We construct the graph $E_{a,b}$ from two vertices a and b and two copies of G such that a is adjacent to the vertices u' of both copies of G and b is adjacent to the vertices v' of both copies of G . There exists a $(1, 1)$ -coloring of $E_{a,b}$ such that a and b have distinct colors and neither a nor b is saturated. There exists a $(1, 1)$ -coloring of $E_{a,b}$ such that a and b have the same color. Moreover, in every $(1, 1)$ -coloring

of $E_{a,b}$ such that a and b have the same color, both a and b are saturated.

The reduction is as follows. Let I be a planar graph. For every edge (p, q) of I , replace (p, q) by a copy of $E_{a,b}$ such that a is identified with p and b is identified with q , to obtain the graph I' . By the properties of $E_{a,b}$, I is $(1, 1)$ -colorable if and only if I' is $(1, 1)$ -colorable. Moreover, I' is planar with girth $g_{1,1}$.

Proof of the case (k, j) .

We consider a graph $G_{k,j}$ as described in Lemma 6, which contains a path uxv where x is a 2-vertex. By minimality, any (k, j) -coloring of $G_{k,j} \setminus \{x\}$ is such that u and v get distinct saturated colors. Let G be the graph obtained from $G_{k,j} \setminus \{x\}$ by adding a vertex u' adjacent to u and a vertex v' adjacent to v . So any (k, j) -coloring of G is such that u' and v' get distinct colors and u' (resp. v') has a color distinct from the color of its (unique) neighbor. Let $t = \min(k - 1, j)$. To prove the NP-completeness, we reduce the (k, j) -coloring to the $(k - t, j - t)$ -coloring. Thus the case (k, k) reduces to the case $(1, 1)$ which is NP-complete, and the case (k, j) with $j < k$ reduces to the case $(k - j, 0)$ which is NP-complete. The reduction is as follows. Let I be a planar graph with girth $g_{k-t, j-t}$. For every vertex s of I , add t copies of G such that the vertices u' and v' of each copy is adjacent to s , to obtain the graph I' . By construction, I is $(k - t, j - t)$ -colorable if and only if I' is (k, j) -colorable. Moreover, I' is planar, and since $g_{k,j} \leq g_{k-t, j-t}$, the girth of I' is $g_{k,j}$.

6 NP-completeness of (k, j, i) -colorings

In this section, we prove Theorem 4 using a reduction from 3-colorability, i.e. $(0, 0, 0)$ -colorability, which is NP-complete for planar graphs [18].

Let E be the graph depicted in Fig 6. The graph E' is obtained from $2k - 1$ copies of E by identifying the edge ab of all copies. Take 4 copies E'_1, E'_2, E'_3 , and E'_4 of E' and consider a triangle T formed by the vertices y_0, x_0, x_1 in one copy of E in E'_1 . The graph E'' is obtained by identifying the edge y_0x_0 (resp. y_0x_1, x_0x_1) of T with the edge ab of E'_2 (resp. E'_3, E'_4). The edge ab of E'_1 is then said to be the edge ab of E'' .

Lemma 7.

1. E'' admits a $(0, 0, 0)$ -coloring.
2. E' does not admit a $(k, k, 1)$ -coloring such that a and b have a same color of impropriety k .
3. E'' does not admit a $(k, k, 1)$ -coloring such that a and b have the same color.

Proof.

