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Near-exact Distributions for Positive Linear Combinations of Independent Non-central Gamma Random Variables

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Abstract. In this paper near-exact distributions for positive linear combinations of independent non-central Gamma random variables are developed. The authors show that through an adequate factorization of the characteristic function it is possible to obtain precise near-exact distributions which correspond to mixtures of Generalized Near-Integer Gamma distributions. Numerical studies are conducted in order to assess the precision of these approximations.

Keywords: Characteristic function, Gamma distribution, Generalized Near-Integer Gamma distribution, Mixtures.
AMS: 62E10, 62E15, 62E20

INTRODUCTION

The non-central Gamma distribution has the Gamma distribution as a particular case which is one of the most important distributions in Statistics. Results on the sum and linear combinations of independent Gamma random variables can be found, for example, in [6] and [9]. In this work the authors study the distribution of the linear combination of independent non-central Gamma random variables. Let us consider p independent random variables, X_i, $i = 1,\dots,p$, with a non-central Gamma distribution, denoted here by $X_i \sim \Gamma(r_i,\lambda_i)(\delta_i)$ where $r_i > 0$ is the shape parameter, $\lambda_i > 0$ is the rate parameter and $\delta_i > 0$ is the non-central parameter with $i = 1,\dots,p$, then the density function of X_i is given by

$$f_{X_i}(x) = \frac{\lambda_i^{r_i}}{\Gamma(r_i)} \exp\{-\lambda_i x\} x^{r_i-1} \exp\{-\delta_i\lambda_i\}\, {}_0F_1(r_i;\delta_i\lambda_i^2 x)\,,\; x > 0$$

where ${}_0F_1$ is the confluent hypergeometric function. As it is clear for $\delta_i = 0$ we have the density of the Gamma distribution. It can be shown that the characteristic function of X_i is given by

$$\Phi_{X_i}(t) = \lambda_i^{r_i}(\lambda_i - \mathrm{i}t)^{-r_i} \exp\left\{\frac{\delta_i\lambda_i \mathrm{i}t}{\lambda_i - \mathrm{i}t}\right\}\,,\; t \in \mathbb{R} \text{ and } \mathrm{i} = (-1)^{1/2}.$$

We should note that a non-central Gamma random variable multiplied by a scalar is still a non-central Gamma random variable with different rate and non-central parameters. For positive $\alpha_1,\dots,\alpha_p$, the characteristic function of the linear combination of p independent non-central Gamma random variables, $W = \alpha_1 X_1 + \cdots + \alpha_p X_p$, is given by

$$\Phi_W(t) = \prod_{i=1}^{p} \lambda_i^{r_i}(\lambda_i - \alpha_i \mathrm{i}t)^{-r_i} \exp\left\{\frac{\alpha_i\delta_i\lambda_i \mathrm{i}t}{\lambda_i - \alpha_i \mathrm{i}t}\right\}\,,\; t \in \mathbb{R}\,. \tag{1}$$

In the following Section, using an adequate factorization of the characteristic function in (1) and by asymptotically approximating a part of it, the authors are able to obtain near-exact distributions for the linear combination of independent non-central Gamma random variables. These near-exact distributions assume the form of mixtures of Generalized Near-Integer Gamma (GNIG) distributions [1, 2], and are built ensuring that a given number of the moments of the approximating distributions are equal to exact ones. In Section “Numerical studies”, numerical studies are conducted in order to analyze the performance of the near-exact distributions developed. The conclusions are presented in Section “Conclusions”.

