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NEAR SOLITON EVOLUTION FOR EQUIVARIANT SCHRÖDINGER MAPS IN TWO SPATIAL DIMENSIONS

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Abstract. We consider the Schrödinger Map equation in $2+1$ dimensions, with values into $\mathbb{S}^2$. This admits a lowest energy steady state Q, namely the stereographic projection, which extends to a two dimensional family of steady states by scaling and rotation. We prove that Q is unstable in the energy space $\dot{H}^1$. However, in the process of proving this we also show that within the equivariant class Q is stable in a stronger topology $X \subset \dot{H}^1$.

1. Introduction

In this article we consider the Schrödinger map equation in $\mathbb{R}^{2+1}$ with values into $\mathbb{S}^2$,

$$u_t = u \times \Delta u, \qquad u(0) = u_0 \tag{1.1}$$

This equation admits a conserved energy,

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx$$

and is invariant with respect to the dimensionless scaling

$$u(t,x) \to u(\lambda^2 t, \lambda x).$$

The energy is invariant with respect to the above scaling, therefore the Schrödinger map equation in $\mathbb{R}^{2+1}$ is *energy critical*.

Local solutions for regular large initial data have been constructed in [24] and [18]. Low regularity small data Schrödinger maps were studied in several works, see [1], [2], [3], [11], [12], [14], [15], [16], [19], [20], [21]. The definitive result for the small data problem was obtained by the authors and collaborators in [4]. There global well-posedness and scattering are proved for initial data which is small in the energy space $\dot{H}^1$.

However, such a result cannot hold for large data. In particular there exists a collection of families $\mathcal{Q}^m$ of finite energy stationary solutions, indexed by integers $m \geq 1$. To describe these families we begin with the maps Q^m defined in polar coordinates by

$$Q^m(r,\theta) = e^{m\theta R}\bar{Q}^m(r), \qquad \bar{Q}^m(r) = \begin{pmatrix} h_1^m(r) \\ 0 \\ h_3^m(r) \end{pmatrix}, \qquad m \in \mathbb{Z} \setminus \{0\}$$

with

$$h_1^m(r) = \frac{2r^m}{r^{2m}+1}, \qquad h_3^m(r) = \frac{r^{2m}-1}{r^{2m}+1}.$$

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Here R is the generator of horizontal rotations, which can be interpreted as a matrix or, equivalently, as the operator below

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Ru = \overrightarrow{k} \times u$$

The families $\mathcal{Q}^m$ are constructed from Q^m via the symmetries of the problem, namely scaling and isometries of the base space $\mathbb{R}^2$ and of the target space $\mathbb{S}^2$. Q^{-m} generates the same family $\mathcal{Q}^m$. The elements of $\mathcal{Q}^m$ are harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^2$, and admit a variational characterization as the unique energy minimizers, up to symmetries, among all maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ within their homotopy class.

In the above context, a natural question is to study Schrödinger maps for which the initial data is close in $\dot{H}^1$ to one of the $\mathcal{Q}^m$ families. One may try to think of this as a small data problem, but in some aspects it turns out to be closer to a large data problem. Studying this in full generality is very difficult. In this article we confine ourselves to a class of maps which have some extra symmetry properties, namely the *equivariant* Schrödinger maps. These are indexed by an integer m called the equivariance class, and consist of maps of the form

$$u(r,\theta) = e^{m\theta R}\bar{u}(r) \tag{1.2}$$

In particular the maps Q^m above are m-equivariant. The case $m = 0$ would correspond to spherical symmetry. Restricted to equivariant functions the energy has the form

$$E(u) = \pi \int_0^\infty \left(|\partial_r \bar{u}(r)|^2 + \frac{m^2}{r^2}(\bar{u}_1^2(r) + \bar{u}_2^2(r)) \right) rdr \tag{1.3}$$

Intersecting the full set $\mathcal{Q}^m$ with the m-equivariant class and with the homotopy class of Q^m we obtain the two parameter family $\mathcal{Q}_e^m$ generated from Q^m by rotations and scaling,

$$\mathcal{Q}_e^m = \{Q_{\alpha,\lambda}^m; \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \lambda \in \mathbb{R}^+\}, \qquad Q_{\alpha,\lambda}^m(r,\theta) = e^{\alpha R}Q^m(\lambda r, \theta)$$

Here $Q_{0,1}^m = Q^m$. Their energy depends on m as follows:

$$E(Q_{\alpha,\lambda}^m) = 4\pi m := E(\mathcal{Q}^m)$$

The study of equivariant Schrödinger maps for m-equivariant initial data close to $\mathcal{Q}_e^m$ was initiated by Gustafson, Kang, Tsai in [7], [8], and continued by Gustafson, Nakanishi, Tsai in [9]. The energy conservation suffices to confine solutions to a small neighborhood of $\mathcal{Q}_e^m$ due to the inequality (see [7])

$$\mathrm{dist}_{\dot{H}^1}(u, \mathcal{Q}_e^m)^2 = \inf_{\alpha,\lambda} \|Q_{\alpha,\lambda}^m - u\|_{\dot{H}^1}^2 \lesssim E(u) - E(\mathcal{Q}^m), \tag{1.4}$$

which holds for all m-equivariant maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ in the homotopy class of $\mathcal{Q}_e^m$ with $0 \leq E(u) - E(\mathcal{Q}^m) \ll 1$. One can interpret this as an orbital stability result for $\mathcal{Q}_e^m$. However, this does not say much about the global behavior of solutions since these soliton families are noncompact; thus one might have even finite time blow-up while staying close to a soliton family.

To track the evolution of an m-equivariant Schrödinger map $u(t)$ along $\mathcal{Q}_e^m$ we use functions $(\alpha(t), \lambda(t))$ describing trajectories in $\mathcal{Q}_e^m$. One may be tempted to try to choose them as minimizers for the infimum in (1.4), but this choice turns out not to be particularly helpful.

Instead, we will allow ourselves more freedom, and be content with any choice $(\alpha(t), \lambda(t))$ satisfying

$$\|u - Q^m_{\alpha(t),\lambda(t)}\|^2_{\dot H^1} \lesssim E(u) - E(\mathcal{Q}^m) \tag{1.5}$$

An important preliminary step in this analysis is the next result concerning both the local wellposedness in $\dot H^1$ and the persistence of higher regularity:

Theorem 1.1. *The equation* (1.1) *is locally well-posed in $\dot H^1$ for m-equivariant initial data u_0 in the homotopy class of $\mathcal{Q}^m_e$ satisfying*

$$E(u) - E(\mathcal{Q}^m) \ll 1$$

If, in addition, $u_0 \in \dot H^2$ then $u \in L^\infty_t \dot H^2$. Furthermore, the $\dot H^1 \cap \dot H^2$ regularity persists for as long as the function $\lambda(t)$ in (1.5) *remains in a compact set.*

This follows from Theorem 1.1 in [7] and Theorem 1.4 in [8]. Given the above result, the main problem remains to understand whether the steady states $Q^m_{\alpha,\lambda}$ are stable or not; in the latter case, one would like to understand the dynamics of the motion of the solutions move the soliton family. The case of large m was considered in prior work:

Theorem 1.2 ([8] for $m \geq 4$, [9] for $m = 3$)**.** *The solitons $Q^m_{\alpha,\lambda}$ are stable in the $\dot H^1$ topology within the m-equivariant class.*

In this article we begin the study of the more difficult case $m = 1$, and establish a very different type of behavior. The soliton Q^1 plays a central role in our analysis, which is why we introduce the notation $Q := Q^1$. Since equivariant functions are easily reduced to their one-dimensional companion via (1.2), we introduce the one dimensional equivariant version of $\dot H^1$,

$$\|f\|^2_{\dot H^1_e} = \|\partial_r f\|^2_{L^2(rdr)} + \|r^{-1} f\|^2_{L^2(rdr)}, \quad \|f\|^2_{H^1_e} = \|f\|^2_{\dot H^1_e} + \|f\|^2_{L^2(rdr)} \tag{1.6}$$

This is natural since for functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $u(r,\theta) = e^{\theta R}\bar u(r)$ we have

$$\|u\|_{\dot H^1} = (2\pi)^{\frac12} \|\bar u\|_{\dot H^1_e}, \qquad \|u\|_{H^1} = (2\pi)^{\frac12}\|\bar u\|_{H^1_e}.$$

For our main result we introduce a slightly stronger topology X with the property that

$$H^1_e \subset X \subset \dot H^1_e. \tag{1.7}$$

This is defined in Section 4 in terms of the spectral resolution of the linearized evolution around the soliton. In a nutshell, the X norm penalizes the behavior near frequency zero. Our first result below asserts that the soliton Q is stable in the X topology (which applies to $\bar Q$).

Theorem 1.3. *Let $m = 1$ and $\gamma \ll 1$. Then for each 1-equivariant initial data u_0 satisfying*

$$\|\bar u_0 - \bar Q\|_X \leq \gamma \tag{1.8}$$

there exists a unique global solution u so that $\bar u - \bar Q \in C(\mathbb{R}; X)$ and

$$\|\bar u - \bar Q\|_{C(\mathbb{R};X)} \lesssim \gamma \tag{1.9}$$

Furthermore, this solution has a Lipschitz dependence on the initial data in X, uniformly on compact time intervals.

We also refer the reader to Theorem 10.1 for a more complete form of this theorem. The above result holds true if $\bar{Q}$ is replaced by $\bar{Q}^1_{\alpha,\lambda}$, which implies that the solitons $Q^1_{\alpha,\lambda}$ are stable in the X topology. However, our second result asserts that the solitons $Q^1_{\alpha,\lambda}$ are unstable in the $\dot{H}^1$ topology:

Theorem 1.4. *For each $\epsilon, \gamma \ll 1$ and (α, λ) so that*

$$|\alpha| + |\lambda - 1| \approx \gamma \tag{1.10}$$

there exists a solution u as in Theorem 1.3 with the additional property that

$$\|u(0) - Q^1_{\alpha,\lambda}\|_{\dot{H}^1} \lesssim \epsilon\gamma \tag{1.11}$$

while (recall that $Q = Q^1_{0,1}$)

$$\lim \sup_{t \to \pm\infty} \|u - Q\|_{\dot{H}^1} \lesssim |\log \epsilon|^{-1}\gamma \tag{1.12}$$

We remark that, in view of (1.7), the solutions u in Theorem 1.3 must satisfy

$$E(u) - E(\mathcal{Q}^1) \lesssim \gamma^2$$

and they can move at most $O(\gamma)$ along the soliton family in the sense of (1.5). For the result in Theorem 1.4 we consider a more restrictive class of solutions, for which

$$\|\bar{u} - \bar{Q}\|_X \approx \gamma$$

while staying closer to the soliton family, (see (1.4)),

$$E(u) - E(\mathcal{Q}^1) \approx \epsilon^2\gamma^2 \tag{1.13}$$

Thus by (1.5) the parameters $(\alpha(t), \lambda(t))$ are restricted to an $O(\epsilon\gamma)$ range for each t. On the other hand, (1.11) and (1.12) show that for the solution in Theorem 1.4 the parameters $(\alpha(t), \lambda(t))$ vary by about $O(\gamma)$ along the flow.

We also remark that if in addition the initial data u_0 is in $\dot{H}^2$ then by Theorem 1.1 the solution u remains in this space at all times. While we do not prove a uniform in time $\dot{H}^2$ bound, such an estimate seems nevertheless likely to hold for solutions as in Theorem 1.3.

To better frame the context of this paper, one should compare the above results with results for the corresponding problem for the Wave-Maps equation in $\mathbb{R}^{2+1}$ with values into $\mathbb{S}^2$. The equivariant families of steady states $\mathcal{Q}^m_e$ are the same there, and they are also orbitally stable. However, in the case of Wave Maps all these steady states are unstable, and blow-up may occur in finite time for all m. We refer the reader to the results in [10], [23], and [22]. Of special relevance to the present paper are some of the spectral techniques developed in [10]; we further develop that circle of ideas in the present paper.

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1.1. **Definitions and notations.** We conclude this section with few definitions and notations. However, the reader should be aware that many objects are defined as the paper progresses; see Section 3 for all gauge elements and their equations, Section 4 for the Fourier analysis and related objects/spaces and Sections 5-6 for the functions spaces used in the analysis of the nonlinear problem.

While at fixed time our maps into the sphere are functions defined on $\mathbb{R}^2$, the equivariance condition allows us to reduce our analysis to functions of a single variable $|x| = r \in [0, \infty)$. One such instance is exhibited in (1.2) where to each equivariant map u we naturally associate its radial component $\bar{u}$. Some other functions will turn out to be radial by definition, see, for instance, all the gauge elements in Section 3. We agree to identify such radial functions with the corresponding one dimensional functions of r. Some of these functions are complex valued, and this convention allows us to use the bar notation with the standard meaning, i.e. the complex conjugate.

Even though we work mainly with functions of a single spatial variable r, they originate in two dimensions. Therefore, it is natural to make the convention that for the one dimensional functions all the Lebesgue integral and spaces are with respect to the rdr measure, unless otherwise specified.

For the Sobolev spaces we have introduced $\dot{H}^1_e$ and H^1_e in (1.6) as the natural substitute for $\dot{H}^1$ and H^1. In a similar fashion we define $\dot{H}^2_e$ and H^2_e by the norms

$$\|f\|^2_{\dot{H}^2_e} = \|\partial_r^2 f\|^2_{L^2} + \|r^{-1}\partial_r f\|^2_{L^2} + \|r^{-2} f\|^2_{L^2}, \qquad \|f\|^2_{\dot{H}^2_e} = \|f\|^2_{\dot{H}^2_e} + \|f\|^2_{L^2}$$

as the as the natural substitute for $\dot{H}^2$ and H^2.

For a real number a we define $a^+ = \max\{0, a\}$ and $a^- = \min\{0, a\}$.

We will use a dyadic partition of $\mathbb{R}^2$ (or $[0, \infty)$ after the dimensional reduction) into sets $\{A_m\}_{m \in \mathbb{Z}}$ given by

$$A_m = \{2^{m-1} < r < 2^{m+1}\}.$$

We will also use the notation $A_{<k} = \cup_{m<k} A_m$ as well as $A_{>k}$, $A_{\geq k}$ which are similarly defined.

Two operators which are often used on radial functions are $[\partial_r]^{-1}$ and $[r\partial_r]^{-1}$ defined as

$$[\partial_r]^{-1} f(r) = -\int_r^\infty f(s) ds, \qquad [r\partial_r]^{-1} f(r) = -\int_r^\infty \frac{1}{s} f(s) ds$$

A direct argument shows that

$$\|[r\partial_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \qquad 1 \leq p < \infty \tag{1.14}$$

We also have a weighted version

$$\|w[r\partial_r]^{-1} f\|_{L^p} \lesssim_p \|wf\|_{L^p}, \qquad 1 \leq p < \infty \tag{1.15}$$

assuming that $g(r) = w(r) r^{\frac{2}{p}}$ is an increasing function satisfying

$$g(r) \leq (1 - \epsilon) g(2r)$$

for some $\epsilon > 0$. The proof is straightforward.

2. An outline of the paper

Due to the complexity of the paper, an overview of the ideas and the organization of the paper is necessary before an in-depth reading.

2.1. **The frame method and the Coulomb gauge.** At first sight the Schrödinger Map equation has little to do with the Schrödinger equation. A good way to bring in the Schrödinger structure is by using the frame method. Precisely, at each point $(x,t) \in \mathbb{R}^{2+1}$ one introduces an orthonormal frame (v,w) in $T_{u(x,t)}\mathbb{S}^2$. This frame is used to measure the derivatives of u, and reexpress them as the complex valued radial *differentiated fields*

$$\psi_1 = \partial_r u \cdot v + i\partial_r u \cdot w, \qquad \psi_2 = \partial_\theta u \cdot v + i\partial_\theta u \cdot w.$$

Here the use of polar coordinates is motivated by the equivariance condition. Thus instead of working with the equation for u, one writes the evolution equations for the differentiated fields. The frame (v,w) does not appear directly there, but only via the real valued radial connection coefficients

$$A_1 = \partial_r v \cdot w, \qquad A_2 = \partial_\theta v \cdot w, \qquad A_0 = \partial_t v \cdot w.$$

A-priori the frame is not uniquely determined. To fix it one first asks that the frame be equivariant, and then that it satisfies an appropriate condition. Here it is convenient to use the *Coulomb gauge*; due to the equivariance this takes a very simple form, $A_1 = 0$. The construction of the Coulomb gauge is the first goal in the next section. In Proposition 3.2 we prove that for $\dot{H}^1$ equivariant maps into $\mathbb{S}^2$ close to Q there exists an unique Coulomb frame (v,w) which satisfies appropriate boundary conditions at infinity, see (3.17). In addition, this frame has a C^1 dependence on the map u.

In the Coulomb gauge the other spatial connection coefficient A_2, while nonzero, has a very simple form $A_2 = u_3$. We will also compute A_0 in terms of ψ_1, ψ_2 and A_2,

$$A_0 = -\frac{1}{2}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) + [r\partial_r^{-1}]\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) \tag{2.1}$$

2.2. **The reduced field ψ.** Due to the equivariance the two fields ψ_1 and ψ_2 are not independent. Hence it is convenient to work with a single field

$$\psi = \psi_1 - ir^{-1}\psi_2$$

which we will call *the reduced field*. The relevance of the variable ψ comes from the following reinterpretation. If $\mathcal{W}$ is defined as the vector

$$\mathcal{W} = \partial_r u - \frac{m}{r} u \times Ru \in T_u(\mathbb{S}^2)$$

then ψ is the representation of W with respect to the frame (v,w). On the other hand, a direct computation, see for instance [8], leads to

$$E(u) = \pi \int_0^\infty \left(|\partial_r \bar{u}|^2 + \frac{m^2}{r^2}|\bar{u} \times R\bar{u}|^2\right) rdr = \pi \|\bar{\mathcal{W}}\|^2_{L^2(rdr)} + 4\pi m$$

where we recall that $u(r,\theta) = e^{m\theta R}\bar{u}(r)$. Therefore $\psi = 0$ is a complete characterization of u being a harmonic map. Moreover the mass of ψ is directly related to the energy of u via

$$\|\psi\|^2_{L^2} = \|\bar{\mathcal{W}}\|^2_{L^2} = \frac{E(u) - 4\pi m}{\pi}. \tag{2.2}$$

A second goal of the next section is to derive an equation for the time evolution of ψ. This is governed by a cubic NLS type equation,

$$(i\partial_t + \Delta - \frac{2}{r^2})\psi = \left(A_0 - 2\frac{A_2}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi})\right)\psi. \tag{2.3}$$

In addition, we show that ψ is connected back to (ψ_2, A_2) via the ODE system

$$\partial_r A_2 = \Im(\psi\bar{\psi}_2) + \frac{1}{r}|\psi_2|^2, \qquad \partial_r\psi_2 = iA_2\psi - \frac{1}{r}A_2\psi_2 \tag{2.4}$$

with the conservation law $A_2^2+|\psi_2|^2 = 1$. However, this does not uniquely determine (ψ_2, A_2) and, by extension, the Schrödinger map u as we are missing a suitable boundary condition.

2.3. **Linearizations and the operators H, $\tilde{H}$.** This is the point in our work where we specialize in the case $m = 1$ and, for convenience, drop the upper-script m from all elements involved, i.e. use h_1, h_3 instead of h_1^1, h_3^1, etc.

A key role in our analysis is played by the linearization of the Schrödinger Map equation around the soliton Q. A solution to the linearized flow is a function

$$u_{lin} : \mathbb{R}^{2+1} \to T_Q S^2.$$

The Coulomb frame associated to Q has the form

$$v_Q(\theta, r) = e^{\theta R}\bar{v}_Q(r), \qquad w_Q(\theta, r) = e^{\theta R}\bar{w}_Q(r)$$

with

$$\bar{v}_Q(r) = \begin{pmatrix} h_3(r) \\ 0 \\ -h_1(r) \end{pmatrix}, \qquad \bar{w}_Q(r) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Expressing u_{lin} in this frame,

$$\phi_{lin} = \langle u_{lin}, v_Q\rangle + i\langle u_{lin}, w_Q\rangle$$

one obtains the Schrödinger type equation

$$(i\partial_t - H)\phi_{lin} = 0 \tag{2.5}$$

where the operator H acting on radial functions has the form

$$H = -\Delta + V, \qquad V(r) = \frac{1}{r^2} - \frac{8}{(1+r^2)^2}.$$

On the other hand linearizing the equation (2.3) around the soliton Q, we obtain a linear Schrödinger equation of the form

$$(i\partial_t - \tilde{H})\psi_{lin} = 0 \tag{2.6}$$

where the operator $\tilde{H}$ acting on radial functions has the form

$$\tilde{H} = -\Delta + \tilde{V}, \qquad \tilde{V}(r) = \frac{2}{r^2}(1 - h_3(r)) = \frac{4}{r^2(r^2+1)}.$$

The operators H and $\tilde{H}$ are conjugate operators and admit the factorizations

$$H = L^*L, \qquad \tilde{H} = LL^*$$

where

$$L = h_1\partial_r h_1^{-1} = \partial_r + \frac{h_3}{r}, \qquad L^* = -h_1^{-1}\partial_r h_1 - \frac{1}{r} = -\partial_r + \frac{h_3 - 1}{r}.$$

The linearized variables ϕ_{lin} and ψ_{lin} are also conjugated variables,

$$\psi_{lin} = L\phi_{lin}. \tag{2.7}$$

The operator H is nonnegative and bounded from $\dot{H}^1$ to $\dot{H}^{-1}$, but it is not positive definite; it has a zero resonance ϕ_0, solving $L\phi_0 = 0$, namely

$$\phi_0(r) = \frac{2r}{1+r^2}.$$

This corresponds to the solution ϕ_{lin} for (2.5) obtained by differentiating the soliton family with respect to either parameter. A consequence of this is that the linear Schrödinger evolution (2.5) does not have good dispersive properties, a fact which is at the heart of our instability result.

The above heuristic linearization argument works for all m, with the proper adjustments. We remark that if $m \geq 2$ then the zero resonance is replaced by a zero eigenvalue. If $m \geq 3$ then this eigenvalue belongs to $\dot{H}^{-1}$, which allows for a clean splitting of the $\dot{H}^1$ space into an eigenvalue mode, which is stationary, and an orthogonal component, which has good dispersive properties. This leads to the stability results in [8], [9]. If $m = 2$ we expect results which are closer to the $m = 1$ case; this will be considered in subsequent work.

If Q is replaced by $Q_{\alpha,\lambda}$ then H and $\tilde{H}$ are replaced by their rescaled versions H_λ and $\tilde{H}_\lambda$ where V and $\tilde{V}$ are replaced by

$$V_\lambda = \lambda^2 V(\lambda r), \qquad \tilde{V}_\lambda = \lambda^2 \tilde{V}(\lambda r).$$

A first goal of Section 4 is to describe the spectral theory for the linear operators H and $\tilde{H}$. The analysis in the case of H has already been done in [10], and it is easily obtained via the L conjugation in the case of $\tilde{H}$. The normalized generalized eigenfunctions for H and $\tilde{H}$ are denoted by ϕ_ξ, respectively ψ_ξ, and satisfy

$$H\phi_\xi = \xi^2\phi_\xi, \qquad \tilde{H}\psi_\xi = \xi^2\psi_\xi, \qquad L\phi_\xi = \xi\psi_\xi.$$

Correspondingly we have a generalized Fourier transform $\mathcal{F}_H$ associated to H and a generalized Fourier transform $\mathcal{F}_{\tilde{H}}$ associated to $\tilde{H}$.

This quickly leads to generalized eigenfunctions for the rescaled operators H_λ and $\tilde{H}_\lambda$. A considerable effort is devoted to the study of the transition from one $\tilde{H}_\lambda$ frame to another. This is closely related to the transference operator introduced in [10].

One reason we prefer to work with the ψ variable is that the operator $\tilde{H}$ has a good spectral behavior at zero, therefore we have favorable decay estimates for the corresponding linear Schrödinger evolution (2.6).

2.4. **The X and LX spaces.** As mentioned before, a stumbling block in formulating a closed evolution equation for ψ is the need for some boundary condition in order to insure uniqueness for the system (2.4). This leads us to introduce a stronger topology $X \subset \dot{H}^1_e$ for $\bar{u} - \bar{Q}$, and therefore also for $\psi_2 - ih_1$ and $A_2 - h_3$. Then the relation (2.7) shows that studying the Schrödinger map equation in the space X corresponds to studying the ψ equation (2.10) in the space LX obtained by applying the operator L to functions in X.

Roughly speaking the space X is maximal with the following properties:

(a) We have the embedding $X \subset \dot{H}^1_e$.
(b) The X norm of u depends only on the the L^2 norms of the dyadic pieces of $\mathcal{F}_H u$.
(c) The operator L is surjective on X.

Part (b) quickly implies a similar property for LX relative to the $\tilde{H}$ Fourier transform $\mathcal{F}_{\tilde{H}}$. It also shows that the linear equations (2.5), respectively (2.6) are well-posed in X, respectively LX.

One of the goals of Section 4 is to define the X and LX spaces. In particular we establish the embedding (1.7) for X, as well as a two sided embedding for LX, namely

$$L^1 \cap L^2 \subset LX \subset L^2. \tag{2.8}$$

We also establish some other simple properties of these spaces.

A key gain due to the fact that we work in the smaller space X is that we can supplement the system (2.4) with a boundary condition at infinity, namely

$$\lim_{r\to\infty} A_2 = 1, \qquad \psi_2 - ih_1 \in X. \tag{2.9}$$

This condition is preserved dynamically along the Schrödinger map flow. Together with (2.3), (2.4) and (2.1) it fully describes the dynamics of ψ. Most of the work in this article is devoted to the study of the evolution of ψ.

2.5. **The elliptic transition between u and its reduced field ψ.** Section 7 is devoted to the study of the elliptic gauge correspondence at fixed time between the map u and its associated reduced field ψ. The main result there asserts that this map is a local C^1 diffeomorphism from a neighborhood of the soliton $\bar{Q}$ in X to a neighborhood of 0 in LX. As an intermediate step we prove that the system (2.4) with the boundary condition (2.9) yields a C^1 map from ψ near 0 in LX to (ψ_2, A_2) near (h_1, h_3) in X.

2.6. **The nonlinear Schrödinger equation for ψ: Take 1 [local].** The equation (2.3) can be rewritten in the form

$$(i\partial_t - \tilde{H})\psi = W\psi, \qquad W = A_0 - 2\frac{A_2 - h_3}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}). \tag{2.10}$$

Ideally, one would hope to be able to solve this equation in the space LX by treating the right hand side perturbatively. This is acceptable for short time, and it provides us with a quick local theory.

The first step toward this goal is achieved in Section 5 we consider the linear Schrödinger evolution (2.6) and prove Strichartz and local energy estimates. Based on these bounds we introduce function spaces $l^2S^\sharp \subset L^\infty L^2$, respectively $l^2N^\sharp \supset L^1L^2$ for L^2 solutions, respectively for the inhomogeneous term in the $\tilde{H}$ Schrödinger equation. Corresponding to LX data we define similar weighted norms $WS^\sharp \subset L^\infty LX$, respectively $WN^\sharp \supset L^1LX$.

In the beginning of Section 8 we use these spaces and a short fixed point argument to prove small data local well-posedness for the equation (2.10) in LX. Unfortunately, such an argument no longer works globally in time; this is due to the failure of the local decay estimates for $A_2 - h_3$. While local decay estimates are valid for ψ, they do not transfer to $\psi_2 - ih_1$ and $A_2 - h_3$ via the ODE (2.4)-(2.9).

2.7. **The functions $(\alpha(t), \lambda(t))$.** A primary goal of this article is to track the drift of Schrödinger maps along the soliton family. For this we need appropriate functions $(\alpha(t), \lambda(t))$ so that (1.5) holds. The role of $(\alpha(t), \lambda(t))$ is roughly to describe the low frequency oscillations of the Schrödinger map u along the family of rescaled solitons.

In the case $m \geq 3$ the parameter λ is defined dynamically via an orthogonality condition with respect to the H eigenvalue ϕ_0, appropriately rescaled (see [8]). Such a strategy cannot work for $m = 1, 2$ as in this case $\phi_0 \notin \dot{H}^{-1}$. Another alternative would be to choose α and λ as the minimizers in the left hand side in (1.4). However the above minimizer plays no other role, and in fact it turns out that choosing it as the "closest" harmonic map to $u(t)$ may not be the best choice for other analytical considerations, see [8] or [9].

In the context of this paper, it is technically convenient to make a choice for $(\alpha(t), \lambda(t))$ which is expressed in terms of (ψ_2, A_2) instead of u. Precisely, we make a dynamic assignation of $(\alpha(t), \lambda(t))$ via the relation

$$A_2(1,t) = h_3(\lambda(t)), \qquad \psi_2(1,t) = ie^{i\alpha(t)}h_1(\lambda(t)) \tag{2.11}$$

which for a soliton simply recovers the soliton parameters. The (small) price to pay is that we need to prove that (1.5) holds; we do this right away in the next section. The choice of $r = 1$ above is arbitrary; different choices of r lead to closely related functions λ.

2.8. **The nonlinear Schrödinger equation for ψ: Take 2 [global].** With $\lambda(t)$ defined as in (2.11), the equation (2.3) can also be rewritten in the form

$$(i\partial_t - \tilde{H}_\lambda)\psi = W_\lambda \psi, \qquad W_\lambda = A_0 - 2\frac{A_2 - h_3^\lambda}{r^2} - \frac{1}{r}\Im(\psi_2 \bar{\psi}). \tag{2.12}$$

The advantage is that, for λ defined as in (2.11), the ODE (2.4)-(2.9) allows us to transfer local energy decay estimates from ψ to $A_2 - h_3^\lambda$ as the latter vanishes now at $r = 1$ instead of infinity. This is achieved in Proposition 7.4.

The price to pay is that we now need to understand the linear evolution

$$(i\partial_t - \tilde{H}_\lambda)\psi = f \tag{2.13}$$

in the space LX, with λ depending on time. We expect λ to stay bounded, but this is far from being enough. Instead we introduce a smaller space $Z_0 \subset L^\infty$ for $\lambda - 1$, defined by

$$Z_0 = \dot{H}^{\frac{1}{2},1} + W^{1,1} + (L^2 \cap L^\infty).$$

Here the last component characterizes the high frequencies (which have good averaged decay), while the first two apply primarily for the low frequencies (and have little decay at infinity).

This is achieved in Section 6, where we consider the global in time linear Schrödinger evolution (2.13) under the assumption that $\lambda - 1$ is small in the space Z_0. We construct function spaces $WS^\sharp[\tilde{\lambda}] \subset L^\infty(LX)$, respectively $WS^\sharp[\tilde{\lambda}] \supset L^1LX$, incorporating also appropriate dispersive information, so that the following linear bound holds for solutions to (2.13):

$$\|\psi\|_{WS^\sharp[\tilde{\lambda}]} \lesssim \|\psi(0)\|_{LX} + \|f\|_{WN^\sharp[\tilde{\lambda}]}. \tag{2.14}$$

Here $\tilde{\lambda}$ is a (somewhat arbitrary) regularization of λ which essentially contains the low frequencies of λ.

Section 8 contains our global in time analysis of the nonlinear equation for ψ. Precisely, we establish a bootstrap estimate for the $WS^\sharp[\tilde{\lambda}]$ size of ψ. This is obtained by combining the linear bound (2.14) with an estimate for the nonlinearity, which has the form

$$\|W_\lambda \psi\|_{WN^\sharp[\tilde{\lambda}]} \lesssim \|\psi\|^2_{WS^\sharp[\tilde{\lambda}]}.$$

We remark that while we are able to prove a bootstrap estimate for solutions ψ to (2.12), we cannot obtain estimates for the difference of two solutions. Hence we can no longer treat the nonlinearity perturbatively globally in time.

In Section 9 we complement the above bootstrap estimate for ψ with a bootstrap estimate for $\|\lambda - 1\|_{Z_0}$. More precisely we show that we recover the regularity of the parameter $\lambda(t)$ from the $WS^\sharp[\tilde{\lambda}]$ regularity of ψ.

Finally, in Section 10 we prove our main stability result in the X topology in Theorem 1.3. This is done via a bootstrap argument, which uses the bootstrap estimates on ψ in $WS^\sharp[\tilde{\lambda}]$, respectively for $\lambda - 1$ in Z_0, from the previous two sections. In addition, we use the results in Section 7 for the transition back and forth between the Schrödinger map u and its reduced field ψ.

2.9. **The instability result.** In the final section of the paper we prove the $\dot{H}^1$ instability result in Theorem 1.4. For this we introduce a second small parameter ϵ and look at maps u for which the reduced field ψ satisfies

$$\|\psi(t)\|_{LX} \approx \gamma, \qquad \|\psi(t)\|_{L^2} \approx \epsilon\gamma$$

The L^2 smallness allows for a better control of the nonlinear effects, and we are able to show that the ψ flow is almost linear,

$$\|\psi(t) - e^{it\tilde{H}}\psi(0)\|_{LX} \lesssim |\log \epsilon|^{-1}\gamma$$

Taking this into account, for each (α, λ) as in (1.10) our strategy is to choose an initial data u_0 which coincides with $Q^1_{\alpha,\lambda}$ for $r < \epsilon^{-1}$ and with $Q = Q^1_{0,1}$ for larger r, with a smooth transition in between. Then we are able to accurately track the Fourier transform $\mathcal{F}_{\tilde{H}}\psi$ of ψ for large t. The decay of the map u to an $O_{\dot{H}^1}(|\log \epsilon|^{-1}\gamma)$ neighborhood of Q is equivalent to the decay of $(\alpha(t), \lambda(t))$ to an an $O(|\log \epsilon|^{-1}\gamma)$ neighborhood of $(0, 1)$, which in turn is a consequence of cancellations due to the oscillations in frequency for $e^{it\tilde{H}}\psi(0)$ as t grows large.

3. The Coulomb gauge representation of the equation

In this section we rewrite the Schrödinger map equation for equivariant solutions in a gauge form. This approach originates in the work of Chang, Shatah, Uhlenbeck [6]. However, our analysis is closer to the one in [3].

