NEAR-OPTIMAL ECHELON-STOCK (*R, nQ*) POLICIES IN MULTISTAGE SERIAL SYSTEMS

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(Received August 1994; revisions received November 1995; December 1996; accepted January 1997)

We study echelon-stock (R, nQ) policies in a multistage, serial inventory system with compound Poisson demand. We provide a simple method for determining near-optimal control parameters. This is achieved in two steps. First, we establish lower and upper bounds on the cost function by over- and under-charging a penalty cost to each upstream stage for holding inadequate stock. Second, we minimize the bounds, which are simple, separable functions of the control parameters, to obtain heuristic solutions. We also provide an algorithm that guarantees an optimal solution at the expense of additional computational effort. A numerical study suggests that the heuristic solutions are easy to compute (even for systems with many stages) and are close to optimal. It also suggests that a traditional approach for determining the order quantities can be seriously suboptimal. All the results can be easily extended to the discrete-time case with independent, identically distributed demands.

Basic models of multistage, production/distribution systems are central to supply chain management, a field that has lately attracted much attention from academics and practitioners alike. This paper considers one such model where the material is processed sequentially before being used to satisfy uncertain customer demand. The model is depicted in Figure 1 where the stages represent the different stocking points in the production-distribution process. Material flow from one stage to the next requires a leadtime and incurs a setup cost (in addition to a variable cost proportional to the flow quantity). Due to the value added, inventory becomes more expensive to carry as it moves closer to the customer. Demand unsatisfied from on-hand inventory is backlogged, incurring penalty costs. The entire supply chain is controlled by a system manager whose goal is to satisfy the customer demand and to minimize the long-run average system-wide cost. (When the different stages are controlled by independent managers, the jointly optimal solution can serve as a benchmark.)

The above model was originally proposed by Clark and Scarf (1962) as a generalization of the now classic model (Clark and Scarf 1960) which does not allow setup costs at any stages except stage N. They introduced the important concept of *echelon stock*. A stage's echelon stock is the inventory position of the subsystem consisting of the stage itself and all its downstream stages.

Our model can also be viewed as a generalization of the deterministic models studied by Roundy (1986), Maxwell and Muckstadt (1985), and Atkins and Sun (1995). They show that the so-called power-of-two policies are close to optimal, under which the reorder intervals (order quantities) at all stages are restricted to be power-of-two multiples of a base time (quantity) unit. This power-of-two structure is designed to facilitate coordination among the different stages.

For our serial model with random demand and setup costs, Clark and Scarf (1962) have pointed out correctly that the optimal policy does *not* have a simple structure. Thus, an optimal policy, even if it exists and is identified, would not be easy to implement. In other words, the "optimal" policy is no longer optimal or even attractive once the managerial effort of implementation is taken into account. Therefore, we turn to simple, cost-effective heuristic policies. Specifically, we consider the echelon-stock (r, nQ) policy, which is a natural generalization of the power-of-two policy. An (r, nQ) policy operates as follows: whenever the inventory position is at or below the reorder point r, order nQ units where n is the minimum integer required to increase the inventory position to above r. We call Q the *base quantity*. Combining the (r, nQ) policy with the echelon-stock concept leads to the echelon-stock (r, nQ) policy whereby every stage uses an (r, nQ) policy based on its echelon stock. (A closely related policy is the installation-stock (r, nQ) policy whereby each stage follows an (r, nQ) policy based on its local inventory position. For serial systems echelon-stock policies are superior to installation-stock policies, see Axsater and Rosling 1993.) To achieve quantity coordination, we require the base quantity of stage $i + 1$ be a positive integer multiple of the base quantity of stage i. Based on the insight from Roundy (1986) and Zheng (1992), we further restrict the base quantities to be of the power-of-two type.

Echelon-stock (r, nQ) policies are easy to implement. Although the initial measurement of a stage's echelon

Subject classifications: Inventory/production: multiechelon, lot-sizing, stochastic, heuristics.
Area of review: Stochastic Models.

0030-364X/98/4604-0592 $05.00

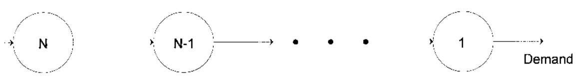

Figure 1. The serial system.

stock requires the inventory information at every downstream stage, its update requires the demand information only at stage 1. Since modern information technologies (e.g., EDI) are capable of effortlessly transmitting the point-of-sale data to the upstream stages of the supply chain, the information infrastructure for implementing echelon-stock policies is in place.

We aim to determine the optimal reorder points and base quantities that minimize the average system-wide cost. Recent developments show that the exact cost of an echelon stock (r, nQ) policy can be computed recursively (Chen and Zheng 1994a) and for fixed base quantities, the optimal reorder points can be determined sequentially (Chen 1995). (In a nutshell, determining the optimal reorder points for fixed base quantities is, after a proper transformation, essentially the same as finding the optimal base-stock levels in the Clark-Scarf model without setup costs.) But it is still unclear how the optimal base quantities can be determined. This paper provides a simple method for determining near-optimal base quantities. We first bound the exact cost function from both above and below by simple functions of the control parameters. These bounds are obtained by over- and under-charging a penalty cost to each upstream stage for holding less-than-adequate stock. Each bound is the sum of N single-stage cost functions. Substituting these bounds for the exact cost function, we effectively decouple the N-stage system into N single-stage systems. Solving these single-stage problems leads to heuristic base quantities. We also provide an algorithm that finds the optimal base quantities at the expense of additional computational effort. A numerical study suggests that the heuristic solutions can be computed efficiently (even for systems with many stages) and more importantly, are close to optimal.

For stochastic inventory systems with fixed ordering costs, it has been widely suggested that the order quantities can be obtained by solving the deterministic counterpart of the problem (see, e.g., Graves and Schwarz 1977). Let us call the order quantities obtained in this way the EOQs. In a numerical study, we observed that our solutions dominate the EOQ solution with substantial savings in examples with high demand volatility. We also observed that the EOQs tend to be too small, and thus should be adjusted upward for stochastic systems. This observation echoes a recent finding from the single-stage (r, Q) model (Zheng 1992).

There is an extensive literature on multiechelon systems with uncertain demand and scale economies, see, e.g., Deuermeyer and Schwarz (1981), De Bodt and Graves (1985), Moinzadeh and Lee (1986), Lee and Moinzadeh (1987a, b), Svoronos and Zipkin (1988), Badinelli (1992), Axsater (1993a, b), and Chen and Zheng (1994a, b, 1997). Most of this literature focuses on evaluating the cost of a heuristic policy with predetermined control parameters, and not on determining the optimal values of the control parameters.

The rest of the paper is organized as follows. Section 1 presents preliminaries. Section 2 bounds the exact cost function. Section 3 describes an algorithm for computing heuristic base quantities. Section 4 outlines a search procedure for determining the optimal base quantities. Section 5 reports a numerical study. Section 6 concludes the paper.

1. PRELIMINARIES

Consider an N-stage, serial system where stage 1 orders from stage 2, 2 from 3, etc., and stage N orders from an outside supplier with unlimited stock. There are economies of scale at each stage for placing orders. The transportation leadtime from stage $i + 1$ to stage i is a constant L_i for $i = 1, \ldots, N$, with stage $N + 1$ being the outside supplier. The demand process is compound Poisson. That is, customers arrive at stage 1 according to a Poisson process with an average rate λ; the demand sizes of different customers are independent and identically distributed, and are independent of the arrival process. We assume that the demand sizes only take integer values. Let μ be the average demand size per customer. Excess demand is backlogged with backorder cost rate p. Let $h_i > 0$ be the echelon holding cost rate at stage i for $i = 1, \cdots, N$. The planning horizon is infinite, and the objective is to minimize the long-run average total cost. (A transportation cost proportional to the quantity shipped can be easily included. We omit it because its long-run average value is constant.)