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If we consider $r_i^\star = \lfloor r_i \rfloor$ and $c_i = r_i - r_i^\star$, the characteristic function of W in (1) may be written as

$$\Phi_W(t) \quad = \quad \underbrace{\prod_{i=1}^{p}\left(\frac{\lambda_i}{\alpha_i}\right)^{r_i^\star}\left(\frac{\lambda_i}{\alpha_i}-\mathrm{i}t\right)^{-r_i^\star}}_{\Phi_{W_1}(t)} \underbrace{\prod_{i=1}^{p}\left(\frac{\lambda_i}{\alpha_i}\right)^{c_i}\left(\frac{\lambda_i}{\alpha_i}-\mathrm{i}t\right)^{-c_i}\exp\left\{\frac{\alpha_i\delta_i\lambda_i\mathrm{i}t}{\lambda_i-\alpha_i\mathrm{i}t}\right\}}_{\Phi_{W_2}(t)} \tag{2}$$

where the characteristic function of W_1 in (2), $\Phi_{W_1}(t)$, is the characteristic function of the sum of independent Gamma random variables with rate parameters $\frac{\lambda_i}{\alpha_i}$ and with integer shape parameters $r_i^\star$, $i=1,\dots,p$. The characteristic function of W_2 in (2), $\Phi_{W_2}(t)$, may also be written as

$$\Phi_{W_2}(t) = \prod_{i=1}^{p}\left(\frac{\lambda_i}{\alpha_i}\right)^{c_i}\left(\frac{\lambda_i}{\alpha_i}-\mathrm{i}t\right)^{-c_i}\exp\left\{\frac{\alpha_i\delta_i\frac{\lambda_i}{\alpha_i}\mathrm{i}t}{\frac{\lambda_i}{\alpha_i}-\mathrm{i}t}\right\} \tag{3}$$

which is the characteristic function of the sum of independent non-central Gamma random variables with rate parameters $\frac{\lambda_i}{\alpha_i}$, non-integer shape parameters c_i and non-central parameters $\alpha_i\delta_i$, $i=1,\dots,p$. Since the exact distribution of the sum of independent Gamma random variables may be represented as an infinite mixture of Gamma random variables all with the same rate parameter and with shape parameters equal to a given parameter successively increased by one [9], in order to compensate for the effect of the exponential part in the characteristic function of a non-central Gamma random variable, we propose as an approximation for the characteristic function in (3) a characteristic function of a finite mixture of Gamma distributions matching $m^\star$ of the first exact moments, also with the same rate parameter, and with shape parameters equal to $r^\star$, successively added by a unit divided by a given number d, that is, $r^\star + j/d$ for $j=0\dots,m^\star$. Thus, we will approximate $\Phi_{W_2}(t)$ in (3) by $\Phi^\star_{W_2}(t)$ which is given by

$$\Phi^\star_{W_2}(t) = \sum_{j=0}^{m^\star}\pi_j(\lambda^\star)^{r^\star+j/d}(\lambda^\star-\mathrm{i}t)^{-(r^\star+j/d)} \tag{4}$$

where $r^\star$ and $\lambda^\star$ are chosen to be equal, respectively, to the smallest shape parameter and to the rate parameter of a mixture of two Gamma distributions matching the first four moments of the exact distribution of W_2, that is the parameters $r^\star$ and $\lambda^\star$ are obtained as solutions of

$$\frac{\partial^h}{\partial t^h}\Phi_{W_2}(t)\bigg|_{t=0} = \frac{\partial^h}{\partial t^h}\left\{p(\lambda^\star)^{s_1}(\lambda^\star-\mathrm{i}t)^{-s_1} + (1-p)(\lambda^\star)^{s_2}(\lambda^\star-\mathrm{i}t)^{-s_2}\right\}\bigg|_{t=0}, h=1,\dots,4 \tag{5}$$

where $r^\star=\min\{s_1,s_2\}$, and the weights π_j in (4) are determined in order to ensure that the approximating distribution has the first $m^\star$ moments equal to the exact moments of W_2, as such π_j $(j=0,\dots,m^\star-1)$, are the real solution of the system

$$\frac{\partial^h}{\partial t^h}\Phi_{W_2}(t)\bigg|_{t=0} = \frac{\partial^h}{\partial t^h}\Phi^\star_{W_2}(t)\bigg|_{t=0}, h=1,\dots,m^\star \quad \text{with} \quad \pi_{m^\star} = 1-\sum_{j=0}^{m^\star-1}\pi_j\,. \tag{6}$$