3.1. **Near soliton maps.** We first investigate some simple properties of maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ which are near a soliton $Q^m_{\alpha,\lambda}$ in the sense that

$$\|u - Q^m_{\alpha,\lambda}\|_{\dot{H}^1} \leq \gamma \ll 1. \tag{3.1}$$

Lemma 3.1. *Let $m \geq 1$ and $u : \mathbb{R}^2 \to \mathbb{S}^2$ be an m-equivariant map which satisfies* (3.1)*. Then*

$$\lim_{r\to 0} u(r, \theta) = -\vec{k}, \qquad \lim_{r\to\infty} u(r, \theta) = \vec{k} \tag{3.2}$$

and

$$\|r^{-1}(u - Q^m_{\alpha,\lambda})\|_{L^2} + \|u - Q^m_{\alpha,\lambda}\|_{L^\infty} \lesssim \gamma. \tag{3.3}$$

Proof. After a rescaling and a rotation the problem reduces to the case $\alpha = 0$ and $\lambda = 1$. We rewrite the $\dot{H}^1$ bound for $u - Q^m$ as in (1.3):

$$\|\partial_r(\bar{u} - \bar{Q}^m)\|_{L^2} + \|r^{-1}(\bar{u}_1 - h_1^m)\|_{L^2} + \|r^{-1}\bar{u}_2\|_{L^2} \lesssim \gamma \tag{3.4}$$

In particular for u_2 we have

$$\|\partial_r \bar{u}_2\|_{L^2} + \|r^{-1}\bar{u}_2\|_{L^2} \lesssim \gamma \tag{3.5}$$

By Sobolev type embeddings

$$\|f\|_{L^\infty} \lesssim \|\partial_r f\|_{L^2} + \|r^{-1} f\|_{L^2} \tag{3.6}$$

this implies that

$$\|\bar{u}_2\|_{L^\infty} \lesssim \gamma$$

Furthermore,

$$\frac{d}{dr}|\bar{u}_2|^2 = 2\bar{u}_2 \partial_r \bar{u}_2 \in L^1(dr)$$

therefore $|\bar{u}_2|^2$ is continuous and has limits as $r \to 0, \infty$. In addition, these limits must be zero in order for the second left hand side norm in (3.5) to be finite. The same argument applies for $\bar{u}_1 - h_1^m$. Thus we have proved that

$$\|\bar{u}_2\|_{L^\infty} + \|\bar{u}_1 - h_1^m\|_{L^\infty} \lesssim \gamma, \qquad \lim_{r \to 0,\infty} \bar{u}_1, \bar{u}_2 = 0 \tag{3.7}$$

To conclude the proof of (3.2) and (3.3) it remains to consider the vertical component $\bar{u}_3(r)$. Integrating the bound

$$\|\partial_r(\bar{u}_3 - h_3^m)\|_{L^2} \lesssim \gamma$$

we obtain (as $h_3^m(1) = 0$)

$$|(\bar{u}_3(r) - h_3^m(r)) - u_3(1)| \lesssim \gamma |\log r|^{\frac{1}{2}}$$

The first part of (3.7) shows that $|\bar{u}_3(1)|^2 \lesssim \gamma$. Since γ is small, it follows that $\bar{u}_3(r)$ is negative for say $r \in [\frac{1}{4}, \frac{1}{2}]$. Since $\bar{u}_3$ is continuous and, by (3.7), cannot vanish for smaller r, it follows that it stays negative for all $r \in (0, \frac{1}{2}]$. Thus for $r < 1/2$ we have

$$\bar{u}_3(r) = -\sqrt{1 - |\bar{u}_1|^2 - |\bar{u}_2|^2}$$

which by (3.7) implies that

$$|\bar{u}_3(r) - h_3^m(r)| \lesssim \gamma, \quad r < \frac{1}{2}, \qquad \lim_{r \to 0} \bar{u}_3(r) = -1.$$

The same argument applies for $r > 2$, where $\bar{u}_3$ is positive. Integrating its r derivative from either side we recover the pointwise bound for $\bar{u}_3 - h_3^m$ for $r \in [\frac{1}{2}, 2]$ and obtain

$$\|\bar{u}_3 - h_3^m\|_{L^\infty} \lesssim \gamma$$

Finally, we consider the L^2 bound for $\bar{u}_3 - h_3^m$. Due to the pointwise bound, it suffices to consider r close to 0 and to infinity. In either case we use the equation of the sphere to write

$$|\bar{u}_3 - h_3^m| \lesssim |\bar{u}_1 - h_1^m| + |\bar{u}_2|$$

and conclude by (3.4). ∎

3.2. **The Coulomb gauge.** We let the differentiation operators $\partial_0, \partial_1, \partial_2$ stand for $\partial_t, \partial_r, \partial_t$ respectively. Our strategy will be to replace the equation for the Schrödinger map u with equations for its derivatives $\partial_1 u$, $\partial_2 u$ expressed in an orthonormal frame $v, w \in T_u\mathbb{S}^2$. To fix the sign in the choice of w, we will assume that

$$u \times v = w$$

Since u is m-equivariant it is natural to work with m-equivariant frames, i.e.

$$v = e^{m\theta R}\bar{v}(r), \qquad w = e^{m\theta R}\bar{w}(r).$$

Given such a frame we introduce the differentiated fields ψ_k and the connection coefficients A_k by

$$\psi_k = \partial_k u \cdot v + i\partial_k u \cdot w, \qquad A_k = \partial_k v \cdot w. \tag{3.8}$$

Due to the equivariance of (u, v, w) it follows that both ψ_k and A_k are spherically symmetric (therefore subject to the conventions made in Section 1.1). Conversely, given ψ_k and A_k we can return to the frame (u, v, w) via the ODE system:

$$\begin{cases} \partial_k u = (\Re\psi_k)v + (\Im\psi_k)w \\ \partial_k v = -(\Re\psi_k)u + A_k w \\ \partial_k w = -(\Im\psi_k)u - A_k v \end{cases} \tag{3.9}$$

If we introduce the covariant differentiation

$$D_k = \partial_k + iA_k, \quad k \in \{0, 1, 2\}$$

it is a straightforward computation to check the compatibility conditions:

$$D_l\psi_k = D_k\psi_l, \quad l, k = 0, 1, 2. \tag{3.10}$$

The curvature of this connection is given by

$$D_lD_k - D_kD_l = \partial_l A_k - \partial_k A_l = \Im(\psi_l\bar{\psi_k}), \quad l, k = 0, 1, 2. \tag{3.11}$$

An important geometric feature is that ψ_2, A_2 are closely related to the original map. Precisely, for A_2 we have:

$$A_2 = m(k \times v) \cdot w = mk \cdot (v \times w) = mk \cdot u = mu_3 \tag{3.12}$$

and, in a similar manner,

$$\psi_2 = m(w_3 - iv_3) \tag{3.13}$$

Since the (u, v, w) frame is orthonormal, the following relations also follow:

$$|\psi_2|^2 = m(u_1^2 + u_2^2), \qquad |\psi_2|^2 + A_2^2 = m^2$$

Now we turn our attention to the choice of the $(\bar{v}, \bar{w})$ frame at $\theta = 0$. Here we have the freedom of an arbitrary rotation depending on t and r. In this article we will use the Coulomb gauge, which for general maps u has the form div $A = 0$. In polar coordinates this is written as $\partial_1 A_1 + r^{-2}\partial_2 A_2 = 0$. However, in the equivariant case A_2 is radial, so we are left with a simpler formulation $A_1 = 0$, or equivalently

$$\partial_r\bar{v} \cdot \bar{w} = 0 \tag{3.14}$$

which can be rearranged into a convenient ODE as follows

$$\partial_r\bar{v} = (\bar{v} \cdot \bar{u})\partial_r\bar{u} - (\bar{v} \cdot \partial_r\bar{u})\bar{u} \tag{3.15}$$

The first term on the right vanishes and could be omitted, but it is convenient to add it so that the above linear ODE is solved not only by v and w, but also by u. Then we can write an equation for the matrix $\mathcal{O} = (\bar{v}, \bar{w}, \bar{u})$:

$$\partial_r \mathcal{O} = M\mathcal{O}, \qquad M = \partial_r \bar{u} \wedge \bar{u} := \partial_r \bar{u} \otimes \bar{u} - \bar{u} \otimes \partial_r \bar{u} \tag{3.16}$$

with an antisymmetric matrix M.

An advantage of using the Coulomb gauge is that it makes the derivative terms in the nonlinearity disappear. Unfortunately, this only happens in the equivariant case, which is why in [4] we had to use a different gauge, namely the caloric gauge.

The ODE (3.15) needs to be initialized at some point. A change in the initialization leads to a multiplication of all of the ψ_k by a unit sized complex number. This is irrelevant at fixed time, but as the time varies we need to be careful and choose this initialization uniformly with respect to t, in order to avoid introducing a constant time dependent potential into the equations via A_0. Since in our results we start with data which converges asymptotically to Q as $r \to \infty$, and the solutions continue to have this property, it is natural to fix the choice of $\bar{v}$ and $\bar{w}$ at infinity,

$$\lim_{r\to\infty} \bar{v}(r,t) = \vec{i}, \qquad \lim_{r\to\infty} \bar{w}(r,t) = \vec{j} \tag{3.17}$$

Before considering the general case we begin with the solitons. The simplest case is when $u = Q^m$ when the triplet $(\bar{v}, \bar{w}, \bar{u})$ is given by

$$\left(\bar{V}^m, \bar{W}^m, \bar{Q}^m\right) = \begin{pmatrix} h_3^m(r) & 0 & h_1^m(r) \\ 0 & 1 & 0 \\ -h_1^m(r) & 0 & h_3^m(r) \end{pmatrix} \tag{3.18}$$

If $m = 1$ we drop the superscript m. More generally, if $u = Q^m_{\alpha,\lambda}$ then from the above, by rescaling and rotation, we obtain the corresponding triplet $\left(\bar{V}^m_{\alpha,\lambda}, \bar{W}^m_{\alpha,\lambda}, \bar{Q}^m_{\alpha,\lambda},\right)$ of the form

$$\begin{pmatrix} h_3^m(\lambda r)\cos^2 m\alpha + \sin^2 m\alpha & (h_3^m(\lambda r) - 1)\sin m\alpha \cos m\alpha & h_1^m(\lambda r)\cos m\alpha \\ (h_3^m(\lambda r) - 1)\sin m\alpha \cos m\alpha & h_3^m(\lambda r)\sin^2 m\alpha + \cos^2 m\alpha & h_3^m(\lambda r)\sin m\alpha \\ -h_1^m(\lambda r)\cos m\alpha & -h_1^m(\lambda r)\sin m\alpha & h_3^m(\lambda r) \end{pmatrix}$$

For later reference we also note the values of ψ_1, ψ_2 and A_2 in this case:

$$\begin{aligned} \psi^m_{\alpha,\lambda,1} = -mr^{-1}h_1^m(\lambda r)e^{im\alpha}, \quad \psi^m_{\alpha,\lambda,2} = imh_1^m(\lambda r)e^{im\alpha}, \\ A^m_{\alpha,\lambda,2} = mh_3^m(\lambda r). \end{aligned} \tag{3.19}$$

To measure the regularity of the frame (v, w) we use the Sobolev type space $\dot{H}^1_C$ of functions $f : \mathbb{R}^2 \to \mathbb{R}^3$, with norm

$$\|f\|_{\dot{H}^1_C} = \|\partial_r \bar{f}\|_{L^2} + \|\bar{f}\|_{L^\infty} + \|r^{-1}\bar{f}_3\|_{L^2}, \qquad f(r,\theta) = e^{m\theta R}\bar{f}(r)$$

The next Lemma shows that the initialization (3.17) is well-defined for arbitrary maps u close to the soliton family:

Proposition 3.2. *a) For each m-equivariant map $u : \mathbb{R}^2 \to \mathbb{S}^2$ satisfying* (3.1) *there exists an unique m-equivariant orthonormal frame (v, w) which satisfies the Coulomb gauge condition* (3.14) *and the boundary condition* (3.17). *This frame satisfies the bounds*

$$\|v - V^m_{\alpha,\lambda}\|_{\dot{H}^1_C} + \|w - W^m_{\alpha,\lambda}\|_{\dot{H}^1_C} \lesssim \gamma. \tag{3.20}$$

b) Furthermore, the maps $u \to v, w$ are C^1 from $\dot{H}^1$ into $\dot{H}^1_C$ as well as from $L^2 \to L^2$.

Proof. a) To construct the Coulomb gauge we use the equation (3.15). The right hand side is linear in $\bar{v}$ and has locally integrable coefficients, therefore by prescribing $\bar{v}$ at $r = 1$ we obtain a unique solution. Also, if the relations

$$|\bar{v}|^2 = 1, \qquad \bar{u} \cdot \bar{v} = 0 \tag{3.21}$$

are enforced at $r = 1$ then they are preserved along the flow. We claim that the limit of $\bar{v}(r)$ as $r \to \infty$ exists. For $\bar{v}_2$ and $\bar{v}_3$ this follows from

$$\|\partial_r \bar{v}_j\|_{L^1(dr)} \lesssim \|\partial_r \bar{u}\|_{L^2(rdr)} \|r^{-1}\bar{u}_j\|_{L^2(rdr)}, \qquad j = 1, 2.$$

On the other hand $\lim_{r\to\infty} \bar{v}_3 = 0$ by orthogonality due to the relation (3.2).

Once we have one solution $\bar{v}$ to (3.15), a second one is obtained by $\bar{w} = \bar{u} \times \bar{v}$. Since (3.15) is linear, it follows that all its solutions are obtained from the initial one by a rotation of a fixed angle in $T_{\bar{u}}\mathbb{S}^2$. This proves the existence and uniqueness of the desired frame (v, w) which satisfies the boundary condition (3.17).

We next prove the pointwise part of the bound (3.20). From (3.15) we obtain

$$\partial_r \bar{v}_2 = -\left(\bar{v} \cdot \partial_r(\bar{u} - \bar{Q}^m)\right)\bar{u}_2 - (\bar{v} \cdot \partial_r \bar{Q}^m)\bar{u}_2$$

Hence using (3.1) we estimate $\|\partial_r \bar{v}_2\|_{L^1(dr)} \lesssim \gamma$, which after integration shows that $|\bar{v}_2| \lesssim \gamma$. Since we also have $|\bar{u}_2| \lesssim \gamma$, it follows that $|\bar{w}_2 - 1| \lesssim \gamma^2$. This in turn shows that $|\bar{w}_1| + |\bar{w}_3| \lesssim \gamma$. Then the pointwise bounds for $\bar{v}_1$ and $\bar{v}_3$ are easily obtained since $v = -u \times w$.

Next we consider the L^2 bounds for $\partial_r \bar{v}$ and $\partial_r \bar{w}$. The easy case is that of $\bar{v}_2$ and $\bar{w}_2$, for which by (3.15) we have

$$\|\partial_r \bar{v}_2\|_{L^2(rdr)} + \|\partial_r \bar{w}_2\|_{L^2(rdr)} \lesssim \|\partial_r \bar{u}_2\|_{L^2(rdr)} \lesssim \gamma$$

For $\bar{v}_1$ we write

$$\partial_r(\bar{v}_1 - h_3^m) = (\bar{v} \cdot \partial_r(\bar{Q}^m - \bar{u}))\bar{u}_1 - (\bar{v} \cdot \partial_r \bar{Q}^m)(\bar{u}_1 - h_1^m) + ((\bar{V}^m - \bar{v}) \cdot \partial_r \bar{Q}^m)h_1^m$$

For the first term we use (3.15) while for the remaining terms we use the pointwise bounds in (3.3) and (3.20) for $\bar{u}_1 - h_1^m$, respectively $\bar{V}^m - \bar{v}$. The same argument applies for $\bar{v}_3$, $\bar{w}_1$ and $\bar{w}_3$.

Finally, we prove the L^2 bounds for $\bar{v}_3$ and $\bar{w}_3$. This is done in a roundabout way using the orthogonality of the $(\bar{u}, \bar{v}, \bar{w})$ frame. For $\bar{w}_3$ we have

$$|\bar{w}_3| = |\bar{u}_1\bar{v}_2 - \bar{u}_2\bar{v}_1| \lesssim |\bar{u}_1 - h_1^m| + h_1^m|\bar{v}_2| + |\bar{u}_2|$$

and we conclude using the L^2 bounds in (3.4) for $\bar{u}_1 - h_1^m$ and $\bar{u}_2$ as well as the pointwise bound for $\bar{v}_2$ in (3.20). Similarly, for $\bar{v}_3$ we have

$$|\bar{v}_3 + h_1^m| = |\bar{u}_1\bar{w}_2 - \bar{u}_2\bar{w}_1 + h_1^m| \lesssim |\bar{u}_1 - h_1^m| + h_1^m|\bar{w}_2 - 1| + |\bar{u}_2|$$

and we conclude as before. The proof of (3.20) is complete.

b) We now prove that the map $u \to v$ is C^1 from $\dot{H}^1$ to $\dot{H}^1_C$. Given an interval I we consider a one parameter family of maps $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$ which are smooth in all variables and agree with Q^m for large r. Then by ODE theory applied to (3.15) it follows that v and w are smooth in all variables away from $r = 0$. The main step is to establish the uniform bounds

$$\|\partial_t v\|_{\dot{H}^1_C} \lesssim \|\partial_t u\|_{\dot{H}^1} \tag{3.22}$$

Having this, the transition to more general maps $u \in C^1(I; \dot H^1)$ is done via a standard density argument, which is omitted. We remark that the $\dot H^1$ convergence for u and the $\dot H^1_C$ convergence for v, w suffice in order to insure that the ODE (3.15) and the boundary condition (3.17) survive in the limit.

To prove (3.22) we differentiate (3.15) with respect to t to obtain an ODE for the covariant time derivative of v,

$$z = \partial_t v + (v \cdot \partial_t u)u$$

We obtain

$$\partial_r \bar z = (\bar z \cdot \bar u)\partial_r \bar u - (\bar z \cdot \partial_r \bar u)\bar u + \bar f, \qquad z(\infty) = 0 \tag{3.23}$$

where

$$f = (\partial_r v \cdot \partial_t u)u + (v \cdot \partial_t u)\partial_r u - (v \cdot \partial_r u)\partial_t u$$

For $\partial_t u$ we use the following bound:

$$\|\partial_t \bar u\|_{L^\infty} + \|r^{-1}\partial_t \bar u\|_{L^2} \lesssim \|\partial_t u\|_{\dot H^1} \tag{3.24}$$

For $\partial_t \bar u_1$ and $\partial_t \bar u_2$ this follows directly due to the form (1.3) of the $\dot H^1$ norm for equivariant functions and to (3.6). On the other hand the bound for $\partial_t \bar u_3$ is obtained indirectly from the orthogonality relation $\partial_t u \cdot u = 0$ (see e.g. the similar argument for u_3 in Lemma 3.1).

From the L^2 bound in (3.24) we obtain

$$\|\bar f\|_{L^1(dr)} \lesssim \|\partial_t u\|_{\dot H^1}$$

therefore integrating (3.23) from infinity we have

$$\|z\|_{L^\infty} \lesssim \|\partial_t u\|_{\dot H^1}$$

Using the L^∞ bound in (3.24) yields

$$\|\bar f\|_{L^2(rdr)} \lesssim \|\partial_t u\|_{\dot H^1}$$

which directly leads to

$$\|\partial_r z\|_{L^2} \lesssim \|\partial_t u\|_{\dot H^1}$$

Further, the orthogonality relation $z \cdot u = 0$ and (3.3) show that

$$\|r^{-1}\bar z_3\|_{L^2(rdr)} \lesssim \|\bar z\|_{L^\infty}(1 + \|r^{-1}\bar u_1\|_{L^2(rdr)} + \|r^{-1}\bar u_2\|_{L^2(rdr)}) \lesssim \|\partial_t u\|_{\dot H^1}$$

Thus we have proved that

$$\|z\|_{\dot H^1_C} \lesssim \|\partial_t u\|_{\dot H^1}$$

Now it is easy to obtain (3.22), estimating the difference via (3.24).

Finally, we prove that the map $u \to v, w$ is C^1 from $L^2 \to L^2$. For this we need the following counterpart of (3.22):

$$\|\partial_t v\|_{L^2} \lesssim \|\partial_t u\|_{L^2} \tag{3.25}$$

Again it suffices to consider the smooth case, since the transition to more general maps $u \in C(I; \dot H^1) \cap C^1(I; L^2)$ is done via a standard density argument.

We begin with

$$\|\bar f\|_{L^1(rdr)} \lesssim \|\partial_t u\|_{L^2}$$

Then integrating (3.23) from infinity we obtain

$$|z(r)| \leq \int_0^\infty 1_{[0,s]}(r)|f(s)|ds$$

and by Minkowski's inequality

$$\|z\|_{L^2(rdr)} \lesssim \int_0^\infty s|f(s)|ds$$

The transition from z to $\partial_t v$ is immediate, therefore (3.25) is proved. □

As a direct consequence of part (a) of the above lemma we can describe the regularity and properties of the differentiated fields ψ_1, ψ_2 and the connection coefficient A_2 at fixed time:

Corollary 3.3. *Let $u : \mathbb{R}^2 \to \mathbb{S}^2$ be an m-equivariant map as in* (3.1)*. Then ψ_1, ψ_2 and A_2 satisfy* (3.10)*,* (3.11) *for $k, l = 1, 2$ as well as the bounds*

$$\|\psi_1 - \psi^m_{\alpha,\lambda,1}\|_{L^2} + \|\psi_2 - \psi^m_{\alpha,\lambda,1}\|_{\dot H^1_e} + \|A_2 - A^m_{\alpha,\lambda,2}\|_{\dot H^1_e} \lesssim \gamma$$

In addition, the map $u \to (\psi_1, \psi_2, A_2)$ from $\dot H^1$ into the above spaces is C^1.

A second step is to consider Schrödinger maps with more regularity, i.e. as in Theorem 1.1. For such maps, if we make the additional decay assumption that $u_0 - Q^m_{\alpha,\lambda} \in L^2$, then this is preserved along the flow. Hence, as a consequence of part (b) of the above lemma we have:

Corollary 3.4. *Let I be a compact interval, and $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$ be an m-equivariant map satisfying* (3.1) *uniformly in I and which has the additional regularity*

$$u - Q^m_{\alpha,\lambda} \in C(I; H^2), \qquad \partial_t u \in C(I; L^2).$$

Then ψ_0, ψ_1, ψ_2 and A_0, A_2 satisfy the relations (3.10)*,* (3.11) *for $k, l = 0, 1, 2$ and have the additional regularity*

$$\begin{aligned} &\psi_0, A_0 \in C(I; L^2), \psi_1 - \psi^m_{\alpha,\lambda,1} \in C(I; H^1_e), \\ &\psi_2 - \psi^m_{\alpha,\lambda,2}, A_2 - A^m_{\alpha,\lambda,2} \in C(I; H^2_e). \end{aligned} \tag{3.26}$$

3.3. **Schrödinger maps in the Coulomb gauge.** We are now prepared to write the evolution equations for the differentiated fields ψ_1 and ψ_2 in (3.8) computed with respect to the Coulomb gauge. To justify the following computations we assume that $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$ is a Schrödinger map as in Theorem 1.1 so that in addition $u_0 - Q^m \in L^2$. Thus the hypothesis of Corollary 3.4 is verified, and we obtain the additional regularity (3.26) for ψ_0, ψ_1, ψ_2 and A_0, A_2. This suffices in order to justify the computations below.

Writing the Laplacian in polar coordinates, a direct computation using the formulas (3.8) shows that we can rewrite the Schrödinger Map equation (1.1) in the form

$$\psi_0 = i\left(D_1\psi_1 + \frac{1}{r}\psi_1 + \frac{1}{r^2}D_2\psi_2\right)$$

Applying the operators D_1 and D_2 to both sides of this equation and using the relations (3.10) and (3.11), we can derive the evolution equations for ψ_m, $m = 1, 2$:

$$\begin{aligned} \partial_t\psi_1 + iA_0\psi_1 = {} & i\Delta\psi_1 - 2A_1\partial_1\psi_1 - \partial_1 A_1\psi_1 - \frac{1}{r}A_1\psi_1 \\ & - iA_1^2\psi_1 - i\frac{1}{r^2}A_2^2\psi_1 - i\frac{1}{r^2}\psi_1 + \frac{2}{r^3}A_2\psi_2 - \frac{1}{r^2}\Im(\psi_1\bar\psi_2)\psi_2 \\ \partial_t\psi_2 + iA_0\psi_2 = {} & i\Delta\psi_2 - 2A_1\partial_1\psi_2 - \partial_1 A_1\psi_2 - \frac{1}{r}A_1\psi_2 \\ & - iA_1^2\psi_2 - i\frac{1}{r^2}A_2^2\psi_2 - \Im(\psi_2\bar\psi_1)\psi_1 \end{aligned}$$

Under the Coulomb gauge $A_1 = 0$ these equations become

$$\begin{aligned}\partial_t \psi_1 + iA_0\psi_1 =& i\Delta\psi_1 - i\frac{1}{r^2}A_2^2\psi_1 - i\frac{1}{r^2}\psi_1 + \frac{2}{r^3}A_2\psi_2 - \frac{1}{r^2}\Im(\psi_1\bar{\psi_2})\psi_2 \\ \partial_t \psi_2 + iA_0\psi_2 =& i\Delta\psi_2 - i\frac{1}{r^2}A_2^2\psi_2 - \Im(\psi_2\bar{\psi_1})\psi_1\end{aligned}$$

while the relations (3.10) and (3.11) become

$$\partial_r A_2 = \Im(\psi_1\bar{\psi_2}), \qquad \partial_r\psi_2 = iA_2\psi_1 \tag{3.27}$$

From the compatibility relations involving A_0, we obtain

$$\partial_r A_0 = -\frac{1}{2r^2}\partial_r(r^2|\psi_1|^2 - |\psi_2|^2) \tag{3.28}$$

from which we derive

$$A_0 = -\frac{1}{2}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) + [r\partial_r]^{-1}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) \tag{3.29}$$

This is where the initialization of the Coulomb gauge at infinity is important. That guarantees that $A_0 \in L^2$, while $|\psi_1|^2 - r^{-2}|\psi_2|^2 \in L^2$. Thus the integrating constant must be zero.

There is quite a bit of redundancy in the equations for ψ_1 and ψ_2; we eliminate this by introducing a single main variable

$$\psi = \psi_1 - i\frac{\psi_2}{r}$$

A direct computation yields the equation for ψ:

$$i\partial_t\psi + \Delta\psi = A_0\psi - 2\frac{A_2}{r^2}\psi + \frac{1}{r^2}\psi + \frac{A_2^2}{r^2}\psi - \frac{1}{r}\Im(\psi_2\bar{\psi_1})\psi$$

By replacing $\psi_1 = \psi + ir^{-1}\psi_2$ and using $A_2^2 + |\psi_2|^2 = 1$, we obtain the key evolution equation we work with in this paper,

$$i\partial_t\psi + \Delta\psi - \frac{2}{r^2}\psi = A_0\psi - 2\frac{A_2}{r^2}\psi - \frac{1}{r}\Im(\psi_2\bar{\psi})\psi \tag{3.30}$$

Our strategy will be to use this equation in order to obtain estimates for ψ. The functions A_2 and ψ_2 are defined in terms of ψ via the system of ODE's

$$\partial_r A_2 = \Im(\psi\bar{\psi_2}) + \frac{1}{r}|\psi_2|^2, \qquad \partial_r\psi_2 = iA_2\psi - \frac{1}{r}A_2\psi_2 \tag{3.31}$$

derived from (3.27). If $m = 1$, the boundary condition for this system will be prescribed at infinity, and it roughly says that (A_2, ψ_2) are close to (h_3, ih_1) as r approaches ∞. In the regular case when $u - Q \in C(I; H^2)$ this is simply the following relation:

$$A_2 - h_3 \in L^2, \qquad \psi_2 - ih_1 \in L^2 \tag{3.32}$$

We will later prove that this condition suffices in order to uniquely determine ψ_2 and A_2 from ψ. This can only work in the 1-equivariant case; indeed, if $m \geq 2$ then nearby solitons cannot be differentiated in this way.

Once (A_2, ψ_2) are computed, the A_0 connection coefficient is given by (3.29) which becomes now

$$A_0(r) = -\frac{1}{2}|\psi|^2 + \frac{1}{r}\Im(\psi_2\bar{\psi}) + [r\partial_r]^{-1}(|\psi|^2 - \frac{2}{r}\Im(\psi_2\bar{\psi})). \tag{3.33}$$

Finally, given ψ, A_2 and ψ_2, we can return to the Schrödinger map u via the system (3.9) with the boundary condition at infinity given by (3.17).

3.4. **The choice of the parameters α, λ.** At this point we already have chosen to work $m = 1$ and drop the upper script m from h_1^m and h_3^m. This allows us to introduce another upper script convention

$$h_1^\lambda(r) = h_1(\lambda r), \qquad h_3^\lambda(r) = h_3(\lambda r)$$

which is very useful due to the key role the parameter λ plays in our analysis.

In order to understand the way a Schrödinger map u evolves along the soliton family, we need to choose a pair of time dependent functions $\alpha(t)$, $\lambda(t)$ so that (1.5) holds. Such a choice is not unique; we will introduce here two alternatives, show that both are suitable and compare them.

Our main choice is analytic, and it is motivated by the equation (3.30), which we want to rewrite as a linear equation with a nonlinear *perturbative* term. This is not the case in (3.30), since A_2 is nonzero if $\psi = 0$. Thus we want to take the bulk part of A_2 and move it into the linear part of the equation. Since A_2 is initialized as h_3 at infinity, one may try to take h_3 as the main part of A_2; this leads to a nonlinear Schrödinger equation governed by the operator $\tilde{H}$, namely

$$(i\partial_t - \tilde{H})\psi = A_0\psi - 2\frac{A_2 - h_3}{r^2}\psi - \frac{1}{r}\Im(\psi_2\bar{\psi})\psi$$

Unfortunately, the second term on the right, though quadratic in ψ, turns out to be nonperturbative on a long time scale; the difficulty is related to the lack of time decay of $A_2 - h_3$ for r in a compact set. To remedy this we instead choose λ so that A_2 is close to h_3^λ for r in a compact set. Precisely, our full choice of parameters is

$$A_2(1,t) = h_3^{\lambda(t)}(1), \qquad \psi_2(1,t) = ie^{i\alpha(t)}h_1^{\lambda(t)}(1) \tag{3.34}$$

which matches (A_2, ψ_2) with $(h_3^{\lambda(t)}, ie^{i\alpha(t)}h_1^{\lambda(t)}(1))$ at $r = 1$. The matching point is arbitrarily chosen; any other one would do. With these parameters, the equation (3.30) takes the form

$$(i\partial_t - \tilde{H}_{\lambda(t)})\psi = A_0\psi - 2\frac{A_2 - h_3^{\lambda(t)}}{r^2}\psi - \frac{1}{r}\Im(\psi_2\bar{\psi})\psi \tag{3.35}$$

With this formulation we are able to track the right hand side perturbatively. The price we pay is that the linear part now has a time dependent operator $H_{\lambda(t)}$, and that in addition to bounds for ψ we also need to bootstrap the appropriate bounds on the parameter λ.

An alternate choice of the parameters α and λ is geometric:

$$u(1,t) = Q_{\tilde{\alpha}(t),\tilde{\lambda}(t)}(1). \tag{3.36}$$

This choice, somewhat related to the one in [9], does not play any role in our analysis, and is given here only for comparison purposes. As a consequence of the pointwise part of the bounds (3.3) and (3.20) we have

Corollary 3.5. *Assume that ψ is small in L^2. Then both $(\alpha(t), \lambda(t))$ and $(\tilde{\alpha}(t), \tilde{\lambda}(t))$ satisfy the condition* (1.5)*. In addition, the two sets of parameters are related by the relations*

$$\lambda(t) = \tilde{\lambda}(t), \qquad |\alpha(t) - \tilde{\alpha}(t)| \lesssim \|\psi\|_{L^2} \tag{3.37}$$

We remark that the first relation in (3.37) follows directly from the identity (3.12). The second part of (3.37) follows from (1.5) since a direct computation shows that

$$\|Q_{\alpha,\lambda} - Q_{\tilde{\alpha},\tilde{\lambda}}\|_{\dot{H}^1} \approx \frac{|\alpha - \tilde{\alpha}| + |\log(\lambda/\tilde{\lambda})|}{1 + |\alpha - \tilde{\alpha}| + |\log(\lambda/\tilde{\lambda})|}$$

4. Spectral analysis for the operators H, $\tilde{H}$; the X, LX spaces

4.1. **Spectral theory for the operator H.** The spectral theory for H was studied in detail by Krieger-Schlag-Tataru in [10]. Here we simply restate the result in [10], in a slightly modified setup. The modification is threefold. Instead of working in $L^2(dr)$, we work with $L^2(rdr)$; this is equivalent to an $r^{-\frac{1}{2}}$ conjugation. Secondly, we prefer to use ξ^2 instead of ξ as the spectral parameter. Finally, we include the spectral measure in the generalized eigenfunctions.

Precisely, we consider H acting as an unbounded selfadjoint operator in $L^2(rdr)$. Then H is nonnegative, and its spectrum $[0, \infty)$ is absolutely continuous. H has a zero resonance, namely $\phi_0 = h_1$,

$$Hh_1 = 0.$$

For each $\xi > 0$ one can choose a normalized generalized eigenfunction ϕ_ξ,

$$H\phi_\xi = \xi^2 \phi_\xi.$$

These are unique up to a ξ dependent multiplicative factor, which is chosen as described below.

To these one associates a generalized Fourier transform $\mathcal{F}_H$ defined by

$$\mathcal{F}_H f(\xi) = \int_0^\infty \phi_\xi(r) f(r) r dr$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \phi_\xi(r) \mathcal{F}_H f(\xi) d\xi$$

The functions ϕ_ξ are smooth with respect to both r and ξ. To describe them one considers two distinct regions, $r\xi \lesssim 1$ and $r\xi \gtrsim 1$.

In the first region $r\xi \lesssim 1$ the functions ϕ_ξ admit a power series expansion of the form

$$\phi_\xi(r) = q(\xi) \left(\phi_0 + \frac{1}{r} \sum_{j=1}^{\infty} (r\xi)^{2j} \phi_j(r^2) \right), \qquad r\xi \lesssim 1 \tag{4.1}$$

where $\phi_0 = h_1$ and the functions ϕ_j are analytic and satisfy

$$|(r\partial_r)^\alpha \phi_j| \lesssim_\alpha \frac{C^j}{(j-1)!} \log(1 + r) \tag{4.2}$$

This bound is not spelled out in [10], but it follows directly from the integral recurrence formula for f_j's (page 578 in the paper). The smooth positive weight q satisfies

$$q(\xi) \approx \begin{cases} \dfrac{1}{\xi^{\frac{1}{2}}|\log\xi|} & \xi \ll 1 \\ \xi^{\frac{3}{2}} & \xi \gg 1 \end{cases}, \qquad |(\xi\partial_\xi)^\alpha q| \lesssim_\alpha q \tag{4.3}$$

Defining the weight

$$m_k^1(r) = \begin{cases} \min\{1, r2^k \dfrac{\ln(1+r^2)}{\langle k\rangle}\} & k<0 \\ \min\{1, r^3 2^{3k}\}, & k \geq 0 \end{cases} \tag{4.4}$$

it follows that the nonresonant part of ϕ_ξ satisfies

$$|(\xi\partial_\xi)^\alpha (r\partial_r)^\beta \left(\phi_\xi(r) - q(\xi)\phi_0(r)\right)| \lesssim_{\alpha\beta} 2^{\frac{k}{2}} m_k^1(r), \qquad \xi \approx 2^k,\ r\xi \lesssim 1 \tag{4.5}$$

In the other region $r\xi \gtrsim 1$ we begin with the functions

$$\phi_\xi^+(r) = r^{-\frac{1}{2}} e^{ir\xi}\sigma(r\xi, r), \qquad r\xi \gtrsim 1 \tag{4.6}$$

solving

$$H\phi_\xi^+ = \xi^2\phi_\xi^+$$

where for σ we have the following asymptotic expansion

$$\sigma(q,r) \approx \sum_{j=0}^{\infty} q^{-j}\phi_j^+(r), \qquad \phi_0^+ = 1, \qquad \phi_1^+ = \frac{3i}{8} + O(\frac{1}{1+r^2})$$

with

$$\sup_{r>0} |(r\partial_r)^k \phi_j^+| < \infty$$

in the following sense

$$\sup_{r>0} |(r\partial r)^\alpha (q\partial_q)^\beta [\sigma(q,r) - \sum_{j=0}^{j_0} q^{-j}\phi_j^+(r)]| \leq c_{\alpha,\beta,j_0} q^{-j_0-1}$$

Then we have the representation

$$\phi_\xi(r) = a(\xi)\phi_\xi^+(r) + \overline{a(\xi)\phi_\xi^+(r)} \tag{4.7}$$

where the complex valued function a satisfies

$$|a(\xi)| = \sqrt{\frac{2}{\pi}}, \qquad |(\xi\partial_\xi)^\alpha a(\xi)| \lesssim_\alpha 1 \tag{4.8}$$

4.2. **Spectral theory for the operator $\tilde{H}$.** The spectral theory for $\tilde{H}$ is derived from the spectral theory for H due to the conjugate representations

$$H = L^*L, \qquad \tilde{H} = LL^*$$

This allows us to define generalized eigenfunctions ψ_ξ for $\tilde{H}$ using the generalized eigenfunctions ϕ_ξ for H,

$$\psi_\xi = \xi^{-1}L\phi_\xi, \qquad L^*\psi_\xi = \xi\phi_\xi$$

It is easy to see that ψ_ξ are real, smooth, vanish at $r = 0$ and solve

$$\tilde{H}\psi_\xi = \xi^2\psi_\xi$$

With respect to this frame we can define the generalized Fourier transform adapted to $\tilde{H}$ by

$$\mathcal{F}_{\tilde{H}}f(\xi) = \int_0^\infty \psi_\xi(r)f(r)rdr$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \psi_\xi(r)\mathcal{F}_{\tilde{H}}f(\xi)d\xi \tag{4.9}$$

To see this we compute, for a Schwartz function f:

$$\begin{aligned}\mathcal{F}_{\tilde{H}}Lf(\xi) &= \int_0^\infty \psi_\xi(r)Lf(r)rdr = \int_0^\infty L^*\psi_\xi(r)f(r)rdr \\ &= \int_0^\infty \xi\phi_\xi(r)f(r)rdr = \xi\mathcal{F}_Hf(\xi)\end{aligned}$$

Hence

$$\|\mathcal{F}_{\tilde{H}}Lf\|_{L^2}^2 = \|\xi\mathcal{F}_Hf(\xi)\|_{L^2}^2 = \langle Hf, f\rangle_{L^2(rdr)} = \|Lf\|_{L^2}^2$$

which suffices since Lf spans a dense subset of L^2.

The representation of ψ_ξ in the two regions $r\xi \lesssim 1$ and $r\xi \gtrsim 1$ is obtained from the similar representation of ϕ_ξ. In the first region $r\xi \lesssim 1$ the functions ψ_ξ admit a power series expansion of the form

$$\psi_\xi = \xi q(\xi)\left(\psi_0(r) + \sum_{j\geq 1}(r\xi)^{2j}\psi_j(r^2)\right)$$

where

$$\psi_j(r) = (h_3 + 1 + 2j)\phi_{j+1}(r) + r\partial_r\phi_{j+1}(r)$$

From (4.2), it follows that

$$|(r\partial_r)^\alpha\psi_j| \lesssim_\alpha \frac{C^j}{(j-1)!}\log(1+r^2)$$

In addition, ψ_0 solves $L^*\psi_0 = \phi_0$ therefore a direct computation shows that

$$\psi_0 = \frac{1}{2}\left(\frac{(1+r^2)\log(1+r^2)}{r^2} - 1\right)$$

In particular, defining the weights

$$m_k(r) = \begin{cases} \min\{1, \dfrac{\ln(1+r^2)}{\langle k \rangle}\}, & \text{if } k < 0 \\ \min\{1, r^2 2^{2k}\}, & \text{if } k \geq 0 \end{cases} \tag{4.10}$$

we have the pointwise bound for ψ_ξ

$$|(r\partial_r)^\alpha (\xi \partial_\xi)^\beta \psi_\xi(r)| \lesssim_{\alpha\beta} 2^{\frac{k}{2}} m_k(r), \qquad \xi \approx 2^k, \ r\xi \lesssim 1 \tag{4.11}$$

On the other hand in the regime $r\xi \gtrsim 1$ we define

$$\psi^+ = \xi^{-1} L \phi^+$$

and we obtain the representation

$$\psi_\xi(r) = a(\xi)\psi_\xi^+(r) + \overline{a(\xi)\psi_\xi^+(r)} \tag{4.12}$$

For ψ^+ we obtain the expression

$$\psi_\xi^+(r) = r^{-\frac{1}{2}} e^{ir\xi} \tilde{\sigma}(r\xi, r), \qquad r\xi \gtrsim 1 \tag{4.13}$$

where $\tilde{\sigma}$ has the form

$$\tilde{\sigma}(q,r) = i\sigma(q,r) - \frac{1}{2} q^{-1} \sigma(q,r) + \frac{\partial}{\partial q}\sigma(q,r) + \xi^{-1} L\sigma(q,r)$$

therefore it has exactly the same properties as σ. In particular, for fixed ξ, we obtain that

$$\tilde{\sigma}(r\xi, r) = i - \frac{7}{8} r^{-1}\xi^{-1} + O(r^{-2}) \tag{4.14}$$

4.3. **The spaces X and LX.** So far we have measured the Schrödinger map u in the space $\dot{H}^1$ (which correspond to $\bar{u} \in \dot{H}^1_e$), while the differentiated field ψ is in L^2. The operator L maps $\dot{H}^1_e$ into L^2. Conversely, if for some $f \in L^2$ we solve

$$Lg = f$$

then we obtain a solution u which is in $\dot{H}^1_e$ and satisfies

$$\|g\|_{\dot{H}^1_e} \lesssim \|f\|_{L^2}$$

However, this solution is only unique modulo a multiple of the resonance ϕ_0. Furthermore, in general it does not make sense to identify u by prescribing its size at infinity. The spaces X and LX are in part introduced in order to remedy this ambiguity in the inversion of L.