For any time t, define

$B(t)$ = backorder level at stage 1,
$I_i(t)$ = echelon inventory at stage i,
= on-hand inventory at stage i plus inventories at, or in transit to, stages $1, \ldots, i - 1$,
$IL_i(t)$ = echelon inventory level at stage i = $I_i(t) - B(t)$,
$IP_i(t)$ = echelon inventory position at stage i
= $IL_i(t)$ plus inventories in transit to stage i, and
$ES_i(t)$ = echelon stock at stage i
= $IL_i(t)$ plus stage i's outstanding orders, in transit or backlogged at stage $i + 1$.

Note that the above variables take integer values only. The difference between $IP_i(t)$ and $ES_i(t)$ is that the former is constrained by $IL_{i+1}(t)$, a variable controlled by the upstream stages, while the latter is controlled by stage i only.

The inventory flow through the system is controlled by an echelon-stock (r, nQ) policy. That is, stage i orders nQ_i units from stage $i + 1$ whenever stage i's echelon stock falls to or below r_i, where n is the minimum integer so that stage i's echelon stock after ordering is above r_i. We call Q_i (a positive integer) the *base quantity*, and r_i (an integer) the *reorder point*, at stage i. The base quantities at the different stages are coordinated in the sense that $Q_{i+1} =$

n_iQ_i, where n_i is a positive integer, for $i = 1, \ldots, N - 1$. Moreover, we assume that the initial on-hand inventory at stage i is an integer multiple of Q_{i-1}, $i = 2, \ldots, N$. (This initial state can always be reached by sending the residual units at each stage, if any, to its immediate downstream stage.) As a result, the on-hand inventory at stage i is always an integer multiple of Q_{i-1}, $i = 2, \ldots, N$.

At stage i we assess a setup cost K_i for each Q_i ordered. Thus, the long-run average setup costs in the system are

$$\sum_{i=1}^{N} \frac{\lambda \mu K_i}{Q_i}. \tag{1}$$

See Zipkin (1995) for a discussion on this convention of charging setup costs. (There are at least two other ways to assess setup costs: one charges a setup cost for each order placed, and the other charges a setup cost for each shipment received. Both lead to more complex expressions for the average setup costs. See Zheng and Chen 1992 and Chen and Zheng 1994a.)

Note that the rate at which the system-wide holding and backorder costs accrue at time t is

$$\sum_{i=1}^{N} h_i I_i(t) + pB(t) = \sum_{i=1}^{N} h_i IL_i(t) + (p + H_1)B(t),$$

where H_1 is the installation holding cost rate at stage 1, i.e., $H_1 = \sum_{i=1}^{N} h_i$. Let IP_i and IL_i represent $IP_i(t)$ and $IL_i(t)$ in steady state, $i = 1, \ldots, N$. The following equation is well known:

$$IL_i(t + L_i) = IP_i(t) - D(t, t + L_i),$$

where $D(t, t + L_i)$ is the total demand in the interval $(t, t + L_i)$. Since the demand process is compound Poisson, $IP_i(t)$ is independent of $D(t, t + L_i)$. Thus,

$$IL_i = IP_i - D_i, \tag{2}$$

where D_i is identically distributed as $D(t, t + L_i)$ and is independent of IP_i. For any integer y, define

$$G_1(y) = E[h_1(y - D_1) + (p + H_1)(y - D_1)^-],$$

where $(x)^- = \max\{0, -x\}$. Let B represent $B(t)$ in steady state. Since $B = (IL_1)^-$ and $IL_1 = IP_1 - D_1$ (see (2)), we have $E[h_1IL_1 + (p + H_1)B] = EG_1(IP_1)$. Therefore, the average total holding-backorder cost is

$$\sum_{i=1}^{N} h_i E(IL_i) + (p + H_1)E(B) = \sum_{i=2}^{N} h_i E(IL_i) + EG_1(IP_1).$$

Adding the average setup costs in (1) to the above expression, we have the long-run average total cost of the echelon-stock (r, nQ) policy:

$$C(\mathbf{r}, \mathbf{Q}) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \frac{\lambda \mu K_i}{Q_i} + \sum_{i=2}^{N} h_i E(IL_i) + EG_1(IP_1), \tag{3}$$

where $\mathbf{r} = (r_1, \ldots, r_N)$ and $\mathbf{Q} = (Q_1, \ldots, Q_N)$. An optimal echelon-stock (r, nQ) policy minimizes the above cost function.

2. BOUNDS

Here we derive upper and lower bounds on the cost function. These bounds have a simple form and will be used later to determine the control parameters.

2.1. Upper-Bound Function

We first define recursively a sequence of functions $G^i(\cdot)$ for $i = 1, \ldots, N$. Let $G^1(y) = G_1(y)$ for any integer y. Let Y_i be the minimum point of $G^i(\cdot)$. For $i = 1, \ldots, N - 1$ and any integer y, define

$$G^{i,i+1}(y) = \begin{cases} G^i(y) - G^i(Y_i), & y \le Y_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G^{i+1}(y) = E[h_{i+1}(y - D_{i+1}) + G^{i,i+1}(y - D_{i+1})].$$

Since $G^1(\cdot)$ is convex, $G^{12}(\cdot)$ is convex (and nonincreasing). Thus, $G^2(\cdot)$ is convex. Repeating this argument, we know that $G^i(\cdot)$ is convex for $i = 1, \ldots, N$. Note that $G^{i,i+1}(IL_{i+1})$ is the induced-penalty cost charged to stage $i + 1$ in the Clark-Scarf model; see Chen and Zheng (1994b).

For any integer r and any positive integer Q, define

$$C^i(r, Q) = \frac{\lambda \mu K_i + \sum_{y=r+1}^{r+Q} G^i(y)}{Q}, \quad i = 1, 2, \ldots, N.$$

One reason why the minimization of the exact cost function is difficult is that the stages are "coupled" in the sense that IP_i depends on not only the control policy at stage i but also the control policies at the upstream stages. The following lemma provides a way to decouple the system since IL_{i+1} is independent of, and ES_i is completely determined by, the control policy at stage i.

Lemma 1. *For $i = 1, \ldots, N - 1$, $G^i(IP_i) \le G^{i,i+1}(IL_{i+1}) + G^i(ES_i)$.*

Proof. By definition, $IP_i \le IL_{i+1}$ and the difference, $IL_{i+1} - IP_i$, is the on-hand inventory at stage $i + 1$. If $IP_i < IL_{i+1}$, i.e., stage $i + 1$ has positive on-hand inventory, then $IP_i = ES_i$. (Note that the echelon stock is the same as the echelon inventory position as long as the upper stage has inventory on hand.) The lemma follows since the induced-penalty cost, $G^{i,i+1}(\cdot)$, is nonnegative. Now suppose $IP_i = IL_{i+1}$. If $IP_i < Y_i$ then $G^i(IP_i) = G^{i,i+1}(IP_i) + G^i(Y_i) = G^{i,i+1}(IL_{i+1}) + G^i(Y_i)$. The lemma follows since $G^i(ES_i) \ge G^i(Y_i)$. On the other hand, if $IP_i \ge Y_i$ then the lemma follows since $G^i(y)$ is nondecreasing for $y \ge Y_i$ and $IP_i \le ES_i$ by definition. □

Corollary 1. *For $i = 1, \ldots, N - 1$, $EG^i(IP_i) \le EG^{i,i+1}(IL_{i+1}) + \sum_{y=r_i+1}^{r_i+Q_i} G^i(y)/Q_i$.*

Proof. Follows directly from Lemma 1 since ES_i is uniformly distributed from $r_i + 1$ to $r_i + Q_i$. □

Theorem 1. *For any feasible echelon-stock* (r, nQ) *policy,* $C(\mathbf{r}, \mathbf{Q}) \leqslant \sum_{i=1}^{N} C^i(r_i, Q_i)$.