Following this procedure we will have, as an approximating characteristic function of Φ_W in (1), the characteristic function

$$\Phi^\star_W(t) \quad = \quad \Phi_{W_1}(t)\times\Phi^\star_{W_2}(t) = \sum_{j=0}^{m^\star}\pi_j\prod_{i=1}^{p}\left(\frac{\lambda_i}{\alpha_i}\right)^{r_i^\star}\left(\frac{\lambda_i}{\alpha_i}-\mathrm{i}t\right)^{-r_i^\star}(\lambda^\star)^{r^\star+j/d}(\lambda^\star-\mathrm{i}t)^{-(r^\star+j/d)}$$

which corresponds to a finite mixture having in each term the sum of p independent Gamma random variables with integer shape parameters with an independent Gamma random variable with a non-integer shape parameter, that is, this characteristic function corresponds to a finite mixture of GNIG distributions. These results give rise to the following theorem, where, using the notation in Appendix 3 in [7], the authors present the probability density and cumulative distribution functions of W .

Theorem 1 *If we replace* $\Phi_{W_2}(t)$ *given in* (3) *by* $\Phi^{\star}_{W_2}(t)$ *in* (4) *we obtain, for r non-integer, near-exact distributions for W with cumulative distribution and density functions given respectively by*

$$F(z)=\sum_{j=0}^{m^\star}\pi_j F^{\mathrm{GNIG}}\left(z\mid r_i^\star, r^\star+\frac{j}{d}\;;\;\lambda_i^\star,\lambda^\star\;;p+1\right)\quad\text{and}\quad f(z)=\sum_{j=0}^{m^\star}\pi_j f^{\mathrm{GNIG}}\left(z\mid r_i^\star, r^\star+\frac{j}{d}\;;\;\lambda_i^\star,\lambda^\star\;;p+1\right)$$

where

$$r_i^\star=\{r_i^\star\,, i=1,\dots,p\}\,,\quad\text{and}\quad \lambda_i^\star=\left\{\frac{\lambda_i}{\alpha_i}\,, i=1,\dots,p\right\},$$

$z>0$, *$r^\star$ and $\lambda^\star$ are obtained as a solution of the system in (5) and the values of π_j are obtained from the numerical solution of the system of equations in* (6).

Some comments are in order:

- As already mentioned, if $X_i\sim\Gamma(r_i,\lambda_i)(\delta_i)$ then $\alpha_i X_i\sim\Gamma\left(r_i,\frac{\lambda_i}{\alpha_i}\right)(\alpha_i\delta_i)$, $i=1,\dots,p$, we may reduce the case of the linear combination of independent non-central Gamma random variables to the case of the sum of independent non-central Gamma random variables. As such, in next section we will consider in all cases $\alpha_i=1$.
- If k shape parameters, r_i, in (1) are integer then $c_i=0$. Without loss of generality if we have the last k shape parameters integers we should write

$$\Phi_{W_2}(t)=\prod_{i=1}^{p-k}\left(\frac{\lambda_i}{\alpha_i}\right)^{c_i}\left(\frac{\lambda_i}{\alpha_i}-\mathrm{i}t\right)^{-c_i}\prod_{i=1}^{p}\exp\left\{\frac{\alpha_i\delta_i\frac{\lambda_i}{\alpha_i}\mathrm{i}t}{\frac{\lambda_i}{\alpha_i}+\mathrm{i}t}\right\}.$$

- The value of d in (4) is chosen as the value that minimizes the proximity between the approximating and the exact distribution according to the measure Δ in (7) of the following Section.
- The system in (5) may, in some cases, provide a negative solution for the shape parameter of interest, $r^\star$, in the mixture of two Gamma distributions. In these cases the authors have chosen to use a single Gamma distribution as a basis to determine the values of the parameters $\lambda^\star$ and $r^\star$.