Definition 4.1. *a) The space X is defined as the completion of the subspace of $L^2(rdr)$ for which the following norm is finite*

$$\|u\|_X = \left(\sum_{k \geq 0} 2^{2k} \|P_k^H u\|_{L^2}^2 \right)^{\frac{1}{2}} + \sum_{k<0} \frac{1}{|k|} \|P_k^H u\|_{L^2}$$

where P_k^H is the Littlewood-Paley operator localizing at frequency $\xi \approx 2^k$ in the H calculus.

b) LX is the space of functions of the form $f = Lu$ with $u \in X$, with norm $\|f\|_{LX} = \|u\|_X$. Expressed in the $\tilde{H}$ calculus, the LX norm is written as

$$\|f\|_{LX} = \left(\sum_{k \geq 0} \|P_k^{\tilde{H}} f\|_{L^2}^2 \right)^{\frac{1}{2}} + \sum_{k<0} \frac{2^{-k}}{|k|} \|P_k^{\tilde{H}} f\|_{L^2}$$

In this article we work with equivariant Schrödinger maps u for which $\|\bar{u} - \bar{Q}\|_X \ll 1$. This corresponds to fields ψ which satisfy $\|\psi\|_{LX} \ll 1$. The simplest properties of the space X are summarized as follows:

Proposition 4.2. *The following embeddings hold for the space X:*

$$H_e^1 \subset X \subset \dot{H}_e^1 \tag{4.15}$$

In addition for f in X we have the following bounds:

$$\|\langle r \rangle^{\frac{1}{2}} f\|_{L^\infty} \lesssim \|f\|_X \tag{4.16}$$

$$\left\| \frac{f}{\ln(1+r)} \right\|_{L^2} \lesssim \|f\|_X \tag{4.17}$$

$$\left\| \langle r \rangle^{\frac{1}{2}} f \right\|_{L^4} \lesssim \|f\|_X \tag{4.18}$$

Proof. We first consider bounds for frequency localized functions in the H frame:

Lemma 4.3. *Assume $f_k \in L^2$ is localized at H-frequency 2^k. Then*

$$|f_k(r)| + |r\partial_r f_k(r)| \lesssim \left(\frac{2^{k^+}}{\langle k^- \rangle} \phi_0(r) + 2^k m_k^1(r) \right) \|f_k\|_{L^2}, \qquad r \lesssim 2^{-k} \tag{4.19}$$

$$|r^{\frac{1}{2}} f_k(r)| \lesssim 2^{\frac{k}{2}} \|f_k\|_{L^2}, \qquad 2^{-k} \lesssim r \tag{4.20}$$

$$\|\partial_r f_k(r)\|_{L^2(A_{\geq -k})} \lesssim 2^k \|f_k\|_{L^2}. \tag{4.21}$$

Proof. The Fourier inversion formula gives

$$f_k(r) = \int p_k(\xi) \mathcal{F}_H f(\xi) \phi_\xi(r) d\xi$$

Then the bound (4.19) follows from (4.5) and the Cauchy-Schwarz inequality. Similarly (4.20) follows from the bound $|\phi_\xi| \lesssim r^{-\frac{1}{2}}$ for $r \gtrsim 2^{-k}$. The estimate (4.21) on the other hand follows directly from

$$\|Lf_k\|_{L^2} \lesssim 2^k \|f_k\|_{L^2} \tag{4.22}$$

□

We now prove the embedding $X \subset \dot{H}_e^1$. Due to the straightforward bound $\|Lf\|_{L^2} \lesssim \|f\|_X$ and the ODE estimate

$$\|\partial_r f\|_{L^2} + \|r^{-1} f\|_{L^2} \lesssim |f(1)| + \|Lf\|_{L^2},$$

it suffices to show that $|f(1)| \lesssim \|f\|_X$. But this is obtained by direct summation from the dyadic pointwise bounds (4.19) and (4.20).

The embedding $H^1_e \in X$ is a consequence of the bound

$$\|f\|_X \lesssim \|f\|_{L^2} + \|Lf\|_{L^2} \tag{4.23}$$

The right hand side above is in effect an equivalent norm in H^1_e. To prove (4.23) we use the L^2 norm of f for low frequencies,

$$\|f_{<0}\|_X \lesssim \sum_{k<0} \frac{1}{|k|}\|f_k\|_{L^2} \lesssim \left(\sum_{k<0} \|f_k\|^2_{L^2(rdr)}\right)^{\frac{1}{2}} \left(\sum_{k<0} \frac{1}{k^2}\right)^{\frac{1}{2}} \lesssim \|f_{<0}\|_{L^2(rdr)}$$

and the L^2 norm of Lf for high frequencies,

$$\|f_{\geq 0}\|_X \approx \|Lf_{\geq 0}\|_{L^2} \lesssim \|Lf\|_{L^2}$$

In view of the above embedding, for (4.16) it suffices to consider $r > 1$. Then (4.16) follows by direct summation from (4.19) and (4.20).

For (4.17) it also suffices to take $r \geq 1$. The high frequencies are bounded directly in L^2,

$$\|f_{\geq 0}\|_{L^2} \lesssim \|f\|_X$$

so it remains to consider a single low frequency component f_k. We have

$$\left\|\frac{f_k}{\ln(1+r)}\right\|^2_{L^2} \lesssim \int_0^{2^{-k}} \frac{|f_k|^2}{|\ln(1+r)|^2} rdr + \frac{1}{k^2}\|f_k\|^2_{L^2}$$

and for the first part we use (4.19).

Finally we prove (4.18). For the high frequencies $f_{\geq 0}$ we interpolate between (4.16) and the L^2 estimate. It remains to consider a fixed low frequency component f_k. If $r \leq 2^{-k}$ then it suffices to perform a direct computation based on (4.19). If $r \geq 2^{-k}$ then we interpolate between (4.20) and the trivial L^2 bound.

□

Now we turn our attention to the space LX.

Lemma 4.4. *If $f \in L^2$ is localized at $\tilde{H}$- frequency 2^k then*

$$|f(r)| \lesssim 2^k m_k(r)(1+2^k r)^{-\frac{1}{2}} \|f\|_{L^2} \tag{4.24}$$

Proof. This follows from the Fourier inversion formula (4.9), the Cauchy-Schwarz inequality and (4.11) for $r \lesssim 2^{-k}$, respectively the bound $|\psi_\xi| \lesssim r^{-\frac{1}{2}}$ for $r \gtrsim 2^{-k}$. □

Proposition 4.5. *The following embeddings hold for LX:*

$$L^1 \cap L^2 \subset LX \subset L^2 \tag{4.25}$$

Proof. The second embedding is trivial. For the first one we use the L^2 norm for high frequencies, and it remains to use the L^1 norm for low frequencies and show that

$$\|P_{\leq 0} f\|_{LX} \lesssim \|f\|_{L^1} \tag{4.26}$$

It suffices to consider the case when f is a Dirac mass, $f = \frac{1}{R}\delta_{r=R}$. For such f we bound its Fourier transform,

$$|\mathcal{F}_{\tilde{H}} f(\xi)| \lesssim \begin{cases} \dfrac{\xi^{\frac{1}{2}}}{|\ln \xi|} \ln(1+R^2) & \xi < R^{-1} \\ R^{-\frac{1}{2}} & \xi > R^{-1} \end{cases}$$

Thus

$$\|P_{\leq 0}f\|_{LX} \lesssim \sum_{k<-|\log R|} \frac{\ln(1+R^2)}{k^2} + \sum_{k>-|\log R|} R^{-\frac{1}{2}} \frac{2^{-\frac{k}{2}}}{k} \lesssim 1$$

and (4.26) follows. □

Based on the above results we can now establish multiplicative properties for X and LX:

Proposition 4.6. *X is an algebra and the following estimates hold:*

$$\|fLg\|_{LX} \lesssim \|f\|_X \|g\|_X \tag{4.27}$$

$$\|h_1 Lf\|_{LX} + \|h_3 Lf\|_{LX} \lesssim \|f\|_X \tag{4.28}$$

Proof. From (4.23) is it enough to prove that if $f, g \in X$ then $f \cdot g \in L^2$ and $L(f \cdot g) \in L^2$. From (4.18) we have

$$\|fg\|_{L^2} \lesssim \|f\|_X \|g\|_X$$

therefore it remains to show that

$$\|L(fg)\|_{L^2} \lesssim \|f\|_X \|g\|_X$$

We write

$$L(fg) = Lf \cdot g + f \cdot Lg + h_3 \frac{f \cdot g}{r}$$

For the first term (and similarly for the second one) we have

$$\|Lf \cdot g\|_{L^2} \lesssim \|Lf\|_{L^2} \|g\|_{L^\infty}$$

and use (4.16). For the third term we have

$$\|r^{-1} f \cdot g\|_{L^2} \lesssim \|r^{-1} f\|_{L^2} \|g\|_{L^\infty} \lesssim \|f\|_{\dot{H}^1_e} \|g\|_{\dot{H}^1_e} \tag{4.29}$$

and the proof of the algebra property is complete.

If $f, g \in X$ then we use (4.17) to estimate

$$\|r^{-1} f \cdot g\|_{L^1} \lesssim \|f\|_X \|g\|_X$$

which combined with (4.29) and (4.25) implies that

$$\|r^{-1} f \cdot g\|_{LX} \lesssim \|r^{-1} f \cdot g\|_{L^1 \cap L^2} \lesssim \|f\|_X \|g\|_X \tag{4.30}$$

Now we are ready to prove (4.27). We have

$$\|f \cdot Lg\|_{L^2} \lesssim \|f\|_{L^\infty} \|Lg\|_{L^2} \lesssim \|f\|_X \|g\|_X$$

and also

$$\|P_{\geq 0} f \cdot Lg\|_{L^1} \lesssim \|P_{\geq 0} f\|_{L^2} \|Lg\|_{L^2} \lesssim \|f\|_X \|g\|_X$$

which places $P_{\geq 0} f \cdot Lg$ in LX due to (4.25). Also

$$P_{\leq 0} f \cdot LP_{\geq 0} g = L(P_{\leq 0} f \cdot P_{\geq 0} g) - LP_{\leq 0} f \cdot P_{\geq 0} g + \frac{h_3}{r} P_{\leq 0} f \cdot P_{\geq 0} g$$

The first term belongs to LX due to the algebra property of X, the second term is treated as above and the third one is estimated as in (4.30). We are then left with estimating the low frequency contributions $P_{\leq 0} f L P_{\leq 0} g$. Due to the l^1 structure of X at low frequencies, this can be reduced to the case of single frequencies, i.e. when f is replaced by $f_k = P_k f$ and g by $g_j = P_j g$ with $k, j < 0$. If $k \geq j$ then

$$\|f_k L g_j\|_{L^1} \lesssim \|f_k\|_{L^2} \|L g_j\|_{L^2} \lesssim 2^j |k||j| \|f_k\|_X \|g_j\|_X \lesssim \|f_k\|_X \|g_j\|_X$$

The case $k \leq j$ is similar after moving L on the lower frequency factor,

$$f_k L g_j = L(f_k \cdot g_j) - L f_k \cdot g_j + \frac{h_3}{r} f_k \cdot g_j.$$

We use a similar argument to prove (4.28). We have

$$h_1 L f = L(h_1 f) - f \partial_r h_1$$

The expression $h_1 f$ is estimated in $H^1_e \subset X$, while $f \partial_r h_1$ trivially belongs to $L^1 \cap L^2$. The same argument applies if h_1 is replaced by $h_3 - 1$, proving the second estimate in(4.28). □

4.4. **A companion space.** Here we define a Sobolev type companion $\tilde{X}$ for X and study some simple properties for it. This space will be used in Section 7 in order to characterize the regularity of the Coulomb frame (v, w).

We begin with the space $[\partial_r]^{-1} l^1 L^2$, defined as the completion of of $H^1_{comp}([0,\infty))$ with respect to the following norm

$$\|f\|_{[\partial_r]^{-1} l^1 L^2} = \|\partial_r f\|_{l^1 L^2} := \sum_m \|\partial_r f\|_{L^2(A_m)}$$

Since $\|\partial_r f\|_{L^1(dr)} \lesssim \|f\|_{[\partial_r]^{-1} l^1 L^2}$, it follows that f has limits both at 0 and ∞; and since it is approximated by functions in $H^1_{comp}([0,\infty))$, it follows that $\lim_{r\to\infty} f(r) = 0$. We also have the following inequality

$$\|\partial_r f\|_{L^2} + \|f\|_{L^\infty} \lesssim \|f\|_{[\partial_r]^{-1} l^1 L^2} \tag{4.31}$$

Now we can define the spaces $\tilde{X}$ and $\partial_r \tilde{X}$,

$$\tilde{X} = \{f : \chi_{r\geq 1} f \in X, \chi_{r\leq 1} f \in X + [\partial_r]^{-1} l^1 L^2\}, \qquad \partial_r \tilde{X} = \{f : f = \partial_r g, g \in \tilde{X}\}$$

with the induced norms. For technical purposes only we also introduce the norm

$$\|f\|^2_{l^2 L^\infty} := \sum_m \|f\|^2_{L^\infty(A_m)}$$

Lemma 4.7. *The following estimates hold:*

$$\|f\|_{\partial_r \tilde{X}} \lesssim \|\chi_{r\leq 1} f\|_{l^1 L^2} + \|r \chi_{r\geq 1} f\|_{L^2} \tag{4.32}$$

$$\|h_1 f\|_{\partial_r \tilde{X}} + \|h_3 f\|_{\partial_r \tilde{X}} \lesssim \|f\|_{\partial_r \tilde{X}} \tag{4.33}$$

$$\|f \cdot g\|_{\partial_r \tilde{X}} \lesssim \|f\|_{\partial_r \tilde{X}} \|g\|_{\tilde{X}} \tag{4.34}$$

$$\|f \cdot g\|_{LX} \lesssim \|f\|_{LX} \|g\|_{\tilde{X}} \tag{4.35}$$

Proof. **Proof of** (4.32)**.** Define g by $\partial_r g = f$, $g(\infty) = 0$. We need to estimate g in $\tilde{X}$. The operator $[r\partial_r]^{-1}$ is L^2 bounded, therefore $\|g\|_{L^2} \lesssim \|rf\|_{L^2}$. Hence by (4.31) we obtain

$$\|\chi_{r\leq 1} g\|_{L^\infty} + \|\partial_r \chi_{r\leq 1} g\|_{l^1 L^2} + \|\chi_{r\geq 1} g\|_{H^1} \lesssim \|\chi_{r\leq 1} f\|_{l^1 L^2} + \|r \chi_{r\geq 1} f\|_{L^2}$$

therefore (4.32) follows by definition and by (4.15).

Proof of (4.33)**.** For the first term we estimate

$$\|\chi_{r\leq 1}h_1 f\|_{l^1L^2} + \|r\chi_{r\geq 1}h_1 f\|_{L^2} \lesssim \|f\|_{L^2} \lesssim \|f\|_{\partial_r\tilde{X}}$$

and conclude by (4.32). A similar argument works for the second term.

Proof of (4.34)**.** We need to show that

$$\|f\cdot\partial_r g\|_{\partial_r\tilde{X}} \lesssim \|f\|_{\tilde{X}}\|g\|_{\tilde{X}} \tag{4.36}$$

We write $f = f_1 + f_2$, $g = g_1 + g_2$ where $f_1, g_1 \in X$ and $f_2, g_2 \in [\partial_r]^{-1}l^1L^2$ are supported in $[0,1]$. The expression $f\partial_r g_2$ inherits the l^1L^2 bound from g_2. For $f_2\partial_r g_1$ we write

$$f_2\partial_r g_1 = \partial_r(f_2 g_1) - g_1\partial_r f_2.$$

We can bound $f_2 g_1$ in $H^1_e \subset X$ while $g_1\partial_r f_2$ belongs to l^1L^2. For the final term we will show

$$\|f_1\cdot\partial_r g_1\|_{\partial_r\tilde{X}} \lesssim \|f_1\|_X\|g_1\|_X \tag{4.37}$$

Starting from the simpler bound

$$\|f_1\|_{l^2L^\infty} \lesssim \|f_1\|_{\dot{H}^1_e} \tag{4.38}$$

we obtain

$$\|f_1\cdot\partial_r g_1\|_{l^1L^2} \lesssim \|f_1\|_X\|g_1\|_X$$

which yields an L^∞ bound for $[\partial_r]^{-1}(f_1\cdot\partial_r g_1)$ and suffices for $r \lesssim 1$. For larger r we consider a dyadic decomposition as in the proof of (4.27). If the first factor has high frequency then we estimate it in L^2 to obtain

$$\|P_{\geq 0}f_1\cdot\partial_r g_1\|_{L^1} \lesssim \|f_1\|_X\|g_1\|_X$$

Combining this with the Sobolev type bound

$$\|[\partial_r]^{-1}h\|_{L^2} \lesssim \|h\|_{L^1} \tag{4.39}$$

we obtain

$$\|[\partial_r]^{-1}(P_{\geq 0}f_1\cdot\partial_r g_1)\|_{L^2} \lesssim \|f_1\|_X\|g_1\|_X$$

therefore

$$\|\chi_{r\geq 1}[\partial_r]^{-1}(P_{\geq 0}f_1\cdot\partial_r g_1)\|_{H^1_e} \lesssim \|f_1\|_X\|g_1\|_X$$

which suffices by (4.15).

If the second factor has high frequency then we switch them

$$f_1\cdot\partial_r P_{\geq 0}g_1 = \partial_r(f_1\cdot P_{\geq 0}g_1) - \partial_r f_1\cdot P_{\geq 0}g_1$$

and use the X algebra property for the first term on the right.

It remains to consider the low frequency interactions $P_k f_1\cdot\partial_r P_j g_1$ with $k, j < 0$. Assuming $k \leq j$, by using (4.19) for small r and (4.21) for large r we obtain

$$\|P_k f_1\cdot\partial_r P_j g_1\|_{L^1} \lesssim \|P_k f_1\|_X\|P_j g_1\|_X$$

and conclude as before. On the other hand if $k > j$ we switch the derivative to the first factor and use again the X algebra property.

Proof of (4.35)**.** We split

$$fg = f\chi_{r\geq 1}g + f\chi_{r\leq 1}g$$

For the first term we use (4.27). The second has compact support, therefore it suffices to estimate it in L^2 and use (4.25). □

4.5. **Littlewood-Paley projectors in the $\tilde{H}$ frame.** The first aim of the following proposition is to characterize the kernels $K_k(r,s)$ of the projectors P_k in the $\tilde{H}$ frame. Secondly, we consider the kernels $K_k^1(r,s)$ of the operators $L^{-1}P_k$, which can be defined as

$$L^{-1}P_k := L^*\tilde{H}^{-1}P_k.$$

We remark that the adjoint operators are given by

$$(L^{-1}P_k)^* = P_k(L^*)^{-1} := P_k\tilde{H}^{-1}L.$$

Proposition 4.8. *a) The kernel $K_k(r,s)$ of P_k satisfies the bounds*

$$|K_k(r,s)| \lesssim \frac{2^{2k}m_k(r)m_k(s)}{(1+2^k|r-s|)^N(1+2^k(r+s))}, \tag{4.40}$$

$$|\partial_r K_k(r,s)| \lesssim \frac{(2^{3k}+2^{2k}r^{-1})m_k(r)m_k(s)}{(1+2^k|r-s|)^N(1+2^k(r+s))}. \tag{4.41}$$

b) If $k \geq 0$ then the kernel $K_k^1(r,s)$ of $L^{-1}P_k$ satisfies the bound

$$|K_k^1(r,s)| \lesssim \frac{2^k m_k^1(r)m_k(s)}{(1+2^k|r-s|)^N(1+2^k(r+s))} \tag{4.42}$$

If $k<0$ then $K_k^1(r,s)$ admits a decomposition

$$K_k^1 = K_{k,reg}^1 + K_{k,res}^1$$

where the regular part $K_{k,reg}^1$ satisfies (4.42) *and the resonant part $K_{k,res}^1$ has the form*

$$K_{k,res}^1(r,s) = h_1(r)\chi_{2^kr\leq 1}c_k(s), \quad |c_k(s)| \lesssim |k|^{-1}m_k(s)(1+2^ks)^{-N} \tag{4.43}$$

and $\chi_{2^kr\leq 1}$ is a smooth bump function supported in $\{2^kr \lesssim 1\}$ which equals 1 in $\{2^kr \ll 1\}$.

Proof. a) We denote by χ_k the symbol of P_k. This is a smooth bump supported at $\xi \approx 2^k$, which is all that we use in the proof. The kernel $K_k(r,s)$ is symmetric and has the form

$$K_k(r,s) = \int \psi_\xi(r)\psi_\xi(s)\chi_k(\xi)d\xi.$$

If $r,s \gtrsim 2^{-k}$ then we use the representation (4.12) to obtain

$$K_k(r,s) = \sum_{j,l=0,1}\int r^{-\frac{1}{2}}s^{-\frac{1}{2}}e^{i((-1)^jr+(-1)^ls)\xi}a_j(\xi)a_l(\xi)\tilde{\sigma}_j(r\xi,r)\tilde{\sigma}_l(r\xi,r)\chi_k(\xi)d\xi$$

where $a_0 = a, a_1 = \overline{a}, \tilde{\sigma}_0 = \tilde{\sigma}, \tilde{\sigma}_1 = \overline{\tilde{\sigma}}$. Using stationary phase together with the bounds on a and the characterization of $\tilde{\sigma}$ gives the bound in (4.40). We note that the stationary phase brings decay factors of type $(1+2^k|r-s|)^{-N}$.

We now consider the case $r \gtrsim 2^{-k}, s \lesssim 2^{-k}$ (and also, by symmetry, the case $r \lesssim 2^{-k}, s \gtrsim 2^{-k}$). Then

$$K_k(r,s) = \sum_{j=0,1}\int \psi_\xi(s)r^{-\frac{1}{2}}e^{i(-1)^jr\xi}a_j(\xi)\tilde{\sigma}_j(r\xi,r)\chi_k(\xi)d\xi$$

The first factor $\psi_\xi(s)$ is smooth in ξ on the dyadic scale; precisely, we have the pointwise bound (4.11). Then we use stationary phase, (4.11), the bounds on a and the characterization of σ to claim (4.40).

Finally, if $r, s \lesssim 2^{-k}$, the arguments for (4.40) and (4.41) follow directly from the pointwise bounds (4.11) on ψ_ξ.

For the estimate (4.41) we write $\partial_r = -L^* + \frac{h_3-1}{r}$. Then

$$|\partial_r K_k(r,s)| \lesssim |L^* K_k(r,s)| + \frac{1}{r}|K_k(r,s)|$$

and $L^* K_k(r,s)$ is of the form $L^* K_k(r,s) = 2^{2k} K_k^1(r,s)$ with K_k^1 as in part (b). Hence it suffices to prove part (b) of the proposition.

b) Since $L^*\psi_\xi = \xi\phi_\xi$, the kernel K_k^1 is given by

$$K_k^1(r,s) = \int \xi^{-1}\phi_\xi(r)\psi_\xi(s)\chi_k(\xi)d\xi$$

If $r \gtrsim 2^{-k}$ then $\phi_\xi(r)$ is similar to $\psi_\xi(r)$ and K_k^1 satisfies the same bounds as K_k with an additional 2^{-k} factor. If $r \lesssim 2^{-k}$ and $k \geq 0$ then ϕ_ξ is smooth on the dyadic scale and has size $r\xi^{\frac{3}{2}}$ therefore we can argue again as in case (a). Finally if $r \lesssim 2^{-k}$ and $k < 0$ then we decompose ϕ_ξ according to (4.1) into

$$\phi_\xi(r) = q(\xi)\phi_0(r) + \phi_\xi^{err}(r)$$

where ϕ_ξ^{err} is smooth on the dyadic scale and has size $q(\xi)r\xi^2\log(1+r)$. The first term yields the resonant component $K_{k,res}^1$ and the second term gives the regular component $K_{k,reg}^1$. ☐

4.6. **Time dependent frames and the transference identity.** Later in the article we need to work with a time dependent parameter λ, and thus with a time dependent Fourier transform associated to the operator $\tilde{H}_\lambda$. By rescaling, its normalized generalized eigenfunctions are

$$\psi_\xi^\lambda(r) = \lambda^{-\frac{1}{2}}\psi_{\lambda\xi}(\lambda^{-1}r), \qquad \tilde{H}_\lambda\psi_\xi^\lambda = \xi^2\psi_\xi^\lambda$$

We denote the associated Fourier transform by $\mathcal{F}_\lambda$; this is an L^2 isometry. To study its λ dependence we use the transference operator $\mathcal{K}_\lambda$, previously introduced and studied in [10]:

$$\mathcal{K}_\lambda = \mathcal{F}_\lambda \frac{d}{d\lambda}\mathcal{F}_\lambda^*$$

By scaling it suffices to analyze the operator $\mathcal{K} = \mathcal{K}_1$.

Proposition 4.9 ([10])**.** *The operator $\mathcal{K}$ is a skew-adjoint Hilbert transform type operator, whose kernel $K(\xi,\eta)$ has the form*

$$K(\xi,\eta) = p.v.\frac{F(\xi,\eta)}{\xi^2-\eta^2}, \qquad F(\xi,\eta) = \langle \frac{1}{(1+r^2)^2}\psi_\xi, \psi_\eta\rangle$$

where the symmetric function F satisfies the following bounds

$$|(\xi\partial_\xi)^\alpha(\eta\partial_\eta)^\beta(\xi\partial_\xi+\eta\partial_\eta)^\sigma F(\xi,\eta)| \lesssim \frac{\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}}{\langle \ln\xi\rangle\langle\ln\eta\rangle} \qquad \xi,\eta \lesssim 1$$

$$|(\xi\partial_\xi)^\alpha\partial_\eta^\beta(\xi\partial_\xi+\eta\partial_\eta)^\sigma F(\xi,\eta)| \lesssim \frac{\xi^{\frac{1}{2}}}{\langle\ln\xi\rangle(1+\eta)^N} \qquad \xi \lesssim 1 \lesssim \eta$$

$$|\partial_\xi^\alpha\partial_\eta^\beta(\xi\partial_\xi+\eta\partial_\eta)^\sigma F(\xi,\eta)| \lesssim \frac{1}{(1+|\xi-\eta|)^N} \qquad 1 \lesssim \xi,\eta$$

where $\sigma, N \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}$ if $|\xi-\eta| \gtrsim \max(\xi,\eta)$ and $\alpha+\beta \leq 2$ if $|\xi-\eta| \ll \max(\xi,\eta)$.

Proof. We merely outline the computation, as a complete proof is given in [10]. Furthermore, the proof is similar to the proof of the next proposition, which is presented in full. Formally the kernel K is given by

$$K(\xi,\eta) = \langle \psi_\xi, \left(\frac{d}{d\lambda}\psi_\eta^\lambda\right)_{|\lambda=1}\rangle$$

We have

$$\left(\frac{d}{d\lambda}\psi_\eta^\lambda\right)_{|\lambda=1} = \frac{d}{d\lambda}\left(\lambda^{-1/2}\psi_{\lambda\eta}(\lambda^{-1}r)\right)_{|\lambda=1} = (\eta\partial_\eta - r\partial_r - \frac{1}{2})\psi_\eta$$

therefore

$$K(\xi,\eta) = \langle \psi_\xi, (\eta\partial_\eta - r\partial_r - \frac{1}{2})\psi_\eta\rangle$$

Hence we obtain:

$$\begin{aligned}
(\xi^2-\eta^2)K(\xi,\eta) =& \langle \xi^2\psi_\xi, (\eta\partial_\eta - r\partial_r - \frac{1}{2})\psi_\eta\rangle - \langle \psi_\xi, \eta^2(\eta\partial_\eta - r\partial_r - \frac{1}{2})\psi_\eta\rangle\\
=& \langle \tilde{H}\psi_\xi, (\eta\partial_\eta - r\partial_r - \frac{1}{2})\psi_\eta\rangle - \langle \psi_\xi, (\eta\partial_\eta - r\partial_r - \frac{1}{2})\eta^2\psi_\eta\rangle + 2\langle \psi_\xi, \eta^2\psi_\eta\rangle\\
=& -\langle \psi_\xi, [\tilde{H}, r\partial_r]\psi_\eta\rangle + 2\langle \psi_\xi, \eta^2\psi_\eta\rangle\\
=& -\langle \psi_\xi, \frac{8}{(1+r^2)^2}\psi_\eta\rangle
\end{aligned}$$

The bounds on F are derived from the representations for ψ_ξ given by (4.11)-(4.13). □

Next we consider the related problem of comparing the Fourier transforms in nearby frames.

Proposition 4.10. *Suppose that $|\lambda_1 - 1| \ll 1$ and $|\lambda_2 - 1| \ll 1$. Then the kernel of the operator $\mathcal{F}_{\lambda_1}\mathcal{F}_{\lambda_2}^*$ has the form*

$$K_{\lambda_1\lambda_2}(\xi,\eta) = a(\lambda_1,\lambda_2,\xi)\delta_{\xi=\eta} + p.v.\frac{2b(\lambda_1,\lambda_2,\xi,\eta)\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}}{\xi^2-\eta^2}$$

where a and b are smooth functions in all variables satisfying

$$a^2(\lambda_1,\lambda_2,\xi) + b^2(\lambda_1,\lambda_2,\xi,\xi) = 1$$

and the following size and regularity:

$$|\partial_{\lambda_{12}}^\alpha(\xi\partial_\xi)^\beta(\eta\partial_\eta)^\gamma(\xi\partial_\xi+\eta\partial_\eta)^\sigma b| \lesssim \frac{1}{\langle\log\xi\rangle\langle\log\eta\rangle} \qquad \xi,\eta\lesssim 1 \tag{4.44}$$

$$|\partial_{\lambda_{12}}^\alpha(\xi\partial_\xi)^\beta\partial_\eta^\gamma(\xi\partial_\xi+\eta\partial_\eta)^\sigma b| \lesssim \frac{1}{\langle\log\xi\rangle(1+\eta)^N} \qquad \xi\lesssim 1\lesssim\eta \tag{4.45}$$

$$|\partial_{\lambda_{12}}^\alpha\partial_\xi^\beta\partial_\eta^\gamma(\xi\partial_\xi+\eta\partial_\eta)^\sigma b| \lesssim \frac{1}{\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}(1+|\eta-\xi|)^N} \qquad 1\lesssim\xi,\eta \tag{4.46}$$

where $\alpha,\sigma,N\in\mathbb{N}$ and $\beta,\gamma\in\mathbb{N}$ if $|\xi-\eta|\gtrsim\max(\xi,\eta)$ and $\beta+\gamma\le 2$ if $|\xi-\eta|\ll\max(\xi,\eta)$.

Proof. Given a smooth radial function f in $(0,\infty)$ which is compactly supported away from zero we have the following integral representation for $\mathcal{F}_{\lambda_1}\mathcal{F}^*_{\lambda_2} f$:

$$\begin{aligned}\mathcal{F}_{\lambda_1}\mathcal{F}^*_{\lambda_2} f(\xi) &= \int_0^\infty \int_0^\infty \psi_\xi^{\lambda_1}(r)\psi_\eta^{\lambda_2}(r) f(\eta) d\eta r dr \\ &= \lim_{R\to\infty} \int_0^\infty \int_0^\infty \chi(r/R)\psi_\xi^{\lambda_1}(r)\psi_\eta^{\lambda_2}(r) f(\eta) d\eta r dr\end{aligned} \tag{4.47}$$

where χ is a smooth radial compactly supported bump function which equals 1 in the unit ball. Then the off-diagonal part of $K_{\lambda_1\lambda_2}(\xi,\eta)$ is given by

$$K_{\lambda_1\lambda_2}(\xi,\eta) = \lim_{R\to\infty} \int_0^\infty \chi(r/R)\psi_\xi^{\lambda_1}(r)\psi_\eta^{\lambda_2}(r) r dr := \langle \psi_\xi^{\lambda_1}, \psi_\eta^{\lambda_2}\rangle$$

This is meaningful if the above limit exists uniformly on compact sets off the diagonal; that is always the case due to the asymptotics for ψ_ξ as $r\to\infty$ in (4.12), (4.13). Multiplying the previous relation by $(\xi^2-\eta^2)$ and integrating by parts gives

$$\begin{aligned}(\xi^2-\eta^2)\langle \psi_\xi^{\lambda_1}, \psi_\eta^{\lambda_2}\rangle &= \langle \xi^2\psi_\xi^{\lambda_1}, \psi_\eta^{\lambda_2}\rangle - \langle \psi_\xi^{\lambda_1}, \eta^2\psi_\eta^{\lambda_2}\rangle = \langle \tilde H_{\lambda_1}\psi_\xi^{\lambda_1}, \psi_\eta^{\lambda_2}\rangle - \langle \psi_\xi^{\lambda_1}, \tilde H_{\lambda_2}\psi_\eta^{\lambda_2}\rangle \\ &= \langle \psi_\xi^{\lambda_1}, (\tilde H_{\lambda_1} - \tilde H_{\lambda_2})\psi_\eta^{\lambda_2}\rangle = \langle \psi_\xi^{\lambda_1}, (\tilde V_{\lambda_1} - \tilde V_{\lambda_2})\psi_\eta^{\lambda_2}\rangle \\ &= \langle \psi_\xi^{\lambda_1}, \frac{4(\lambda_2^2-\lambda_1^2)}{(1+\lambda_1^2 r^2)(1+\lambda_2^2 r^2)}\psi_\eta^{\lambda_2}\rangle\end{aligned}$$

which leads to the following formula for b:

$$b(\lambda_1,\lambda_2,\xi,\eta) = \langle \psi_\xi^{\lambda_1}, \frac{4(\lambda_2^2-\lambda_1^2)}{(1+\lambda_1^2 r^2)(1+\lambda_2^2 r^2)}\psi_\eta^{\lambda_2}\rangle$$

We note that the above computation should be done with the cutoff $\chi(r/R)$ included, and then pass to the limit $R\to\infty$; This computation is tedious but routine, so we omit it. The bounds (4.44)-(4.46) are obtained from this formula using again the representation (4.11), (4.12) and (4.13) for the functions ψ_ξ.

Next we identify the behavior of the kernel $K_{\lambda_1\lambda_2}$ near the diagonal by using the representation in (4.12) and (4.13) for $r\xi \gtrsim 1$,

$$\psi_\xi^\lambda(r) = \Re\left(r^{-\frac12} e^{ir\xi}(i - \frac{1}{8r\xi}) a(\lambda\xi)\right) + O(r^{-\frac52})$$

Since we have already identified the off-diagonal kernel of $K_{\lambda_1\lambda_2}$, for this purpose we can freely neglect any part of $K_{\lambda_1\lambda_2}$ which has a locally bounded kernel. For large r we have

$$\begin{aligned}\psi_\xi^{\lambda_1}(r)\psi_\eta^{\lambda_2}(r) &= \frac1r \Re\left(e^{ir\xi}(i - \frac{7}{8r\xi}) a(\lambda_1\xi)\right)\Re\left(e^{ir\eta}(i - \frac{7}{8r\eta}) a(\lambda_2\eta)\right) + O(\frac{1}{r^3}) \\ &= I_{res} - I_{nr} + II_{res} - II_{nr} + O(\frac{1}{r^3})\end{aligned}$$

where

$$I_{res} = \frac{1}{2r}\Re\left(a(\lambda_1\xi)\bar a(\lambda_2\eta)e^{ir(\xi-\eta)}\right), \qquad I_{nr} = \frac{1}{2r}\Re\left(a(\lambda_1\xi)a(\lambda_2\eta)e^{ir(\xi+\eta)}\right)$$

$$II_{res} = \frac{7}{16r^2}\frac{\xi-\eta}{\xi\eta}\Im\left(a(\lambda_1\xi)\bar a(\lambda_2\eta)e^{ir(\xi-\eta)}\right),$$

$$II_{nr} = \frac{7}{16r^2}\frac{\xi+\eta}{\xi\eta}\Im\left(a(\lambda_1\xi)a(\lambda_2\eta)e^{ir(\xi+\eta)}\right)$$

Hence returning to (4.47), for ξ in a compact set we have

$$\mathcal{F}_{\lambda_1}\mathcal{F}_{\lambda_2}^* f(\xi) = \int_0^\infty\int_0^\infty (I_{res} - I_{nr} + II_{res} - II_{nr}) f(\eta) d\eta r dr + O(\|f\|_{L^1})$$

In the nonresonant terms I_{nr} and II_{nr} the phase is uniformly oscillatory, so integration by parts in r allows for a gain of arbitrarily many powers of r^{-1}.