Proof. Apply Corollary 1 (with $i = 1$) to the right side of (3). Since $IL_2 = IP_2 - D_2$, we have $E[h_2 IL_2 + G^{12}(IL_2)] = EG^2(IP_2)$. Now apply Corollary 1 again (with $i = 2$), etc. □

2.2. Lower-Bound Function

We first define recursively a sequence of functions $G_i(\cdot)$, $i = 1, \ldots, N$. $G_1(\cdot)$ is given above. Suppose we have $G_i(\cdot)$. For any integer r and any positive integer Q, define

$$C_i(r, Q) = \frac{\lambda\mu K_i + \sum_{y=r+1}^{r+Q} G_i(y)}{Q}.$$

For fixed Q, let $C_i(r, Q)$ be minimized at $r = r_i(Q)$. Let $C_i(r_i(Q), Q)$ be minimized at Q_i^0. Set $r_i^0 = r_i(Q_i^0)$ and $C_i^0 = C_i(r_i^0, Q_i^0)$. Then, for any integer y, define

$$G_{ii}(y) = \begin{cases} G_i(y), & r_i^0 + 1 \leqslant y \leqslant r_i^0 + Q_i^0, \\ C_i^0, & \text{otherwise}, \end{cases}$$

$$G_{i,i+1}(y) = \begin{cases} G_i(y) - C_i^0, & y \leqslant r_i^0, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$G_{i+1}(y) = E[h_{i+1}(y - D_{i+1}) + G_{i,i+1}(y - D_{i+1})].$$

Note that $G_{i,i+1}(\cdot)$ is the induced-penalty cost used by Chen and Zheng (1994b) to construct a lower bound on the average costs of all feasible policies for several production/inventory networks. Here, we use it to derive a lower-bound function.

Since $G_1(\cdot)$ is convex, $C_1(r, Q)$ has the form of the cost function of a single-stage (r, Q) policy, which has been thoroughly studied by Federgruen and Zheng (1992). Below are some of its properties:

(i) $G_1(r_1^0 + 1) \leqslant C_1^0 \geqslant G_1(r_1^0 + Q_1^0)$,
(ii) $r_1^0 < y_1 \leqslant r_1^0 + Q_1^0$,
(iii) $r_1(Q) < y_1 \leqslant r_1(Q) + Q$, for any positive integer $\mathbf{Q}$,

where y_1 is the minimum point of $G_1(\cdot)$. Using these properties, one can easily verify that $-G_{11}(\cdot)$ is unimodal, and that $G_{12}(\cdot)$ and $G_2(\cdot)$ are both convex. Thus, the above properties still hold with subscript 1 replaced by subscript 2. Repeating the above argument, we have that $-G_{ii}(\cdot)$ is unimodal for $i = 1, \ldots, N - 1$ and that $G_i(\cdot)$ is convex for $i = 1, \ldots, N$.

For $i = 1, \ldots, N - 1$, let $H_i(y)$ be the yth smallest value of $G_{ii}(\cdot)$, $y = 1, 2, \ldots$. Let $H_N(y)$ be the yth smallest value of $G_N(\cdot)$, $y = 1, 2, \ldots$. For any positive integer Q, define

$$C_i(Q) = \frac{\lambda\mu K_i + \sum_{y=1}^{Q} H_i(y)}{Q}, \quad i = 1, \ldots, N.$$

Lemma 2. (i) *For* $i = 1, \ldots, N - 1$, $EG_{ii}(IP_i) \geqslant \sum_{y=1}^{Q_i} H_i(y)/Q_i$.
(ii) $EG_N(IP_N) \geqslant \sum_{y=1}^{Q_N} H_N(y)/Q_N$.

Proof. (ii) is essentially a single-location result. The proof of (i) is harder. The first step is to show $\sum_{m=-\infty}^{+\infty} \Pr(IP_i = x + mQ_i) = 1/Q_i$ for any integer x. Then show that $\sum_{m=-\infty}^{+\infty} \Pr(IP_i = x + mQ_i)G_{ii}(x + mQ_i) \geqslant G_{ii}(z)/Q_i$ where $z = x + m'Q_i$ for some integer m' and $r_i(Q_i) + 1 \leqslant z \leqslant r_i(Q_i) + Q_i$. We leave the details to the reader. □

Take any $i = 1, \ldots, N - 1$. By definition, $G_i(y) \geqslant G_{i,i+1}(y) + G_{ii}(y)$ for any integer y. This, together with the fact that $G_{i,i+1}(\cdot)$ is nonincreasing and $IP_i \leqslant IL_{i+1}$, leads to $EG_i(IP_i) \geqslant EG_{i,i+1}(IL_{i+1}) + EG_{ii}(IP_i)$. From Lemma 2, we have

$$EG_i(IP_i) \geqslant EG_{i,i+1}(IL_{i+1}) + \sum_{y=1}^{Q_i} H_i(y)/Q_i, \quad i = 1, \ldots, N - 1. \tag{4}$$

Note that the first term on the right side is independent of the control policy at stage i, while the second term depends on Q_i only. This enables us to decouple the system.

Theorem 2. *For any feasible echelon-stock* (r, nQ) *policy,* $C(\mathbf{r}, \mathbf{Q}) \geqslant \sum_{i=1}^{N} C_i(Q_i)$.

Proof. Apply (4) (with $i = 1$) to the right side of (3). Since $IL_2 = IP_2 - D_2$, we have $E[h_2 IL_2 + G_{12}(IL_2)] = EG_2(IP_2)$. Now apply (4) (with $i = 2$) again, etc. The final step uses Lemma 2 (ii). □

2.3. Alternative Lower-Bound Functions

By allocating the setup costs among the stages, we may be able to obtain a better lower-bound function. To see the intuition, consider a two-stage system where K_1 is much larger than K_2 so that $Q_1^0 > Q_2^0$. In this case, it is conceivable that the optimal base quantities at the two stages must be the same due to the constraint $Q_1 \leqslant Q_2$. Now allocate part of K_1 to K_2. This reduces Q_1^0 and thus increases the induced-penalty cost charged to stage 2. But for any feasible policy with $Q_1 = Q_2$, the allocation does not change the average total setup cost. The result is a better lower-bound function. Below, we state a condition which must be satisfied by the allocated setup costs in order to have an alternative lower-bound function. A specific allocation will be given in Section 3.3.

