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Near Sets: An Introduction

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Abstract This article gives a brief overview of near sets. The proposed approach in introducing near sets is to consider a set theory-based form of nearness (proximity) called *discrete proximity*. There are two basic types of near sets, namely, spatially near sets and descriptively near sets. By endowing a nonempty set with some form of a nearness (proximity) relation, we obtain a structured set called a proximity spaces. Let $\mathcal{P}(X)$ denote the set of all subsets of a nonempty set X. One of the oldest forms of nearness relations p (later denoted by δ) was introduced by E. Čech during the mid-1930s, which leads to the discovery of spatially near sets, *i.e.*, those sets that have elements in common. That is, given a proximity space (X, δ), for any subset $A \in \mathcal{P}(X)$, one can discover nonempty nearness collections $\xi(A) = \{B \in \mathcal{P}(X) : A\ \delta\ B\}$. Recently, descriptively near sets were introduced as a means of solving classification and pattern recognition problems arising from disjoint sets (*i.e*, sets with empty spatial intersections) that resemble each other. One discovers descriptively near sets by choosing a set of probe functions Φ that represent features of points in a set and endowing the set of points with a descriptive proximity relation δ_Φ and obtaining a descriptively structured set (called descriptive proximity space). Given a descriptive proximity spaces (X, δ_Φ), one can discover collections of subsets that resemble each other. This leads to the discovery of descriptive nearness collections $\xi_\Phi(A) = \{B \in \mathcal{P}(X) : A\ \delta_\Phi\ B\}$. That is, if $B \in \xi_\Phi(A)$, then $A\ \delta_\Phi\ B$ (relative to the chosen features of points in X, A resembles B). The focus of this tutorial is on descriptively near sets.

Keywords Description · Near sets · Proximity space · Spatial

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Fig. 1 A Bit More, Punch, 1845

1 Introduction

A nonempty set X endowed with a nearness relation is a structured set. The choice of an appropriate nearness relation in defining a structured set is a bit like choosing how far to lower the knight in Fig. 1, so that the knight fits in the saddle. A good choice for the knight leads to a good ride. In terms of set structures, there are many choices and a good choice makes it easier to find interesting nearness patterns.

This article gives a brief overview of near sets. The basic approach in this introduction to near sets is to consider a set theory-based form of nearness (proximity) called *discrete proximity* (see, *e.g.*, [1–3] and [4, Ch. 21]). A pair of nonempty sets are considered near, provided the intersection of the sets is not empty. In other words, in the discrete form of proximity, near sets have elements in common. A more general form of proximity considers *metric proximity* based on the distance between sets (see, *e.g.*, [1,2]). In that context, a pair of nonempty sets are near, provided the distance between the sets is zero. Many other forms of proximity have also been introduced (for a summary, see [2, pp. 93–94]). We only consider the discrete proximity case in this article. We have chosen discrete proximity as a basis for introducing near sets because discrete proximity is perhaps the simplest form of proximity to implement and provides a straightforward basis for applications. For the more general forms of proximity, there are many sources (see, *e.g.*, [1–3,5].

There are two basic types of near sets, namely, spatially near sets and descriptively near sets. The study of the nearness of sets spans more than 100 years, starting with the address by Riesz at the International Congress of Mathematicians in Rome in 1908 [6], recently commented on by Naimpally [7] and Di Concilio [2,8,9]. One of the earliest introductions to nearness (proximity) relations was given by Čech during a 1936–1939 Brno seminar, published in 1966 [10, §25.A.1]. Čech used the symbol p to denote a proximity relation defined on a nonempty set X, which Čech axiomatized. Čech's work on proximity spaces started two years after Efremovič's work (in 1933), who introduced a widely considered axiomatization of proximity, which was not published until 1951 [11]. For a detailed presentation of Efremovič's proximity axioms, see, *e.g.*, [1,2] and for applications, see, *e.g.*, [3,12,13].

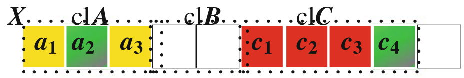


Fig. 2 $\mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset$ implies $A\ \delta\ B$

2 Spatially Near Sets

Let $\mathcal{P}(X)$ denote the collection of all subsets of X. Subsets $A, B \in \mathcal{P}(X)$ are spatially near (denoted by $A\ p\ B$), if the intersection of A and B is nonempty, which implies $A\ p\ B$. Čech was one of the first to axiomatize a proximity relation. Later, a spatial nearness relation was denoted by δ instead of the Čech symbol p (see, *e.g.*, [1]). The proximity relation δ is a spatial relation, since $A\ \delta\ B$ means that A and B have points in common. To understand what it means to say that a pair of finite sets are spatially near each other, consider the *closure* of a subset $A \in \mathcal{P}(X)$ (denoted by $\mathrm{cl}A$), defined by

$$\mathrm{cl}A = \{x \in X : x\ \delta\ A\},$$

i.e., $\mathrm{cl}A$ is the set of all points x in X that are near A. In effect, this means $\mathrm{cl}A$ contains all of the boundary points of A as well as all of the interior points of A.