In the second resonant term II_{res} the phase may be stationary. However, the factor of $\xi - \eta$ allows for one integration by parts in r which gain an r^{-1} factor, sufficient to insure absolute convergence in the integral. Thus we are left with

$$\begin{aligned}\mathcal{F}_{\lambda_1}\mathcal{F}_{\lambda_2}^* f(\xi) &= \int_0^\infty\int_0^\infty I_{res} f(\eta) d\eta r dr + O(\|f\|_{L^1}) \\ &= \int_0^\infty\int_0^\infty \frac{1}{2}\Re\left(a(\lambda_1\xi)\bar{a}(\lambda_2\eta)e^{ir(\xi-\eta)}\right) f(\eta) d\eta dr + O(\|f\|_{L^1})\end{aligned}$$

Using elementary properties of the Fourier transform and the notation $\mathbf{H}$ for the Heaviside function, the last integral is expressed in the form

$$\begin{aligned}&\frac{1}{4}\Re\iint_{\mathbb{R}^2} e^{ir(\xi-\eta)} a(\lambda_1\xi)\overline{a}(\lambda_2\eta)(1+\mathbf{H}(r)) f(\eta) d\eta dr \\ &\qquad = \frac{\pi}{2}\Re(a(\lambda_1\xi)\overline{a}(\lambda_2\xi)) f(\xi) + \frac{\pi}{2} p.v. \int \Im(a(\lambda_1\xi)\overline{a}(\lambda_2\eta))\frac{1}{\xi-\eta} f(\eta) d\eta\end{aligned}$$

Hence for ξ, η in a compact set we obtain

$$K_{\lambda_1\lambda_2}(\xi,\eta) = \frac{\pi}{2}\Re(a(\lambda_1\xi)\overline{a}(\lambda_2\xi))\delta(\xi-\eta) + \frac{\pi}{2}\Im(a(\lambda_1\xi)\overline{a}(\lambda_2\xi)) p.v.\frac{1}{\xi-\eta} + O(1)$$

Comparing this with the off-diagonal representation of $K_{\lambda_1\lambda_2}$, we obtain the representation in the proposition with

$$a(\lambda_1,\lambda_2,\xi) = \frac{\pi}{2}\Re(a(\lambda_1\xi)\overline{a}(\lambda_2\xi)), \qquad b(\lambda_1,\lambda_2,\xi,\xi) = \frac{\pi}{2}\Im(a(\lambda_1\xi)\overline{a}(\lambda_2\xi))$$

Using (4.8), it then follows that on the diagonal we have

$$a^2(\lambda_1,\lambda_2,\xi) + b^2(\lambda_1,\lambda_2,\xi,\xi) = |\frac{\pi}{2}a(\lambda_1\xi)\overline{a}(\lambda_2\eta)|^2 = 1.$$

□

4.7. **Compositions of Littlewood-Paley projectors.** We first consider dyadic bump functions in the Fourier space, and we estimate their inverse Fourier transforms:

Proposition 4.11. *Let $k \in \mathbb{Z}$ and χ_k be a unit size bump function supported in the $\{\xi \approx 2^k\}$ dyadic region. Then for $|\lambda - 1| \ll 1$ its inverse Fourier transform satisfies the bounds*

$$|\partial_\lambda^\alpha (r\partial_r)^\beta (\mathcal{F}_\lambda^* \chi_k)(r)| \lesssim_{\alpha,\beta} 2^{\frac{3k}{2}} m_k(r)(1+2^k r)^{-N} \tag{4.48}$$

Proof. We have

$$\mathcal{F}_\lambda^*\chi_k(r) = \int \psi_\xi^\lambda(r)\chi_k(\xi) d\xi$$

If $r \lesssim 2^{-k}$, then using the pointwise estimate (4.11) gives (4.11). If $r \geq 2^{-k}$, then using (4.12), (4.13) and stationary phase gives (4.11). $\square$

The next step is to consider the composition of two dyadically separated projectors associated with different frames.

Proposition 4.12. *Let $j, k \in \mathbb{Z}$ with $|j-k| \gg 1$, and λ in a compact subset of $(0, \infty)$. Then the kernels $K_{jk}(r,s)$ of $P_j P_k^\lambda$ satisfy the bounds*

$$|K_{jk}(r,s)| \lesssim \frac{2^{j+k-|j-k|-N(j^+ + k^+)}}{\langle j^- \rangle \langle k^- \rangle} \frac{m_j(r) m_k(s)}{(1+2^j r)^N (1+2^k s)^N} \tag{4.49}$$

Proof. The kernel $K_{jk}(r,s)$ of $P_j \tilde{P}_k^\lambda$ is given by

$$K_{jk}(r,s) = \int \psi_\xi(r) \chi_j(\xi) K_{1\lambda}(\xi, \eta) \psi_\eta^\lambda(s) \chi_k(\eta) d\mu d\eta$$

By Proposition 4.10, the symbol $\chi_j(\xi) K_{1\lambda}(\xi,\eta) \chi_k(\eta)$ is smooth on the dyadic scales and has size

$$|\chi_j(\xi) K_{1\lambda}(\xi,\eta)\chi_k(\eta)| \lesssim \frac{1}{\langle k^- \rangle \langle j^- \rangle} 2^{-\frac{k+j}{2} - |k-j|} 2^{-N(k^+ + j^+)}$$

Hence (4.49) follows from Proposition 4.11. $\square$

Given a frequency localized function in one frame, the above proposition allows us to relocalize it in a different frame with good pointwise error bounds. For this we consider a projector $\tilde{P}_k$ whose symbol $\tilde{\chi}_k$ equals 1 within the support of χ_k.

Corollary 4.13. *Let $\psi \in L^2$ and $k \in \mathbb{Z}$. Then we have*

$$P_k^\lambda \psi = \tilde{P}_k P_k^\lambda \psi + \psi_k^{err}, \qquad |\psi_k^{err}| \lesssim \frac{2^{k^-} 2^{-Nk^+} \langle (\log(2+r) + k)^- \rangle}{\langle k^- \rangle^2 (1 + 2^{k^-} r)^2} \|P_k^\lambda \psi\|_{L^2} \tag{4.50}$$

Proof. We write

$$(1 - \tilde{P}_k) P_k^\lambda = \sum_{|j-k| \gg 1} P_j \tilde{P}_k^\lambda P_k^\lambda$$

A direct estimate using (4.49) gives

$$|P_j P_k^\lambda \psi(r)| \lesssim \frac{2^{j-|j-k|-N(j^+ + k^+)}}{\langle j^- \rangle \langle k^- \rangle} \frac{m_j(r)}{(1+2^j r)^N} \|P_k^\lambda \psi\|_{L^2}$$

It remains to evaluate the sum with respect to j of the coefficients on the right. Because of the rapid decay for positive j, k, it suffices to assume that $k < 0$ and restrict the sum to $j < 0$. Then we are left with the sum

$$\frac{2^k}{|k|} \sum_{j<0} \log(1+r) \frac{2^{j-k-|j-k|}}{j^2 (1+2^j r)^N}$$

There are two thresholds for j in this sum, namely $-\log(1+r)$ and k. If $2^k r > 1$ then the sum is given by the summand at $j = -\log(1+r)$. Else, the sum is bounded by

$$\frac{2^k}{|k|} \sum_{j=k}^{-\log(1+r)} \frac{\log(1+r)}{j^2} \frac{1}{(1+2^j r)^N}$$

The bound (4.50) easily follows. $\square$

Finally, we consider the product of three projectors:

Proposition 4.14. *Let $j, h, k \in \mathbb{Z}$ and λ in a compact subset of $(0, \infty)$.*

a) Assume that $|j-k| \gg 1$ and $|h-k| \gg 1$. Then the kernels $K_{jkh}(r,s)$ of $P_j P_k^\lambda P_h$ can be represented as the sum of a rapidly convergent series of terms $K^l_{jkh}(r,s)$ of the form

$$(4.51)\qquad \begin{aligned} K^l_{jkh}(r,s) &= c_{jkh}\, l^{-N} g^l(\lambda)\phi_j(r)\phi_h(s), \\ c_{jkh} &= \frac{2^{-|j-k|-|k-h|-N(j^+ + k^+ + h^+)}}{\langle j\rangle\langle k\rangle^2\langle h\rangle} \\ |\phi_j(r)| &\le 2^j(1+2^j r)^{-N}, \quad |\phi_h(s)| \le 2^h(1+2^h s)^{-N}. \end{aligned}$$

with g^l uniformly bounded in C^N.

b) Assume that either $|j-k| \lesssim 1$ and $|h-k| \gg 1$ or $|j-k| \gg 1$ and $|h-k| \lesssim 1$. Then the kernels $K_{jkh}(r,s)$ of $P_j P_k^\lambda P_h$ can be represented as above but with

$$(4.52)\qquad c_{jkh} = \frac{2^{-|j-h|-N(j^+ + h^+)}}{\langle j\rangle\langle h\rangle}$$

c) Assume that $|j-h| \lesssim 1$ and $|h-k| \lesssim 1$. Then the operators $P_j P_k^\lambda P_h$ can be represented as sum of a rapidly convergent series of the form

$$(4.53)\qquad P_j P_k^\lambda P_h = P_j P_k P_h + c_{jkh} \sum l^{-N} g^l(\lambda) Q^l_{jkh}, \qquad c_{jkh} = \frac{1}{2^{k^+}\langle k^-\rangle^2}$$

with g^l uniformly bounded in C^N and $\|Q^l_{jkh}\|_{L^2 \to L^2} \le 1$.

Proof. a) We use Proposition 4.10 to estimate the Fourier kernel $\hat{K}_{jkh}$ of $P_j \tilde{P}_k^\lambda \tilde{P}_h$, given by

$$\hat{K}_{jkh}(\xi,\zeta) = \int \chi_j(\xi) K_{1\lambda}(\xi,\eta)\chi_k(\eta) K_{\lambda 1}(\eta,\zeta)\chi_h(\zeta) d\eta$$

It follows that $\hat{K}_{jkh}$ is smooth in ξ, ζ on the dyadic scale, smooth in λ and has size

$$(4.54)\qquad |\hat{K}_{jkh}(\xi,\zeta)| \lesssim 2^{-\frac{j+h}{2}} c_{jkh}$$

Separating variables, it suffices to consider kernels $\hat{K}_{jkh}$ of the form

$$\hat{K}_{jkh}(\xi,\zeta) = \sum_l 2^{-\frac{j+h}{2}} c_{jkh} l^{-N} g^l(\lambda)\chi^l_j(\xi)\chi^l_h(\eta)$$

with χ^l_j, χ^l_h smooth dyadic bump functions, and g smooth. Then the conclusion follows using the bounds for the inverse Fourier transforms of χ^l_j and χ^l_h given by Proposition 4.11.

b) The proof is similar to the one in case (a), with the only difference that we need to consider the contribution of the diagonal term in exactly one of the kernels $K_{1\lambda}(\xi,\eta)$ and $K_{\lambda 1}(\eta,\zeta)$. Here we take advantage of the factor $(\xi\partial_\xi + \eta\partial_\eta)^\sigma$ with $\sigma \in \mathbb{N}$ in (4.44)-(4.46) in order to claim that if s is smooth in η then

$$\int \frac{b(\lambda_1,\lambda_2,\xi,\eta)}{\xi^2-\eta^2} s(\eta) d\eta$$

is smooth in ξ.

c) In this case we need to allow near diagonal contributions from both kernels $K_{1\lambda}(\xi,\eta)$ and $K_{\lambda 1}(\eta,\zeta)$. For each of them we can use Proposition 4.11 to write

$$K_{1\lambda}(\xi,\eta) = \delta_{\xi=\eta} + (1 - a(1,\lambda,\xi,\eta))\,\delta_{\xi=\eta} + p.v.\frac{2b(1,\lambda,\xi,\eta)\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}}{\xi^2-\eta^2}$$

In the region $\xi,\eta,\zeta \approx 2^k$ the functions $1-a$ and b are smooth in the λ variable and have size $\langle k^-\rangle^{-2}2^{-k^+}$. Hence separating the variable λ and performing the remaining compositions we arrive at the desired conclusion.

□

4.8. **Nonresonant quadrilinear forms.** Here we prove bounds for quadrilinear expressions in nonresonant situations. Precisely, we consider four dyadic frequencies

$$j < k_3 < k_2 = k_1, \qquad j < 0$$

and corresponding frequencies $\xi \approx 2^j$, $\xi_l \approx 2^{k_l}, l \in \{1,2,3\}$ which are subject to one of the two additional conditions:

i) $k_2 - k_3 \gg 1$ and $|\xi_1 - \xi_2| \ll 2^{k_3}$.

ii) $|k_3 - k_2| \lesssim 1$ and $|\xi_1^2 + \xi_2^2 - \xi_3^2| \ll 2^{2k_3}$.

In both cases ξ_1 and ξ_2 may be close but ξ_3 is dyadically separated from them. To the quadruplet of generalized eigenfunctions ψ_ξ, ψ_{ξ_1} ψ_{ξ_2} and ψ_{ξ_3} we associate the quadrilinear expressions:

$$G_0(\xi_1,\xi_2,\xi_3,\xi) = \int \psi_\xi(r)\psi_{\xi_1}(r)\psi_{\xi_2}(r)\psi_{\xi_3}(r)rdr$$

$$G_1(\xi_1,\xi_2,\xi_3,\xi) = \int_0^\infty \psi_\xi(r)\psi_{\xi_3}(r)\int_r^\infty \frac{1}{s}\psi_{\xi_1}(s)\psi_{\xi_2}(s)dsrdr$$

$$G_2(\xi_1,\xi_2,\xi_3,\xi) = \int_0^\infty \psi_\xi(r)\psi_{\xi_1}(r)\int_r^\infty \frac{1}{s}\psi_{\xi_2}(s)\psi_{\xi_3}(s)dsrdr$$

Also we consider the truncated integrals

$$G_0^m(\xi_1,\xi_2,\xi_3,\xi) = \int \chi_{>m}(r)\psi_\xi(r)\psi_{\xi_1}(r)\psi_{\xi_2}(r)\psi_{\xi_3}(r)rdr$$

$$G_1^m(\xi_1,\xi_2,\xi_3,\xi) = \int_0^\infty \psi_\xi(r)\psi_{\xi_3}(r)\int_r^\infty \chi_{>m}(s)\frac{1}{s}\psi_{\xi_1}(s)\psi_{\xi_2}(s)dsrdr$$

where $\chi_{>m}$ is a smooth approximation of the characteristic function of $[2^m,\infty)$. We denote $D^{\alpha\beta\gamma\sigma} = (\xi_1\partial_{\xi_1})^\alpha(\xi_3\partial_{\xi_1})^\beta(\xi_3\partial_{\xi_2})^\gamma(\xi_3\partial_{\xi_3})^\sigma$ and

$$g_{jk_1k_2k_3} = \frac{2^{\frac{j}{2}}}{|j|}\frac{\langle k_3^-\rangle 2^{-2k_3^+}}{2^{\frac{k_3}{2}}} \tag{4.55}$$

and estimate these integrals as follows:

Proposition 4.15. *For ξ, ξ_1, ξ_2 and ξ_3 as above we have the bounds*

$$|D^{\alpha\beta\gamma\sigma}G_{0,1}(\xi_1,\xi_2,\xi_3,\xi)| \lesssim_{\alpha\beta\gamma\sigma} g_{jk_1k_2k_3}, \tag{4.56}$$

$$|D^{\alpha\beta\gamma\sigma}G_2(\xi_1,\xi_2,\xi_3,\xi)| \lesssim_{\alpha\beta\gamma\sigma} 2^{k_3-k_1}g_{jk_1k_2k_3}. \tag{4.57}$$

In addition, if $m + k_3 \geq 0$ then we have

$$|D^{\alpha\beta\gamma\sigma}G_{0,1}^m(\xi_1,\xi_2,\xi_3,\xi)| \lesssim_{\alpha\beta\gamma\sigma} 2^{k_3-k_1-N(m+k_3)}g_{jk_1k_2k_3} \tag{4.58}$$

Proof. Given the conditions (i),(ii) above and the asymptotic expansions for the functions ψ_ξ, it follows that the integral defining G_0 is oscillatory with frequencies $\pm\xi_1 \pm \xi_2 \pm \xi_3 \pm \xi$ of size 2^{k_3} and larger. Hence the contributions of regions A_m decay rapidly for $m > -k_3$. Only the region $A_{<-k_3}$ has a nontrivial contribution, which we estimate directly. If $k_3 < 0$ then we obtain

$$|G_0| \lesssim \ 2^{\frac{k_1}{2}} 2^{\frac{k_2}{2}} \frac{2^{\frac{k_3}{2}}}{|k_3|} \frac{2^{\frac{j}{2}}}{|j|} \int_{r<2^{-k_3}} \frac{|\log(1+r^2)|^2}{1+2^{k_1}r} r dr \approx \frac{2^{\frac{j}{2}}}{|j|} \frac{|k_3|}{2^{\frac{k_3}{2}}}$$

If $k_3 \geq 0$ then there is some further gain, as ψ_ξ no longer reaches the logarithmic part before the oscillatory regime. In that case we obtain

$$|G_0| \lesssim 2^{\frac{k_1}{2}} 2^{\frac{k_2}{2}} 2^{\frac{k_3}{2}} \frac{2^{\frac{j}{2}}}{|j|} \int_{r\leq 2^{-k_3}} \frac{(r2^{k_3})^2 r^2}{1+2^{k_1}r} r dr \approx \frac{2^{\frac{j}{2}}}{|j|} 2^{-\frac{5k_2}{2}}$$

Adding the differentiation operator $D^{\alpha\beta\gamma\sigma}$ does not alter the pointwise bounds used above. The estimate for the cut-off G_0^m also follows from the above considerations.

In the case of G_2 the inner integral is oscillatory with frequencies of size 2^{k_1}. Hence we obtain

$$\int_r^\infty \frac{1}{s} \psi_{\xi_2}(s)\psi_{\xi_3}(s) ds \approx \frac{1}{2^{k_1}r} \psi_{\xi_2}(r)\psi_{\xi_3}(r), \quad r \gg 2^{-k_1}$$

$$\left| \int_r^\infty \frac{1}{s} \psi_{\xi_2}(s)\psi_{\xi_3}(s) ds \right| \lesssim 2^{\frac{k_1}{2}} 2^{\frac{k_3}{2}} \frac{\langle k_1^- \rangle \langle (k_1 - \ln(1+r))^- \rangle}{2^{2k_1^+}} \frac{2^{2k_3^+}}{\langle k_3^- \rangle}, \quad r \lesssim 2^{-k_1}$$

Then we conclude as in the case of G_0.

Finally we consider G_1. Then we no longer want to estimate the inner integral. Instead we integrate by parts,

$$G_1 = \int_0^\infty \frac{1}{r} \int_0^r \psi_\xi(s)\psi_{\xi_3}(s) s ds \psi_{\xi_1}(r)\psi_{\xi_2}(r) dr$$

Now the inner integral is again oscillatory, and using the orthogonality of ψ_ξ and ψ_{ξ_3} we can switch the inner integration to $[r, \infty)$ and estimate

$$\int_0^r \psi_\xi(s)\psi_{\xi_3}(s) s ds \approx \frac{r}{2^{k_3}} \psi_\xi(r)\psi_{\xi_3}(r), \quad r \gg 2^{-k_3}$$

$$| \int_0^r \psi_\xi(s)\psi_{\xi_3}(s) s ds| \lesssim \frac{2^{\frac{j}{2}}}{|j|} \frac{2^{\frac{k_3}{2}} 2^{2k_3^+}}{\langle k_3^- \rangle} r^2 \ln^2(1+r^2), \quad r \lesssim 2^{-k_3}$$

Then a similar argument to the one used for G_0 leads to the same bound, as the main contribution arising from the region $r\xi_3 \approx 1$ rests unchanged. A similar argument gives the estimate for G_1^m.

□

5. The linear $\tilde{H}$ Schrödinger equation

Here we consider bounds and function spaces associated to the linear $\tilde{H}$ evolution

$$(i\partial_t - \tilde{H})\psi = f, \qquad \psi(0) = \psi_0 \tag{5.1}$$

restricted to radial functions. We recall that the operator $\tilde{H}$ has the form

$$\tilde{H} = -\Delta + \tilde{V}, \qquad \tilde{V} = \frac{4}{r^2(1+r^2)}$$

and, restricted to radial functions, admits the factorization $\tilde{H} = LL^*$. In the first part of the section we introduce several relevant function spaces associated to this evolution, and in the second part we prove that (5.1) is well-posed in these spaces.

5.1. **Function spaces.**

5.1.1. *Globally defined spaces.* To measure solutions we will use the energy norm $L_t^\infty L_x^2$, the Strichartz norm $L_{t,x}^4$ whose one-dimensional correspondents are $L_t^\infty L_r^2$, respectively $L_{t,r}^4$. We also use the local energy norm defined by

$$\|\psi\|_{LE} = \|[r\log(2+r)]^{-1}\psi\|_{L^2}$$

Combining these norms we define the space S for solutions to (5.1) and the dual type space N (precisely, $S = N^*$) for the inhomogeneous term in (5.1).

$$S = L_t^\infty L_r^2 \cap L_{tr}^4 \cap LE, \qquad N = L_t^1 L_r^2 + L_{tr}^{\frac{4}{3}} + LE^*$$

5.1.2. *Frequency localized spaces.* For many of our estimates we need to be more precise and work with a dyadic Littlewood-Paley decomposition in the $\tilde{H}$-frequency,

$$\psi = P_k\psi$$

To measure frequency 2^k waves we define a local energy space LE_k,

$$\|\psi\|_{LE_k} = 2^k\|\psi\|_{L^2(A_{<-k})} + \sup_{m>-k} 2^{\frac{k-m}{2}}\|\psi\|_{L^2(A_m)}$$

as well as an adapted L_k^4 norm (allowed due to the radial symmetry):

$$\|\psi\|_{L_k^4} = \sup_m \max\{2^{-\frac{m+k}{2}}, 2^{\frac{m+k}{8}}\}\|\psi\|_{L^4(A_m)}.$$

The dual norms are denoted by LE_k^*, respectively $L_k^{\frac{4}{3}}$. The frequency adapted versions of the S and N norms are

$$S_k = L_t^\infty L_r^2 \cap L_k^4 \cap LE_k, \qquad N_k = L_t^1 L_r^2 + L_k^{\frac{4}{3}} + LE_k^*, \qquad S_k = N_k^*$$

Square summing these norms we obtain the spaces l^2S and l^2N with norms

$$\|\psi\|_{l^2S}^2 = \sum_{k\in\mathbb{Z}} \|P_k\psi\|_{S_k}^2, \qquad \|f\|_{l^2N}^2 = \sum_{k\in\mathbb{Z}} \|P_k f\|_{N_k}^2,$$

Given the nice bound (4.40) on the kernel of the projectors P_k it is easy to see that these are dual spaces, thus justifying our notation.

We remark that in the frequency localized setting one has the usual Bernstein type estimates, with an additional improvement near $r = 0$. Precisely, from the pointwise bounds (4.40) for the spectral projector kernels we obtain

Lemma 5.1. *The following frequency localized pointwise bounds hold:*

$$\|m_k^{-1}P_k\psi\|_{L_t^2L_r^\infty} \lesssim \|P_k\psi\|_{LE_k} \tag{5.2}$$

$$\|(1+2^kr)^{\frac{1}{2}}m_k^{-1}P_k\psi\|_{L_{t,r}^\infty} \lesssim 2^k\|P_k\psi\|_{L^\infty L^2} \tag{5.3}$$

5.1.3. *The Z spaces.* The Z spaces, which are used later in the paper for the parameter λ which tracks the evolution of the Schrödinger map along the soliton family, are defined by

$$Z = \dot{W}^{1,1} + \dot{H}^{\frac{1}{2},1}, \qquad Z_0 = Z + L^2 \cap L^\infty$$

Concerning these spaces we need the following

Lemma 5.2. *The spaces Z and Z_0 are algebras.*

The proof is not very difficult and left to the reader.

5.1.4. *The $X^{s,b}$ spaces.* For a function $\psi : \mathbb{R} \times (0,\infty) \to \mathbb{C}$, we define its space-time Fourier transform by $\hat{\psi} = \mathcal{F}_t \mathcal{F}_{\tilde{H}} \psi$, where $\mathcal{F}_t$ is the time Fourier transform. We define the modulation localization operators $\{Q_j\}_{j\in\mathbb{Z}}$ by $\widehat{Q_j\psi}(\tau,\xi) = \chi_j(|\tau + \xi^2|)\hat{\psi}(\tau,\xi)$ where χ_j is a smooth characteristic function of the set $\{\tau \sim 2^j\}$. We define $Q_{<j}, Q_{\leq j}, Q_{>j}, Q_{\geq j}$ in a similar way.

The $X^{s,b}$ type spaces $\dot{X}^{0,\pm\frac{1}{2},1}$ and $\dot{X}^{0,\pm\frac{1}{2},\infty}$ associated to the $\tilde{H}$ flow are defined as

$$\|\psi\|_{\dot{X}^{0,\pm\frac{1}{2},1}} = \sum_j 2^{\pm\frac{j}{2}} \|Q_j \psi\|_{L^2}, \qquad \|\psi\|_{\dot{X}^{0,\pm\frac{1}{2},\infty}} = \sup_j 2^{\pm\frac{j}{2}} \|Q_j\psi\|_{L^2}$$

These spaces play a less prominent role in this paper, as they are used only at high modulations $j > 2k$ where 2^k is the frequency of ψ. Precisely, we use them to define the dyadic space S_k^r with norms

$$\|\psi\|_{S_k^r} = \|\psi\|_{S_k} + \|Q_{>2k}\psi\|_{\dot{X}^{0,\frac{1}{2},1} + W^{1,1}L^2}$$

We observe that due to the truncation to high modulations in the second term above, we can replace the norm by an equivalent one and write

$$\|\psi\|_{S_k^r} = \|\psi\|_{S_k} + \|Q_{>2k}\psi\|_{ZL^2}$$

Somewhat similarly, for the inhomogeneous term we define the space N_k^r by

$$N_k^r = L^1L^2 + Q_{>2k}\dot{X}^{0,-\frac{1}{2},1}$$

Summing up dyadic contributions in l^2 we obtain the spaces l^2S^r and l^2N^r:

$$\|\psi\|^2_{l^2S^r} = \sum_k \|P_k\psi\|^2_{S_k^r}, \qquad \|f\|_{l^2N^r} = \sum_k \|P_kf\|^2_{N_k^r}$$

These norms, used only on frequency 2^k functions, represent a modest strengthening of the S_k norms but only for high modulations. Their role in this paper is twofold. On one hand, they represent all the information we are able to transfer from the time dependent frame setting in the next section back into the fixed frame setting; on the other hand, they are critically used in Section 9 to recover the regularity of the parameter λ describing the evolution of the Schrödinger map along the soliton family.

5.1.5. *The U^2 and V^2 spaces.* Given a Hilbert space $\mathcal{H}$ (which in our case will be $L^2(rdr)$), and $1 \leq p < \infty$, we define the spaces $U^p\mathcal{H}$ and $V^p\mathcal{H}$ as follows:

a) $U^p\mathcal{H} \subset L^\infty\mathcal{H}$ is an atomic space, where the atoms are step functions

$$a = \sum_k 1_{[t_k,t_{k+1})}u_k, \qquad \sum_k \|u_k\|^p_{\mathcal{H}} \leq 1$$

with t_k arbitrary finite increasing sequence in $[-\infty,\infty)$.

b) $V^pH \subset L^\infty H$ is the space of all right continuous H valued functions for which the following norm is finite:

$$\|u\|^p_{V^pH} = \sup_{t_k \nearrow} \sum_k \|u(t_{k+1} - u(t_k))\|^p_H$$

where the sup norm is over all increasing sequences $\{t_k\}$ as above.

In our case we use the above definitions to construct the $U^p_{\tilde{H}}L^2$ and $V^p_{\tilde{H}}L^2$ spaces associated to the $\tilde{H}$ flow by

$$\|\psi\|_{U^p_{\tilde{H}}L^2} = \|e^{-it\tilde{H}}\psi(t)\|_{U^pL^2}, \qquad \|\psi\|_{V^p_{\tilde{H}}L^2} = \|e^{-it\tilde{H}}\psi(t)\|_{V^pL^2} \tag{5.4}$$

Such spaces were introduced in the study of dispersive equations in unpublished work of the second author. For more details we refer the reader to [17], [5] and [13]. In the context of Schrödinger maps such spaces were also used in [4].

We are primarily interested in the case $p = 2$. There we have the embeddings

$$\dot{X}^{0,\frac{1}{2},1} \subset U^2_{\tilde{H}}L^2 \subset V^2_{\tilde{H}}L^2 \subset \dot{X}^{0,\frac{1}{2},\infty} \tag{5.5}$$

Another favorable property of these spaces is that they are stable with respect to modulation truncations:

$$Q_{<j}P_k : U^2_{\tilde{H}}L^2 \to U^2_{\tilde{H}}L^2, \qquad Q_{<j}P_k : V^2_{\tilde{H}}L^2 \to V^2_{\tilde{H}}L^2 \tag{5.6}$$

For the inhomogeneous term we also define the space $DU^2_{\tilde{H}}L^2$ as

$$DU^2_{\tilde{H}}L^2 = \{(i\partial_t - \tilde{H})\psi; \psi \in U^2_{\tilde{H}}L^2\}$$

with the induced norm. Here the derivatives are interpreted as distributional derivatives. This satisfies

$$\dot{X}^{0,-\frac{1}{2},1} \subset DU^2_{\tilde{H}}L^2 \subset \dot{X}^{0,-\frac{1}{2},\infty} \tag{5.7}$$

5.1.6. *The sharp spaces.* Here we define our strongest dyadic spaces, namely $S^\sharp_k$ for frequency 2^k solutions, with norm

$$\|\psi\|^2_{S^\sharp_k} = \|\psi\|^2_{(S_k \cap V^2_{\tilde{H}}L^2)} + \|Q_{>2k}\psi\|^2_{W^{1,1}L^2 + \dot{X}^{0,\frac{1}{2},1}}$$

as well as the space $N^\sharp_k$ for the inhomogeneous term, with norm

$$\|f\|^2_{N^\sharp_k} = \|f\|^2_{(N_k + DU^2_{\tilde{H}}L^2)} + \|Q_{>2k}f\|^2_{L^1L^2 + \dot{X}^{0,-\frac{1}{2},1}}$$

As before, we also define the full norms $l^2S^\sharp$ and $l^2N^\sharp$ by

$$\|\psi\|^2_{l^2S^\sharp} = \sum_k \|P_k\psi\|^2_{l^2S^\sharp_k}, \qquad \|\psi\|^2_{l^2N^\sharp} = \sum_k \|P_k\psi\|^2_{l^2N^\sharp_k}$$

The V^2 and DU^2 spaces have been added in in order to allow for a harmless transition between the high and low modulations, and also to simplify some proofs in this section. Otherwise, the DU^2 norm above plays no role. The V^2 space does play a role though, namely to allow for better bounds when truncating in modulation.

The $S^\sharp$ and $N^\sharp$ type spaces are needed at two crucial points in the article. First, we use them to establish the well-posedness of the non-autonomous $\tilde{H}_\lambda$ Schrödinger flow in the next section. Secondly, we use them for the bootstrapping estimates in the nonlinear problem in Section 8.

5.1.7. *Restrictions to compact intervals.* For the purpose of bootstrap arguments, many of our estimates need to be proved first on compact time intervals $I = [0, T]$. Thus we need to define our function spaces also on such intervals. This is done in a standard manner, in terms of extensions to the full real line, by setting

$$\|f\|_{\mathcal{X}(I)} = \inf\{\|f^{ext}\|_{\mathcal{X}};\ f^{ext} \text{ extends } f \text{ from } I \text{ to } \mathbb{R}\}$$

where $\mathcal{X}$ is any of the spaces previously introduced in this section. In fact, it is only the use of the $\dot{X}^{0,\pm\frac{1}{2},1}$ structure at high modulations which requires the use of extensions.

We say that an extension f^{ext} of f is suitable if $\|f^{ext}\|_{\mathcal{X}} \sim \|f\|_{\mathcal{X}(I)}$. Some ways of producing suitable extensions are described next:

- For f in Z or Z_1 a suitable extension is given by

$$f^{ext}(t) = \begin{cases} f(a), & t \leq a \\ f(t), & a \leq t \leq b \\ f(b), & b \leq t \end{cases}$$

- For f in Z_0 a suitable extension is the zero extension.
- For f in S, N, S_k, N_k, l^2S and l^2N a suitable extension is the zero extension.
- For ψ in the spaces S_k^r, l^2S^r, $S_k^\sharp$, $l^2S^\sharp$ a suitable extension is obtained by solving the homogeneous equation outside I,

$$\psi^{ext}(t) = \begin{cases} e^{-i(t-a)\tilde{H}}\psi(0), & t \leq 0 \\ \psi(t), & a \leq t \leq b \\ e^{i(t-b)\tilde{H}}\psi(T), & t \geq T \end{cases} \tag{5.8}$$

  Here a nonzero extension is required due to the high modulation structure of the sharp spaces. This high modulation structure is of the form ZL^2, therefore this extension can be thought of as a direct counterpart of the Z extension.
- For the spaces N_k^r, l^2N^r, $N_k^\sharp$, $l^2N^\sharp$ a suitable extension is the zero extension. This is less straightforward, and it involves proving estimates of the type

$$\|1_{[0,T]}\psi\|_{l^2N^\sharp} \lesssim \|\psi\|_{l^2N^\sharp} \tag{5.9}$$

We outline the proof of (5.9). It suffices to consider its dyadic counterpart. Of all components of the $N_k^\sharp$ norm, only the high modulation part is not trivially stable with respect to time truncations. But at high modulation $N_k^\sharp$ has a $(\dot{H}^{-\frac{1}{2},1} + L^1)L^2$ structure, therefore (5.9) follows from an one-dimensional estimate

$$\|\chi_{[0,T]}f\|_{\dot{H}^{-\frac{1}{2},1}+L^1} \lesssim \|f\|_{\dot{H}^{-\frac{1}{2},1}+L^1}$$

In fact we only need to show $\|\chi_{[0,T]}f\|_{H^{-\frac{1}{2},1}+L^1} \lesssim \|f\|_{H^{-\frac{1}{2},1}}$, which can be further reduced to $\|\chi_{[0,T]}f_k\|_{H^{-\frac{1}{2},1}+L^1} \lesssim 2^{-\frac{k}{2}}\|f_k\|_{L^2}$, where $f_k = P_kf$ and P_k are the standard Little-Paley projectors. It is obvious that

$$\|P_{\gtrsim k}(\chi_{[0,T]}f_k)\|_{H^{-\frac{1}{2}}} \lesssim 2^{-\frac{k}{2}}\|f_k\|_{L^2}$$

so it is enough to show that

$$\|P_{\ll k}(\chi_{[0,T]}f_k)\|_{L^1} \lesssim 2^{-\frac{k}{2}}\|f_k\|_{L^2}$$

But this follows from the straightforward estimate $\|P_k\chi_{[0,T]}\|_{L^2} \lesssim 2^{-\frac{k}{2}}$.

5.1.8. *Relations between spaces.* We summarize the relations between the spaces we have defined so far, as well as some other simple properties for them, in the following

Proposition 5.3. *a) The following dyadic embeddings hold*

$$S_k^\sharp \subset S_k^r \subset S_k, \qquad N_k, N_k^r \subset N_k^\sharp \tag{5.10}$$

b) The following embeddings hold:

$$l^2 S^\sharp \subset l^2 S^r \subset l^2 S \subset S, \qquad N \subset l^2 N \subset l^2 N^\sharp, \tag{5.11}$$

c) Modulation localizations:

$$\|Q_{<j} P_k \psi\|_{S_k^\sharp} \lesssim \langle (2k-j)^+ \rangle \|P_k \psi\|_{S_k^\sharp} \tag{5.12}$$

Proof. a) The first (sequence of) embeddings follows directly from the definitions. The only nontrivial part of the second embedding is due to the contribution of the $L^{\frac{4}{3}}$ component of N at high modulation. There we write on dyadic pieces

$$L^{\frac{4}{3}} \subset L^1 L^2 + L^2 L^1 \subset L^1 L^2 + 2^k L^2$$

In the first step we can preserve the frequency localization since by Proposition 4.8 the spectral projectors are bounded in all L^p spaces. In the second step we use Bernstein's inequality, which is valid in our setting due to the kernel bounds (4.40) for the spectral projectors.

b) Given part (a), it remains to show that $l^2 S \subset S$ and $N \subset l^2 N$. By duality it suffices to establish the first embedding. By the fixed time almost orthogonality of the P_k's we have

$$\|\psi\|_{L^\infty L^2}^2 \lesssim \sum_{k \in \mathbb{Z}} \|P_k \psi\|_{L^\infty L^2}^2$$

The L^4 norms are similarly easy to add,

$$\|\psi\|_{L^4(A_m)} \lesssim \sum_{k \in \mathbb{Z}} 2^{-\frac{|m+k|}{8}} \|P_k \psi\|_{L_k^4}$$

which leads to

$$\sum_{m \in \mathbb{Z}} \|\psi\|_{L^2(A_m)}^2 \lesssim \sum_{k \in \mathbb{Z}} \|P_k \psi\|_{L_k^4}^2 \lesssim \|\psi\|_{l^2 S}^2$$

from which $\|\psi\|_{L^4} \lesssim \|\psi\|_{l^2 S}$ follows. Finally we consider the local energy norms, for which we need to show that

$$\|\psi\|_{LE}^2 \lesssim \sum_{k \in \mathbb{Z}} \|P_k \psi\|_{LE_k}^2 \tag{5.13}$$

By a direct summation in the regions $r > 2^{-k}$ and by summing the better bounds in (5.2) in the regions $r < 2^{-k}$ one obtains

$$\begin{aligned} \|\frac{1}{r \ln(2+r)} \sum_k \psi_k\|_{L^2(A_j)} &\lesssim \sum_{k \gtrsim -j} \|\frac{\psi_k}{r}\|_{L^2(A_j)} + \sum_{k \lesssim -j} \frac{1}{\langle k^- \rangle} \|\psi_k\|_{L_t^2 L_r^\infty(A_j)} \\ &\lesssim \sum_{k \gtrsim -j} 2^{-\frac{j+k}{2}} \|\psi_k\|_{LE_k} + \sum_{k \lesssim -j} \frac{1}{\langle k^- \rangle} \|\psi_k\|_{LE_k} \end{aligned}$$

By using the following two estimates on sequences

$$\|(\sum_{k\geq j}\frac{a_k}{\langle k\rangle})_{j\geq 1}\|_{l^2}\lesssim\|(a_k)_{k\geq 1}\|_{l^2},\qquad \|(\sum_{k\geq j}2^{\frac{j-k}{2}}a_k)_{j\in\mathbb{Z}}\|_{l^2}\lesssim\|(a_k)_{k\in\mathbb{Z}}\|_{l^2}$$

we obtain the desired estimate (5.13) in the region $r\gtrsim 1$. A slight variation of the above argument gives also (5.13) in the region $r\lesssim 1$.

c) From (5.5), $\|Q_lP_k\psi\|_{S_k^\sharp}\lesssim\|Q_lP_k\psi\|_{X^{0,\frac{1}{2},1}}\lesssim\|P_k\psi\|_{V^2\tilde{H}}\lesssim\|P_k\psi\|_{S_k^\sharp}$ for any $j\leq l\leq 2k$. Then (5.12) follows, as there is nothing to prove if $j\geq 2k$.