Let $\tilde{K}_i$ be the new setup cost at stage i, $i = 1, \ldots, N$. Suppose

$$K_1 + \cdots + K_i \geqslant \tilde{K}_1 + \cdots + \tilde{K}_i, \quad i = 1, \ldots, N. \tag{5}$$

Since $Q_1 \leqslant \cdots \leqslant Q_N$, we have

$$\sum_{i=1}^{N} \frac{K_i}{Q_i} = \sum_{i=1}^{N-1} \left(\frac{1}{Q_i} - \frac{1}{Q_{i+1}}\right) \sum_{j=1}^{i} K_j + \frac{1}{Q_N} \sum_{j=1}^{N} K_j$$
$$\geqslant \sum_{i=1}^{N-1} \left(\frac{1}{Q_i} - \frac{1}{Q_{i+1}}\right) \sum_{j=1}^{i} \tilde{K}_j + \frac{1}{Q_N} \sum_{j=1}^{N} \tilde{K}_j$$
$$= \sum_{i=1}^{N} \frac{\tilde{K}_i}{Q_i}.$$

Therefore,

$$C(\mathbf{r}, \mathbf{Q}) \geq \sum_{i=1}^{N} \frac{\lambda\mu\tilde{K}_i}{Q_i} + \sum_{i=2}^{N} h_i E(IL_i) + EG_1(IP_1).$$

Now treat the right side of the above inequality as a new cost function and follow the approach in Section 2.2. This leads to a new lower-bound function. (Many lower bounds have been derived by setup-cost allocations, see, e.g., Atkins and Iyogun 1987, Atkins 1990, and Rosling 1993.)

3. HEURISTICS

The upper- and lower-bound functions developed in the previous section have a simple form. We suspect that they are reasonably close to the exact cost function. By minimizing the upper- and lower-bound functions, we hope to identify near-optimal control parameters.

3.1. Heuristic I

Consider the upper-bound function established in Section 2.1. Let $r^i(Q_i)$ be the optimal r_i that minimizes $C^i(r_i, Q_i)$ or equivalently $\sum_{y=r_i+1}^{r_i+Q_i} G^i(y)$ for fixed Q_i. Define $C^i(Q_i) = C^i(r^i(Q_i), Q_i)$. The minimization of the upper-bound function can be formulated as:

$$P_u\text{: } \min \sum_{i=1}^{N} C^i(Q_i)$$
$$\text{s.t. } Q_{i+1} = n_i Q_i$$
$$n_i \geq 1, \text{ integer, } i = 1, \ldots, N-1.$$

This problem can be solved in two steps.

First consider the following relaxation of P_u:

$$P_u^-\text{: } \min \sum_{i=1}^{N} C^i(Q_i)$$
$$\text{s.t. } Q_{i+1} \geq Q_i, \; i = 1, \ldots, N-1.$$

This problem can be solved by a simple clustering technique. Let $S = \{1, 2, \ldots, N\}$. For any $i, j \in S$ with $i \leq j$, the set $\{i, i+1, \ldots, j\}$ is called a *cluster*. For any cluster c, define

$$Q_c = \text{argmin}_Q \sum_{i \in c} C^i(Q).$$

(The minimization is over all positive integers. Thus Q_c is a positive integer.) A *partition* of S is a set of disjoint clusters whose union is S. A partition, $\{c(1), \ldots, c(n)\}$, is optimal if and only if

- $Q_{c(1)} \leq Q_{c(2)} \leq \cdots \leq Q_{c(n)}$, and
- for each cluster $c(k) = \{l_1, \ldots, l_2\}$, there does not exist an l with $l_1 \leq l < l_2$ so that $Q_{c^-(k)} < Q_{c^+(k)}$ where $c^-(k) = \{l_1, \ldots, l\}$ and $c^+(k) = \{l+1, \ldots, l_2\}$.

(An algorithm for finding an optimal partition is in Muckstadt and Roundy 1993.) Let $\{c(1), \ldots, c(n)\}$ be an optimal partition. Let $\bar{Q}_i = Q_{c(k)}$ for $i \in c(k)$, $k = 1, 2, \ldots, n$. Then $(\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_N)$ is the optimal solution to P_u^-.

The above solution to P_u^- can be rounded to power-of-two integers: $Q_i^u = 2^{m_i}$, $i = 1, \ldots, N$, where m_i is the unique integer with $2^{m_i}/\sqrt{2} \leq \bar{Q}_i < 2^{m_i}\sqrt{2}$. (Since $\bar{Q}_i$ is a positive integer, $m_i \geq 0$ or $Q_i^u \geq 1$.) For example, if $N = 3$ and $(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3) = (1, 3, 9)$ then $(Q_1^u, Q_2^u, Q_3^u) = (1, 4, 8)$. Now use Q_i^u as the base quantity at stage i, $i = 1, \ldots, N$. (Clearly these base quantities satisfy the constraint in P_u.) Given these base quantities, determine the optimal reorder points by using the sequential algorithm in Chen (1995). The resulting heuristic policy is called *Heuristic I*. (An alternative is to use $r^i(Q_i^u)$ as the reorder point at stage i. This turns out to be near optimal for the given Qs, see Section 5.)

Remark. Although the above power-of-two quantities may not be the optimal solution to P_u, it should be close. The reason is that the function $C^i(\cdot)$ is rather flat near its minimum (Zheng 1992). In fact, if we allow the decision variables to be continuous, then $(Q_1^u, \ldots, Q_N^u)$ is within 6 percent of the optimal solution to P_u.

3.2. Heuristic II

A different heuristic solution can be obtained by minimizing the lower-bound function established in Theorem 2. Replace $C^i(\cdot)$ in P_u and P_u^- with $C_i(\cdot)$, and call the resulting problems P_l and P_l^-. The clustering algorithm solves P_l^-. Let $(\underline{Q}_1, \ldots, \underline{Q}_N)$ be the optimal solution. Define

$$C_l^- = \sum_{i=1}^{N} C_i(\underline{Q}_i).$$

Clearly, C_l^- is a lower bound on the average costs of all feasible echelon-stock (r, nQ) policies. It can be used as a benchmark for any heuristic solution.

Now let $Q_i^l = 2^{m_i}$, $i = 1, \ldots, N$, where m_i is the unique integer with $2^{m_i}/\sqrt{2} \leq \underline{Q}_i < 2^{m_i}\sqrt{2}$. Use Q_i^l as the base quantity at stage i, $i = 1, \ldots, N$, and determine the corresponding optimal reorder point for each stage. The resulting heuristic policy is *Heuristic II*. (We can also use $r_i^l = r_i(Q_i^l)$ as the reorder point at stage i, $i = 1, \ldots, N$; see Section 5 for details.)

Remarks. (1) The lower-bound function has some special properties. Consider $C_i(Q)$, $i = 1, \ldots, N-1$. First note that $C_i(Q_i^0) = C_i^0$, which is the minimum value of $C_i(\cdot)$. Since $H_i(y) = C_i^0$ for $y > Q_i^0$, we have for $Q > Q_i$

$$C_i(Q) = \frac{Q_i^0}{Q} C_i(Q_i^0) + \frac{Q - Q_i^0}{Q} C_i^0 = C_i^0.$$

Therefore $C_i(Q)$ is flat for $Q \geq Q_i^0$. Now take any cluster c with $N \notin c$. Let $Q_c = \text{argmin}_Q \sum_{i \in c} C_i(Q)$. This problem has an infinite number of solutions. One of them is $Q_c = \max\{Q_i^0, i \in c\}$, with $\sum_{i \in c} C_i(Q_c) = \sum_{i \in c} C_i^0$.

(2) Note that $\sum_{i=1}^{N} C_i^0$ is the induced-penalty bound established in Chen and Zheng (1994b), which is a lower bound on the average costs of *all* feasible policies. Now suppose $Q_1^0 \leq \cdots \leq Q_N^0$. In this case, $(Q_1^0, \ldots, Q_N^0)$ is an optimal solution to P_l^- and C_l^- reduces to the induced-penalty bound. In fact, C_l^- is equal to the induced-penalty

bound as long as $Q_N^0 \geqslant Q_i^0$, $i = 1, \ldots, N-1$. Otherwise, C_l^- is larger.