Example 1 Spatially Near Sets
Let the set of points X be represented by the weave cells in Fig. 2 and let the closures of sets $A, B \in \mathcal{P}(X)$ be represented by $\mathrm{cl}A, \mathrm{cl}B$ in Fig. 2. Observe that $\mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset$. Hence, $A\ \delta\ B$.

A spatial nearness relation δ (called a *discrete* proximity) is defined by

$$\delta = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}A\ \cap\ \mathrm{cl}B \neq \emptyset\}.$$

Whenever sets A and B have no points in common, the sets are *far* from each other (denoted by $A\ \underline{\delta}\ B$), where

$$\underline{\delta} = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta.$$

Example 2 Spatially Non-Near Sets
Let the set of points X be represented by the picture in Fig. 1. Let $A, B \in \mathcal{P}$ be the set of points in the knight's horse and set of points in the suspended knight, respectively. $A\ \underline{\delta}\ B$, since there are no points in A that are touching points in B. That is, the closure of A (denoted $\mathrm{cl}A$) has no points in common with the closure of B. A point $x \in B$ lies in the closure of A, provided x is near B. In effect, $A\ \underline{\delta}\ B$.

A comprehensive introduction to spatially near sets is given by Naimpally [**?**]. The role of *near* and *far* in both the theory and applications in topology, proximity spaces, and uniform spaces is given in great detail in [3]. The history of near sets and their applications can be found in [13] (see, also, [15]).

A proximity space (X, δ) (set X is endowed with a nearness relation δ) is structured by δ. This means that by virtue of the nearness relation defined X, one can find nearness collections $\xi \in \delta$ such that

$$\xi(A) = \{B \in X : A\ \delta\ B\}.$$

In effect, one can then identify nearness patterns in a proximity space, *i.e.*, points in the closure one subset in a space that are in the intersection of the closure of other subsets in the space and, hence, such subsets are *spatially near* each other. In a more general setting, collections of near sets lead to what is known as a *nearness space* [16] and the category **Near** [17]. For nearness collections, we obtain the following result.

Lemma 1 (Spatial Nearness Collections). *For a proximity space (X, δ) with $A \in \mathcal{P}(X)$ and nearness collection $\xi(A)$, the set $A \in \xi(A)$.*

Proof For nonempty $A \in \mathcal{P}(X)$, observe that $A \cap A = A$. Then $A\ \delta\ A$ (A is near itself). Hence, $A \in \xi(A)$. □

3 Descriptively Near Sets

Recently, descriptively near sets were introduced as a means of solving classification and pattern recognition problems arising from disjoint sets (*i.e*, sets with empty spatial intersections) that resemble each other. Descriptively near sets were introduced in 2007 [18,19], stemming from the 2002 correspondence between J.F. Peters and Z. Pawlak in 2002, eventually leading to the publication of a philosophic poem about the nearness of objects such as snowflakes and trees [20] and collaboration between Peters, Skowron and Stepaniuk in 2006 [21], leading to the publication of a paper on the nearness of objects in 2007 [22]. Recently, the connections between spatially near sets and descriptively near sets have been explored by Peters and Naimpally in [13,3].

Let X be a nonempty set, x a member of X, $\Phi = \{\phi_1, \ldots, \phi_n\}$ a set of probe functions that represent features of each x. A *probe function* $\phi : X \to \mathbb{R}$ is real-valued and represents a feature of an object such as a sample point (pixel) in a picture. Let $\Phi(x)$ denote a feature vector for the object x, *i.e.*, a vector of feature values that describe x. A feature vector provides a description of an object and subsets of X. To obtain a descriptive proximity relation (denoted by δ_Φ), one first chooses a set of probe functions, which provide a basis for describing points in a set. Let $A, B \in \mathcal{P}(X)$. Let $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. For example,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\},$$

Let $x \in A \cup B$. Then $\Phi(x) \in \mathcal{Q}(A)$, if and only if, $\Phi(x) = \Phi(a)$ for some $a \in A$. In other words, there is a point a in A with a description that matches the description of x. For example, assume A, B are disjoint as in Fig. 2. For $x = a_2 \in A$, $\Phi(a_2) = \Phi(b_4)$. Hence, $\Phi(x)$ is in $\mathcal{Q}(A)$ and in $\mathcal{Q}(B)$.