□

5.2. **Estimates for the linear $\tilde{H}$ Schrödinger flow.** Our main well-posedness result concerning the linear $\tilde{H}$ equation is as follows:

Proposition 5.4. *The solution ψ to* (5.1) *satisfies the bound:*

$$\|\psi\|_{l^2S^\sharp}\lesssim\|\psi_0\|_{L^2}+\|f\|_{l^2N^\sharp} \tag{5.14}$$

Proof. The bound (5.14) follows by dyadic summation from its frequency localized version:

$$\|\psi\|_{S_k^\sharp}\lesssim\|\psi_0\|_{L^2}+\|f\|_{N_k^\sharp} \tag{5.15}$$

whenever ψ is localized at $\tilde{H}$-frequency 2^k. The proof of (5.15) proceeds in several steps:

STEP 1: Frequency localized local energy decay. Here we consider functions f, ψ_0 which are localized at frequency 2^k, and prove that the solution of (5.1) obeys the following bound

$$\|\psi\|_{LE_k}+2^{-k}\|\partial_r\psi\|_{LE_k}\lesssim\|\psi_0\|_{L^2}+\|f\|_{LE_k^*}. \tag{5.16}$$

Our approach is in the spirit of the one used by the second author in [25], using the positive commutator method.

First we say that a sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is slowly varying if

$$|\ln\alpha_j-\ln\alpha_{j-1}|\leq 2^{-10},\qquad\forall j\in\mathbb{Z}.$$

Based on such a sequence we introduce the normed space $X_{k,\alpha}$ and its dual $X'_{k,\alpha}$ as follows

$$\|u\|^2_{X_{k,\alpha}}=2^{2k}\|u\|^2_{L^2(A_{<-k})}+2^k\sum_{j\geq -k}\alpha_j2^{-j}\|u\|^2_{L^2(A_j)}$$

$$\|u\|^2_{X'_{k,\alpha}}=2^{-2k}\|u\|^2_{L^2(A_{<-k})}+2^{-k}\sum_{j\geq -k}\alpha_j^{-1}2^{j}\|u\|^2_{L^2(A_j)}$$

For all slowly varying sequences $\{\alpha_n\}_{n\in\mathbb{Z}}$ with $\sum_n\alpha_n=1$, we claim that

$$\|\psi\|_{X_{k,\alpha}}+2^{-k}\|\partial_r\psi\|_{X_{k,\alpha}}\lesssim\|\psi_0\|_{L^\infty_tL^2_r}+\|f\|_{X'_{k,\alpha}} \tag{5.17}$$

Assuming that (5.17) is true, then we can consider another slowly varying sequence $\{\beta_n\}_{n\in Z}$ with $\sum_n\beta_n=1$ and apply the result in (5.17) for $\{\alpha_n+\beta_n\}_{n\in Z}$ to obtain

$$\|\psi\|_{X_{k,\alpha+\beta}}+2^{-k}\|\partial_r\psi\|_{X_{k,\alpha+\beta}}\lesssim\|\psi_0\|_{L^\infty_tL^2_r}+\|f\|_{X'_{k,\alpha+\beta}}$$

from which we derive the weaker estimate

$$\|\psi\|_{X_{k,\alpha}}+2^{-k}\|\partial_r\psi\|_{X_{k,\alpha}}\lesssim\|\psi_0\|_{L^\infty_tL^2_r}+\|f\|_{X'_{k,\beta}} \tag{5.18}$$

Since any l^1 sequence can be dominated by a slowly varying sequence with a comparable l^1 size, we can drop the assumption in (5.18) that α and β are slowly varying. By maximizing the right-hand side with respect to $\alpha \in \lambda^1$ and by minimizing the left-hand side with respect to $\beta \in l^1$, we obtain (5.16).

The remaining part of this step is devoted to the proof of (5.17). We start by introducing the antisymmetric multiplier

$$Qu = \chi(2^k r) r\partial_r u + r\partial_r(\chi(2^k r)u)$$

where χ will be chosen to be a smooth function related to the slowly varying sequence α_n. Note that if the problem had a scale invariance then one could rescale it to $k = 1$ and discard the factor of 2^k in the construction of k. But this is not the case for (5.1).

Using the equation for ψ we obtain

$$\begin{aligned}
\Re \int_0^T \langle Q\psi, f\rangle ds &= \Re \int_0^T \langle Q\psi, (i\partial_t - \tilde{H})\psi\rangle ds \\
&= -\Im \int_0^T \langle Q\psi, \partial_t \psi\rangle - \Re \int \langle Q\psi, \tilde{H}\psi\rangle \\
&= -\frac{1}{2}\Im \int_0^T \partial_t \langle Q\psi, \psi\rangle - \Re \int \langle Q\psi, \tilde{H}\psi\rangle
\end{aligned}$$

which, by rearranging terms, becomes

$$-\Re \int_0^T \langle Q\psi, f\rangle ds - \frac{1}{2}\Im\langle Q\psi, \psi\rangle|_0^T = \Re \int \langle Q\psi, \tilde{H}\psi\rangle \tag{5.19}$$

The right hand side can be expanded as follows

$$\begin{aligned}
\Re \int \langle Q\psi, \tilde{H}\psi\rangle &= \Re \int \langle Q\psi, -\Delta\psi + \tilde{V}\psi\rangle \\
&= \Re \int \langle Q\partial_r\psi, \partial_r\psi\rangle + \Re \int \langle [\partial_r, Q]\psi, \partial_r\psi\rangle + \Re \int \langle Q\psi, \tilde{V}\psi\rangle \\
&= \Re \int \langle [\partial_r, Q]\psi, \partial_r\psi\rangle + \frac{1}{2} \int \langle [\tilde{V}, Q]\psi, \psi\rangle
\end{aligned}$$

where we have used twice the antisymmetry of Q. We now compute the commutators and start with the easier one,

$$\frac{1}{2}[\tilde{V}, Q] = -r\chi(2^k r)\partial_r \tilde{V} = \chi(2^k r)\frac{4}{r^2+1}(\frac{1}{r^2} + \frac{1}{r^2+1}) > 0$$

The other commutator is

$$[\partial_r, Q] = 2(2^k r\chi'(2^k r) + \chi(2^k r))\partial_r + (2^k\chi'(2^k r) + 2^{2k} r\chi''(2^k r))$$

We now impose a first condition on the function χ

$$|(r\chi')'| \leq \delta(r\chi)' \tag{5.20}$$

for some $0 < \delta \ll 1$. Using this and the Cauchy-Schwartz inequality we obtain

$$\Re \int \langle Q\psi, \tilde{H}\psi\rangle \gtrsim \int a_k(r)(|\partial_r\psi|^2 - \delta 2^{2k}|\psi|^2) r dr dt + \frac{1}{2}\int [\tilde{V}, Q]|\psi|^2 r dr dt$$

where $a_k(r) = \chi(2^k r) + 2^k r\chi'(2^k r)$. Hence, by (5.19) we have

$$LHS((5.19)) \gtrsim \int a_k(r)(|\partial_r\psi|^2 - \delta 2^{2k}|\psi|^2) r dr dt + \frac{1}{2}\int [\tilde{V}, Q]|\psi|^2 r dr dt \tag{5.21}$$

We claim that given a slowly varying sequence α_n and $\delta > 0$ we can find χ satisfying (5.20), so that

$$a_k(r) \gtrsim \frac{\alpha_{n+k}}{1 + 2^{n+k}}, \qquad r \approx 2^n \tag{5.22}$$

and the following three fixed time bounds hold for functions localized at frequency 2^k:

$$\|Q\psi\|_{L^2} \lesssim \|\psi\|_{L^2} \tag{5.23}$$

$$\|Q\psi\|_{X_{k,\alpha}} \lesssim \|\psi\|_{X_{k,\alpha}} \tag{5.24}$$

$$\int_{\mathbb{R}} a_k(r)|\partial_r\psi|^2 r dr + \frac{1}{2}\int [\tilde{V}, Q]|\psi|^2 r dr dt \gtrsim 2^{2k}\int_{\mathbb{R}} a_k(r)|\psi|^2 r dr \tag{5.25}$$

Using these three relations in the above integral estimate we obtain

$$\|\psi\|^2_{X_{k,\alpha}} \lesssim \|\psi\|^2_{L^\infty L^2} + \|\psi\|_{X_{x,\alpha}}\|f\|_{X'_{k,\alpha}}$$

when all terms are restricted to the time interval $[0, T]$, but with the a constant independent of T. This implies (5.17).

We now proceed with the construction of χ satisfying (5.20), (5.22) and (5.23)-(5.25). We first increase α_n the so that it remains slowly varying and, in addition, satisfies

$$\alpha_n = 1, \quad \text{for } n \leq n_0 - k; \qquad \sum_{n \geq n_0 - k} \alpha_n \approx 1 \tag{5.26}$$

Here n_0 is a positive number to be chosen later. Based on this, we construct a slowly varying function α such that

$$\alpha(s) \approx \alpha_n \ \text{ if } \ s \approx 2^n$$

and with symbol regularity

$$|\partial^k \alpha(s)| \lesssim (1+s)^{-k}\alpha(s)$$

Due to the first condition in (5.26) we can take α such that $\alpha(s) = 1$ for $s \leq 2^{n_0 - k}$. We then construct the function χ by

$$s\chi(s) = \int_0^s \alpha(2^{-k}s)h(s)ds$$

where h_{n_0} is a smooth adapted variant of r^{-1}, namely $h_{n_0}(s) = 1$ for $r \leq 2^{n_0}$ and $h(s) \approx 2^{n_0}s^{-1}$ for $s \geq 2^{n_0+1}$. One easily verifies the pointwise bounds

$$\chi(s) \approx (1 + 2^{-n_0}s)^{-1}, \qquad |\chi^{(k)}(s)| \lesssim 2^{-kn_0}(1 + 2^{-n_0}s)^{-k-1}, \quad k \leq 4 \tag{5.27}$$

Furthermore, we have

$$(s\chi(s))' = \alpha(2^k s)h_{n_0}(s) \gtrsim (1 + 2^{-n_0}s)^{-1.1} \qquad |(s\chi'(s))'| \lesssim 2^{-n_0}(1 + 2^{-n_0}s)^{-2}$$

It is a straightforward exercise to verify that χ satisfies (5.22). Furthermore, by taking n_0 large enough, depending on δ, we insure that χ satisfies also the bound (5.20).

Next we turn our attention to the estimates (5.23)-(5.25). For proving (5.23)-(5.24) we start by rearranging

$$Q\psi = 2\chi(2^k r) r\partial_r \psi + 2^k r\chi'(2^k r)\psi$$

and using (5.27) we obtain $|2^k r\chi'(2^k r)| \lesssim 1$ therefore

$$\|2^k r\chi'(2^k r)\psi\|_{L^2} \lesssim \|\psi\|_{L^2}, \qquad \|2^k r\chi'(2^k r)\psi\|_{X_{k,\alpha}} \lesssim \|\psi\|_{X_{k,\alpha}}$$

Using again (5.27), we conclude the proof of (5.23)-(5.24) by showing that

$$\|\chi(2^k r) r\partial_r \psi\|_{L^2} \lesssim \|\psi\|_{L^2}, \qquad \|\chi(2^k r) r\partial_r \psi\|_{X_{k,\alpha}} \lesssim \|\psi\|_{X_{k,\alpha}} \tag{5.28}$$

Since ψ is localized at frequency 2^k we use an operator P_k as in Proposition 4.8, localizing at frequency 2^k and such that $P_k\psi = \psi$. Then we use the characterization of $\partial_r K_k(r,s)$ from (4.41) for the kernel K_k of P_k; precisely, by (4.41) and (5.27) we obtain

$$|\chi(2^k r) r\partial_r K_k(r,s)| \lesssim \frac{2^{2k}}{(1+2^k|r-s|)^N(1+2^k(r+s))}$$

Then (5.28) follows since the above kernel has rapid off-diagonal decay while the weights α_k are slowly varying.

For (5.25) we claim the following estimate

$$\int a_k(r)|L^*\psi(r)|^2 rdr \gtrsim 2^{2k}\int a_k(r)|\psi(r)|^2 rdr \tag{5.29}$$

Assuming (5.29), we can now complete the argument for (5.25). We write

$$\partial_r\psi = -L^*\psi + \frac{h_3-1}{r}\psi = -L^*\psi - \frac{2\psi}{r(r^2+1)}$$

which shows that

$$|L^*\psi|^2 \lesssim |\partial_r\psi|^2 + \frac{1}{r^2(r^2+1)^2}|\psi|^2$$

Since $a_k(r) \lesssim (1+2^k r)^{-1} \approx \chi(2^k r)$, it follows that

$$\int a_k(r)|L^*\psi(r)|^2 rdr \lesssim \int a_k(r)|\partial_r\psi|^2 rdr + \frac{1}{2}\int [\tilde{V}, Q]|\psi|^2$$

Thus (5.29) implies (5.16).

We finish this subsection with the proof of (5.29). For this we write ψ in terms of $L^*\psi$ as

$$\psi = P_k\psi = P_k\tilde{H}^{-1}LL^*\psi = (L^{-1}P_k)^* L^*\psi$$

where the kernel $K_k^1(r,s)$ of $L^{-1}P_k$ was estimated in Proposition 4.8(b). We need to distinguish two cases:

i) $k \geq 0$. Then, by (4.42), K_k^1 satisfies the symmetric bound

$$|K_k^1(r,s)| \lesssim \frac{2^k}{(1+2^k|r-s|)^N(1+2^k(r+s))}$$

and (5.29) directly follows.

ii) $k < 0$. The regular part $K_{k,reg}^1$ still satisfies the above bound, and causes no difficulties. For the resonant part $K_{k,res}^1$ we use (4.43) to derive the estimate

$$|K_{k,res}^1(r,s)| \lesssim \frac{h_1(r)}{k(1+2^k r)^N}\frac{1}{(1+2^k s)^N}$$

which has a 2^{-k} bound from $L^2 \to L^2$ and decays rapidly above the $r, s = 2^{-k}$ threshold. Thus (5.29) again follows.

STEP 2: A dyadic $L^4_t L^\infty_r$ bound for the homogeneous problem
Here we establish the bound

$$2^{-\frac{k}{2}} \|\psi\|_{L^4_t L^\infty_r(A_{<k})} + \sup_{j \geq -k} 2^{\frac{j}{2}} \|\psi\|_{L^4_t L^\infty_r(A_j)} \lesssim \|\psi_0\|_{L^2} \tag{5.30}$$

for ψ_0 localized at frequency 2^k. The first term is easily bounded by interpolating between (5.3) and (5.2). Consider now $j \geq -k$, and χ_j a smooth bump function supported in A_j. The function $\psi_j = \chi_j \psi$ solves the equation

$$(i\partial_t - \tilde{H})\psi_j = 2\partial_r \chi_j \partial_r \psi + \delta\chi_j \cdot \psi := f_j$$

From the local energy decay estimate for ψ we obtain the following bounds

$$\|\psi_j(0)\|_{L^2} + 2^{\frac{k-j}{2}} \|\psi_j\|_{L^2} + 2^{\frac{j-k}{2}} \|f_j\|_{L^2} \lesssim \|\psi(0)\|_{L^2}$$

We conjugate by $r^{\frac{1}{2}}$ and set $v_j(t, r) = r^{\frac{1}{2}} \psi_j$. A direct computation shows that v_j solves a one dimensional Schrödinger equation

$$(i\partial_t - \partial_r^2) v_j = r^{\frac{1}{2}} f_j + (r^{-2} + \tilde{V}(r)) v_j := g_j$$

where

$$\|v_j(0)\|_{L^2} + 2^{\frac{k-j}{2}} \|v_j\|_{L^2} + 2^{\frac{j-k}{2}} \|g_j\|_{L^2} \lesssim \|\psi(0)\|_{L^2}$$

Applying the one dimensional $L^4 L^\infty$ Strichartz estimate over each time interval of size 2^{j-k} we obtain

$$\|v_j\|_{L^4 L^\infty} \lesssim \|\psi(0)\|_{L^2}$$

Returning to ψ this yields

$$\|\psi\|_{L^4 L^\infty(A_j)} \lesssim 2^{-\frac{j}{2}} \|\psi(0)\|_{L^2}$$

Hence (5.30) is proved.

STEP 3: The dyadic L^4 bound for the homogeneous problem
Here we establish the bound

$$\|\psi\|_{L^4_k} \lesssim \|\psi(0)\|_{L^2} \tag{5.31}$$

for ψ localized at frequency 2^k. The L^4 bound in A_j with $j \leq -k$ follows from the first term in (5.30) by Hölder's inequality. The L^4 bound in A_j with $j > -k$ is obtained by interpolating between the $L^4 L^\infty$ bound in the second term in (5.30), the $L^2_{t,x}$ bound in LE_k and the $L^\infty L^2$ energy estimate.

STEP 4: The role of the U^p and V^p spaces Here we show that

$$\|\psi\|_{S_k \cap V^2_{\tilde{H}} L^2} \lesssim \|\psi(0)\|_{L^2} + \|f\|_{N_k + DU^2_{\tilde{H}} L^2} \tag{5.32}$$

By Steps 1 and 3 we know that for the homogeneous problem we have

$$\|\psi\|_{S_k} \lesssim \|\psi_0\|_{L^2}, \qquad f = 0 \tag{5.33}$$

which implies (5.32) in this case. By duality this shows that for the inhomogeneous problem we have

$$\|\psi\|_{L^\infty L^2} \lesssim \|\psi_0\|_{L^2} + \|f\|_{N_k} \tag{5.34}$$

Applying (5.33) for each step of each U^2 atom, we further obtain

$$\|\psi\|_{S_k} \lesssim \|\psi\|_{U^2_{\tilde{H}} L^2} \tag{5.35}$$

which suffices for $f \in DU^2_{\tilde{H}} L^2$. It remains to consider $f \in N_k$, which we further split in two.

i) $f \in L^{\frac{4}{3}}_k + L^1 L^2$. For any partition $\mathbb{R} = \cup I_l$ of the time interval into subintervals we have

$$\sum_l \|f\|^{\frac{4}{3}}_{(L^{\frac{4}{3}}_k + L^1 L^2)(I_l)} \lesssim \|f\|^{\frac{4}{3}}_{L^{\frac{4}{3}}_k + L^1 L^2}$$

which combined to (5.34) yields

$$\|\psi\|_{V^{\frac{4}{3}}_{\tilde{H}} L^2} \lesssim \|f\|_{L^{\frac{4}{3}}_k + L^1 L^2}$$

Since $V^{\frac{4}{3}} \subset U^2$, the proof is concluded in this case.

ii) $f \in LE^*_k$. By Step 1 we have the LE_k and $L^\infty L^2$ bounds for ψ. On the other hand arguing as in case (i) above we obtain

$$\|\psi\|_{V^2_{\tilde{H}} L^2} \lesssim \|f\|_{LE^*_k}$$

This concludes the proof since $V^2 \subset U^4$, and we have the following variation of (5.35),

$$\|\psi\|_{L^4_k} \lesssim \|\psi\|_{U^4_{\tilde{H}} L^2} \tag{5.36}$$

STEP 5: The high modulation bound. Given (5.32), to conclude the proof of the proposition it remains to prove the high modulation bound

$$\|Q_{>2k}\psi\|_{W^{1,1}L^2 + \dot{X}^{0,\frac{1}{2},1}} \lesssim \|Q_{>2k} f\|_{L^1 L^2 + \dot{X}^{0,-\frac{1}{2},1}}$$

This is straightforward; details are left for the reader. □

6. The time dependent linear evolution

Here we consider the linear equation

$$(i\partial_t - \tilde{H}_\lambda) u = f, \qquad u(0) = u_0 \tag{6.1}$$

where

$$\tilde{H} = -\Delta + \tilde{V}_\lambda, \qquad \tilde{V}_\lambda = \frac{4}{r^2(1 + \lambda^2 r^2)}$$

In the space L^2 this can be viewed as a small perturbation of the $\lambda = 1$ problem in (5.1):

Proposition 6.1. *Assume that $|\lambda - 1|_{L^\infty} \ll 1$. Then the equation (6.1) is well-posed in L^2, and the following bound holds:*

$$\|u\|_{l^2 S^\sharp} \lesssim \|u(0)\|_{L^2} + \|f\|_{l^2 N^\sharp} \tag{6.2}$$

Proof. Since $|V_\lambda - V_1| \lesssim |\lambda - 1|(1 + r^2)^{-2}$ it follows that

$$\|(V_\lambda - V_1) h\|_{LE^*} \lesssim \|h\|_{LE}$$

Therefore we can rewrite the (6.1) as

$$(i\partial_t - \tilde{H}) u = (V_\lambda - V_1) u + f, \qquad u(0) = u_0$$

and treat the $(V_\lambda - V_1) u$ as a perturbation. The result follows then from (5.14). □

Our main goal in this section is to study the equation (6.1) in the smaller space LX. The condition $\|\lambda - 1\|_{L^\infty} \ll 1$ is no longer sufficient for the analysis in LX. Instead we use the stronger topology Z for λ (see Section 5.1.3), and work with

$$\|\lambda - 1\|_Z \leq \gamma \ll 1 \tag{6.3}$$

Our aim will be achieved in two steps.

- The spaces of type $l^2 S^\sharp$, $l^2 N^\sharp$ associated to the $\lambda = 1$ flow are not robust enough for the variable λ flow. Hence we introduce some modified spaces $WS^\sharp[\lambda]$, $WN^\sharp[\lambda]$ adapted to the time dependent frame. To simplify some of the analysis, we also provide some partial characterizations of functions in $WS^\sharp[\lambda]$ and $WN^\sharp[\lambda]$ with respect to the time independent frame $\lambda = 1$.
- We prove that if (6.3) holds, then the evolution (6.1) is globally well-posed for initial data in LX and inhomogeneous term in $WN^\sharp[\lambda]$, and the solution ψ is uniformly bounded in LX, and further it belongs to $WS^\sharp[\lambda]$.

6.1. **The spaces** $WS^\sharp[\lambda]$**,** $WN^\sharp[\lambda]$**.** Here we define the $[\lambda]$ type spaces as counterparts of the spaces from the previous section which take into account time-variable Littlewood-Paley projectors. We begin as usual with a dyadic decomposition, but with respect to the λ dependent frame,

$$\psi = \sum_k P_k^\lambda \psi.$$

For $\mathcal{X} \in \{S, N, S^\sharp, N^\sharp\}$ we define the space $l^2\mathcal{X}[\lambda]$ with norm

$$\|\psi\|_{l^2\mathcal{X}[\lambda]} = \left(\sum_k \|P_k^\lambda \psi\|_{l^2\mathcal{X}}^2 \right)^{\frac{1}{2}}$$

This gives rise to the spaces $l^2 S[\lambda]$, $l^2 N[\lambda]$, $l^2 S^\sharp[\lambda]$, $l^2 N^\sharp[\lambda]$ which correspond to L^2 initial data in (6.1). For LX initial data, on the other hand, we need to replace the l^2 dyadic summation with the same summation as in the LX norm. Hence we define

$$\|\psi\|_{W\mathcal{X}[\lambda]} = \sum_{k<0} \frac{1}{2^k |k|} \|P_k^\lambda \psi\|_{l^2\mathcal{X}} + \left(\sum_{k\geq 0} \|P_k^\lambda \psi\|_{l^2\mathcal{X}}^2 \right)^{\frac{1}{2}}$$

All of the above spaces $l^2\mathcal{X}[\lambda]$ and $W\mathcal{X}[\lambda]$ have their finite time interval counterpart $l^2\mathcal{X}[\lambda](I)$ and $W\mathcal{X}[\lambda](I)$, which are obtained by using $l^2\mathcal{X}(I)$ instead of $l^2\mathcal{X}$ in the above definitions. We especially remark that they are **not** obtained by restricting to I a similar class of functions over the entire real line; such a definition would be dependent on specifying an extension of λ, which we wish to avoid. The spaces $WS^\sharp[\lambda], WN^\sharp[\lambda]$ play a fundamental role in our analysis.

We remark that the functions $P_k^\lambda \psi$ are frequency localized in the time dependent frame but not in the fixed $\lambda = 1$ frame. This will cause considerable technical difficulties later on. Because of this, it is useful to transfer as much information as possible back to the fixed frame.

Proposition 6.2 (Characterizations of $WS^\sharp[\lambda]$ and $WN^\sharp[\lambda]$ functions)**.** *Suppose that λ takes values in a compact subset of $(0, \infty)$. Then* *a) The following S type norms are equivalent:*

$$\|P_k^\lambda \psi\|_{l^2 S} \approx \|P_k^\lambda \psi\|_{S_k}, \qquad \|P_k^\lambda g\|_{l^2 N} \approx \|P_k^\lambda g\|_{N_k} \tag{6.4}$$

as well as

$$\|\psi\|_{l^2 S[\lambda]} \approx \|\psi\|_{l^2 S}, \quad \|\psi\|_{l^2 N[\lambda]} \approx \|\psi\|_{l^2 N}, \quad \|\psi\|_{WS[\lambda]} \approx \|\psi\|_{WS[1]}. \tag{6.5}$$

Furthermore, we have the improved local energy decay

$$\left\| \frac{\psi}{\langle r\rangle^{\frac12} \ln(1+r)} \right\|_{L^2} \lesssim \|\psi\|_{WS^\sharp[\lambda]} \tag{6.6}$$

b) Assume in addition that $\lambda - 1 \in Z$. Then the following inclusions hold:

$$WS^\sharp[\lambda] \subset WS^r[1] \tag{6.7}$$

$$WN^r[1] \subset WN^\sharp[\lambda] \tag{6.8}$$

We remark that all of the above bounds with the exception of (6.7) also hold trivially in any interval; this is because all the norms involved can be measured in an interval by taking the zero extension outside it. The bound (6.7) also holds in any interval, but this is a more delicate matter which we will only be able to consider after we prove Proposition 6.3 below.

Proof. **The estimate (6.4).** This is trivial if $\lambda = 1$. Otherwise, by definition,

$$\|P_k^\lambda \psi\|_{l^2 S}^2 = \sum_{j\in\mathbb{Z}} \|P_j P_k^\lambda \psi\|_{S_j}^2$$

If $|j-k| \lesssim 1$ then the S_k and S_j norms are equivalent, and the multipliers P_j are bounded in S_k. Thus we have

$$\|P_j P_k^\lambda \psi\|_{S_k} \lesssim \|P_j P_k^\lambda \psi\|_{S_j} \lesssim \|P_k^\lambda \psi\|_{S_k}, \qquad |j-k| \lesssim 1 \tag{6.9}$$

Consider now the case $|j-k| \gg 1$. We write $P_j P_k^\lambda = P_j \tilde P_k^\lambda P_k^\lambda$. For the kernel $K_{jk}(r,s)$ of $P_j \tilde P_k^\lambda$ we use the estimate (4.49). Then a direct computation shows that

$$\begin{aligned} \|P_j P_k^\lambda \psi\|_{S_j} &\lesssim \frac{2^{(j-k)^-} 2^{-N(k^+ + j^+)}}{\langle k^-\rangle \langle j^-\rangle} \|P_k^\lambda \psi\|_{S_k}, \\ \|P_j P_k^\lambda \psi\|_{S_k} &\lesssim \frac{2^{-\frac{|k-j|}{2}} 2^{-N(k^+ + j^+)}}{\langle k^-\rangle \langle j^-\rangle} \|P_k^\lambda \psi\|_{S_k} \end{aligned} \tag{6.10}$$

We use (6.9) and (6.10) to conclude the proof of the first estimate in (6.4). On one hand we have

$$\|P_k^\lambda \psi\|_{l^2 S}^2 \lesssim (1 + \sum_j \langle k^-\rangle^{-2} \langle j^-\rangle^{-2} 2^{-2N(k^+ + j^+)}) \|P_k^\lambda \psi\|_{S_k}^2 \lesssim \|P_k^\lambda \psi\|_{S_k}^2$$

Conversely, we denote the separation threshold by k_0 and compute

$$\begin{aligned} \|P_k^\lambda \psi\|_{S_k} &\lesssim \sum_{|j-k|\le k_0} \|P_j P_k^\lambda \psi\|_{S_k} + \sum_{|j-k|>k_0} \|P_j P_\mu^\lambda \psi\|_{S_k} \\ &\lesssim c_{k_0} \|P_\mu^\lambda \psi\|_{l^2 S} + 2^{-\frac{k_0}{2}} \|P_k^\lambda \psi\|_{S_k} \end{aligned}$$

By appropriately adjusting k_0, the last term on the right can be absorbed by the the first term $\|P_k^\lambda \psi\|_{S_k}$, thus giving us the reverse inequality

$$\|P_k^\lambda \psi\|_{S_k} \lesssim \|P_k^\lambda \psi\|_{l^2 S}$$

This completes the proof of the first estimate in (6.4). The second follows from a similar argument.

The estimate (6.6)**.** The proof of (6.6) is almost identical to the proof of (5.11). The fact that λ is not equal to 1 makes no difference there.

The estimates (6.5)**.** We only prove the third bound, which is more important in this article. The first two are similar but simpler. The proofs of the two inclusions are identical, so it suffices to show one of them, say $WS[\lambda] \subset WS[1]$. For fixed frequency j we decompose into a diagonal and an off-diagonal part

$$P_j \psi = \sum_{|j-k|\lesssim 1} P_j P_k^\lambda \psi + \sum_{|j-k|\gg 1} P_j P_k^\lambda \psi := (P_j\psi)_{\rm diag} + (P_j \psi)_{\rm offd} \tag{6.11}$$

For the diagonal part it suffices to use the S_j boundedness of P_j. For the off-diagonal part we use the first part of (6.10) to obtain

$$\|(P_j\psi)_{\rm offd}\|_{S_j} \lesssim \sum_{|k-j|\lesssim 1} \|P_k^\lambda \psi\|_{S_k} + \sum_{|k-j|\gg 1} \frac{2^{(j-k)^-} 2^{-N(k^+ + j^+)}}{\langle k^-\rangle \langle j^-\rangle} \|P_k^\lambda \psi\|_{S_k}$$

To conclude it suffices to sum up the second term on the right with respect to j and the weights $2^{-j^-}\langle j^-\rangle^{-1}$. Indeed we have

$$\begin{aligned}\sum_j \frac{2^{-j^-}}{\langle j^-\rangle} \|(P_j\psi)_{\rm offd}\|_{S_j} &\lesssim \sum_j \frac{2^{-j^-}}{\langle j^-\rangle} \sum_k \frac{2^{(j-k)^- - N(k^+ + j^+)}}{\langle k^-\rangle\langle j^-\rangle} \|P_k^\lambda \psi\|_{S_k} \\ &\lesssim \sum_k \frac{2^{-k^- - Nk^+}}{\langle k^-\rangle^2} \|P_k^\lambda \psi\|_{S_k} \lesssim \|\psi\|_{WS[\lambda]}\end{aligned}$$

where (6.4) was used in the last step.

The estimate (6.7)**.** Given (6.5), it remains to bound the additional high modulation component in the $S^r[1]$ norm. We decompose $P_j\psi$ again as in (6.11). The diagonal part is estimated directly in $S_j^\sharp$. The nontrivial part of the argument is to estimate the off-diagonal part. For these we decompose further

$$Q_{\gtrsim 2j}(P_j f)_{\rm offd} = \sum_{|k-j|\gg 1} \sum_h Q_{\gtrsim 2j} P_j \tilde{P}_k^\lambda \tilde{P}_h P_h P_k^\lambda f$$

The definition of $WS^\sharp[\lambda]$ gives us good estimates on $P_h P_k^\lambda f$ in $S_h^\sharp$, and, after applying the operator $P_j \tilde{P}_k^\lambda \tilde{P}_h$, we need to estimate the (high modulation) output in $(X^{0,\frac{1}{2},1} + W^{1,1})L^2 = ZL^2$. For the kernel K_{jkh} of $P_j \tilde{P}_k^\lambda \tilde{P}_h$ we use the representation in Proposition 4.14 (a) or (b). Hence it suffices to consider kernels K_{jkh} of the form

$$K_{jkh}(r,s) = c_{jkh} g(\lambda) \phi_j(r) \phi_h(s)$$

where $|\phi_j(r)| \lesssim 2^j(1+2^j r)^{-N}$, and similarly for ϕ_h. For such K_{jkh} we write

$$P_j \tilde{P}_k^\lambda \tilde{P}_h P_h f P_k^\lambda f = c_{jkh} g(\lambda) \phi_j(r) \langle \phi_h, P_h P_k^\lambda f\rangle$$

We first estimate the last inner product. Globally we use local energy to obtain

$$\|\langle \phi_h, P_h P_k^\lambda f\rangle\|_{L^2} \lesssim 2^{-h}\|P_h P_k^\lambda f\|_{LE_h}$$

while at high modulation we have

$$\|Q_{>2h}\langle \phi_h, P_h P_k^\lambda f\rangle\|_Z \lesssim \|Q_{>2h} P_h P_k^\lambda f\|_{ZL^2}$$

Combining the last two estimates we obtain

$$\|\langle \phi_h, P_h P_k^\lambda f\rangle\|_Z \lesssim \|P_h P_k^\lambda f\|_{S_h^\sharp}$$

Due to the Z algebra property this bound is not affected by multiplication by $g(\lambda)$. Estimating ϕ_j in L^2 we obtain

$$\|Q_{>2j} P_j \tilde{P}_k^\lambda \tilde{P}_h P_h P_k^\lambda f\|_{ZL^2} \lesssim c_{jkh}\|P_h P_k^\lambda f\|_{S_h^\sharp} \tag{6.12}$$

where the modulation truncation $Q_{>2j}$ was harmlessly added at the end. Hence in order to estimate the high modulation component of $\|f\|_{WS^r[1]}$ we add up the dyadic pieces

$$\begin{aligned}
\sum_j \frac{2^{-j^-}}{\langle j^-\rangle}\|Q_{>2j}(P_j f)_{\text{offd}}\|_{ZL^2} &\lesssim \sum_{j,k,h} \frac{2^{-j^-}}{\langle j^-\rangle} c_{jkh}\|P_h P_k^\lambda f\|_{S_h^\sharp}\\
&\lesssim \sum_{k,h} \frac{2^{-k^- - |k-h| - N(k^+ + h^+)}}{\langle k^-\rangle^2}\|P_h P_k^\lambda f\|_{S_h^\sharp}\\
&\lesssim \sum_k \frac{2^{-k^- - Nk^+}}{\langle k^-\rangle^2}\|P_k^\lambda f\|_{l^2 S^\sharp} \lesssim \|f\|_{WS^\sharp[\lambda]}
\end{aligned}$$

which completes the proof.