3.3. Heuristic III

First, we suggest an allocation of setup costs based on the solution to a deterministic problem. Second, we verify that the new setup costs satisfy condition (5) and thus can be used to derive a new lower-bound function. We then propose a new heuristic solution.

Define

$$P_d\colon \min \sum_{i=1}^{N} (K_i/T_i + \hbar_i T_i)$$
$$\text{s.t. } T_{i+1} \geqslant T_i,\ i = 1, \ldots, N-1,$$

where

$$\hbar_i = \frac{\lambda\mu p^2 h_i}{2(p+H_i)(p+H_{i+1})}, \quad i = 1, \ldots, N,$$

and $H_i = \sum_{j=i}^N h_j$ for $i = 1, \ldots, N$ and $H_{N+1} = 0$. (The solution to P_d provides a lower bound on the average costs of any feasible policies in the deterministic counterpart of our serial system where demand arrives at a constant rate $\lambda\mu$. See Atkins and Sun 1995 and Chen 1998.) The problem can again be solved by the clustering algorithm. Let $\{c(1), \ldots, c(n)\}$ be the optimal partition and $(T_1^*, \ldots, T_N^*)$ the optimal solution to P_d. Consequently,

$$T_i^* = T_{c(k)} = \sqrt{\frac{\sum_{j\in c(k)} K_j}{\sum_{j\in c(k)} \hbar_j}},\ \forall\, i \in c(k),\ k = 1, \ldots, n.$$

The above solution suggests the following new setup costs:

$$\tilde{K}_i = \hbar_i (T_i^*)^2, \quad i = 1, \ldots, N. \tag{6}$$

We next verify that the new setup costs satisfy (5). Take any cluster in the optimal partition, say, $c(k) = \{l_1, \ldots, l_2\}$. Notice that

$$\sum_{i\in c(k)} \tilde{K}_i = (T_{c(k)})^2 \sum_{i\in c(k)} \hbar_i = \sum_{i\in c(k)} K_i. \tag{7}$$

Take any l with $l_1 \leqslant l < l_2$. Let $c^-(k) = \{l_1, \ldots, l\}$ and $c^+(k) = \{l+1, \ldots, l_2\}$. Since $T_{c^-(k)} \geqslant T_{c^+(k)}$ or

$$\frac{\sum_{i\in c^-(k)} K_i}{\sum_{i\in c^-(k)} \hbar_i} \geqslant \frac{\sum_{i\in c^+(k)} K_i}{\sum_{i\in c^+(k)} \hbar_i},$$

we have

$$\frac{\sum_{i\in c^-(k)} K_i}{\sum_{i\in c^-(k)} \hbar_i} \geqslant \frac{\sum_{i\in c(k)} K_i}{\sum_{i\in c(k)} \hbar_i} = (T_{c(k)})^2,$$

or

$$\sum_{i\in c^-(k)} K_i \geqslant \sum_{i\in c^-(k)} \tilde{K}_i. \tag{8}$$

From (7) and (8) we have (5).

Now allocate setup costs according to (6). These new setup costs lead to a new lower-bound function (Section 2.3). Use the new lower-bound function to re-define the problems P_l and P_l^- in Section 3.2. The solution to the new P_l^- leads to a new lower bound on the average costs of all feasible echelon-stock (r, nQ) policies, which is denoted by C_a^-. Let $(Q_1^a, \ldots, Q_N^a)$ be the power-of-two solution to the new P_l. Now use Q_i^a as the base quantity at stage i, $i = 1, \ldots, N$, and determine the optimal reorder point for each stage. The resulting heuristic policy is *Heuristic III*.

4. THE OPTIMAL SOLUTION

Let $\mathbf{Q}^* = (Q_1^*, \ldots, Q_N^*)$ be the optimal base quantities. Here we present an algorithm that finds $\mathbf{Q}^*$. We begin by deriving bounds on $\mathbf{Q}^*$.

Let H_i be the installation holding cost rate at stage i. Thus, $H_i = \sum_{j=i}^N h_j$. For any integer y, define

$$G_i^i(y) = E[H_i(y - D_i) + G_{i-1,i}(y - D_i)], \qquad i = 1, \ldots, N,$$

where $G_{0,1}(y) = (p + H_1)y^-$ and $G_{i-1,i}(\cdot)$ is defined in Section 2.2 for $i = 2, \ldots, N$. It is easy to see that $G_i^i(\cdot)$ is convex, $i = 1, \ldots, N$. For any integer r and any positive integer Q, define

$$C_i^i(r, Q) = \frac{\lambda\mu K_i + \sum_{y=r+1}^{r+Q} G_i^i(y)}{Q}.$$

For fixed Q, let $C_i^i(r, Q)$ be minimized at $r = r_i^i(Q)$. Set $C_i^i(Q) = C_i^i(r_i^i(Q), Q)$.

Consider an arbitrary unit of inventory. It travels from the outside supplier to stage N, then to stage $N-1$, etc. Take any $i = 1, \ldots, N$. We want to determine a lower bound on the total holding cost incurred by this unit before it reaches stage i. Note that holding costs start to accumulate as soon as the unit enters the system (or stage N). While traveling from stage N to stage $N-1$, the unit is counted as the (installation) on-hand inventory at stage N. Since it takes L_{N-1} units of time to go from stage N to stage $N-1$ and the installation holding cost rate at stage N is H_N, the total holding cost accumulated from stage N to stage $N-1$ is $H_N L_{N-1}$. Repeating this argument, we know that the total holding cost incurred by the unit before reaching stage i is at least $\sum_{j=i+1}^N H_j L_{j-1}$. (The unit may pause before reaching stage i.) Since inventory flows through the system at an average rate of $\lambda\mu$ units per unit of time, a lower bound on the average total holding cost incurred in the subsystem of stages $i+1, \ldots, N$ is

$$\sum_{j=i+1}^{N} \lambda\mu H_j L_{j-1}. \tag{9}$$

Now consider the subsystem of stages $1, \ldots, i$. Imagine that stage $i+1$ is an outside supplier with unlimited stock. The subsystem becomes an i-stage serial system. For this serial system, one can show that a lower bound on the average total cost is