The expression $A\ \delta_\Phi\ B$ reads *A is descriptively near B*. Similarly, $A\ \underline{\delta}_\Phi\ B$ denotes that A is descriptively far (remote) from B. The descriptive proximity of A and B is defined by

$$A\ \delta_\Phi\ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset.$$

The descriptive intersection $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

That is, $x \in A \cup B$ is in $A \underset{\Phi}{\cap} B$, provided there is are $a \in A, b \in B$ such that $\Phi(x) = \Phi(a) = \Phi(b)$. Observe that A and B can be disjoint and yet $A \underset{\Phi}{\cap} B$ can be nonempty.

Example 3 Descriptive Intersection of Disjoint Sets
Choose Φ to be a set of probe functions representing weave cell colours. Let the set of cells X in Fig. 2 be endowed with δ_Φ. Sets $A, C \in \mathcal{P}(X)$ are disjoint. Let $x \in A \cup C$ be a weave cell named a_2. Observe that $\Phi(x)$ is in $\mathcal{Q}(A)$ and $\Phi(x)$ is in $\mathcal{Q}(C)$, since $\Phi(a_2) = \Phi(c_4)$. Again, choose weave cell $x = c_4$ and $\Phi(x) \in \mathcal{Q}(A)$ as well as in $\mathcal{Q}(C)$. There are no other cells in A and B with matching descriptions. Hence, $A \underset{\Phi}{\cap} C = \{a_2, c_4\}$.

The descriptive proximity relation δ_Φ is defined by

$$\delta_\Phi = \left\{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \text{cl}A \underset{\Phi}{\cap} \text{cl}B \neq \emptyset\right\}.$$

Descriptively near sets can be spatially disjoint sets.

Example 4 Descriptively Near Disjoint Sets
Again, choose Φ to be a set of probe functions representing weave cell colours. Let the set of cells X in Fig. 2 be endowed with the descriptive proximity relation δ_Φ. Again, observe that sets $A, C \in \mathcal{P}(X)$ are disjoint, *i.e.*, $A \cap C = \emptyset$. Sets A and C contain cells with matching colours, namely, cells $a_2 \in A$ and $c_4 \in C$. Then $\text{cl}A \underset{\Phi}{\cap} \text{cl}B \neq \emptyset$. Hence, $A\ \delta_\Phi\ C$.

In classifying subsets of point samples (briefly, *points*) in a digital image or in a drawing, it is helpful to choose a set of probe functions that make it possible to compare shapes. In either a digital image or in a drawing, a *point sample* is a number representing intensity of light [*e.g.*, 0 = lowest intensity (black) and 255 = highest intensity (white)]. For example, let $\phi \in \Phi$ be a probe function that represents the gradient direction of an edge in a drawing.

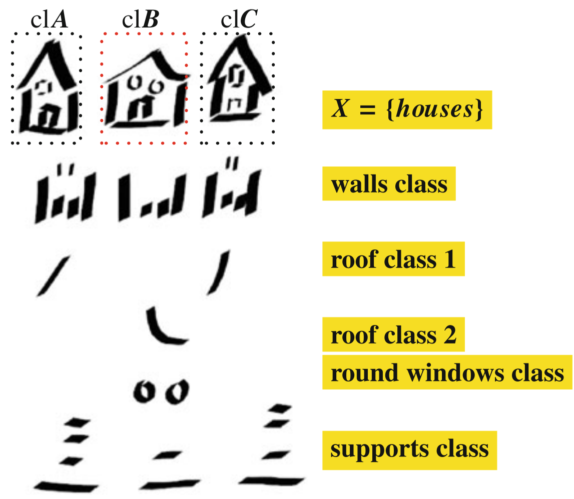


Fig. 3 House Classes

Example 5 Shape Near Sets
Let X be a finite set of point samples for the drawing of house fronts[1] in Fig. 3 and let Φ be a set of probe functions that includes a gradient direction probe ϕ. Further, let cl$A$ be the set of point samples in the leftmost house and cl$C$ be the set of point samples in the rightmost house in Fig. 3. Since the points in the walls in cl$A$ and cl$C$ have almost the same gradient direction (*i.e.*, $\text{cl}A \underset{\Phi}{\cap} \text{cl}C \neq \emptyset$), then $A\ \delta_\Phi\ C$. In addition, for the same reason, $A\ \delta_\Phi\ B$. By collecting together in separate sets containing points with the same gradient direction, we obtain point equivalence classes like those shown in Fig. 3.