NUMERICAL STUDIES

To analyze the performance of the approximation proposed for the linear combination of independent non-central Gamma random variables we use a measure that gives an upper bound on the proximity between two distribution functions and which is based on the respective characteristic functions. The measure has already been used in many related studies, [5, 3, 4, 8], and is given by

$$\Delta=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\Phi_W(t)-\Phi^\star_W(t)}{t}\right|\mathrm{d}t\,,\tag{7}$$

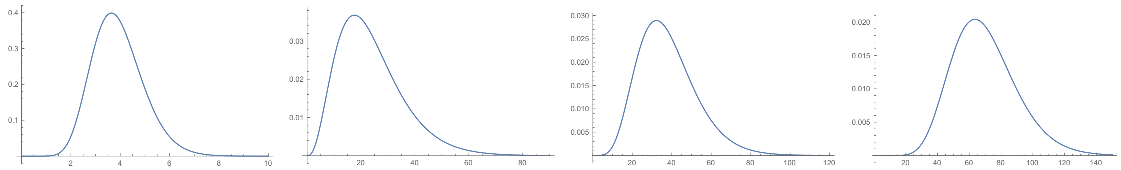
where $\Phi_W(t)$ represents the exact characteristic function of the linear combination of independent non-central Gamma random variables and $\Phi^\star_W(t)$ represents the approximating characteristic function of $\Phi_W(t)$. In Table 1 we consider four different scenarios:

I) $r_i=\{\frac{14}{4},\frac{32}{3}\}$, $\lambda_i=\{\frac{30}{7},\frac{25}{6}\}$ and $\delta_i=\{\frac{1}{5},\frac{1}{3}\}$;
II) $r_i=\{\frac{10}{4},\frac{5}{4}\}$, $\lambda_i=\{\frac{1}{5},\frac{1}{6}\}$ and $\delta_i=\{\frac{3}{2},\frac{7}{3}\}$;
III) $r_i=\{\frac{10}{4},\frac{5}{4},\frac{17}{6}\}$, $\lambda_i=\{\frac{1}{5},\frac{1}{6},\frac{3}{12}\}$ and $\delta_i=\{\frac{3}{2},\frac{7}{3},\frac{14}{5}\}$;
IV) $r_i=\{\frac{10}{4},\frac{7}{2},\frac{5}{4},\frac{9}{5},\frac{17}{6}\}$, $\lambda_i=\{\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{4},\frac{1}{2}\}$ and $\delta_i=\{\frac{3}{2},\frac{7}{3},\frac{14}{5},3,5\}$.

In Table 1 it is possible to observe that the values of the measure Δ in (7) are quite small which indicates that the approximations developed are quite accurate. As expected, higher values of $m^\star$ provide better approximations. However, it was not possible to identify why and how the value of d varies.

TABLE 1. Values of Δ for the different scenarios

Scenario	$m^\star = 4$	d	$m^\star = 10$	d	$m^\star = 15$	d
I	2.3×10^{-5}	4	9.5×10^{-9}	4	2.0×10^{-11}	3
II	3.0×10^{-5}	5	3.4×10^{-8}	3	7.1×10^{-10}	3
III	6.2×10^{-6}	3	6.4×10^{-9}	2	1.4×10^{-10}	3
IV	4.1×10^{-6}	13	2.0×10^{-8}	6	4.7×10^{-10}	7



scenario I scenario II scenario III scenario IV

FIGURE 1. Near-exact densities for positive linear combinations of independent non-central Gamma random variables

In Figure 1 we present the plots of the near-exact densities for the four different scenarios when 15 moments are matched and for the corresponding value of d in Table 1.

CONCLUSIONS

Near-exact distributions in the form of mixtures of GNIG distributions were developed for linear combinations of independent non-central Gamma distributions. In order to better control the precision of these approximations a parameter, d, was introduced in the shape parameters of the GNIG distributions involved in the mixtures. The numerical studies conducted illustrate the high precision of these approximations in different scenarios.

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