$$\sum_{j=1}^{i-1} C_j(Q_j) + C_i^i(Q_i). \tag{10}$$

Table I
Heuristic vs. Optimal Solutions: Simple Poisson Demand

					Heuristic Solution							Optimal Solution								Lower		
No.	λ	K_1	K_3	p	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	Lower Bound I	Bound II	Heuristic Dev	Bound Dev
1	1	10	10	10	0	8	0	8	1	8	18.1737	0	6	1	6	2	6	17.7390	17.5064	17.5542	2.45%	1.05%
2	1	10	10	20	1	4	2	4	3	4	20.6137	1	5	2	5	3	5	19.9160	19.7038	19.7267	3.50%	0.96%
3	1	10	100	10	0	8	0	8	0	16	26.5296	0	7	1	7	0	14	26.4164	26.0826	26.1126	0.43%	1.16%
4	1	10	100	20	1	4	2	4	2	16	29.2051	1	5	2	5	2	15	28.6695	28.4559	28.4608	1.87%	0.73%
5	1	100	10	10	0	8	0	8	1	8	29.4237	0	11	0	11	0	11	28.2272	26.5410	28.0909	4.24%	0.49%
6	1	100	10	20	1	8	1	8	2	8	31.7901	0	10	1	10	2	10	30.9436	29.5157	30.8671	2.74%	0.25%
7	1	100	100	10	−1	16	−1	16	−1	16	35.7187	−1	14	−1	14	−1	14	35.4643	33.3376	35.2861	0.72%	0.51%
8	1	100	100	20	0	16	0	16	1	16	39.0118	0	13	1	13	1	13	38.4416	36.2356	38.1725	1.48%	0.70%
9	10	10	10	10	10	16	19	16	28	16	76.7482	9	19	18	19	27	19	76.2283	75.5898	75.7323	0.68%	0.65%
10	10	10	10	20	11	16	21	16	31	16	82.7607	11	18	21	18	30	18	82.5368	82.0675	82.1183	0.27%	0.51%
11	10	10	100	10	10	16	19	16	22	64	106.1548	9	22	17	22	23	44	104.0117	103.0929	103.1260	2.06%	0.86%
12	10	10	100	20	11	16	21	16	26	64	112.6840	11	17	21	17	27	51	110.6577	109.9992	110.0139	1.83%	0.59%
13	10	100	10	10	7	32	15	32	23	32	110.1429	7	34	15	34	22	34	109.9492	104.4476	109.5840	0.18%	0.33%
14	10	100	10	20	9	32	18	32	27	32	118.0247	9	33	18	33	27	33	118.0225	113.2123	117.7525	0.00%	0.23%
15	10	100	100	10	7	32	15	32	20	64	136.5550	6	44	13	44	20	44	132.8222	125.8141	132.2654	2.81%	0.42%
16	10	100	100	20	9	32	18	32	25	64	144.7779	8	42	17	42	25	42	142.0274	134.8886	141.2871	1.94%	0.52%

(The proof is essentially the same as in Section 2.2. The only difference is that the echelon holding cost rate at stage i is now H_i. We omit the details. The same idea is also used in Chen 1995.) Combining (9) and (10), we have the following theorem.

Theorem 3. *For* $i = 1, \ldots, N$, $C(\mathbf{r}, \mathbf{Q}) \geq \sum_{j=i+1}^{N} \lambda\mu H_j L_{j-1} + \sum_{j=1}^{i-1} C_j(Q_j) + C_i^i(Q_i)$.

Let C_f be the average cost of a feasible echelon-stock (r, nQ) policy. This can be the average cost of one of the heuristic policies identified in Section 4. For $i = 1, \ldots, N$, define

$$\underline{Q}_i = \min\left\{ q \geq 1, \text{ integer} \middle| C_i^i(q) \leq C_f - \sum_{j=i+1}^{N} \lambda\mu H_j L_{j-1} - \sum_{j=1}^{i-1} C_j^0 \right\},$$

and

$$\bar{Q}_i = \max\left\{ q \geq 1, \text{ integer} \middle| C_i^i(q) \leq C_f - \sum_{j=i+1}^{N} \lambda\mu H_j L_{j-1} - \sum_{j=1}^{i-1} C_j^0 \right\}.$$

Corollary 2. $\underline{Q}_i \leq Q_i^* \leq \bar{Q}_i$, $i = 1, \ldots, N$.

Proof. From Theorem 3, we have

$$C_f \geq \sum_{j=i+1}^{N} H_j \lambda\mu L_{j-1} + \sum_{j=1}^{i-1} C_j(Q_j^*) + C_i^i(Q_i^*).$$

The corollary follows since the minimum value of $C_j(\cdot)$ is C_j^0.

Corollary 2 defines a bounded region that contains $\mathbf{Q}^*$. Clearly, the bounds become tighter as C_f decreases. To determine the optimal base quantities, it suffices to search the entire region. Of course, we only need to consider those base quantities that satisfy the integer-ratio constraint.

5. NUMERICAL EXAMPLES

The main objective here is to test the performance of the heuristic solutions developed in Section 3. This is achieved by comparing the heuristic solutions with the optimal solutions if they are available or the lower bounds otherwise. Another objective is to compare our solution method with a widely recommended approach which determines the order quantities by solving a deterministic model.

We assume that the demand-size distribution is geometric, i.e.,

$$\Pr(D = x) = (1 - \alpha)^{x-1}\alpha, \quad x = 1, 2, \ldots,$$

where D is the demand size of a customer and $0 < \alpha \leq 1$. (This demand process is also called a stuttering Poisson process. When $\alpha = 1$, it reduces to simple Poisson.) The coefficient of variation of the total demand in one unit of time is $\sqrt{(2 - \alpha)/\lambda}$. The examples used have a coefficient of variation ranging from below to above one.

The numerical examples are divided into six groups. The first two groups are used to illustrate the performance of the heuristics and the lower bounds, using the optimal solution as a benchmark. Groups 3 and 4 are used to test the performance of the heuristics and the lower bounds as the number of stages increases. The purpose of the final two groups is to compare our approach with a traditional one for finding the order quantities.

Group 1. The examples in this group have the following in common: $N = 3$, $\alpha = 1$, $K_2 = 10$, $h_i = 1$ and $L_i = 1$ for $i = 1, 2, 3$. We varied the remaining parameters, λ, K_1, K_3 and p, to generate 16 examples. For these examples, Table I reports the best heuristic among Heuristics I, II and III; the optimal echelon-stock (r, nQ) policy; and the two

Table II
Heuristic vs. Optimal Solutions: Compound Poisson Demand

					Heuristic Solution							Optimal Solution								Lower Bound	Heu- ristic	
No.	λ	K_1	K_3	p	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	Lower Bound I	II	Dev	Bound Dev
1	1	10	10	10	9	16	13	16	16	16	77.3892	8	19	12	19	15	19	77.1160	76.0334	76.1636	0.35%	1.25%
2	1	10	10	20	13	16	18	16	23	16	93.0702	12	18	17	18	22	18	92.8886	92.0098	92.0651	0.20%	0.89%
3	1	10	100	10	9	16	13	16	12	32	93.7445	8	20	11	20	10	40	92.7435	91.7937	91.8117	1.08%	1.01%
4	1	10	100	20	13	16	18	16	19	32	109.4850	12	19	17	19	17	38	108.6316	107.8263	107.8375	0.79%	0.74%
5	1	100	10	10	5	32	7	32	10	32	95.0581	5	31	8	31	10	31	95.0137	91.0108	93.7259	0.05%	1.37%
6	1	100	10	20	9	32	13	32	17	32	111.5689	9	30	14	30	18	30	111.4384	107.9957	110.3237	0.12%	1.01%
7	1	100	100	10	5	32	7	32	10	32	109.1206	3	38	6	38	8	38	108.0819	103.8674	106.7689	0.96%	1.23%
8	1	100	100	20	9	32	13	32	17	32	125.6314	8	36	12	36	16	36	124.9496	120.9207	123.7624	0.55%	0.96%
9	10	10	10	10	59	64	107	64	153	64	335.1443	61	56	110	56	156	56	334.3862	330.6367	331.0603	0.23%	1.00%
10	10	10	10	20	68	64	121	64	172	64	373.5888	70	53	125	53	176	53	371.9675	368.9920	369.1689	0.44%	0.76%
11	10	10	100	10	59	64	107	64	139	128	386.9145	60	60	109	60	141	120	386.5565	383.0998	383.1586	0.09%	0.89%
12	10	10	100	20	68	64	121	64	159	128	426.1509	69	58	123	58	162	116	425.2815	422.4206	422.4440	0.20%	0.67%
13	10	100	10	10	47	128	88	128	129	128	403.5995	53	92	98	92	142	92	394.3214	380.8020	390.3738	2.35%	1.01%
14	10	100	10	20	68	64	121	64	172	64	443.9013	63	88	114	88	163	88	435.3080	423.9342	432.1055	1.97%	0.74%
15	10	100	100	10	47	128	88	128	129	128	438.7558	48	116	91	116	133	116	437.5073	423.6013	433.8099	0.29%	0.85%
16	10	100	100	20	57	128	105	128	151	128	483.7613	59	109	109	109	157	109	480.8285	467.4089	477.5360	0.61%	0.69%

lower bounds (Lower Bound I is C_l^- and Lower Bound II is C_a^-). The last two columns indicate the relative differences between of the heuristic and the optimal solution and between the optimal solution and the better lower bound.

