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Near-rings in which each element is a power of itself

Howard E. Bell

Let R denote a near-ring such that for each $x \in R$, there exists an integer $n(x) > 1$ for which $x^{n(x)} = x$. We show that the additive group of R is commutative if $0.x = 0$ for all $x \in R$ and every non-trivial homomorphic image $\bar{R}$ of R contains a non-zero idempotent e commuting multiplicatively with all elements of $\bar{R}$. As the major consequence, we obtain the result that if R is distributively-generated, then R is a ring - a generalization of a recent theorem of Ligh on boolean near-rings.

1. Introduction

In [6], Ligh proved that a distributively-generated boolean near-ring is a ring and asked whether the same can be said of distributively-generated near-rings satisfying the identities $x^p = x$ and $px = 0$, where p is a prime. We give here an affirmative answer to this question, and we obtain some more general results on additive commutativity in near-rings in which $x^{n(x)} = x$. The major theorems are

THEOREM 1. *Let R be a non-trivial near-ring satisfying the following properties:*

(i) $0.x = 0$ *for all* $x \in R$ *;*

(ii) for each $x \in R$ *, there exists an integer* $n(x) > 1$ *such that* $x^{n(x)} = x$ *;*

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(iii) every non-trivial homomorphic image of R *contains a non-zero central idempotent.*

Then the additive group of R *is commutative.*

THEOREM 2. *Let* R *be a distributively-generated near-ring such that for each* $x \in R$ *there is an integer* $n(x) > 1$ *for which* $x^{n(x)} = x$. *Then* R *is a commutative ring.*

2. Definitions and preliminary results

Our definitions of near-ring, distributive element, distributively-generated near-ring, and ideal are as in [6]. A near-ring ideal P will be called *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$. An element a of the near-ring R will be called *central* if $xa = ax$ for all $x \in R$.

The left distributive law implies

(1) $$x.0 = 0 \quad \text{for all} \quad x \in R$$

and

(2) $$x(-y) = -xy \quad \text{for all} \quad x, y \in R \, ;$$

moreover, if d is a distributive element of R , we have

(3) $$(-x)d = -xd \quad \text{for all} \quad x \in R \, .$$

Property (2) permits left cancellation of elements which are not zero-divisors; and from (1) it follows that in near-rings satisfying *(i)*, the notion of nilpotent element may be borrowed from ring theory, with nilpotent elements behaving as we would expect. In particular, we have the readily-proved

LEMMA 1. *If* R *is a near-ring satisfying (i) and having no non-zero nilpotent elements, then* $ab = 0$ *implies that* $ba = 0$ *and that* $arb = 0$ *for all* $r \in R$.

We shall refer to the second conclusion of this lemma as IFP (insertion-of-factors property).

The elementary proofs of the "$x^n = x$ theorem" for rings use the fact that in rings with no non-zero nilpotent elements, idempotents are central. This result does not extend to near-rings satisfying *(i)* (note counterexamples in [2]); however, we obtain a partial generalization as follows:

LEMMA 2. *Let* R *be a near-ring satisfying (i) and having no non-zero nilpotent elements. Then we have*

(A) *every distributive idempotent is central;*

(B) *for every idempotent* e *and every element* $x \in R$ *,* $ex^2 = (ex)^2$ *;*

(C) *if* R *has a multiplicative identity element, then all idempotents are central.*

Proof. We first show that for each $x \in R$ and idempotent e , $xe = exe$. Since $e(xe-exe) = 0$, Lemma 1 guarantees that $(xe-exe)e = 0 = (xe-exe)e(-xe)$; hence, we have $(xe-exe)^2 = (xe-exe)xe + (xe-exe)(-exe) = 0$, so that $xe - exe = 0$.

If e is a distributive idempotent, we also have $(ex-exe)e = exe + (-exe)e$; hence by (3) $(ex-exe)e = 0$. It follows that $e(ex-exe) = ex - exe = 0$; and the proof of (A) is complete.

To establish (B), note that for any idempotent e , $xe(x-ex) = 0$, so that by IFP we get $ex(x-ex)$ nilpotent and hence zero.

To establish (C), we need only show that if R has 1 , then $ex = exe$ for all $x \in R$ and arbitrary idempotents e . Now $e(1-e) = 0$, so $(1-e)e = 0$ as well; moreover, $e(ex-exe) = ex - exe$ and $ex(1-e) = ex - exe$. Therefore, $(ex-exe)^2 = ex(1-e)e(ex-exe) = 0 = ex - exe$.

The standard proofs of the "$x^n = x$ theorem" for rings involve ideals which are not easily shown to be normal subgroups of R^+ ; we overcome this obstacle by use of a kind of annihilator ideal introduced in [1].