Whenever sets A and B have no points with matching (or almost near) descriptions, the sets are *descriptively far* from each other (denoted by $A\ \underline{\delta}_\Phi\ B$), where

$$\underline{\delta}_\Phi = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta_\Phi.$$

Theorem 1 (Descriptive Nearness Collections) *Let Φ be a set of probe functions representing features of points x in a nonempty set X. For a descriptive proximity space (X, δ_Φ) with $A \in \mathcal{P}(X)$ and nearness collection $\xi_\Phi(A)$, the set $A \in \xi_\Phi(A)$.*

Proof Symmetric with the proof of Lemma 1. □

A descriptive proximity space (X, δ_Φ) (set X is endowed with a descriptive nearness relation δ_Φ) is structured by δ_Φ. This means that by virtue of the nearness relation defined on X, one can find descriptive nearness collections $\xi_\Phi(A) \in \delta$ defined by

$$\xi_\Phi(A) = \{B \in X : A\ \delta_\Phi\ B\}.$$

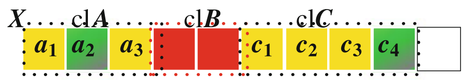
Example 6 Sample Descriptive Nearness Collection
Choose Φ to be a set of probe functions representing the slope of points in lines in a drawing such as the one in Fig. 3. In addition, let X be the set of points for the house fronts in Fig. 3 and let $A, B, C \in \mathcal{P}$. Then we obtain the nearness pattern

$$\xi_\Phi(A) = \{A, B, C\},$$

since the walls in each of these sets have almost the same slope.

[1] This drawing was made using Inkscape, a public domain, vector graphics system that makes it possible to make freehand drawings that can be saved as a LaTeX (.tex) file containing a pspicture environment for the drawing. Many thanks to Mario Liziér, Universidade Federal de São Carlos (UFSCar), for suggesting the use of Inkscape in this way.

Fig. 4 cl$A \underset{\Phi}{\cap}$ cl$B = \emptyset$ implies $A \, \underline{\delta}_\Phi \, B$



In effect, one can then identify descriptive nearness patterns in a proximity space, *i.e.*, points in one subset in a space that are in the descriptive intersection of one or more other subsets in the space and, hence, such subsets are *descriptively near* each other. In finding subsets $A, B \in \mathcal{P}(X)$ that are descriptively near, one considers descriptive intersection of the closure of A and the closure of B. That is, cl$A \underset{\Phi}{\cap}$ cl$B$ implies $A \; \delta_\Phi \; B$. To understand what it means to say that the cl$A$ is descriptively near the cl$B$, consider the idea of a descriptive boundary point.

A *descriptive boundary* point x of a set A is one that is descriptively near A as well as descriptively near its complement A^c. So x belongs to the descriptive intersection of cl$A$ and clA^c, which is called the descriptive boundary (frontier) of A, denoted by A^b_Φ. The descriptive boundary A^b_Φ belongs to the closure of A and B^b_Φ belongs to the closure of B. Recall that a *spatial boundary point* x of a set A is a point that is spatially near A as well as spatially near A^c. That is, x belongs to the intersection of cl$A$ and clA^c, which is called the *spatial boundary* (frontier) of A (denoted by A^b). This leads to the following result.

Theorem 2 (Descriptive Boundary Sets) *Let Φ be a set of probe functions representing features of points x in a non-empty set X, subsets $A, B \in \mathcal{P}(X)$ and let (X, δ), (X, δ_Φ) be spatial and descriptive proximity spaces, respectively. Then $A^b \subseteq B^b_\Phi$.*

Descriptively far sets can be spatially near sets.

Example 7 Descriptively Near Disjoint Sets
Choose Φ to be a set of probe functions representing weave cell colours. Let the set of cells X in Fig. 4 be endowed with the descriptive proximity relation δ_Φ. Observe that sets $A, B \in \mathcal{P}(X)$ are spatially near, *i.e.*, cl$A \cap$ cl$B = \emptyset$. However, sets A and B contain no cells with matching colours. Then cl$A \underset{\Phi}{\cap}$ cl$B = \emptyset$. Hence, $A \; \underline{\delta}_\Phi \; B$, *i.e.*, A is descriptively far from B.

4 Concluding Remarks

The basics of spatially near sets and descriptively near sets are briefly presented in this introduction to near sets. For a variety of applications of near sets, both spatial and descriptive, see [3,12,13] and in this special issue, see, *e.g.*, [23–25].

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