Group 2. Same as Group 1 except that $\alpha = 0.2$. The results are in Table II.

Group 3. The examples in this group have: $\lambda = 1$, $\alpha = 1$, $p = 10$, and $K_i = 10$, $h_i = 1$, and $L_i = 1$ for $i = 1, \ldots, N$, where N ranges from 2 to 12. Figure 2 summarizes the results. We only include the best heuristic and lower bound.

Group 4. Same as Group 3 except that $\alpha = 0.2$. The results are in Figure 3.

Group 5. The examples in this group have the following in common: $N = 3$, $\alpha = 1$, $p = 30$, $h_i = 1$ and $L_i = 1$ for $i = 1, 2, 3$. We varied the remaining parameters, λ, K_1, K_2 and K_3, to generate 24 examples. For these examples, we computed the best heuristic, the EOQ solution, and the two lower bounds. To compute the EOQ solution, we first determined the base quantities by solving a deterministic serial model and then identified the corresponding optimal reorder points. The deterministic model assumes that demand arrives at constant rate λ and the system uses nested, stationary policies (Chen 1998). The results are in Table III, which also provides the relative deviations of the heuristic and the EOQ solutions from the better lower bound.

Group 6. Same as Group 5 except that $\alpha = 0.2$. The results are in Table IV. (Here the deterministic model assumes that demand arrives at constant rate $\lambda\mu$.)

Observations

(1) Our numerical experience suggests that there is no clear dominance among the three heuristics. The heuristic reorder points, i.e., $r^i(Q_i^u)$ (resp., $r_i(Q_i^l)$ and $r_i(Q_i^a)$) for Heuristic I (resp., II and III) are near optimal. Using the optimal reorder points, for the given Qs, only leads to improvements which are typically less than 1 percent. (The heuristic reorder points are not reported here.)

(2) The heuristics are very easy to compute. For the examples in Table I (2), the average time on a 486 PC is

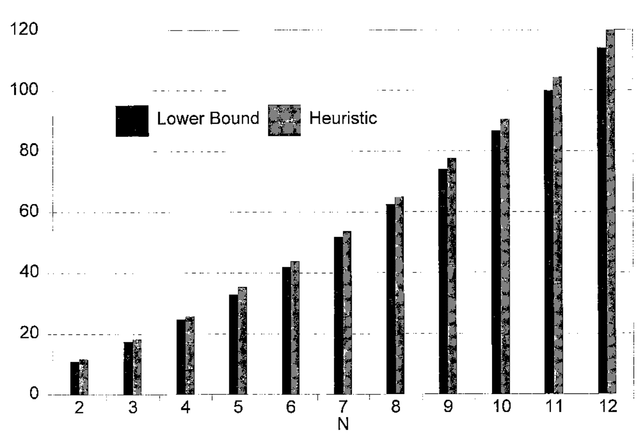

Figure 2. Simple Poisson Examples.

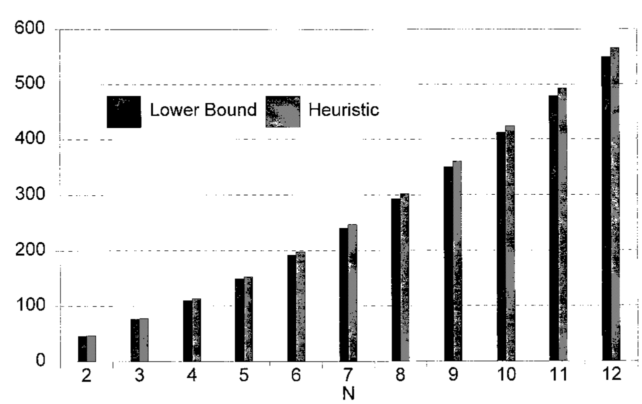

Figure 3. Compound Poisson Examples.

Table III
Heuristic vs. EOQ Solutions: Simple Poisson

					Heuristic Solution							EOQ Solution										
No.	λ	K_1	K_2	K_3	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	Lower Bound I	Lower Bound II	Heu-ristic Dev	EOQ Dev
1	0.1	10	10	10	0	2	0	2	0	2	6.1376	0	2	0	2	0	2	6.1376	6.1143	6.1173	0.33%	0.33%
2	0.1	10	10	1000	0	2	0	2	−1	16	18.0669	0	2	0	2	−1	16	18.0669	17.9738	17.9745	0.51%	0.51%
3	0.1	10	1000	10	0	2	−1	8	−1	8	22.4009	0	2	−1	8	−1	8	22.4009	21.5823	21.6235	3.60%	3.60%
4	0.1	10	1000	1000	0	2	−1	16	−1	16	29.7292	0	2	−1	16	−1	16	29.7292	29.5680	29.5671	0.55%	0.55%
5	0.1	1000	10	10	−1	8	−1	8	−1	8	24.0731	−1	8	−1	8	−1	8	24.0731	21.4457	23.9872	0.36%	0.36%
6	0.1	1000	10	1000	−1	8	−1	8	−1	16	34.0403	−1	8	−1	8	−1	16	34.0403	29.4624	33.1991	2.53%	2.53%
7	0.1	1000	1000	10	−1	16	−1	16	−2	16	34.8553	−1	16	−1	16	−2	16	34.8553	33.5251	33.5939	3.75%	3.75%
8	0.1	1000	1000	1000	−1	16	−1	16	−2	16	41.0428	−1	16	−1	16	−2	16	41.0428	41.0001	41.0020	0.10%	0.10%
9	1	10	10	10	1	4	3	4	4	4	21.8985	1	4	3	4	4	4	21.8985	20.9064	20.8986	4.75%	4.75%
10	1	10	10	1000	1	4	2	8	0	64	62.2741	1	4	3	4	1	32	62.8649	59.1110	59.1042	5.35%	6.35%
11	1	10	1000	10	1	4	1	32	0	32	71.6989	1	4	1	32	0	32	71.6989	70.7729	71.1667	0.75%	0.75%
12	1	10	1000	1000	1	4	−1	64	−2	64	100.7495	1	4	1	32	0	32	102.6364	95.9278	95.9390	5.01%	6.98%
13	1	1000	10	10	0	32	0	32	0	32	80.0156	0	32	0	32	0	32	80.0156	69.7383	79.1786	1.06%	1.06%
14	1	1000	10	1000	0	32	0	32	−1	64	110.5859	0	32	0	32	0	32	110.9531	94.8869	107.8111	2.57%	2.91%
15	1	1000	1000	10	0	32	0	32	0	32	110.9531	0	32	0	32	0	32	110.9531	108.1585	108.8462	1.94%	1.94%
16	1	1000	1000	1000	−1	64	−2	64	−3	64	137.8828	−1	64	−2	64	−3	64	137.8828	131.8266	131.8723	4.56%	4.56%
17	10	10	10	10	12	16	22	16	32	16	86.0823	12	16	22	16	32	16	86.0823	85.5558	85.5939	0.57%	0.57%
18	10	10	10	1000	12	16	22	16	25	128	208.5540	12	16	22	16	25	128	208.5540	206.8252	206.8380	0.83%	0.83%
19	10	10	1000	10	12	16	16	128	21	128	249.5842	12	16	16	128	21	128	249.5842	243.9881	245.4230	1.70%	1.70%
20	10	10	1000	1000	12	16	16	128	21	128	326.9280	12	16	16	128	21	128	326.9280	323.6436	323.6884	1.00%	1.00%
21	10	1000	10	10	8	64	16	64	25	64	283.5951	8	64	16	64	25	64	283.5951	240.3115	271.1053	4.61%	4.61%
22	10	1000	10	1000	6	128	12	128	18	128	365.3680	6	128	12	128	18	128	365.3680	319.9833	361.6762	1.02%	1.02%
23	10	1000	1000	10	6	128	12	128	18	128	365.3680	6	128	12	128	18	128	365.3680	362.4384	364.9963	0.10%	0.10%
24	10	1000	1000	1000	6	128	12	128	18	128	442.7117	6	128	12	128	18	128	442.7117	437.5647	437.7054	1.14%	1.14%