The estimate (6.8)**.** We need to show that for f at frequency $h < 0$ and modulation $\sigma > 2h$ we have

$$\|f\|_{WN^\sharp[\lambda]} \lesssim \frac{2^{-h-\frac{\sigma}{2}}}{|h|}\|f\|_{L^2} \tag{6.13}$$

We decompose f as follows,

$$f = f_0 + f_1 + f_2 + f_3, \qquad f_i = \sum_{k,j \in \mathcal{A}_i} P_j P_k^\lambda \tilde{P}_h f$$

$$\mathcal{A}_0 = \{|j-k| \gg 1,\ |k-h| \gg 1\}, \qquad \mathcal{A}_1 = \{|j-k| \lesssim 1,\ |k-h| \gg 1\}$$
$$\mathcal{A}_2 = \{|j-k| \gg 1,\ |k-h| \lesssim 1\}, \qquad \mathcal{A}_3 = \{|j-k| \lesssim 1,\ |k-h| \lesssim 1\}$$

For indices in the set $\mathcal{A}_0$ we have the representation of $P_j P_k^\lambda \tilde{P}_h$ given in Proposition 4.14(a), as a rapidly convergent series of operators of the form

$$T_{jkh} = c_{jkh} g(\lambda)\phi_j(r)\langle \phi_h(s), \cdot\rangle$$

with c_{jkh} as in (4.51). For the inner product we have

$$\|\langle \phi_h(s), f\rangle\|_{L^2} \lesssim \|f\|_{L^2}$$

which immediately leads to

$$\|P_j P_k^\lambda \tilde{P}_h f\|_{L^2} + 2^j\|P_j P_k^\lambda \tilde{P}_h f\|_{LE_j^*} \lesssim \frac{2^{-|j-k| - |k-h| - N(j^+ + k^+ + h^+)}}{\langle j^-\rangle\langle k^-\rangle^2\langle h^-\rangle}\|f\|_{L^2}$$

We will use the LE_j^* bound at high frequencies, $2j > \sigma - 8$. For smaller frequencies we use the L^2 bound at high modulation ($\geq \sigma - 4$). For low modulations we instead obtain an L^1L^2 bound. Here the idea is that f is localized at high modulation, and the only way to arrive to low modulations is to use high modulations of $g(\lambda)$. Precisely we have

$$Q_{<\sigma-4} P_j P_k^\lambda \tilde{P}_h f = \sum c_{jkh} [Q_{>\sigma-2} g(\lambda)] \phi_j(r) \langle \phi_h(s), f \rangle$$

Since $g(\lambda) \in Z$, we have $\|Q_{>\sigma-2} g(\lambda)\|_{L^2} \lesssim 2^{-\sigma/2}$. Hence we arrive at

$$\|Q_{<\sigma-4} P_j P_k^\lambda \tilde{P}_h f\|_{L^1L^2} \lesssim 2^{-\frac{\sigma}{2}} \frac{2^{-|j-k|-|k-h|-N(j^+ + k^+ + h^+)}}{\langle j^- \rangle \langle k^- \rangle^2 \langle h^- \rangle} \|f\|_{L^2}$$

Thus in all cases it follows that

$$\|P_j P_k^\lambda \tilde{P}_h f\|_{N_j^\sharp} \lesssim \min\{2^{-j}, 2^{-\frac{\sigma}{2}}\} \frac{2^{-|j-k|-|k-h|-N(j^+ + k^+ + h^+)}}{\langle j^- \rangle \langle k^- \rangle^2 \langle h^- \rangle} \|f\|_{L^2}$$

Summing up, we obtain for f_0

$$\begin{aligned} \|f_0\|_{WN^\sharp[\lambda]} &\lesssim \sum_{j,k} \frac{2^{-j}}{\langle j^- \rangle} \min\{2^{-j}, 2^{-\frac{\sigma}{2}}\} \frac{2^{-|j-k|-|k-h|-N(j^+ + k^+ + h^+)}}{\langle j^- \rangle \langle k^- \rangle^2 \langle h^- \rangle} \|f\|_{L^2} \\ &\lesssim \sum_{j} \frac{2^{-j}}{\langle j^- \rangle} \min\{2^{-j}, 2^{-\frac{\sigma}{2}}\} \frac{\langle h - j \rangle 2^{-|j-h|-N(j^+ + h^+)}}{\langle j^- \rangle^2 \langle h^- \rangle^2} \|f\|_{L^2} \\ &\lesssim 2^{-\frac{\sigma}{2}} \frac{2^{-h-Nh^+}}{\langle h^- \rangle^3} \|f\|_{L^2} \end{aligned}$$

which is slightly stronger than needed.

For the terms in f_1 and f_2 the computation is almost identical. Using (4.52) instead of (4.51) we obtain

$$\|P_j P_k^\lambda \tilde{P}_h f\|_{N_j^\sharp} \lesssim \min\{2^{-j}, 2^{-\frac{\sigma}{2}}\} \frac{2^{-|j-h|-N(j^+ + h^+)}}{\langle j^- \rangle \langle h^- \rangle} \|f\|_{L^2}$$

The summation with respect to k is trivial in this case. The j summation is as above, and we obtain

$$\|f_0\|_{WN^\sharp[\lambda]} \lesssim 2^{-\frac{\sigma}{2}} \frac{2^{-h-Nh^+}}{\langle h^- \rangle^3} \|f\|_{L^2}$$

Finally, we consider the last component f_4 of f. Owing to the different form of (4.53), in this case we do not have the option of using any local energy decay estimate. The first part of $P_j P_k^\lambda \tilde{P}_h$ is essentially the identity, but does not depend on λ so the high modulations are preserved,

$$\|P_j P_k \tilde{P}_h f\|_{N_j^\sharp} \lesssim \frac{2^{-h}}{\langle h^- \rangle} 2^{-\frac{\sigma}{2}} \|f\|_{L^2}$$

For the second part we have the product of a time independent L^2 bounded operator with $g(\lambda)$, so we can apply the same L^1L^2 estimate as above for low modulations. We obtain

$$\|P_j (P_k^\lambda - P_k) \tilde{P}_h f\|_{N_j^\sharp} \lesssim \frac{2^{-h}}{\langle h^- \rangle} 2^{-\frac{\sigma}{2}} \frac{2^{-h^+}}{\langle h^- \rangle^2} \|f\|_{L^2}$$

In both cases the k and j summations are trivial, so the proof is concluded. □

6.2. **The time dependent linear flow.** We are now ready to consider the well-posedness of the equation (6.1) in the smaller space LX.

Proposition 6.3. *Let* λ *be as in* (6.3)*. Then the equation* (6.1) *is well-posed in* LX*, and the following estimate holds:*

$$\|\psi\|_{WS^\sharp[\lambda]} \lesssim \|\psi(0)\|_{L^2} + \|g\|_{WN^\sharp[\lambda]} \tag{6.14}$$

Proof. For ψ solving (6.1) we consider its dyadic decomposition

$$\psi = \sum_k \psi_k, \qquad \psi_k = P_k^\lambda \psi$$

To write an equation for ψ_k we use the transference operator described earlier in Proposition 4.9,

$$K_\lambda = \lambda \mathcal{F}_\lambda \frac{d}{d\lambda} \mathcal{F}_\lambda^* = -\lambda \frac{d}{d\lambda} \mathcal{F}_\lambda \mathcal{F}_\lambda^*$$

For $P_k^\lambda = \mathcal{F}_\lambda^* \chi_k \mathcal{F}_\lambda$ we compute

$$\lambda \frac{d}{d\lambda} P_k^\lambda = \mathcal{F}_\lambda^* [\mathcal{K}, \chi_k] \mathcal{F}_\lambda \psi$$

Hence the FT of the components ψ_k solve the equations

$$(i\partial_t - \tilde{H}_\lambda)\psi_k = i\frac{\lambda'}{\lambda} \mathcal{F}_\lambda^* [\mathcal{K}, \chi_k] \mathcal{F}_\lambda \psi + g_k$$

After a further dyadic decomposition on the right, we obtain the infinite coupled system

$$(i\partial_t - \tilde{H}_\lambda)\psi_k = \sum_j K_{kj}^\lambda \psi_j + g_k, \qquad K_{kj}^\lambda v = i\frac{\lambda'}{\lambda} \mathcal{F}_\lambda^* [\mathcal{K}, \chi_k] \chi_j \mathcal{F}_\lambda v \tag{6.15}$$

For the left hand side we use Proposition 6.1 to treat each equation in this system in L^2 where $\tilde{H}_\lambda$ can be viewed as a small perturbation of $\tilde{H}$. We claim that the first term $f_k = \sum_j K_{kj}^\lambda \psi_j$ in the right hand side is perturbative,

$$\sum_{k<0} \frac{1}{2^k|k|} \|f_k\|_{l^2 N^\sharp} + \left(\sum_{k \geq 0} \|f_k\|_{l^2 N^\sharp}^2 \right)^{\frac{1}{2}} \ll \sum_{k<0} \frac{1}{2^k|k|} \|\psi_k\|_{l^2 S^\sharp} + \left(\sum_{k \geq 0} \|\psi_k\|_{l^2 S^\sharp}^2 \right)^{\frac{1}{2}} \tag{6.16}$$

This is a consequence of the following estimate:

$$\|K_{kj}^\lambda \psi_j\|_{l^2 N^\sharp} \lesssim \gamma a_{kj} \|\psi_j\|_{l^2 S^\sharp} \tag{6.17}$$

where

$$a_{kj} = \frac{2^{-|j-k|}}{\langle j^- \rangle \langle k^- \rangle} (1 + 2^j + 2^k)^{-N} \text{ if } |k-j| \gg 1, \qquad a_{kj} = \frac{1}{\langle k^- \rangle \langle j^- \rangle} \text{ if } |k-j| \lesssim 1$$

It is easy to see that (6.17) implies (6.16). Harmlessly neglecting the case when either $j > 0$ or $k > 0$, where we have rapid decay or $|k-j| \approx 1$ which sums directly, it suffices to verify that for $j \leq 0$ we have

$$\sum_{k \leq 0} \frac{1}{2^k \langle k \rangle} a_{kj} \lesssim \frac{1}{2^j \langle j \rangle}$$

But this is straightforward.

It remains to prove (6.17). The Fourier kernel of K_{kj}^λ in the $\tilde H_\lambda$ frame (i.e. the kernel of $\mathcal{F}_\lambda K_{kj}^\lambda \mathcal{F}_\lambda^*$) has the form $\lambda' C_{kj}^\lambda$ where

$$C_{kj}^\lambda(\xi,\eta,\lambda) = \frac{1}{\lambda} F(\lambda\xi,\lambda\eta)\frac{\chi_k(\xi)-\chi_k(\eta)}{\xi^2-\eta^2}\chi_j(\eta)$$

The simpler case is when $|k-j| \gg 1$; then C_{kj}^λ is dyadically localized in its two arguments at frequency 2^k, respectively 2^j, has size (see Proposition 4.9)

$$|\partial_\lambda^a(\xi\partial_\xi)^b(\eta\partial_\eta)^c(\xi\partial_\xi+\eta\partial_\eta)^d C_{kj}^\lambda| \lesssim \frac{2^{-\frac{k+j}{2}-|k-j|}}{\langle k^-\rangle\langle j^-\rangle}(1+2^k+2^j)^{-N}. \tag{6.18}$$

The other case is when $|k-j| \lesssim 1$. Then η is still localized at frequency 2^j but ξ ranges over all the positive real axis. In this case we decompose smoothly

$$C_{kj}^\lambda = \sum_l C_{kj}^{\lambda l}, \qquad C_{kj}^{\lambda l}(\xi,\eta) = \chi_l(\xi) C_{kj}^\lambda(\xi,\eta) \tag{6.19}$$

Then $C_{kj}^{\lambda l}$ has the same size and regularity as C_{lj}^λ above in the case $|l-j| \gg 1$.

If $|l-j| \lesssim 1$, by Proposition 4.9, we have that

$$\begin{aligned} &|\partial_\lambda^a(\xi\partial_\xi)^b(\eta\partial_\eta)^c(\xi\partial_\xi+\eta\partial_\eta)^d C_{kj}^{\lambda l}| \lesssim \frac{2^{-k}}{\langle k^-\rangle^2}, \qquad k \lesssim 0 \\ &|\partial_\lambda^a\partial_\xi^b\partial_\eta^c(\xi\partial_\xi+\eta\partial_\eta)^d C_{kj}^{\lambda l}| \lesssim 2^{-2k}\langle \xi-\eta\rangle^{-N}, \qquad k \gtrsim 0 \end{aligned} \tag{6.20}$$

where $a, d, N \in \mathbb{N}$ and $b+c \le 2$.

Our main $l^2S^\sharp \to l^2N^\sharp$ bound is:

Lemma 6.4. *Let λ be as in* (6.3). *Let K_{kj}^0 be an operator whose Fourier kernel in the $\tilde H_\lambda$ frame has the form $\lambda' C_{kj}^0(\lambda,\xi,\eta)$ where C_{kj}^0 is dyadically localized in the region $\xi \approx 2^k$, $\eta \approx 2^j$ and satisfies the uniform bounds* (6.18) *if $|k-j| \gg 1$ and* (6.20) *if $|k-j| \lesssim 1$, then*

$$\|K_{kj}^0\|_{l^2S^\sharp \to l^2N^\sharp} \lesssim \gamma a_{kj}. \tag{6.21}$$

By (6.18), this implies (6.17) directly $|k-j| \gg 1$, and after an l summation corresponding to the decomposition (6.19) if $|k-j| \lesssim 1$. The proof of Proposition 6.3 is concluded. ☐

Proof of Lemma 6.4. The first step in the proof is to reduce the problem to the case when the Fourier kernel of K_{kj}^0 in the $\tilde H_1$ frame is as in the lemma. To switch from the $\tilde H_\lambda$ frame to the $\tilde H_1$ frame we write

$$K_{kj}^0 = \sum_{k_1,j_1} P_{k_1} K_{kj}^0 P_{j_1} := \sum_{k_1,j_1} K_{kj}^{k_1 j_1}$$

where the Fourier kernels $C_{kj}^{k_1j_1}$ of $K_{kj}^{k_1j_1}$ are the kernels of

$$\chi_{k_1}\mathcal{F}_1 K_{kj}^0 \mathcal{F}_1^* \chi_{j_1} = \chi_{k_1}(\mathcal{F}_1\mathcal{F}_\lambda^*)(\mathcal{F}_\lambda K_{kj}^0\mathcal{F}_\lambda^*)(\mathcal{F}_\lambda\mathcal{F}_1^*)\chi_{j_1}$$

For the operator $\mathcal{F}_1\mathcal{F}_\lambda^*$ and its adjoint above we use Proposition 4.10. Thus the bounds for $C_{kj}^{k_1j_1}$ are the bounds for C_{kj}^0 corrected on the right hand side with the factor $\alpha_{kj}^{k_1j_1} = \alpha_k^{k_1}\alpha_j^{j_1}$

where

$$\alpha_k^{k_1} = \begin{cases} 1 & |k-k_1| \lesssim 1 \\ \dfrac{2^{\frac{k-k_1}{2} - |k-k_1| - N(k^+ + k_1^+)}}{\langle k^- \rangle \langle k_1^- \rangle} & |k-k_1| \gg 1 \end{cases}$$

Suppose we know that the lemma applies for the operators $K_{kj}^{k_1 j_1}$ then it is easily seen that it also applies to K_{kj}^0 since

$$\sum_{k_1, j_1} \alpha_{kj}^{k_1 j_1} \lesssim 1$$

It remains to prove the lemma in the simpler case where the Fourier kernel of K_{kj}^0 is given in the $\tilde{H}_1$ frame.

If $|k-j| \gg 1$, by separating the variables λ, ξ and η in C_{kj}^0 we reduce the problem to the case when the kernel C_{kj}^0 has the form

$$C_{kj}^0(\lambda, \xi, \eta) = g(\lambda)\chi_k(\xi)\chi_j(\eta)$$

where χ_k and χ_j represent smooth unit bumps with dyadic localization. With these notations, the operator K_{kj}^0 takes the form

$$K_{kj}^0 u_j = \lambda' g(\lambda) \phi_k \langle \phi_j, u_j \rangle \qquad \phi_k = \mathcal{F}^* \chi_k, \ \phi_j = \mathcal{F}^* \chi_j$$

For ϕ_k and ϕ_j we have the bound (4.11) which we repeat here for convenience,

$$|\phi_k(r)| \lesssim 2^{\frac{3k}{2}} (1 + 2^k r)^{-N} \tag{6.22}$$

In particular we have the bounds

$$\|\phi_k\|_{L^2} \lesssim 2^{\frac{k}{2}}, \qquad \|\phi_j\|_{L^2} \lesssim 2^{\frac{j}{2}} \tag{6.23}$$

as well as

$$\|f\phi_k\|_{LE_k} \lesssim 2^{-\frac{k}{2}} \|f\|_{L^2}, \qquad \|f\phi_j\|_{LE_k^*} \lesssim 2^{-\frac{j}{2}} \|f\|_{L^2} \tag{6.24}$$

where f represents a function of time.

If $|k-j| \lesssim 1$ and $k \lesssim 0$, then we separate the variable λ and write

$$C_{kj}^0(\lambda, \xi, \eta) = g(\lambda) G(\xi, \eta)$$

where G is dyadically localized at $\xi \approx 2^k$ and $\eta \approx 2^j$, and has size conditions

$$|(\xi \partial_\xi)^\alpha (\eta \partial_\eta)^\beta G(\xi, \eta)| \lesssim \frac{2^{-k}}{\langle k^- \rangle^2}, \qquad \alpha + \beta \leq 2 \tag{6.25}$$

If $|k-j| \lesssim 1$ and $k \gg 0$, then the fast decay away from the diagonal allows us to simplify the problem to the case

$$C_{kj}^0(\lambda, \xi, \eta) = g(\lambda) \sum G_n(\xi, \eta)$$

where the sum runs over the positive integers $\approx 2^k$ and G_n satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta G_n(\xi, \eta)| \lesssim 2^{-2k} \langle \xi - n \rangle^{-N} \langle \eta - n \rangle^{-N}, \qquad \alpha + \beta \leq 2 \tag{6.26}$$

Since λ belongs to the algebra Z, we can further simplify the expression $\lambda' g(\lambda)$ occurring in K_{kj}^0 and simply replace it by λ'. By the definition of the space Z we have two possibilities to consider:

Case A: $\lambda \in W^{1,1}$. This is the easier case. Then $\|\lambda'\|_{L^1} \lesssim \gamma$ therefore by (6.23) we obtain

$$\|K^0_{kj} u_j\|_{L^1L^2} \lesssim \gamma 2^{\frac{k+j}{2}} \|u_j\|_{L^\infty L^2} \tag{6.27}$$

which suffices for (6.21).

Case B: $\lambda \in \dot{H}^{\frac{1}{2},1}$. In this case we split λ into a low frequency part and a high frequency part,

$$\lambda = \lambda_{\leq m_0+4} + \lambda_{>m_0+4}, \qquad m_0 = 2\max\{k,j\}$$

Case B1: The contribution of $\lambda_{\leq m_0+4}$. The low frequency part of λ' satisfies a favorable L^2 bound

$$\|\lambda'_{\leq m_0+4}\|_{L^2} \lesssim \gamma 2^{\frac{m_0}{2}}$$

Suppose $|j-k| \gg 1$. Using (6.24) for ϕ_k, (6.23) for ϕ_j and the energy of u_j we obtain

$$\|K^0_{kj} u_j\|_{l^2LE^*} \lesssim \gamma 2^{\frac{m_0-k+j}{2}} \|u_j\|_{L^\infty L^2}$$

which is favorable if $j<k$. In the opposite case $j>k$ we use (6.24) for ϕ_j and (6.23) for ϕ_k to obtain the better dual type bound

$$\|K^0_{kj} u_j\|_{L^1L^2} \lesssim \gamma 2^{\frac{m_0+k-j}{2}} \|u_j\|_{l^2LE}$$

Consider now the case $|j-k| \lesssim 1$ and $k \lesssim 0$. There we have

$$K^0_{kj} u_j = \lambda'_{\leq m_0+4}(t) f, \qquad \hat{f}(t,\xi) = \int G(\xi,\eta)\hat{u}_j(t,\eta)d\eta$$

For $\hat{f}$ we can use (6.25) to estimate up to two derivatives at fixed time

$$|(\xi\partial_\xi)^\alpha \hat{f}(t,\xi)| \lesssim \frac{2^{-\frac{k}{2}}}{\langle k^-\rangle^2}\|u_j(t)\|_{L^2}, \qquad \alpha \leq 1$$

Then a variation of Proposition 4.11 shows that

$$|f(t,r)| \lesssim \frac{2^k}{\langle k^-\rangle^2} m_k(r)(1+2^k r)^{-\frac{5}{2}}\|u_j(t)\|_{L^2}$$

which allows us to estimate

$$\|K^0_{kj}u_j\|_{LE^*_k} \lesssim \|\lambda'_{\leq m_0+4}\|_{L^2}\frac{2^{-k}}{\langle k^-\rangle^2}\|u_j(t)\|_{L^\infty L^2} \lesssim \gamma a_{kj}\|u_j(t)\|_{L^\infty L^2}$$

In the case $|j-k| \lesssim 1$ and $k \gg 0$ we have

$$K^0_{kj} u_j = \lambda'_{\leq m_0+4}(t)\sum_n f_n, \qquad \hat{f}_n(t,\xi) = \int G_n(\xi,\eta)\hat{u}_j(t,\eta)d\eta$$

For $\hat{f}_n$ we can use (6.26) to estimate up to two derivatives at fixed time

$$|\partial^\alpha_\xi \hat{f}_n(t,\xi)| \lesssim 2^{-\frac{3k}{2}}\langle \xi - n\rangle^{-N}\|u_j(t)\|_{L^2}, \qquad \alpha \leq 2$$

Then a variation of Proposition 4.11 shows that

$$|f_n(t,r)| \lesssim m_k(r)(1+2^k r)^{-\frac{5}{2}}\|u_j(t)\|_{L^2}$$

which allows us to estimate

$$\|K^0_{kj}u_j\|_{LE^*_k} \lesssim \|\lambda'_{\leq m_0+4}\|_{L^2}\sum_n \|f_n\|_{LE^*_k} \lesssim \gamma 2^k \sum_n 2^{-2k}\|u_j(t)\|_{L^\infty L^2} \lesssim \gamma\|u_j(t)\|_{L^\infty L^2}$$

where we have used that the range of summation has cardinal $\approx 2^k$.

Case B2: The contribution of λ_m, $m > m_0 + 4$. The idea in this case is that a large modulation for λ forces either a large modulation in the input or a large modulation in the output. Precisely, if $|k-j| \gg 1$ then we have

$$\lambda'_m \phi_k \langle \phi_j u_j \rangle = Q_{>m-4}(\lambda'_m \phi_k \langle \phi_j, u_j \rangle) + \lambda'_m \phi_k \langle \phi_j, Q_{>m-4} u_j \rangle \\ - Q_{>m-4}(\lambda'_m \phi_k \langle \phi_j, Q_{>m-4} u_j \rangle)$$

The first (as well as the last) term is at high modulation so it suffices to bound it in L^2,

$$\|Q_{>m-4}(\lambda'_m \phi_k \langle \phi_j, u_j \rangle)\|_{L^2} \lesssim 2^{\frac{k+j}{2}} \|\lambda'_m\|_{L^2} \|u_j\|_{L^\infty L^2} \lesssim \gamma 2^{\frac{k+j}{2}} 2^{\frac{m}{2}} \|u_j\|_{L^\infty L^2}$$

On the other hand in the second term the function u_j is restricted to high modulations, where we have a good L^2 bound:

$$\|\lambda'_m \phi_k \langle \phi_j, Q_{>m-4} u_j \rangle\|_{L^1 L^2} \lesssim 2^{\frac{k+j}{2}} \|\lambda'_m\|_{L^2} \|Q_{>m-4} u_j\|_{L^2} \lesssim \gamma 2^{\frac{k+j}{2}} \|u_j\|_{S_j^\sharp}$$

A similar argument also applies for $|k-j| \lesssim 1$. The proof of the lemma is concluded. □

The global in time result in the previous proposition easily implies its compact interval counterpart:

Corollary 6.5. *Let $T > 0$ and λ so that*

$$\|\lambda - 1\|_{Z([0,T])} \lesssim \gamma \ll 1 \tag{6.28}$$

Then the solution of (6.1) *in* $[0,T]$ *satisfies:*

$$\|\psi\|_{WS^\sharp[\lambda][0,T]} \lesssim \|\psi(0)\|_{L^2} + \|g\|_{WN^\sharp[\lambda][0,T]} \tag{6.29}$$

Proof. Consider an admissible extension λ^{ext} for λ in Z, so that

$$\|\lambda^{ext} - 1\|_Z \lesssim \gamma \ll 1$$

Consider also the zero extension g^{ext} of g. This satisfies

$$\|g^{ext}\|_{WN^\sharp[\lambda^{ext}]} \lesssim \|g\|_{WN^\sharp[\lambda][0,T]}$$

Now solve (6.1) with λ^{ext}, g^{ext} instead of λ, g. By the previous proposition this yields a global solution ψ satisfying

$$\|\psi\|_{WS^\sharp[\lambda^{ext}]} \lesssim \|\psi(0)\|_{L^2} + \|g\|_{WN^\sharp[\lambda][0,T]}$$

The conclusion follows by restricting ψ to the time interval $[0,T]$. □

Proposition 6.3 also allows us to prove the interval counterpart to (6.7):

Corollary 6.6. *Let I be an interval and λ with*

$$\|\lambda - 1\|_{Z(I)} \lesssim \gamma \ll 1$$

Then the following inclusion holds:

$$WS^\sharp[\lambda](I) \subset WS^r[1](I) \tag{6.30}$$

Proof. Set $I = [a, b]$. Consider an admissible extension λ^{ext} for λ in Z. Given $\psi \in WS^\sharp[\lambda](I)$, we extend it to the real line as a solution to

$$(i\partial_t - \tilde{H}_{\lambda^{ext}})\psi = 0 \qquad \text{in } \mathbb{R} \setminus I,$$

matching Cauchy data at $t = a, b$. By the previous corollary we have

$$\|\psi\|_{WS^\sharp[\lambda^{ext}](-\infty,a)} + \|\psi\|_{WS^\sharp[\lambda^{ext}](b,\infty)} \lesssim \|\psi(a)\|_{LX} + \|\psi(b)\|_{LX} \lesssim \|\psi\|_{WS^\sharp[\lambda](I)}$$

Due to the Cauchy data matching at $t = a, b$ this implies the global bound

$$\|\psi\|_{WS^\sharp[\lambda^{ext}]} \lesssim \|\psi\|_{WS^\sharp[\lambda](I)}$$

Now we apply (6.7) to obtain

$$\|\psi\|_{WS^r[\lambda^{ext}]} \lesssim \|\psi\|_{WS^\sharp[\lambda](I)}$$

The conclusion follows by restricting the LHS to I. $\square$

6.3. **The autonomous vs nonautonomous flow.** Here, under a suitable L^2 smallness condition, we show that the solution to the non-autonomous homogeneous equation

$$(i\partial_t - \tilde{H}_\lambda)\psi = 0, \qquad \psi(0) = \psi_0 \tag{6.31}$$

stays close to the solution of the corresponding autonomous homogeneous equation

$$(i\partial_t - \tilde{H})\tilde{\psi} = 0, \qquad \tilde{\psi}(0) = \psi_0 \tag{6.32}$$

Proposition 6.7. *Let λ be as in* (6.3)*. Suppose that*

$$\|\psi(0)\|_{LX} \leq 1, \qquad \|\psi(0)\|_{L^2} \leq \epsilon \ll 1. \tag{6.33}$$

Then the solutions to (6.31) *and* (6.32) *stay close,*

$$\|\psi - \tilde{\psi}\|_{L^\infty LX} \lesssim |\log \epsilon|^{-1} \tag{6.34}$$

Proof. From the L^2 bound for the initial data we obtain

$$\|\psi\|_S \lesssim \epsilon, \qquad \|\tilde{\psi}\|_S \lesssim \epsilon$$

Because of the L^2 bounds above we have

$$\|P^\lambda_{\gtrsim[\log\epsilon]}\psi\|_{L^\infty LX} \lesssim |\log\epsilon|^{-1}, \qquad \|P_{\gtrsim[\log\epsilon]}\tilde{\psi}\|_{L^\infty LX} \lesssim |\log\epsilon|^{-1}$$

so it remains to consider the low frequencies. For $k < \log\epsilon$ we will compare

$$\psi_k = P^\lambda_k \psi, \qquad \tilde{\psi}_k = P_k\tilde{\psi}$$

With the notations from the proof of Proposition 6.3, we have the following system for $\{\psi_k\}$

$$(i\partial_t - \tilde{H}_\lambda)\psi_k = \sum_j K^\lambda_{kj}\psi_j := g_k$$

By (6.17), using the L^2 bound for frequencies larger than ϵ and the LX bound for frequencies smaller than ϵ we obtain

$$\sum_{k<\log\epsilon} \frac{1}{2^k|k|}\|g_k\|_{l^2N^\sharp} \lesssim \frac{1}{|\log\epsilon|} \tag{6.35}$$

For the initial data we claim to have a similar relation

$$\sum_{k<\log\epsilon} \frac{1}{2^k|k|}\|\psi_k(0) - \tilde{\psi}_k(0)\|_{L^2} \lesssim \frac{1}{|\log\epsilon|} \tag{6.36}$$

To prove this we write

$$\mathcal{F}_{\tilde{H}}(\psi_k(0) - \tilde{\psi}_k(0)) = \sum_{j,h} (P_j P_k \tilde{P}_h - P_j P_k^\lambda \tilde{P}_h) P_h \psi(0)$$

and use Proposition 4.14 to estimate each term,

$$\begin{aligned} LHS(6.36) &\lesssim \sum_{k<\log\epsilon} \frac{1}{2^k|k|} \sum_{j,h} c_{jkh} \|P_h\psi(0)\|_{L^2} \approx \sum_h \sum_{k<\log\epsilon} \frac{1}{2^k|k|} c_{kkh} \|P_h\psi(0)\|_{L^2} \\ &\approx \sum_h \sum_{k<\log\epsilon} \frac{1}{2^k|k|} \frac{2^{-|k-h|-Nh^+}}{|k|\langle h\rangle} \|P_h\psi(0)\|_{L^2} \\ &\approx \sum_{h<\log\epsilon} \frac{1}{2^h|h^2|} \|P_h\psi(0)\|_{L^2} + \sum_{h>\log\epsilon} \frac{1}{|\log\epsilon|} \frac{2^{-Nh^+}}{2^h\langle h\rangle} \|P_h\psi(0)\|_{L^2} \end{aligned}$$

Hence (6.36) follows.

Finally we consider the effect of the change in the potential,

$$\|(V_\lambda - V)\psi_k\|_{LE^*} \lesssim \frac{1}{|k|}\|\psi_k\|_{LE_k} \lesssim \frac{1}{|k|}\|\psi_k\|_{l^2LE} \tag{6.37}$$

(see (6.4) and (5.2)). Thus, comparing ψ_k and $\tilde{\psi}_k$ along the $\tilde{H}$ flow we obtain

$$\sum_{k<\log\epsilon} \frac{1}{2^k|k|} \|\psi_k - \tilde{\psi}_k\|_S \lesssim \frac{1}{|\log\epsilon|} \tag{6.38}$$

We need to turn this into an $L^\infty LX$ bound. We will use only the $L^\infty L^2$ part of the S norm. At fixed time we write

$$\|\psi_k - \tilde{\psi}_k\|_{LX} \lesssim \frac{1}{2^k|k|} \|\tilde{P}_k(\psi_k - \tilde{\psi}_k)\|_{L^2} + \|(1-\tilde{P}_k)\psi_k\|_{LX}$$

For the second term we use Proposition 4.10. We obtain

$$\|\psi_k - \tilde{\psi}_k\|_{LX} \lesssim \frac{1}{2^k|k|} \|\psi_k - \tilde{\psi}_k\|_{L^2} + \frac{1}{2^k|k|^2} \|\psi_k\|_{L^2}$$

which combined with (6.38) leads to

$$\sum_{k<\log\epsilon} \|\psi_k - \tilde{\psi}_k\|_{LX} \lesssim \frac{1}{|\log\epsilon|}$$

The proof is concluded.

□

7. Analysis of the gauge elements in X, LX

In Section 3 we have studied the forward transition from the Schrödinger map u to its Coulomb gauge v, w, its coordinates ψ_1, ψ_2, A_2 and finally to its reduced field ψ in the setup where $\|u - Q\|_{\dot{H}^1} \leq \gamma \ll 1$, which corresponds to $\psi \in L^2$. However, the reverse process is not uniquely determined in this context. The easiest way to see this is that if $\psi = 0$ then all we can say is that u is one of the solitons $Q_{\alpha,\lambda}$. In some sense, this is the only possible ambiguity.

Here we consider again the transition between u and its reduced field ψ but in the more regular setting where $\bar{u} - \bar{Q} \in X$ and $\psi \in LX$. An advantage in doing this is that it allows us to impose a natural boundary condition at infinity for the system (3.31), namely

$$\lim_{r\to\infty} A_2 = 1, \qquad \psi_2 - ih_1 \in X \tag{7.1}$$

We will see that on one hand this condition is dynamically preserved along the Schrödinger map flow, while, on the other hand, it allows for an unique identification of u in terms of ψ. We remark that, in view of (1.7), this condition is satisfied for maps u for which $u - Q \in L^2$. The main result of this section is as follows:

Theorem 7.1. *a) Let $u : \mathbb{R}^2 \to \mathbb{S}^2$ be an 1-equivariant map which satisfies $\|u - Q\|_{\dot{H}^1} \ll 1$ and $\|\bar{u} - \bar{Q}\|_X \leq \gamma \ll 1$. Then the Coulomb gauge constructed in Section 3 satisfies the additional properties*

$$\|\bar{v} - \bar{V}\|_{\tilde{X}} + \|\bar{w} - \bar{W}\|_{\tilde{X}} \leq \gamma \tag{7.2}$$

$$\|\bar{v}_3 - \bar{V}_3\|_X + \|\bar{w}_3 - \bar{W}_3\|_X \leq \gamma \tag{7.3}$$

$$\|\psi_2 - ih_1\|_X + \|A_2 - h_3\|_X \leq \gamma \tag{7.4}$$

$$\|\psi\|_{LX} \lesssim \gamma \tag{7.5}$$

Furthermore, the map from $\bar{u} - \bar{Q} \in X$ to $\psi \in LX$ is of class C^1.

b) Let $\psi : \mathbb{R}^2 \to \mathbb{C}$ be a function which satisfies $\|\psi\|_{LX} \leq \gamma \ll 1$. Then there exists an unique 1-equivariant map $u : \mathbb{R}^2 \to \mathbb{S}^2$ satisfying

$$\|\bar{u} - \bar{Q}\|_X \lesssim \gamma \tag{7.6}$$

so that ψ is the reduced field for u, and the map from $\psi \in LX$ to $\bar{u} - \bar{Q} \in X$ is of class C^1. Furthermore, the uniqueness of u is also valid in the class of maps u with $\|u - Q\|_{\dot{H}^1} \ll 1$ which satisfy the additional qualitative condition $u - Q \in L^2$.

As a quick reminder, $\bar{V}, \bar{W}, \bar{Q}$ were introduced in (3.18), and $\bar{V}_3, \bar{W}_3$ stand for the third component of the vectors $\bar{V}$, respectively $\bar{W}$.

The plan of this section is as follows. We first prepare for the proof with an ODE result which will be applied later to the system for the orthogonal matrix $\mathcal{O} = (v, w, u)$ in both parts (a) and (b). Then we prove part (a) in two stages, beginning with the ODE construction of $\mathcal{O}$ and continuing with the algebraic derivation of ψ_2, A_2 and ψ. Finally, we prove part (b) also in two stages, namely the recovery of (ψ_2, A_2) via the ODE system (3.31) with the boundary condition (7.1), and then a second ODE construction for the matrix $\mathcal{O}$.

7.1. **An ODE result.**

Lemma 7.2. *Consider the ODE*

$$\partial_r Z = NZ + F, \qquad \lim_{r\to\infty} Z(r) = 0 \tag{7.7}$$

If N is small in $\partial_r \tilde{X}$ then the above equation has a unique solution $Z \in \tilde{X}$ satisfying

$$\|Z\|_{\tilde{X}} \lesssim \|F\|_{\partial_r \tilde{X}} \tag{7.8}$$

Furthermore, the map from $N, F \in \partial_r \tilde{X}$ to $Z \in \tilde{X}$ is analytic.

Proof. The solution Z is obtained via a Picard iteration in the space $\tilde{X}$. Indeed, the results of Lemma 4.7 show that

$$\|NZ + F\|_{\partial_r X} \lesssim \|N\|_{\partial_r \tilde{X}} \|Z\|_{\tilde{X}} + \|F\|_{\partial_r \tilde{X}}$$

and the convergence of the iterations is insured due to the smallness of $\|N\|_{\partial_r \tilde{X}}$. □

7.2. **The transition from u to (v, w).** We use the equation (3.16) for the matrix $\mathcal{O} = (\bar{v}, \bar{w}, \bar{u})$, namely

$$\partial_r \mathcal{O} = M(\bar{u})\mathcal{O}, \qquad \mathcal{O}(\infty) = I_3, \qquad M(\bar{u}) = \partial_r \bar{u} \wedge \bar{u} \tag{7.9}$$

If $u = Q$ them $M(\bar{u})$ has the form

$$M(\bar{Q}) = \begin{pmatrix} 0 & 0 & -\frac{h_1}{r} \\ 0 & 0 & 0 \\ \frac{h_1}{r} & 0 & 0 \end{pmatrix} \tag{7.10}$$

For the difference we claim that

$$\|M(\bar{u}) - M(\bar{Q})\|_{\partial_r \tilde{X}} \lesssim \|\bar{u} - \bar{Q}\|_X \tag{7.11}$$

Indeed, we write

$$M(\bar{u}) - M(\bar{Q}) = \partial_r(\bar{u} - \bar{Q}) \wedge (\bar{u} - \bar{Q}) + 2\partial_r(\bar{u} - \bar{Q}) \wedge \bar{Q} + \partial_r \left(\bar{Q} \wedge (\bar{u} - \bar{Q})\right).$$

For the first term we use (4.34) and for the second we use (4.33). For the third term we have

$$\bar{Q} \wedge (\bar{u} - \bar{Q}) = \vec{k} \wedge (\bar{u} - \bar{Q}) + (\bar{Q} - \vec{k}) \wedge (\bar{u} - \bar{Q})$$

where the first term is trivially in $\tilde{X}$ while the second belongs to $H^1_e \subset X$ due to the r^{-1} decay of $\bar{Q} - \vec{k}$ at infinity. Hence (7.11) is proved.