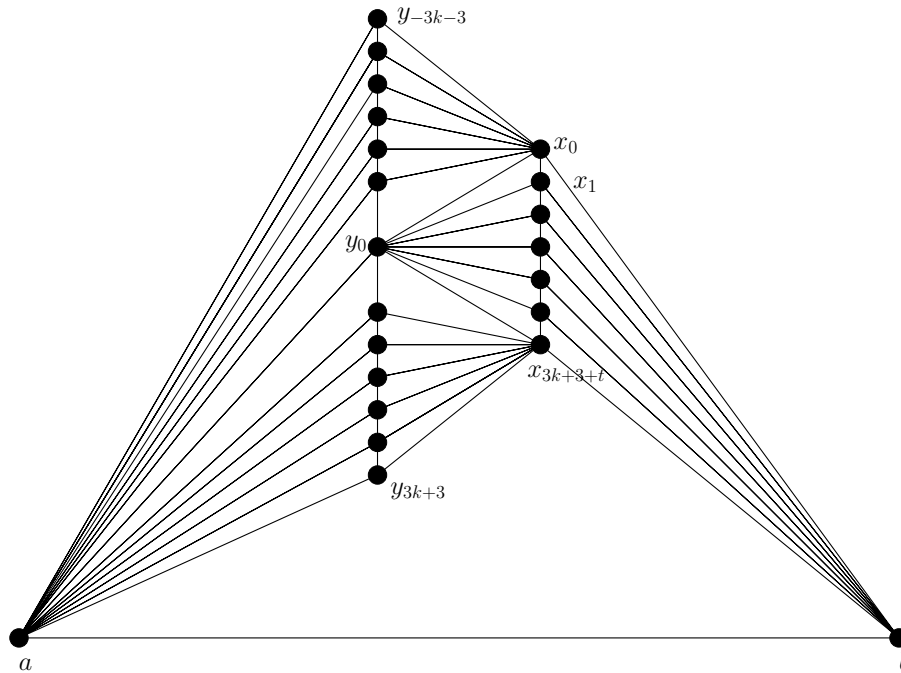


Figure 6: The graph E . We take $t = 0$ if k is odd and $t = 1$ if k is even, so that $3k + 3 + t$ is even.

1. The following $(0, 0, 0)$ -coloring c of E is unique up to permutation of colors: $c(a) = c(x_i) = 1$ for even i , $c(b) = c(y_i) = 2$ for even i , and $c(x_i) = c(y_i) = 3$ for odd i . This coloring can be extended into a $(0, 0, 0)$ -coloring of E' and E'' .
2. Let k_1 , k_2 , and 1 denote the colors in a potential $(k, k, 1)$ -coloring c of E' such that $c(a) = c(b) = k_1$. By the pigeon-hole principle, one of the $2k - 1$ copies of E in E' , say E^* , is such that a and b are the only vertices with color k_1 . So, one of the vertices x_0 , y_0 , and x_{3k+3+t} in E^* must get color k_2 since they cannot all get color 1. We thus have a vertex $v_1 \in \{a, b\}$ colored k_1 and vertex $v_2 \in \{x_0, y_0, x_{3k+3+t}\}$ colored k_2 in E^* which both dominate a path on at least $3k + 3$ vertices. This path contains no vertex colored k_1 since it is in E^* . Moreover, it contains at most k vertices colored k_2 . On the other hand, every set of 3 consecutive vertices in this path contains at least one vertex colored k_2 , so it contains at least $\frac{3k+3}{3} = k + 1$ vertices colored k_2 . This contradiction shows that E' does not admit a $(k, k, 1)$ -coloring such that a and b have a same color of impropriety k .
3. By the previous item and by construction of E'' , if E'' admits a $(k, k, 1)$ -coloring c such that $c(a) = c(b)$, then $c(a) = c(b) = 1$. We thus have that $\{c(y_0), c(x_0), c(x_1)\} \subset \{k_1, k_2\}$. This implies that at least one edge of the triangle T is monochromatic with a color of impropriety k . By

the previous item, the coloring c cannot be extended to the copy of E' attached to that monochromatic edge. This shows that E'' does not admit a $(k, k, 1)$ -coloring such that a and b have the same color.

□

For every fixed integer k , we give a polynomial construction that transforms every planar graph G into a planar graph G' such that G' is $(0, 0, 0)$ -colorable if G is $(0, 0, 0)$ -colorable and G' is not $(k, k, 1)$ -colorable otherwise. The graph G' is obtained from G by identifying every edge of G with the edge ab of a copy of E'' . If G is $(0, 0, 0)$ -colorable, then this coloring can be extended into a $(0, 0, 0)$ -coloring of G' by Lemma 7.1. If G is not $(0, 0, 0)$ -colorable, then every $(k, k, 1)$ -coloring G contains a monochromatic edge uv , and then the copy of E'' corresponding to uv (and thus G') does not admit a $(k, k, 1)$ -coloring by Lemma 7.3. The instance graph G in the proof that $(0, 0, 0)$ -coloring is NP-complete [18] is 3-degenerate. Then by construction, G' is also 3-degenerate.

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