LEMMA 3. *Let* R *be a non-trivial near-ring satisfying (i) and having no non-zero nilpotent elements. Then* R *contains a family of completely prime ideals with trivial intersection.*

Proof. Since R has no non-zero nilpotent elements, there must

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exist multiplicative subsemigroups which do not contain zero, and an application of Zorn's Lemma shows that any such subsemigroup is contained in a subsemigroup maximal with respect to excluding zero. Let M be any such maximal subsemigroup, and define

$$A(M) = \{x \in R \mid ax = 0 \text{ for at least one } a \in M\} .$$

If $u, v \in A(M)$, there exist $a, b \in M$ such that $au = bv = 0$. By IFP, we then have $abu = 0$, and thus $ab(u-v) = 0$; moreover, for arbitrary $x \in R$, $a(x+u-x) = 0$, so $A(M)$ is a normal subgroup of R^+. Also, if $x, y \in R$, we have $axu = 0$ and
$a[(x+u)y - xy] = a(x+u)y - axy = (ax+au)y - axy = (ax+0)y - axy = 0$;
hence $A(M)$ is an ideal.

Now if $x \notin M$, the multiplicative subsemigroup generated by M and x must contain 0; and since R has no non-zero nilpotent elements, some finite product containing x as at least one factor and having at least one factor from M must be zero. Repeated application of IFP establishes the existence of an $m \in M$ such that mx is nilpotent and hence 0. Therefore the set-theoretic complement of $A(M)$ is M, and $A(M)$ is a completely prime ideal. Clearly every non-zero element of R is excluded from at least one of the ideals $A(M)$.

3. Proofs of Theorems 1 and 2 and some corollaries

Proof of Theorem 1. A near-ring satisfying *(i)* and *(ii)* obviously has no non-zero nilpotent elements, hence Lemma 3 applies. For each $P = A(M)$, the near-ring $\bar{R} = \frac{R}{P}$ satisfies *(i)* and *(ii)*, has no zero-divisors, and contains a non-trivial central idempotent e_0. From part (B) of Lemma 2, we see that every idempotent of $\bar{R}$ is a left identity element, hence e_0 is the only non-zero idempotent and is an identity element. Now $a^n = a$ implies a^{n-1} is idempotent, hence non-zero elements in $\bar{R}$ have inverses and $\bar{R}$ is then a near-field. Thus $\bar{R}$ has commutative addition [5, 7]; and additive commutators in R lie in each of the completely prime ideals $A(M)$, hence are zero.

Proof of Theorem 2. All distributively-generated near-rings satisfy

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(i). Moreover, if a is a distributive element and $a^n = a$, then a^{n-1} is a distributive idempotent, which is central by part (A) of Lemma 2. Thus, by Theorem 1, R^+ is commutative. But by a theorem of Fröhlich [3, p. 93], additive commutativity in a distributively-generated near-ring R implies that R is a ring. That R is also a commutative ring is the well-known "$x^n = x$ theorem" of Jacobson [4].

Two corollaries of Theorem 1 are

THEOREM 3. *Let R be a near-ring with identity satisfying (i) and (ii). Then R^+ is commutative.*

THEOREM 4. *Let R be a finite near-ring; and suppose R is embeddable in a near-ring with identity which satisfies (i) and has no non-zero nilpotent elements. Then R^+ is commutative.*

Theorem 3 is obvious; Theorem 4 follows from Theorem 1 and part (C) of Lemma 2 once we note that a finite near-ring with *(i)* and without nilpotent elements satisfies *(ii)*.

4. Remarks

In the class of near-rings satisfying *(i)* and *(ii)*, condition *(iii)* is sufficient for additive commutativity; but it is not necessary, as we see by considering [2], example 53 with additive group Z_6. Lemma 2 and Theorems 3 and 4 point out an apparent difference in behaviour depending on whether R does or does not have an identity element. This difference is real, as is shown by [2], example 34 with additive group S_3.

References

[1] Howard E. Bell, "Duo rings: some applications to commutativity theorems", *Canad. Math. Bull.* 11 (1968), 375-380.

[2] James R. Clay, "The near-rings on groups of low order", *Math. Z.* 104 (1968), 364-371.

[3] A. Fröhlich, "Distributively generated near-rings, (I. Ideal Theory)",

https://doi.org/10.1017/S0004972700042052 Published online by Cambridge University Press

Proc. London Math. Soc. (3), 8 (1958), 76-94.

[4] I.N. Herstein, "An elementary proof of a theorem of Jacobson", *Duke Math. J.* 21 (1954), 45-48.

[5] Steve Ligh, "On distributively generated near-rings", *Proc. Edinburgh Math. Soc.* 16 (1969), 239-243.

[6] Steve Ligh, "On boolean near-rings", *Bull. Austral. Math. Soc.* 1 (1969), 375-379.

[7] J.L. Zemmer, "The additive group of an infinite near-field is abelian", *J. London Math. Soc.* 44 (1969), 65-67.

Brock University,
St Catharines,
Ontario, Canada.

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