Returning to (7.9), we start with the solution $\mathcal{O}_0$ for the case $u = Q$, which is given by

$$\mathcal{O}_0 = \begin{pmatrix} h_3 & 0 & h_1 \\ 0 & 1 & 0 \\ -h_1 & 0 & h_3 \end{pmatrix}, \qquad \mathcal{O}_0^{-1} = \mathcal{O}_0^t \tag{7.12}$$

Then we write the solution to (7.9) is of the form

$$\mathcal{O}(r) = \mathcal{O}_0(r)(I + Y(r)) \tag{7.13}$$

where Y solves the differential equation

$$\partial_r Y = NY + G, \qquad \mathcal{O}(\infty) = 0 \qquad N = G = \mathcal{O}_0^{-1}(M(\bar{u}) - M(\bar{Q}))\mathcal{O}. \tag{7.14}$$

The bound (7.11) combined with (4.33) shows that we can apply Lemma 7.2 for Y. The bound (7.2) follows after another application of (4.33).

For the extra improvement in (7.3) we still need to estimate $\|r^{-1}\bar{v}_3\|_{L^2}$ and $\|r^{-1}\bar{w}_3\|_{L^2}$. Consider for instance the latter. Writing

$$\bar{w}_3 = \bar{u}_1\bar{v}_2 - \bar{u}_2\bar{v}_1 = (\bar{u}_1 - h_1)\bar{v}_2 - \bar{u}_2\bar{v}_1 + h_1\bar{v}_2$$

the desired bound easily follows.

7.3. **The transition from** (u, v, w) **to** ψ_2, A_2 **and** ψ**.** By (3.13), the bound (7.4) for ψ_2 is exactly (7.3), while by (3.12), the bound (7.4) for A_2 follows from the hypothesis.

It remains to consider ψ, which is represented as

$$\psi = \mathcal{W} \cdot v + i\mathcal{W} \cdot w, \qquad \mathcal{W} = \partial_r \bar{u} - \frac{1}{r}\bar{u} \times R\bar{u}$$

In view of the bound (7.3) for (v, w) and of the LX multiplicative estimates (4.28) and (4.35), it suffices to show that

$$\|\mathcal{W}\|_{LX} \lesssim \gamma \tag{7.15}$$

Since $\mathcal{W}$ vanishes if $u = Q$, we can write

$$\begin{aligned}\mathcal{W} =& \partial_r(\bar{u} - \bar{Q}) - \frac{1}{r}(\bar{u} - \bar{Q}) \times R(\bar{u} - \bar{Q}) - \frac{1}{r}\bar{Q} \times R(\bar{u} - \bar{Q}) - \frac{1}{r}(\bar{u} - \bar{Q}) \times R\bar{Q} \\ =& L(\bar{u} - \bar{Q}) - \frac{1}{r}(\bar{u} - \bar{Q}) \times R(\bar{u} - \bar{Q}) + \tilde{\mathcal{W}}\end{aligned}$$

The first term is in LX by definition and the second belongs to the smaller space $L^1 \cap L^2$ by (4.16) and (4.17). It remains to consider the last component

$$\tilde{\mathcal{W}} = -\frac{h_3}{r}(\bar{u} - \bar{Q}) - \frac{1}{r}\bar{Q} \times R(\bar{u} - \bar{Q}) - \frac{1}{r}(\bar{u} - \bar{Q}) \times R\bar{Q}$$

A direct computation shows that the components of $\tilde{\mathcal{W}}$ contain the expressions $r^{-1}h_1(\bar{u}_3 - h_3)$, $r^{-1}h_1(\bar{u}_1 - h_1)$ and $r^{-1}h_3(\bar{u}_3 - h_3)$; we will estimate all of them in $L^1 \cap L^2$. The L^2 bound is obtained directly from the $\dot{H}^1$ norm of $u - Q$. The L^1 bound for the first two expressions is a consequence of (4.17). This also applies to the third expression but only for $r \lesssim 1$ On the other hand for $r \gg 1$ we can use the equation of the sphere to obtain

$$|\bar{u}_3 - h_3| \lesssim (u_1 - h_1)^2 + u_2^2 + h_1|u_1 - h_1|$$

at which point we can use again (4.17).

7.4. **The transition from** ψ **to** (ψ_2, A_2)**.** This is achieved by solving the ODE system (3.31) with the boundary condition (7.1) at infinity. We note that by Proposition 3.2, if $u - Q \in L^2$ then we have $\psi_2 - ih_1 \in L^2$, which implies (7.1). For convenience we recall (3.31) here:

$$\begin{cases} \partial_r A_2 = \Im(\psi\bar{\psi}_2) + \frac{1}{r}|\psi_2|^2, \\ \partial_r \psi_2 = iA_2\psi - \frac{1}{r}A_2\psi_2 \end{cases} \tag{7.16}$$

We are only interested in solutions which belong to the sphere

$$A_2^2 + |\psi_2|^2 = 1 \tag{7.17}$$

A straightforward computation shows that this sphere is invariant with respect to the (7.16) flow. Thus given any Cauchy data on this sphere at any point $r_0 \in \mathbb{R}^+$ and any $\psi \in L^2$, there exists an unique global solution to this ODE. Our challenge here is to instead prescribe the asymptotic behavior at infinity via (7.1). To achieve this we will take advantage of the additional information that $\psi \in LX$. We state our main result here separately for later use:

Proposition 7.3. *Assume that $\psi \in LX$, small. Then the system* (7.16) *admits a unique solution* (ψ_2, A_2) *which satisfies* (7.1). *Furthermore, this solution satisfies the bound*

$$\|\psi_2 - ih_1\|_X + \|A_2 - h_3\|_X \lesssim \|\psi\|_{LX}, \tag{7.18}$$

and it has Lipschitz dependence on ψ,

$$\|\psi_2 - \tilde{\psi}_2\|_X + \|A_2 - \tilde{A}_2\|_X \lesssim \|\psi - \tilde{\psi}\|_{LX}. \tag{7.19}$$

In addition, the above solution satisfies the following $\dot{H}^1_e$ bounds:

$$\|\psi_2 - ih_1\|_{\dot{H}^1_e} + \|A_2 - h_3\|_{\dot{H}^1_e} \lesssim \|L^{-1}\psi\|_{\dot{H}^1_e}. \tag{7.20}$$

We remark that from (7.18) and (7.17) one can get better decay for $A_2 - h_3$ both near 0 and near infinity.

Proof. We carry out this proof in several steps:

Step 1: Here we assume that a solution (ψ_2, A_2) to (7.16) which satisfies the boundary condition (7.1) exists, and we study further its a-priori regularity. In what follows C will denote a large positive constant which may vary from line to line.

Since $\psi_2 - ih_1 \in X$ then, by (4.17) and (4.16), we must have

$$|\psi_2| \leq C\langle r\rangle^{-\frac{1}{2}}, \qquad \|\frac{r^{\frac{1}{2}}}{\log(1+r)}\psi_2\|_{L^2(dr)} < \infty \tag{7.21}$$

and similar bounds for $A_2 - h_3$. By virtue of the compatibility relation (7.17), we can improve the bounds for $A_2 - h_3$ to

$$|A_2 - h_3| \leq C\langle r\rangle^{-1}, \qquad \|\frac{r^{\frac{1}{2}}\langle r\rangle^{\frac{1}{2}}}{\log(1+r)}(A_2 - h_3)\|_{L^2(dr)} < \infty \tag{7.22}$$

We rewrite the second equation in (7.16) as

$$L\psi_2 = i\psi + f, \qquad f = i(A_2 - 1)\psi + \frac{h_3 - A_2}{r}\psi_2 \tag{7.23}$$

For large r it suffices to consider this equation, since A_2 is uniquely determined as $A_2 = \sqrt{1 - |\psi_2|^2}$ due to (7.17). Since $\psi \in L^2$, using also (7.21) and (7.22) we obtain the decay of f at infinity,

$$\|r^{\frac{1}{2}}\langle r\rangle f\|_{L^2(dr)} + \|\frac{r\langle r\rangle^{\frac{1}{2}}}{\log(1+r)}f\|_{L^1(dr)} < \infty \tag{7.24}$$

In particular by (4.25) it follows that $f \in LX$. As $\psi_2 - h_1 \in X$, the solution ψ to (7.23) must have the form

$$\psi_2 = ih_1 + ig + \Psi, \qquad g = L^{-1}\psi \in X, \qquad \Psi = L^{-1}f \in X.$$

Since f has the better decay at infinity given by (7.24), we can express Ψ in the integral form

$$\Psi(r) = -h_1 \int_r^\infty h_1(r_1)^{-1} f(r_1) dr_1$$

By (7.24) this integral is absolutely convergent, and we have the pointwise bound

$$|\Psi(r)| \leq C\frac{\log(2+r)}{\langle r\rangle^{\frac{3}{2}}} \tag{7.25}$$

We can recast the equation (7.23) as an equation for Ψ,

$$\Psi = N(\Psi, \psi) \tag{7.26}$$

where the nonlinear expression on the right has the integral form

$$N(\Psi, \psi) = -h_1(r) \int_r^\infty h_1^{-1}(i(A_2 - 1)\psi + \frac{h_3 - A_2}{s}(ih_1 + ig + \Psi))ds,$$

and A_2 and g are dependent variables given by

$$A_2 = \sqrt{1 - |ih_1 + ig + \Psi|^2}, \qquad g = L^{-1}\psi$$

We have proved that solving the system (7.16) with the boundary condition (7.1) is equivalent to solving the equation (7.26) for $\Psi \in X$ satisfying the decay condition (7.25).

Step 2: Here we will use the contraction principle to show that for r near infinity, $r \in [R, \infty)$ there exists a unique solution Ψ to (7.26) satisfying (7.25), which depends in a Lipschitz manner on $\psi \in LX$. For this we will prove that the nonlinearity N satisfies the Lipschitz bound

$$\|N(\Psi, \psi) - N(\tilde{\Psi}, \tilde{\psi})\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))} \leq \frac{C_0}{R^{\frac{1}{4}}}(\|\psi - \tilde{\psi}\|_{LX} + \|\Psi - \tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))}) \tag{7.27}$$

where

$$C_0 = C_0(\|\psi\|_{LX}, \|\tilde{\psi}\|_{LX}, \|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}[R,\infty)}, \|\tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}[R,\infty)})$$

For the existence and Lipschitz dependence part we start with ψ satisfying $\|\psi\|_X \leq 1$, choose R so that

$$R^{-\frac{1}{4}} C_0(1, 1, 1, 1) \leq \frac{1}{4}$$

and apply the contraction principle in the unit ball in $L^\infty_{r^{\frac{5}{4}}}[R, \infty)$. This yields a unique solution Ψ which satisfies

$$\|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))} \lesssim \|\psi\|_{LX} \tag{7.28}$$

with the Lipschitz dependence

$$\|\Psi - \tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))} \lesssim \|\psi - \tilde{\psi}\|_{LX} \tag{7.29}$$

For the uniqueness part we use (7.27) but with a larger $\tilde{R}$ chosen so that

$$\tilde{R}^{-\frac{1}{4}} C_0(\|\psi\|_{LX}, \|\tilde{\psi}\|_{LX}, \|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}[R,\infty)}, \|\tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}[R,\infty)}) < 1$$

It suffices to prove uniqueness on a smaller interval $[\tilde{R}, \infty)$ since the equation (7.26) is equivalent to the original ODE system (7.16), for which uniqueness holds for r in a compact interval in $(0, \infty)$.

We now continue with the proof of (7.27). From the formulas for $A_2, \tilde{A}_2$, we have

$$|A_2 - \tilde{A}_2| \lesssim r^{-\frac{1}{2}}(|g - \tilde{g}| + |\Psi - \tilde{\Psi}|)$$

$$|A_2 - 1| \lesssim h_1^2 + r^{-\frac{1}{2}}(|g| + |\Psi|), \quad |A_2 - h_3| \lesssim r^{-\frac{1}{2}}(|g| + |\Psi|)$$

which implies that (recall the definition of f from (7.23))

$$\begin{aligned} |f-\tilde{f}| &\lesssim |A_2-\tilde{A}_2||\tilde{\psi}|+|A_2-1||\psi-\tilde{\psi}| \\ &\quad + r^{-1}|A_2-h_3||\psi_2-\tilde{\psi}_2|+r^{-1}|A_2-\tilde{A}_2||\tilde{\psi}_2| \\ &\lesssim \delta f_1+\delta f_2+\delta f_3 \end{aligned} \tag{7.30}$$

where

$$\delta f_1 = r^{-\frac{1}{2}}|\psi|(|g-\tilde{g}|+|\Psi-\tilde{\Psi}|), \qquad \delta f_2 = (h_1^2+r^{-\frac{1}{2}}(|\tilde{g}|+|\tilde{\Psi}|))|\psi-\tilde{\psi}|$$
$$\delta f_3 = r^{-2}(|g-\tilde{g}|+|\Psi-\tilde{\Psi}|)$$

Correspondingly we derive a bound for $N(\Psi,\psi)-N(\tilde{\Psi},\tilde{\psi})$,

$$|N(\Psi,\psi)-N(\tilde{\Psi},\tilde{\psi})| \lesssim \delta N_1+\delta N_2+\delta N_3, \quad \delta N_i(r) = h_1(r)\int_r^\infty h_1^{-1}(s)\delta f_i(s)ds$$

Now we successively consider the three contributions. For δf_3 by (4.16) we have the pointwise bound

$$|\delta f_3(r)| \lesssim r^{-\frac{5}{2}}(\|g-\tilde{g}\|_X+\|\Psi-\tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})$$

therefore its contribution δN_3 satisfies:

$$\delta N_3(r) \lesssim r^{-\frac{3}{2}}(\|g-\tilde{g}\|_X+\|\Psi-\tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})$$

For δf_2 we use instead (4.17), to get an L^1 bound

$$\|r^{\frac{1}{2}}|\log r|^{-1}\delta f_2\|_{L^1(rdr)} \lesssim (\|\tilde{g}\|_X+\|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})\|\psi-\tilde{\psi}\|_{L^2}$$

which leads to

$$\delta N_2(r) \lesssim r^{-\frac{3}{2}}|\log r|^{-1}(\|\tilde{g}\|_X+\|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})\|\psi-\tilde{\psi}\|_{L^2}$$

Finally, for δf_1 we use again (4.17) to estimate

$$\|r^{\frac{1}{2}}|\log r|^{-1}\delta f_1\|_{L^1(rdr)} \lesssim (\|g-\tilde{g}\|_X+\|\Psi-\tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})\|\psi\|_{L^2}$$

which yields

$$\delta N_1(r) \lesssim r^{-\frac{3}{2}}|\log r|^{-1}(\|g-\tilde{g}\|_X+\|\Psi-\tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})\|\psi\|_{L^2}$$

The proof of (7.27) is concluded.

Step 3: Now we consider the solution Ψ to (7.26) obtained in the previous step in the interval $[R,\infty)$, and we supplement the pointwise bounds (7.28) and (7.29) with H_e^1 bounds

$$\|\Psi\|_{H_e^1[R,\infty)} \lesssim \|\psi\|_{LX} \tag{7.31}$$

$$\|\Psi-\tilde{\Psi}\|_{H_e^1[R,\infty)} \lesssim \|\psi-\tilde{\psi}\|_{LX} \tag{7.32}$$

In view of the embedding $H_e^1 \subset X$, these will be useful later to establish the Lipschitz dependence in X. Returning to (ψ_2, A_2) these bounds imply that

$$\|(\psi_2-g)-(\tilde{\psi}_2-\tilde{g})\|_{H_e^1[R,\infty)} \lesssim \|\psi-\tilde{\psi}\|_{LX} \tag{7.33}$$

respectively

$$\|A_2-\tilde{A}_2\|_{H_e^1[R,\infty)} \lesssim \|\psi-\tilde{\psi}\|_{LX} \tag{7.34}$$

The L^2 part of (7.31) and (7.32) follow trivially from (7.28) and (7.29). Consider now the estimate for the L^2 norm for $\partial_r\Psi$. Given that we already have an L^2 bound for Ψ, we can freely replace $\partial_r\Psi$ by $L\Psi = f$. It remains to show that

$$\|f - \tilde{f}\|_{L^2} \lesssim \|\psi - \tilde{\psi}\|_{LX} + \|\Psi - \tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))} \tag{7.35}$$

Consider the bound (7.30) for $\|f - \tilde{f}\|_{L^2}$. The estimate for δf_3 proved in the previous step is already good enough. We only need to revisit the bounds on δf_2 and δf_1, which we do by using (4.16) instead of (4.17). For δf_2 we obtain

$$\|r\delta f_2\|_{L^2(rdr)} \lesssim (\|\tilde{g}\|_X + \|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})\|\psi - \tilde{\psi}\|_{L^2}$$

Similarly for δf_1 we have

$$\|r\delta f_1\|_{L^2(rdr)} \lesssim (\|g - \tilde{g}\|_X + \|\Psi - \tilde{\Psi}\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))})\|\psi\|_{L^2}$$

Both bounds are much stronger than we need.

Step 4: Here we prove the large r part of (7.20), which with respect to Ψ takes the form

$$\|N(\Psi, \psi)\|_{\dot{H}^1_e([R,\infty))} \lesssim \|g\|_{\dot{H}^1_e} \tag{7.36}$$

This implies that the solution ψ_2 constructed above in $[R, \infty)$ satisfies

$$\|\psi_2 - ih_1\|_{\dot{H}^1_e([R,\infty))} + \|A_2 - h_3\|_{\dot{H}^1_e([R,\infty))} \lesssim \|g\|_{\dot{H}^1_e} \tag{7.37}$$

In proving (7.36) we can assume that the following bounds are valid:

$$\|g\|_X + \|\Psi\|_{L^\infty_{r^{\frac{5}{4}}}([R,\infty))} \lesssim 1$$

To establish (7.36), we use the following pointwise estimate on f:

$$\begin{aligned} |f| &\lesssim |A_2 - 1||\psi| + \frac{|h_3 - A_2|}{r}||\psi_2| \\ &\lesssim (h_1^2 + r^{-\frac{1}{2}}(|g| + |\Psi|))|\psi| + r^{-\frac{3}{2}}(|g| + |\Psi|)(h_1 + |g| + |\Psi|) \end{aligned}$$

Then for $\Psi = N(\Psi, \psi) = L^{-1}f$, using(4.17), we obtain

$$\begin{aligned} \|\frac{\Psi}{r}\|_{L^2([R,\infty))} &\lesssim \int_R^\infty s|f(s)|ds \\ &\lesssim \int_R^\infty (s^{-2} + s^{-\frac{1}{2}}(|g(s)| + |\Psi(s)|))|\psi(s)|sds \\ &\quad + \int_R^\infty s^{-\frac{3}{2}}(|g(s)| + |\Psi(s)|)(h_1 + |g(s)| + |\Psi(s)|)sds \\ &\lesssim R^{-\frac{1}{2}+\epsilon}(\|\psi\|_{L^2} + \|\frac{g}{r}\|_{L^2} + \|\frac{\Psi}{r}\|_{L^2([R,\infty))}) \end{aligned}$$

By taking R large and using $\|\psi\|_{L^2} \lesssim \|g\|_{\dot{H}^1_e}$ we obtain

$$\|\frac{\Psi}{r}\|_{L^2([R,\infty))} \lesssim \|g\|_{\dot{H}^1_e}$$

From this estimate and the above pointwise bound for f, it also follows that

$$\|f\|_{L^2([R,\infty))} \lesssim \|\psi\|_{L^2} + \|\frac{g}{r}\|_{L^2} + \|\frac{\Psi}{r}\|_{L^2([R,\infty))} \lesssim \|g\|_{\dot{H}^1_e}$$

Finally from the last two estimates we obtain

$$\|\partial_r \Psi\|_{L^2([R,\infty))} \lesssim \|g\|_{\dot H^1} + \|f\|_{L^2} \lesssim \|g\|_{\dot H^1_e}$$

which concludes the proof of (7.36) and of the full characterization of Ψ on $[R,\infty)$.

Step 5: The bounds for ψ_2, A_2 on $I = [R^{-1}, R]$. On the interval $[R^{-1}, R]$ we can no longer use only the second equation in (7.16). However, in this interval there is no singularity so a standard ODE analysis allows us to extend the solution with Lipschitz pointwise bounds. Precisely, a straightforward application of Gronwall's inequality shows that as long as $\|\psi\|_{L^2}, \|\tilde\psi\|_{L^2} \ll 1$ we have

$$\|(\psi_2, A_2) - (\tilde\psi_2, \tilde A_2)\|_{L^\infty(I)} \lesssim \|\psi - \tilde\psi\|_{L^2(I)} + |(\psi_2, A_2)(R) - (\tilde\psi_2, \tilde A_2)(R)|$$

Reusing this in (7.16) we can also estimate the r derivatives,

$$\|\partial_r(\psi_2, A_2) - \partial_r(\tilde\psi_2, \tilde A_2)\|_{L^2(I)} \lesssim \|\psi - \tilde\psi\|_{L^2(I)} + |(\psi_2, A_2)(R) - (\tilde\psi_2, \tilde A_2)(R)|$$

Estimating the second term on the right by (7.33) and (7.34) we obtain

$$\|(\psi_2, A_2) - (\tilde\psi_2, \tilde A_2)\|_{H^1_e(I)} \lesssim \|\psi - \tilde\psi\|_{LX} \tag{7.38}$$

If $\tilde\psi = 0$ then using (7.37) instead we get

$$\|(\psi_2, A_2) - (ih_1, h_3)\|_{H^1_e(I)} \lesssim \|L^{-1}\psi\|_{\dot H^1_e} \tag{7.39}$$

Step 6 : The bounds for (Ψ_2, A_2) on $(0, R^{-1}]$. On the interval $(0, R^{-1}]$, A_2 is expected to be negative, so we can use again only the second equation in (7.16) with $A_2 = -\sqrt{1 - |\psi_2|^2}$. We repeat the fixed point argument as we did on $[R,\infty)$. We rewrite the second equation in (7.16) as

$$L\psi_2 = -i\psi + f, \qquad f = i(A_2+1)\psi + \frac{h_3 - A_2}{r}\psi_2 \tag{7.40}$$

We introduce $\psi_2 - ih_1 + ig = \Psi$ and rewrite the problem as

$$\Psi(r) = \frac{h_1(r)}{h_1(R^{-1})}\Psi(R^{-1}) + N(\Psi, \psi) \tag{7.41}$$

where the nonlinearity N is defined as

$$N(\Psi,\psi) = -h_1(r)\int_r^{R^{-1}} h_1(s)^{-1} f(s)ds, \qquad f = i(A_2+1)\psi + \frac{h_3 - A_2}{s}(ih_1 - ig + \Psi)$$

with A_2 and g as dependent variables,

$$A_2 = -\sqrt{1 - |h_1 - ig + \Psi|^2}, \qquad g = L^{-1}\psi.$$

The value for $\Psi(R^{-1}) = (\psi_2 - ih_1 - ig)(R^{-1})$ is collected from Step 5 and satisfies

$$|\Psi(R^{-1}) - \tilde\Psi(R^{-1})| \lesssim \|\psi - \tilde\psi\|_{LX} \ll 1, \qquad \Psi(R^{-1}) \lesssim \|g\|_{\dot H^1_e} \ll 1. \tag{7.42}$$

For this new nonlinearity N we claim the following bound:

$$\|N(\Psi,\psi) - N(\tilde\Psi, \tilde\psi)\|_{\dot H^1_e((0,R])} \leq \frac{1}{2}(\|\Psi - \tilde\Psi\|_{\dot H^1_e(0,R]} + \|g - \tilde g\|_{\dot H^1_e(0,R]}) \tag{7.43}$$

under the assumption that

$$\|\psi\|_{LX}, \|\tilde\psi\|_{LX}, \|\Psi\|_{\dot H^1_e(0,R]}, \|\tilde\Psi\|_{\dot H^1_e(0,R]}, R^{-1} \ll 1 \tag{7.44}$$

As in the large r case, the Lipschitz bound (7.43) and (7.42) allows us use the contraction principle to obtain a solution $\Psi \in \dot{H}^1_e(0, R^{-1}]$ to (7.41) satisfying

$$\|\Psi\|_{\dot{H}^1_e((0,R])} \lesssim \|L^{-1}\psi\|_{\dot{H}^1_e}, \quad \|\Psi - \tilde{\Psi}\|_{\dot{H}^1_e((0,R])} \lesssim \|\psi - \tilde{\psi}\|_{LX}$$

Returning to ψ_2 and A_2 this gives

$$\|\psi_2 - ih_1\|_{\dot{H}^1_e((0,R])} \lesssim \|L^{-1}\psi\|_{\dot{H}^1}, \quad \|\psi_2 - \tilde{\psi}_2\|_{\dot{H}^1_e((0,R])} \lesssim \|\psi - \tilde{\psi}\|_{LX} \tag{7.45}$$

$$\|A_2 - h_3\|_{\dot{H}^1_e((0,R])} \lesssim \|L^{-1}\psi\|_{\dot{H}^1_e}, \quad \|A_2 - \tilde{A}_2\|_{\dot{H}^1_e((0,R])} \lesssim \|\psi - \tilde{\psi}\|_{LX} \tag{7.46}$$

It remains to establish (7.43). We start with the inequalities

$$|A_2 - \tilde{A}_2| \lesssim |g - \tilde{g}| + |\Psi - \tilde{\Psi}|, \quad |A_2 + 1| + |A_2 - h_3| \lesssim r + |g| + |\Psi|$$

which are derived from (7.44) and the formulas from $A_2, \tilde{A}_2$. From these estimates we derive a pointwise bound for f,

$$\begin{aligned} |f - \tilde{f}| \lesssim &(|g - \tilde{g}| + |\Psi - \tilde{\Psi}|)|\psi| + (r + |\tilde{g}| + |\tilde{\Psi}|)|\psi - \tilde{\psi}| \\ &+ (r + |g| + |\Psi| + |\tilde{g}| + |\tilde{\Psi}|)\frac{|g - \tilde{g}| + |\Psi - \tilde{\Psi}|}{r} \end{aligned}$$

This directly leads to the L^2 bound

$$\|f - \tilde{f}\|_{L^2} \lesssim C(\|g - \tilde{g}\|_{\dot{H}^1_e((0,R])} + \|\Psi - \tilde{\Psi}\|_{\dot{H}^1_e((0,R])})$$

$$C = R^{-1} + \|g\|_{\dot{H}^1_e} + \|\Psi\|_{\dot{H}^1_e((0,R])} + \|\tilde{g}\|_{\dot{H}^1_e} + \|\tilde{\Psi}\|_{\dot{H}^1_e((0,R])}$$

For small r we have $h_1(r) \sim r$ therefore

$$|N(\Psi, \psi) - N(\tilde{\Psi}, \tilde{\psi})| \lesssim r[r\partial_r]^{-1}|f - \tilde{f}|$$

Hence combining the above L^2 bound for $f - \tilde{f}$ with the Hardy estimate (1.14) and with (7.44) we obtain

$$\|r^{-1}(N(\Psi, g) - N(\tilde{\Psi}, \tilde{g}))\|_{L^2((0,R])} \lesssim C(\|g - \tilde{g}\|_{\dot{H}^1_e((0,R])} + \|\Psi - \tilde{\Psi}\|_{\dot{H}^1_e((0,R])})$$

Finally, using

$$\partial_r N(\Psi, g) = -\frac{h_3}{r} N(\Psi, g) + f$$

we also bound $\partial_r N(\Psi, g)$ in L^2, completing the proof of (7.43).

Step 7: Conclusion

In the end, based on (7.33), (7.34), (7.38), (7.45), (7.46) and (4.15), we upgrade the solution constructed above to $\psi_2 - ih_1, A_2 - h_3 \in X$ with the bounds (7.18)-(7.19). In addition (7.20) follows from (7.36), (7.39) and (7.45).

□

7.5. **The transition from ψ, ψ_2 and A_2 to (u, v, w).** To achieve this we use the system (3.9), which we recast in a matrix form as an equation for $\mathcal{O} = (v, w, u)$ as follows

$$\partial_r \mathcal{O} = \mathcal{O} R(\psi) \tag{7.47}$$

with

$$R = \begin{pmatrix} 0 & 0 & \Re\psi_1 \\ 0 & 0 & \Im\psi_1 \\ -\Re\psi_1 & -\Im\psi_1 & 0 \end{pmatrix}$$

If $\psi = 0$ then $\psi_2 = ih_1$, which yields $\psi_1 = -\frac{h_1}{r}$. Hence $R(0) = M(\bar{Q})$ as in (7.10). We will prove that

$$\|R(\psi) - R(0)\|_{\partial_r \tilde{X}} \lesssim \|\psi\|_{LX} \tag{7.48}$$

Suppose this is done. Then the same argument as in Section 7.2 leads to the bound (7.2), as well as

$$\|u - Q\|_{\tilde{X}} \lesssim \gamma$$

To upgrade the above norm to an X norm we need an additional bound for $\|r^{-1}(u-Q)\|_{L^2}$. We first remark that the last row of $\mathcal{O}$ is a-priori known, namely $(v_3, w_3, u_3) = (\Im\psi_2, \Re\psi_2, A_2)$; this already shows that

$$\|r^{-1}(v_3 - h_1)\|_{L^2} + \|r^{-1}w_3\|_{L^2} + \|r^{-1}(u_3 - h_3)\|_{L^2} \lesssim \gamma$$

To transfer this information to u_1 and u_2 we use again the orthogonality of $\mathcal{O}$. For u_1 for instance we have

$$u_1 = v_2 w_3 - v_3 w_2 = v_2 w_3 - (v_3 - h_1) w_2 - h_1(w_2 - 1) + h_1$$

which suffices.

It remains to prove the bound (7.48). Using the second relation in (7.16) we have

$$\begin{aligned} \psi_1 =& \psi + i\frac{\psi_2}{r} = -iA_2\partial_r\psi_2 + |\psi_2|^2\psi + i\frac{\psi_2}{r}|\psi_2|^2 \\ =& -\frac{h_1}{r} - iA_2\partial_r(\psi_2 - ih_1) + (A_2 - h_3)\partial_r h_1 + |\psi_2|^2\psi + \frac{1}{r}(i\psi_2|\psi_2|^2 + h_1^3) \end{aligned}$$

The first term is the value that corresponds to $\psi = 0$. The second is placed in $\partial_r\tilde{X}$ by (4.34) and (4.33). The remaining terms are estimated in $\partial_r\tilde{X}$ just based on their size, via (4.32). The third term is pointwise bounded by $\gamma\langle r\rangle^{-3}$. For the third one we use the L^2 bound for ψ, combined with the l^2L^∞ bound on $\psi_2 - ih_1$ for small r and the pointwise $r^{-\frac{1}{2}}$ bound on $\psi_2 - ih_1$ for large r. The fourth one is similar, only the L^2 bound for ψ is replaced by the L^2 bound for $r^{-1}(\psi_2 - ih_1)$.

7.6. **Local energy bounds.** A key role in the study of the Schrödinger type equation (3.35) for ψ is played by the dispersive estimates for ψ, most notably the local energy decay, which allows us to control a norm for ψ which is of the form

$$\|\psi\|_{LE[\lambda]} = \sum_{k<0} \frac{1}{k2^k}\|P_k^\lambda\psi\|_{LE_k} + \left(\sum_{k\geq 0}\|P_k^\lambda\psi\|_{LE_k}^2\right)^{\frac{1}{2}}$$

where λ is a function time for which

$$|\lambda - 1| \ll 1 \tag{7.49}$$

This is always satisfied in the context of this paper, as the functions (ψ_2, A_2) given by Proposition 7.3 satisfy

$$|\lambda - 1| + |\alpha| \lesssim |A_2(1) - h_3(1)| + |\psi_2(1) - h_1(1)| \lesssim \|\psi\|_{LX} \tag{7.50}$$

A-priori this norm depends on the choice of λ. However, using Proposition 4.10 and Proposition 4.11 it is easy to prove that different choices of λ subject to (7.49) yield equivalent norms.

In this section we study to what extent the local energy decay bounds for ψ can be transferred to (ψ_2, A_2) via the system (7.16). At first one might attempt to prove local energy decay bounds for $\psi_2 - ih_1$ and $A_2 - h_3$. If that were true, it would imply square integrability for $\lambda(t) - 1$ and $\alpha(t)$, where $\lambda(t)$ and $\alpha(t)$ are the parameters defined in (3.34) describing (ψ_2, A_2) at $r = 1$. However, such decay estimates turn out not to hold.

Our remedy for this difficulty is to start with λ and α defined in (3.34) and to compare (ψ_2, A_2) with their value associated to the harmonic map $Q_{\alpha,\lambda}$. Precisely, with λ and α given by

$$A_2(1) = h_3^\lambda(1), \psi_2(1) = ie^{i\alpha} h_1^\lambda(1) \tag{7.51}$$

we seek to estimate the differences

$$\delta^{\lambda,\alpha}\psi_2 = \psi_2 - ie^{i\alpha}h_1^\lambda, \qquad \delta^\lambda A_2 = A_2 - h_3^\lambda. \tag{7.52}$$

For λ and α in (7.51) we assume that

$$\|\alpha\|_{L^\infty} + \|\lambda - 1\|_{L^\infty} \ll 1 \tag{7.53}$$

In the context of Proposition 7.3 this is a consequence of the bound

$$\|\psi\|_{L^\infty LX} \ll 1$$

The main result of this section is the following

Proposition 7.4. *a) Suppose that $\psi \in L^2$, small. Let (ψ_2, A_2) be the solutions to* (7.16) *with initial data as in* (7.51), (7.53)*. Then we have the fixed time bound*

$$\|\delta^{\lambda,\alpha}\psi_2\|_{\dot H^1_e} + \|\delta^\lambda A_2\|_{\dot H^1_e} \lesssim \|\psi\|_{L^2} \tag{7.54}$$

b) Assume in addition that ψ is small in $L^\infty LX$ and that (7.53) *is valid. Then the following space-time bound holds:*

$$\|\frac{\langle r\rangle^{-\epsilon}}{r}\delta^{\lambda,\alpha}\psi_2\|_{L^2} + \|\frac{\langle r\rangle^{\frac12-\epsilon}}{r}\delta^\lambda A_2\|_{L^2} \lesssim \|\psi\|_{LE[1]} \tag{7.55}$$

We remark that heuristically (7.54) can be viewed as a consequence of the estimate (1.5) and the relation (2.2).

Proof. a) By (7.53), λ is close to 1. Solving (7.16) on the time interval $[1, \lambda]$ we obtain the bound

$$|\delta^{\lambda,\alpha}\psi_2(\lambda)| + |\delta^\lambda A_2(\lambda)| \lesssim \|\psi\|_{L^2_{comp}} \tag{7.56}$$

Then we can use a rotation and scaling to set $\lambda = 1$ and $\alpha = 0$ in (7.52) at the expense of replacing (7.54) by

$$\|\delta\psi_2\|_{\dot H^1_e} + \|\delta A_2\|_{\dot H^1_e} \lesssim \|\psi\|_{L^2} + |\delta\psi_2(1)| + |\delta A_2(1)| \tag{7.57}$$

under a smallness assumption on the right hand side. Here we make the convention that if $\lambda = 1, \alpha = 0$ then we drop the upper-scripts from δ. Using (7.17) we rewrite the equation (7.16) in the equivalent form

$$\left\{ \begin{array}{l} L\delta\psi_2 = iA_2\psi - \frac{1}{r}\delta A_2\psi_2, \\ \\ L_1\delta A_2 = \Im(\psi\bar{\psi}_2) + \frac{1}{r}|\delta A_2|^2 \end{array} \right. \tag{7.58}$$

where the operators L and L_1 are given by

$$L = \partial_r + \frac{h_3}{r}, \qquad L_1 = \partial_r + 2\frac{h_3}{r}$$

The functions h_1, respectively h_1^2 solve the homogeneous equations $Lf = 0$, respectively $L_1 f = 0$. Then the inverses T, respectively T_1 of L, respectively L_1 with zero Cauchy data at $r = 1$ have the form

$$Tf(r) = h_1(r)\int_1^r \frac{f(s)}{h_1(s)}ds, \qquad T_1 f(r) = h_1^2(r)\int_1^r \frac{f(s)}{h_1^2(s)}ds$$

Then we have:

Lemma 7.5. *The operators T and T_1 satisfy the bounds*

$$\|r^\alpha Tf\|_{\dot{H}_e^1(1,r_0)} \lesssim \|r^\alpha f\|_{L^2(1,r_0)}$$

where the range of α is $\alpha < 1$ if $r_0 > 1$, respectively $\alpha > -1$ if $r_0 < 1$.

The proof of the lemma is straightforward, and is left for the reader. To continue with the proof of the proposition we rewrite (7.58) as

$$\left\{ \begin{array}{l} \delta\psi_2 = h_1\delta\psi_2(1) + T(iA_2\psi - \frac{1}{r}\delta A_2\psi_2), \\ \\ \delta A_2 = h_1^2\delta A_2(1) + T_1(\Im(\psi\bar{\psi}_2) + \frac{1}{r}|\delta A_2|^2) \end{array} \right. \tag{7.59}$$

and solve this equation using the contraction principle in $\dot{H}_e^1 \times \dot{H}_e^1$. Given Lemma 7.5 it suffices to show that the map

$$(\psi, \delta\psi_2, \delta A_2) \to (iA_2\psi - \frac{1}{r}\delta A_2\psi_2, \Im(\psi\bar{\psi}_2) + \frac{1}{r}|\delta A_2|^2)$$

is locally Lipschitz from $L^2 \times \dot{H}_e^1 \times \dot{H}_e^1$ to $L^2 \times L^2$. This follows easily since $\dot{H}_e^1 \subset L^\infty \cap rL^2$. We note that the requisite smallness in the contraction principle comes from the smallness of the right hand side in (7.57), while the small Lipschitz constant is produced by unbalancing the norms

$$\|(\psi, \delta\psi_2, \delta A_2)\|_{L^2\times\dot{H}_e^1\times\dot{H}_e^1} = \|\psi\|_{L^2} + \|\delta\psi_2\|_{\dot{H}_e^1} + M\|\delta A_2\|_{\dot{H}_e^1}$$

with large M (and similarly for $L^2 \times L^2$).

b) After a time dependent rescaling and rotation we can assume that $\lambda = 1$ and $\alpha = 0$ in (7.52). The price we pay is twofold:

i) As in part (a), the initial condition becomes $\delta\psi_2(\lambda) = 0$, $\delta A_2(\lambda) = 0$. However, we can shift back to $r = 1$ using the time integrated form of (7.56).

ii) The norm $LE[1]$ in (7.55) is replaced by $LE[\lambda^{-1}]$. However, these two norms are equivalent.