about seven seconds (one minute) per example for computing the three heuristics. Moreover, the computational time grows only linearly as N increases. For the examples in Figure 2 (3), the time increases about 10 (25) seconds as N increases by one. In contrast, the computational effort for the optimal solution is much greater. For the examples in Table I (2), the average time is about 10 minutes (two hours). (Since no special effort has been devoted to

Table IV
Heuristic vs. EOQ Solutions: Compound Poisson

					Heuristic Solution							EOQ Solution										
No.	λ	K_1	K_2	K_3	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	R_1	Q_1	R_2	Q_2	R_3	Q_3	Cost	Lower Bound I	Lower Bound II	Heu-ristic Dev	EOQ Dev
1	0.1	10	10	10	2	8	2	8	2	8	34.1279	4	4	4	4	4	4	34.9230	33.9818	33.9868	0.42%	2.75%
2	0.1	10	10	1000	2	8	2	8	−1	32	56.5721	4	4	4	4	−1	32	57.2681	56.3065	56.3085	0.47%	1.70%
3	0.1	10	1000	10	2	8	−1	32	−1	32	62.5413	4	4	1	16	0	16	68.0167	61.1993	61.3614	1.92%	10.85%
4	0.1	10	1000	1000	2	8	−1	32	−1	32	78.0100	4	4	−1	32	−1	32	78.3513	77.6430	77.6498	0.46%	0.90%
5	0.1	1000	10	10	−1	32	−1	32	−2	32	67.8887	−1	16	−1	16	−1	16	68.3526	60.4385	64.1636	5.81%	6.53%
6	0.1	1000	10	1000	−1	32	−1	32	−2	32	83.3575	−1	32	−1	32	−2	32	83.3575	76.9267	82.4456	1.11%	1.11%
7	0.1	1000	1000	10	−1	32	−1	32	−2	32	83.3575	−1	32	−1	32	−2	32	83.3575	82.4680	82.9947	0.44%	0.44%
8	0.1	1000	1000	1000	−1	32	−1	32	−2	32	98.8262	−1	32	−1	32	−2	32	98.8262	98.1219	98.1472	0.69%	0.69%
9	1	10	10	10	15	16	21	16	27	16	102.4332	18	8	25	8	30	8	109.1891	101.4970	101.5286	0.89%	7.55%
10	1	10	10	1000	15	16	21	16	12	128	180.8281	18	8	25	8	12	128	185.4027	178.4009	178.4086	1.36%	3.92%
11	1	10	1000	10	15	16	13	64	15	64	202.7745	15	16	13	64	15	64	202.7745	197.0042	198.3647	2.22%	2.22%
12	1	10	1000	1000	15	16	7	128	7	128	253.4318	15	16	7	128	7	128	253.4318	250.4091	250.4588	1.19%	1.19%
13	1	1000	10	10	6	64	10	64	13	64	211.7500	6	64	10	64	13	64	211.7500	194.2092	210.5454	0.57%	0.57%
14	1	1000	10	1000	6	64	10	64	9	128	277.3369	6	64	10	64	9	128	277.3369	247.6107	270.1098	2.68%	2.68%
15	1	1000	1000	10	6	64	10	64	13	64	289.0937	6	64	10	64	13	64	289.0937	269.8533	271.9096	6.32%	6.32%
16	1	1000	1000	1000	1	128	3	128	4	128	323.9484	1	128	3	128	4	128	323.9484	320.3702	320.5030	1.07%	1.07%
17	10	10	10	10	73	64	129	64	182	64	395.4860	81	32	140	32	194	32	401.6991	390.7442	390.8422	1.19%	2.78%
18	10	10	10	1000	73	64	129	64	156	256	656.9875	81	32	140	32	157	256	661.4803	641.7312	641.7519	2.37%	3.07%
19	10	10	1000	10	73	64	105	256	146	256	714.0915	81	32	105	256	146	256	715.6901	706.5210	711.1186	0.42%	0.64%
20	10	10	1000	1000	73	64	105	256	146	256	907.4509	81	32	105	256	146	256	909.0495	878.1704	878.3418	3.31%	3.50%
21	10	1000	10	10	51	256	96	256	139	256	767.8306	51	256	96	256	139	256	767.8306	697.9401	755.9829	1.57%	1.57%
22	10	1000	10	1000	51	256	96	256	139	256	961.1900	51	256	96	256	139	256	961.1900	869.7933	948.5653	1.33%	1.33%
23	10	1000	1000	10	51	256	96	256	139	256	961.1900	51	256	96	256	139	256	961.1900	947.4553	954.5282	0.70%	0.70%
24	10	1000	1000	1000	51	256	96	256	139	256	1154.5494	51	256	96	256	139	256	1154.5494	1110.1104	1110.5356	3.96%	3.96%

improving the efficiency of the computer programs, these times should only be interpreted in relative terms.)

(3) The heuristic solution is close to optimal. It is remarkable that as the number of stages increases, the performance of the heuristic solution does not deteriorate.

(4) C_a^- is often larger (thus better) than C_l^-. When C_l^- is larger, the difference is very small. Therefore the setup-cost allocation suggested in Section 3.3 indeed improves the lower bound. Sometimes the improvement is substantial. This happens when the setup cost at a downstream stage is much larger than the setup cost at an upstream stage, as expected.

(5) The heuristic solution dominates the EOQ solution. In one example (No. 3 in Table IV), the average cost of the EOQ solution is almost 10 percent higher than that of the heuristic solution. For this example, the coefficient of variation is 4.24 (highest among all the examples). This seems to suggest that the EOQ solution may perform badly in systems with high demand volatility.

(6) The heuristic base quantities are always larger than or equal to the EOQs. This observation echoes a recent finding in the single-stage (r, Q) model that the optimal Q is larger than the EOQ (Zheng 1992). It suggests that the EOQs should be adjusted upward for stochastic systems, especially those with high demand volatility.

6. CONCLUSION

This paper provides an efficient algorithm for determining near-optimal control parameters of echelon-stock (r, nQ) policies in multi-stage, serial, production/distribution systems. The algorithm is based on simple lower and upper bounds on the exact cost function. The bounds are separable functions of the control parameters, whose minimization leads to heuristic control parameters. We also provide an algorithm that is more time consuming but finds the optimal solution. Numerical experience suggests that the order quantities based on the solution to a deterministic problem can be seriously suboptimal, especially when the demand is volatile. Although the entire paper focuses on the continuous-time model with compound Poisson demand, all the results can be easily extended to the discrete-time case with independent, identically distributed demands.

ACKNOWLEDGMENT

The authors thank Paul Zipkin and the anonymous reviewers for their helpful suggestions. This research was supported in part by the Faculty Research Fund of the Columbia Business School.

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