After this reduction, it remains to prove the estimate

$$\| \frac{\langle r\rangle^{-\epsilon}}{r} \delta\psi_2\|_{L^2} + \|\frac{\langle r\rangle^{\frac12-\epsilon}}{r}\delta A_2\|_{L^2} \lesssim \|\psi\|_{LE[1]} + \|\delta\psi_2(1)\|_{L^2} + \|\delta A_2(1)\|_{L^2} \tag{7.60}$$

In the interval $[0,R]$ this is obtained directly from (7.59) via Lemma 7.5 with $\alpha = 0$. This yields

$$\begin{cases} \|\delta\psi_2\|_{\dot H^1_e[0,1]} \lesssim |\delta\psi_2(1)| + \|\psi\|_{L^2[0,1]} + \|\delta A_2\|_{\dot H^1_e[0,1]}(1 + \|\delta\psi_2\|_{\dot H^1_e[0,1]}) \\ \\ \|\delta A_2\|_{\dot H^1_e[0,1]} \lesssim |\delta A_2(1)| + \|\psi\|_{L^2[0,1]} + \|\delta A_2\|^2_{\dot H^1_e[0,1]} \end{cases}$$

From part (a) we have $\|\delta\psi_2\|_{\dot H^1_e} + \|\delta A_2\|_{\dot H^1_e} \lesssim \epsilon$, which allows us to close and obtain

$$\|\delta\psi_2\|_{\dot H^1_e[0,1]} + \|\delta A_2\|_{\dot H^1_e[0,1]} \lesssim |\delta\psi_2(1)| + |\delta A_2(1)| + \|\psi\|_{L^2[0,1]}$$

We square this and integrate in time.

It remains to consider the interval $[1,\infty)$. Here we apply Lemma 7.5 with $-\frac12 < \alpha < 0$ to obtain

$$\begin{aligned} \|r^{\alpha-1}\delta\psi_2\|_{L^2[1,\infty)} \lesssim & \ |\delta\psi_2(1)| + \|r^{\alpha-1}T(A_2\psi)\|_{L^2[1,\infty)} \\ & + \|r^{\alpha-1}\delta A_2\|_{L^2[1,\infty)}(1 + \|\delta\psi_2\|_{\dot H^1_e}) \end{aligned}$$

respectively

$$\begin{aligned} \|r^{\alpha-1}\delta A_2\|_{L^2[1,\infty)} \lesssim & \ |\delta A_2(1)| + \|r^{\alpha-1}T_1(\psi_2\psi)\|_{L^2[1,\infty)} \\ & + \|r^{\alpha-1}\delta A_2\|_{L^2[1,\infty)}\|\delta A_2\|_{\dot H^1_e} \end{aligned}$$

At this point we use the assumption that ψ is small in $L^\infty LX$; by Proposition 7.3 this implies $|\delta\psi_2| \lesssim r^{-\frac12}$ and $|\delta A_2| \lesssim r^{-1}$. These bounds allow us to obtain a favorable bound for most of the contributions of ψ, namely

$$\|r^{\alpha-1}T((A_2-1)\psi)\|_{L^2[1,\infty)} + \|r^{\alpha-1}T_1(\psi_2\psi)\|_{L^2[1,\infty)} \lesssim \|r^{\alpha-\frac12}\psi\|_{L^2}$$

It remains to prove the linear estimate

$$\|r^{\alpha-1}T\psi\|_{L^2[1,\infty)} \lesssim \|\psi\|_{LE[1]} \tag{7.61}$$

For this we write

$$T\psi = L^{-1}\psi - h_1 L^{-1}\psi(1)$$

We consider a dyadic decomposition of ψ, $\psi = \sum \psi_k$. For $k \geq 0$ the kernels K^1_k of $L^{-1}P_k$, described in Proposition 4.8, satisfy

$$|K^1_k(r,s)| \lesssim \frac{1}{(1+2^k|r-s|)^N(r+s)},$$

This gives the L^2 bound

$$\|r^{\alpha-1}L^{-1}\psi_k\|_{L^2[1,\infty)} \lesssim 2^{-\frac{3k}{2}}\|\psi_k\|_{LE_k}$$

and the pointwise bound

$$\|L^{-1}\psi_k(1)\|_{L^2_t} \lesssim 2^{-k}\|\psi_k\|_{LE_k}$$

which are easy to sum up.

For $k < 0$ the kernels K^1_k split into

$$K^1_k = K^1_{k,reg} + K^1_{k,res}$$

where the regular part satisfies

$$|K^1_{k,reg}(r,s)| \lesssim \frac{2^k \log(1+r)}{|k|(1+2^k|r-s|)^N(1+2^k(r+s))},$$

while the resonant part, present only for $k<0$, satisfies

$$|K^1_{k,res}(r,s)| = \frac{1}{2^k|k|} h_1(r)\chi_{2^k r\leq 1}(r) c_k(s)$$

where $\chi_{2^k r\leq 1}$ is a bump which equals 1 for $2^k r \ll 1$, and $|c_k(s)| \lesssim (1+2^k s)^{-N}$. For the regular part we have

$$\|r^{\alpha-1} L^{-1}_{reg}\psi_k\|_{L^2[1,\infty)} + \|L^{-1}_{reg}\psi_k(1)\|_{L^2_t} \lesssim \frac{1}{|k|2^k}\|\psi_k\|_{LE_k}$$

which suffices due to the extra $|k|2^k$ weight in the definition of $LE[1]$.

Finally for the resonant part we take advantage of the cancellation of the resonance. We have

$$L^{-1}_{res}\psi_k = \frac{1}{2^k|k|} h_1(r)\chi_{2^k r\leq 1}(r) f_k(t), \qquad \|f\|_{L^2} \lesssim 2^{-k}\|\psi_k\|_{LE_k}$$

Then for the corresponding part of T we have

$$T_{res}\psi_k = \frac{1}{2^k|k|} h_1(r)(\chi_{2^k r\leq 1}(r)-1) f_k(t)$$

which leads to the stronger bound

$$\|r^{\alpha-1} T_{res}\psi_k\|_{L^2} \lesssim 2^{-\alpha k}\frac{1}{2^k|k|}\|\psi_k\|_{LE_k}$$

This concludes the argument for the part of (7.61) concerning $\delta\psi_2$; however we need to improve on the decay for the δA_2 term. We make the following general observation which will be of use later too. In some estimates we need better decay bounds for δA_2 near spatial infinity. For that we observe that for large r the function δA_2 can be algebraically estimated as

$$|\delta A_2| \lesssim h_1|\delta\psi_2| + |\delta\psi_2|^2 \tag{7.62}$$

For the first term on the right we have an extra order of decay. For the second we can either use the LX norm of ψ to get another half unit of decay, or we can get almost an $L^\infty L^1$ bound. In particular, this justifies the second part of (7.61). □

8. The nonlinear equation for ψ

8.1. **A short time result.** We write the nonlinear equation for ψ as

$$(i\partial_t - \tilde{H})\psi = W\psi, \qquad \psi(0)=\psi_0, \qquad W = A_0 - 2\frac{\delta A_2}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}) \tag{8.1}$$

with A_2 and ψ_2 uniquely determined by ψ, see Proposition 7.3, and $\delta A_2 = A_2 - h_3$. A_0 is given by (3.33) which we recall for convenience

$$A_0(r) = -\frac{1}{2}|\psi|^2 + \frac{1}{r}\Im(\psi_2\bar{\psi}) + [r\partial_r]^{-1}(|\psi|^2 - \frac{2}{r}\Im(\psi_2\bar{\psi}))$$

Treating the right hand side perturbatively, we prove a local in time well-posedness result for (8.1):

Theorem 8.1. *For each initial data ψ_0 satisfying*

$$\|\psi_0\|_{LX} \leq \gamma \ll 1$$

there is an unique solution ψ for (8.1) *in the time interval $I = [0,1]$, satisfying*

$$\|\psi\|_{WS^\sharp[1](I)} \lesssim \gamma \tag{8.2}$$

Furthermore, the solution map $\psi_0 \to \psi$ is Lipschitz from LX to $WS^\sharp[1](I)$.

Proof. By Proposition 5.4 it suffices to show that the map $\psi \to W\psi$ is Lipschitz from $WS^\sharp[1](I)$ to $WN^\sharp[1](I)$ with a small Lipschitz constant for ψ as in (8.2). We consider each term in W and use Proposition 7.3 to describe the dependence of A_2 and ψ_2 on ψ. Some but not all of the estimates below depend on the size of the time interval I. To identify those we use the $\lesssim_I$ notation. For convenience, we use $WS^\sharp, WN^\sharp$ below instead of $WS^\sharp[1](I), WN^\sharp[1](I)$.

1. The A_2 term is estimated using the bound

$$\|r^{-2}fg\|_{WN^\sharp} \lesssim_I \|f\|_{L^\infty X}\|g\|_{WS^\sharp} \tag{8.3}$$

For high frequencies in the output we use the local energy norms,

$$\|r^{-2}fg\|_{LE^*} \lesssim \|\langle r\rangle^{\frac12} f\|_{L^\infty}\|g\|_{LE} \lesssim \|f\|_{L^\infty X}\|g\|_S \tag{8.4}$$

For low frequencies in the output we use (4.26) and an L^1 bound

$$\begin{aligned}\|r^{-2}fg\|_{L^1} \lesssim_I \|r^{-2}fg\|_{L^2L^1} &\lesssim \||\log(1+r)|^{-1}f\|_{L^\infty L^2}\|r^{-2}\log(1+r)g\|_{L^2}\\ &\lesssim \|f\|_{L^\infty X}\|g\|_S\end{aligned}$$

2. The ψ_2 term is estimated using the bounds

$$\|r^{-1}fgh\|_{WN^\sharp} + \|r^{-1}h_1 gh\|_{WN^\sharp} \lesssim (1+\|f\|_{L^\infty X})\|g\|_{WS^\sharp}\|h\|_{WS^\sharp} \tag{8.5}$$

We only discuss the first bound; the second is similar but easier. For high frequencies it suffices to write

$$\|r^{-1}fgh\|_{LE^*} \lesssim \|\langle r\rangle^{\frac12} f\|_{L^\infty}\|g\|_{L^4}\|h\|_{L^4} \lesssim \|f\|_{L^\infty X}\|g\|_S\|h\|_S \tag{8.6}$$

For low frequencies we use (4.26) and an L^1 bound derived using (6.6):

$$\begin{aligned}\|r^{-1}fgh\|_{L^1} &\lesssim \|\langle r\rangle^{\frac12} f\|_{L^\infty}\|r^{-\frac12}\langle r\rangle^{-\frac14} g\|_{L^2}\|r^{-\frac12}\langle r\rangle^{-\frac14} h\|_{L^2}\\ &\lesssim \|f\|_{L^\infty X}\|g\|_{WS^\sharp}\|h\|_{WS^\sharp}\end{aligned} \tag{8.7}$$

3. The $|\psi|^2$ part of the A_0 term is estimated using the bounds

$$\|fgh\|_{WN^\sharp} \lesssim_I \|f\|_{WS^\sharp}\|g\|_{WS^\sharp}\|h\|_{WS^\sharp} \tag{8.8}$$

Indeed, for the high frequency part we have

$$\|fgh\|_N \lesssim \|fgh\|_{L^{\frac43}} \lesssim \|f\|_{L^4}\|g\|_{L^4}\|h\|_{L^4} \lesssim \|f\|_S\|g\|_S\|h\|_S \tag{8.9}$$

while for the low frequency part we write

$$\|fgh\|_{L^1} \lesssim_I \|fgh\|_{L^2L^1} \lesssim \|f\|_{L^4}\|g\|_{L^4}\|h\|_{L^\infty L^2} \lesssim \|f\|_S\|g\|_S\|h\|_S \tag{8.10}$$

4. The $[r\partial_r]^{-1}|\psi|^2$ part of the A_0 term is estimated as in Case 3 using in addition the Hardy type inequality (1.14) for $[r\partial_r]^{-1}$.

5. The $[r\partial_r]^{-1}(r^{-1}\Im(\psi_2\bar{\psi}))$ part of the A_0 term is estimated as in Case 2: using (1.14) with $p = 2$ for the corresponding case to (8.6) and using (1.15) with $p = 1$ and $w = \langle r\rangle^{-\frac{1}{2}}$ for the corresponding case to (8.7). ☐

8.2. **The long time result.** We rewrite the equation for ψ in the form

$$(i\partial_t - \tilde{H}_\lambda)\psi = W_\lambda\psi, \quad \psi(0) = \psi_0, \quad W_\lambda = A_0 - 2\frac{\delta^\lambda A_2}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}) \tag{8.11}$$

with A_0, A_2 and ψ_2 as well as the time dependent parameter λ uniquely determined by ψ (see (7.51) and (7.52)). Our main long time bootstrap result is as follows:

Theorem 8.2. *Let $T \in (0,\infty]$, $\epsilon \leq 1$ and $\gamma \leq \gamma_0 \ll 1$. Suppose that the initial data for ψ satisfies*

$$\|\psi(0)\|_{L^2} \leq \epsilon\gamma, \qquad \|\psi(0)\|_{LX} \leq \gamma \tag{8.12}$$

Assume that the parameter λ and the function ψ satisfy the following bootstrap assumptions:

$$\|\lambda - 1\|_{Z_0[0,T]} \leq \gamma_0. \tag{8.13}$$

respectively

$$\|\psi\|_{l^2S^\sharp[0,T]} \leq \epsilon\gamma_0, \qquad \|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T]} \leq \gamma_0, \tag{8.14}$$

where $\tilde{\lambda}$ is any function with the following properties:

$$\|\tilde{\lambda} - 1\|_{Z[0,T]} + \|\lambda - \tilde{\lambda}\|_{(L^2\cap L^\infty)[0,T]} \lesssim \gamma_0 \tag{8.15}$$

Then the functions ψ and λ must satisfy the stronger bounds

$$\|\psi\|_{l^2S^\sharp[0,T]} \lesssim \epsilon(\gamma + \gamma_0^2), \qquad \|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T]} \lesssim \gamma + \gamma_0^2, \tag{8.16}$$

respectively

$$\|\lambda - 1\|_{Z_0[0,T]} \lesssim \gamma + \gamma_0^2. \tag{8.17}$$

To close the bootstrap it suffices to choose $\gamma_0 = C\gamma$ for a fixed large universal constant C.

We remark that for the global well-posedness result it suffices to take $\epsilon = 1$. However, the parameter ϵ, along with the stronger bounds in (8.21), is needed for the proof of the instability result.

The additional parameter $\tilde{\lambda}$ is needed because the spaces $WS^\sharp[\lambda]$, $WN^\sharp[\lambda]$ and the linear result in Proposition 6.3 require $\lambda - 1 \in Z$, while above we only have $\lambda - 1 \in Z_0$. There is some flexibility in the choice of $\tilde{\lambda}$. An acceptable choice would be for instance $\tilde{\lambda} = Q_{\leq 1}\lambda^{ext}$ where λ^{ext} is any suitable extension of λ in Z_0.

For brevity we omit the time interval $[0,T]$ in the notations in this section. For the most part this plays no role. At one point in the proof this requires an additional discussion.

For the first bound in (8.16) we use Theorem 6.1. Hence it suffices to estimate the nonlinear expression $W\psi$, for which we will prove

$$\|W\psi\|_{l^2N^\sharp} \lesssim \epsilon\gamma_0^2. \tag{8.18}$$

For the second bound (8.16) for ψ we rewrite the equation in the form

$$(i\partial_t - \tilde{H}_{\tilde{\lambda}})\psi = (V_\lambda - V_{\tilde{\lambda}})\psi + W_\lambda\psi, \quad \psi(0) = \psi_0 \tag{8.19}$$

and use Theorem 6.3. Hence it suffices to estimate the linear term $(V_\lambda - V_{\tilde\lambda})\psi$ and the nonlinear expression $W\psi$. Precisely, we will prove the bounds

$$\|(V_\lambda - V_{\tilde\lambda})\psi\|_{WN^\sharp[\tilde\lambda]} \lesssim \epsilon\gamma_0^2 \tag{8.20}$$

$$\|W_\lambda\psi\|_{WN^\sharp[\tilde\lambda]} \lesssim \epsilon\langle\log\epsilon\rangle^2\gamma_0^2. \tag{8.21}$$

The three bounds above follow from Propositions 8.3, 8.4, 8.5 below. The last part of the theorem, namely the λ bound (8.17), is proved in the next section. We begin with the nonlinear bound in L^2:

Proposition 8.3. *Suppose that ψ satisfies* (8.14). *Then*

$$\|Wu\|_N \lesssim \gamma_0\|u\|_S \tag{8.22}$$

Proof. We consider each of the terms in W as in the five cases in the proof of Theorem 8.1. The bound (8.22) follows by applying (8.4), (8.6), (8.9) and their counterparts in the last two cases. It is essential that none of these bounds depend on the size of the time interval. □

Next we consider the linear bound (8.20):

Proposition 8.4. *Suppose that λ and $\tilde\lambda$ satisfy* (8.13) *and* (8.15). *Then*

$$\|(V_\lambda - V_{\tilde\lambda})u\|_{WN^\sharp[\tilde\lambda]} \lesssim \gamma_0\|u\|_{l^2S^\sharp} \tag{8.23}$$

Proof. Since both λ and $\tilde\lambda$ are close to 1, it follows that

$$|V_{\lambda_1} - V_{\lambda_2}| \lesssim |\lambda_1 - \lambda_2|(1+r^2)^{-2}$$

Hence, using the embedding (4.25) and the LE norm for u, we obtain

$$\begin{aligned}\|(V_{\lambda_1} - V_{\lambda_2})u\|_{L^1LX} \lesssim & \|(V_{\lambda_1} - V_{\lambda_2})u\|_{L^1(L^1\cap L^2)} \lesssim \|\langle r\rangle^{3-}(V_{\lambda_1} - V_{\lambda_2})u\|_{L^1L^2} \\ \lesssim & \|u\|_{LE}\|\lambda_1 - \lambda_2\|_{L^2}\end{aligned}$$

Thus (8.23) follows. □

Next we consider the bound (8.21), which corresponds to initial data in the smaller space $LX \subset L^2$.

Proposition 8.5. *Suppose that ψ satisfies* (8.14) *and λ satisfies* (8.13). *Then for $\tilde\lambda$ as in* (8.15) *we have*

$$\|W_\lambda u\|_{WN^\sharp[\tilde\lambda]} \lesssim \epsilon\langle\log\epsilon\rangle^2\gamma_0\|u\|_{WS^\sharp[\tilde\lambda]} + \gamma\|u\|_{l^2S^\sharp} \tag{8.24}$$

Proof. The difference in the potentials $W_\lambda - W = 2\dfrac{h_3^\lambda - h_3}{r^2}$ decays rapidly enough at ∞ so that, in a similar manner to (8.22), one easily derives

$$\|(W_\lambda - W)u\|_N \lesssim |\lambda - 1|\|u\|_S \lesssim \gamma\|u\|_S$$

Combining this with (8.22) gives us

$$\|W_\lambda u\|_N \lesssim \gamma\|u\|_S \tag{8.25}$$

By the inclusion $N \subset l^2N$ and the first part of (6.5), we can use the above results to estimate the high frequency output $\|P_{\geq 0}^{\tilde\lambda}(W_\lambda u)\|_{WN^\sharp[\tilde\lambda]}$.

It remains to estimate the low frequency output $\|P^{\tilde\lambda}_{<0}(W_\lambda u)\|_{WN^\sharp[\tilde\lambda]}$. We divide the potential W_λ into three parts, $W_\lambda = W_0 + W_1 + W_2$ where

$$W_0 = \frac{\delta^\lambda A_2}{r^2}, \qquad W_1 = -\int \frac{1}{r^2}\Im(\psi_2\bar\psi), \qquad W_2 = -\frac{1}{2}|\psi|^2 - \int \frac{1}{r}|\psi|^2 dr$$

The contribution of W_0 can be estimated directly by using the local energy decay for u and $\delta^\lambda A_2$, see (6.4) and (7.55), to write

$$\|W_0 u\|_{L^1 LX} \lesssim \|W_0 u\|_{L^1} \lesssim \left\|\frac{\log(2+r)}{r}\delta^\lambda A_2\right\|_{L^2}\left\|\frac{1}{r\log(2+r)}u\right\|_{L^2} \lesssim \gamma_0\|u\|_S$$

A similar argument applies for the contribution of W_1. Indeed, consider first the simpler potential $\tilde W_1 = \dfrac{\psi_2}{r}\bar\psi$ and use the pointwise bound for ψ_2, $|\psi_2| \lesssim \langle r\rangle^{-\frac12}$. Using (6.6) and (8.14) we obtain

$$\begin{aligned}\|\tilde W_1 u\|_{L^1} &\lesssim \|\langle r\rangle^{\frac12}\psi_2\|_{L^\infty}\left\|\frac{\ln(2+r)}{r}\psi\right\|_{L^2}\left\|\frac{1}{\langle r\rangle^{\frac12}\ln(2+r)}u\right\|_{L^2}\\ &\lesssim \epsilon\langle\log\epsilon\rangle^2\gamma\|u\|_{WS^\sharp[\tilde\lambda]}\end{aligned}$$

A similar bound is obtained for W_1 by using (1.15) with $p=2$ and $w = \langle r\rangle^{\frac12}\ln(2+r)$.

The bulk of the proof is devoted to the low frequency estimate for $W_2 u$, which is independent of λ. It suffices to show that

$$\|P^{\tilde\lambda}_{<0}(W_2\ u)\|_{WN^\sharp[\tilde\lambda]} \lesssim \epsilon\gamma_0^2\|u\|_{WS^\sharp[\tilde\lambda]} + \gamma_0^2\|u\|_{l^2S^\sharp}$$

The expression $W_2 u$ is a trilinear expression in (ψ,ψ,u),

$$W_2 u = N(\psi,\psi,u) + \tilde N(\psi,\psi,u),$$

where

$$N(\psi_1,\psi_2,\psi_3) = -\frac12\psi_1\overline{\psi}_2\psi_3 \qquad \tilde N(\psi_1,\psi_2,\psi_3) = -\psi_3[r\partial_r]^{-1}(\psi_1\overline{\psi}_2)$$

Given the bounds (8.14) on ψ, the above inequality can be rewritten in a more symmetric way as

$$\|P^{\tilde\lambda}_{<0}N(\psi^1,\psi^2,\psi^3)\|_{WN^\sharp[\tilde\lambda]} \lesssim \sum_{\sigma\in S_3}\|\psi^{\sigma(1)}\|_{l^2S^\sharp}\|\psi^{\sigma(2)}\|_{WS^\sharp[\tilde\lambda]}\|\psi^{\sigma(3)}\|_{WS^\sharp[\tilde\lambda]} \tag{8.26}$$

and the similar bound for $\tilde N$. To avoid repetition, we focus on establishing this inequality for N. Every step in the analysis for N has its counterpart for $\tilde N$. As a general rule, the estimate for $\tilde N$ is similar to the one for N by the use of (1.14) and (1.15), with one exception which require separate analysis.

We decompose in the time dependent frame $\psi^i = \sum \psi^i_k$, i.e. $\psi^i_k = P^{\tilde\lambda}_k\psi^i$ For a large constant n_0 we expand

$$\begin{aligned}P^{\tilde\lambda}_{<0}N(\psi^1,\psi^2,\psi^3) &= \sum_{\sigma\in S_3}\sum_{k_1\ge k_2\ge k_3} P^{\tilde\lambda}_{<k_3^- - 2n_0}N^\sigma(\psi^{\sigma(1)}_{k_1},\psi^{\sigma(2)}_{k_2},\psi^{\sigma(3)}_{k_3})\\ &+\sum_{j<0}P^{\tilde\lambda}_{[j-2n_0,0]}\left(N(\psi^1_j,\psi^2,\psi^3) + N(\psi^1_{\ge j},\psi^2_j,\psi^3) + N(\psi^1_{\ge j},\psi^2_{\ge j},\psi^3_j)\right)\end{aligned}$$

where $N^\sigma(f^1,f^2,f^3) := N(f^{\sigma^{-1}(1)}, f^{\sigma^{-1}(2)}, f^{\sigma^{-1}(3)})$. We observe that the second term has a favorable frequency balance and is estimated directly using only the Strichartz norms,

$$\begin{aligned}\|P^{\tilde\lambda}_{[j-2n_0,0]} N(\psi^1_j,\psi^2,\psi^3)\|_{WN^\sharp[\tilde\lambda]} &\lesssim \frac{1}{|j|2^j}\|N(\psi^1_j,\psi^2,\psi^3)\|_N \\ &\lesssim \sum_{k<j} \frac{1}{|j|2^j}\|\psi^1_j\|_S\|\psi^2\|_S\|\psi^3\|_S\end{aligned}$$

followed by a straightforward summation with respect to j. The other two nonlinear factors in the second term are treated similarly. Using (1.14) the same argument works for $\tilde N$.

The first term requires considerably more work; in what follows, we will prove that for all $k_1 \geq k_2 \geq k_3$ and $\sigma \in S^3$ the following bound holds:

$$\|P^{\tilde\lambda}_{<k_3^- -2n_0} N^\sigma(\psi^1_{k_1},\psi^2_{k_2},\psi^3_{k_3})\|_{l^2N^\sharp} \lesssim \frac{\|\psi^1_{k_1}\|_{l^2S^\sharp}\|\psi^2_{k_2}\|_{l^2S^\sharp}\|\psi^3_{k_3}\|_{l^2S^\sharp}}{2^{k_1^+/8}\langle k_2^-\rangle 2^{k_2^-}\langle k_3^-\rangle 2^{k_3^-}} \tag{8.27}$$

as well as the similar one for $\tilde N$.

One difficulty is that the functions $\psi^{\sigma(j)}_{k_j}$ are only localized in the frequency dependent frame. To deal with this we relocalize them in the fixed frame,

$$\psi^{\sigma(j)}_{k_j} = \tilde P_{k_j}\psi^{\sigma(j)}_{k_j} + \psi^{\sigma(j),err}_{k_j}$$

and estimate pointwise the error $\psi^{j,err}_{k_j}$ using Corollary 4.13, see (4.50),

$$|\psi^{j,err}_{k_j}(r)| \lesssim \frac{2^{-k_j^+/2}}{r\log^2(2+r)}\|\psi^j_{k_j}\|_{L^\infty L^2}$$

Then for the part of (8.27) containing at least one error term we combine this with the local energy decay estimates and (5.2), (6.4) and the low frequency part of (4.25). One such term is

$$\begin{aligned}\|N^\sigma(\psi^1_{k_1},\psi^2_{k_2},\psi^3_{k_3})\|_{L^1} &\lesssim \|r\log^2(2+r)\psi^{1,err}_{k_1}\|_{L^\infty}\prod_{j=2,3}\left\|\frac{\psi^j_{k_j}}{r^{\frac12}\log(2+r)}\right\|_{L^2} \\ &\lesssim \frac{\|\psi^1_{k_1}\|_{l^2S^\sharp}\|\psi^2_{k_2}\|_{l^2S^\sharp}\|\psi^3_{k_3}\|_{l^2S^\sharp}}{2^{k_1^+/2}2^{\frac{k_2}{2}}2^{\frac{k_3}{2}}}\end{aligned}$$

The other terms are treated in a similar way. In the case of $\tilde N$, the same argument works when combined with (1.15) as follows: $p=2$ and $w = r^{\frac12}\log(2+r)$ if the *err* term is inside $[r\partial_r]^{-1}$ or $p=1$ and $w=\frac{1}{r\log^2(2+r)}$ otherwise.

After the above reduction we can assume that the functions $\psi^j_{k_j}$ in (8.27) are localized in the fixed frame. Thus the inputs $\psi^j_{k_j}$ no longer bear any relation to $\tilde\lambda$.

So far the time interval has played no role. At this point we consider suitable frequency localized[1] extensions for both $\psi^j_{k_j}$ and $\tilde\lambda$, and prove that (8.27) holds over the entire real line. This directly implies the similar bound over each subinterval.

[1]with respect to the fixed $\lambda = 1$ frame

A second difficulty is that the output N is also localized in the time dependent frame, while it is more natural at this point to project the output in a fixed frame. We denote

$$g = N^\sigma(\psi^1_{k_1}, \psi^2_{k_2}, \psi^3_{k_3})$$

and normalize

$$\|\psi^{\sigma(i)}_{k_i}\|_{l^2 S^\sharp} = 1, \tag{8.28}$$

We partition g in frequency/modulation and estimate each piece as in (8.27).

Case 1: The high frequency part $P_{\geq k_3^- - n_0} g$. Using (5.2), we obtain

$$\|(1+2^{k_3} r)^{-\frac{1}{2}} g\|_{L^1} \lesssim 2^{-\frac{k_1+k_2+6k_3}{8}} \|\psi^{\sigma(1)}_{k_1}\|_{L^4_{k_1}} \|\psi^{\sigma(2)}_{k_2}\|_{L^4_{k_2}} \|\psi^{\sigma(3)}_{k_3}\|_{LE_{k_3}} \tag{8.29}$$

which is satisfactory for $r \lesssim 2^{-k_3}$ but not for larger r. We use this bound to estimate $P^{\tilde{\lambda}}_j P_{\geq k_3^- - n_0}$ in $L^1 L^2$ for $j < k_3^- - 2n_0$. From Proposition 4.11 and Proposition 4.10 it follows that for $j < k_3^- - 2n_0$ the kernel of $P^{\tilde{\lambda}}_j P_{\geq k_3^- - n_0}$ satisfies

$$\begin{aligned} |K(r,s)| \lesssim & \frac{2^{2j}}{j^2} \frac{\log(1+r)}{(1+2^j r)^N} \left(\sum_{k=k_3^- - n_0}^{0} \frac{1}{\langle k \rangle^2} \frac{\log(1+s)}{(1+2^k s)^N} + \sum_{k=1}^{\infty} \frac{2^{-Nk} s^2}{(1+s)^N} \right) \\ \lesssim & \frac{2^{2j}}{j^2} \frac{\log(1+r)}{(1+2^j r)^N} \frac{1}{(1+2^{k_3^-} s)^N} \end{aligned}$$

Thus by (8.29) we obtain

$$\|P^{\tilde{\lambda}}_j P_{\geq k_3^- - n_0} g\|_{L^1 L^2} \lesssim \frac{2^j}{|j|} 2^{-\frac{k_1+k_2+6k_3}{8}} 2^{\frac{k_3^+}{2}}$$

Summing in j with weights $(2^j |j|)^{-1}$ we estimate $P_{\geq k_3^- - n_0} g$ as in (8.27). The argument for $\tilde{N}$ is obtained by using (1.15) with $p = 2$ and $w = (1+2^{k_3} r)^{\frac{1}{8}}$ if the k_3 frequency term is outside $[r\partial_r]^{-1}$ and with $p = \frac{4}{3}$ and $w = (1+2^{k_3} r)^{-\frac{1}{2}}$ if the k_3 frequency term is inside $[r\partial_r]^{-1}$.

Case 2: Low frequency, high modulations: $Q_{\geq k_1+k_3-n_0} P_{<k_3^- - n_0} g$. For this term we prove an L^2 bound

$$\sum_{j < k_3^- - n_0} \frac{1}{2^j |j|} \|Q_{\geq k_1+k_3-n_0} P_j g\|_{L^2} \lesssim \langle k_2^- \rangle \tag{8.30}$$

which leads to an estimate as in (8.27) by using (6.8) (precisely, its high modulation part from (6.13)). We have

$$\|(1+r2^{k_2})^{\frac{1}{8}} g\|_{L^2 L^1} \lesssim \|\psi^1_{k_1}\|_{L^\infty L^2} \|\psi^2_{k_2}\|_{L^4_{k_2}} \|\psi^3_{k_3}\|_{L^4} \lesssim 1.$$

Hence, arguing as in the proof of (4.26), it follows that for $j < 0$ we have

$$\frac{1}{2^j |j|} \|P_j g\|_{L^2} \lesssim \sum_m \frac{2^{m^-} \langle m^+ \rangle}{j^2} \|g\|_{L^2 L^1(A_m)} \lesssim \frac{\langle k_2^- \rangle}{j^2}$$

and (8.30) follows after a j summation. The bound for $\tilde{N}$ is obtained by using (1.14) with $p = \frac{4}{3}$ if the k_2 frequency term is outside $[r\partial_r]^{-1}$ and (1.15) when the k_2 frequency term is inside $[r\partial_r]^{-1}$ as follows: with $w = (1+r2^{k_2})^{\frac{1}{8}}$, $p = 2$ when the k_3 frequency term is also inside $[r\partial_r]^{-1}$ and with $w = (1+r2^{k_2})^{\frac{1}{8}}$, $p = \frac{4}{3}$ when the k_3 frequency term is outside $[r\partial_r]^{-1}$.

Case 3: The low frequency, low modulations $Q_{<k_1+k_3-n_0} P_{<k_3^- - n_0} g$. We will prove that

$$\|Q_{<k_1+k_3-n_0} P_{<k_3^-} g\|_{L^1 LX} \lesssim \frac{1}{2^{k_1^+/8} \langle k_2^- \rangle 2^{k_2^-} \langle k_3^- \rangle 2^{k_3^-}} \tag{8.31}$$

which, by (6.8), suffices for (8.27). We separate this into two cases:

Case 3A: One input has high modulation $\geq 2^{k_1+k_3-n_0}$. Say that factor is $\psi^1_{k_1}$; the other cases are similar. Then we bound

$$\begin{aligned} &\|N^\sigma(Q_{\geq k_1+k_3-n_0}\psi^1_{k_1}, \psi^2_{k_2}, \psi^3_{k_3})\|_{L^1} \\ \lesssim &\|Q_{\geq k_1+k_3-n_0}\psi^1_{k_1}\|_{L^2} \|\psi^2_{k_2}\|_{L^4} \|\psi^3_{k_3}\|_{L^4} \lesssim 2^{-\frac{k_3+k_1}{2}} \end{aligned}$$

and conclude with (4.26). We note that the L^4 bound is stable under cut-offs in modulation ($\leq 2^{k_1+k_3-n_0}$) due to the $V^2_{\tilde{H}} L^2$ structure, see (5.6). This works for $\tilde{N}$ as well, by using (1.14).

Case 3B: All factors $\psi^i_{k_i}$ have low modulation. By duality, it suffices to estimate the quadrilinear integral

$$I_0 = \int N^\sigma(Q_{<k_1+k_3-n_0}\psi^1_{k_1}, Q_{<k_1+k_3-n_0}\psi^2_{k_2}, Q_{<k_1+k_3-n_0}\psi^3_{k_3}) \overline{Q_{<k_1+k_3-n_0}\psi_j}\; r dr dt$$

with frequency localized inputs and show that

$$|I_0| \lesssim \frac{2^j}{|j|} \frac{1}{2^{k_1^+/8} \langle k_2^- \rangle 2^{k_2^-} \langle k_3^- \rangle 2^{k_3^-}} \|\psi_j\|_{L^\infty L^2} \tag{8.32}$$

We begin with a short frequency/modulation analysis. If the frequencies in the four factors are ξ_1, ξ_2, ξ_3 and ξ and all modulations are $\ll 2^{k_1+k_3}$ then the time frequency in the I integrand is

$$\phi = \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi^2 + m$$

where m is the sum of the four modulations involved, hence $m \ll 2^{k_1+k_3}$. Hence the time integral vanishes unless

$$(\xi_1, \xi_2, \xi_3, \xi) \in D = \{\xi_i \approx 2^{k_i}, \xi \approx 2^j; |\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi^2| \ll 2^{k_1+k_3}\} \tag{8.33}$$

Given the dyadic localization of ξ_1, ξ_2, ξ_3 and ξ, this leads to one of the following two scenarios (recall that $j + n_0 \leq k_3 \leq k_2 \leq k_1$):

(i) Equal frequency inputs,

$$|k_1 - k_2| \lesssim 1, \quad |k_2 - k_3| \lesssim 1, \qquad |\xi_1^2 + \xi_2^2 - \xi_3^2| \ll 2^{2k_3}.$$

(ii) Unbalanced frequency inputs,

$$|k_1 - k_2| \lesssim 1, \quad k_3 \ll k_2, \qquad |\xi_1 - \xi_2| \ll 2^{k_3}.$$

On the other hand, resonant interactions can only occur when

$$\xi_1 \pm \xi_2 \pm \xi_3 \pm \xi = 0 \tag{8.34}$$

where the $\pm$ signs correspond to outgoing/incoming waves. But this is precluded in both cases (i) (ii). We will strongly exploit this fact in our analysis.

In proving (8.32) we only use the S_{k_3} norm for $\psi^3_{k_3}$, together with boundedness of $Q_{<k_1+k_3-n_0}$ in S_{k_3}. For $Q_{<k_1+k_3-n_0}\psi^1_{k_1}$ and $Q_{<k_1+k_3-n_0}\psi^2_{k_2}$ we would also like to use the norms S_{k_1} respectively S_{k_2}. Unfortunately, the operator $Q_{<k_1+k_3-n_0}$ is not always bounded in these spaces; instead, from (5.12), we have

$$\|Q_{\leq k_1+k_3-n_0}\psi^i_{k_i}\|_{S_{k_i}} \lesssim (1+k_1-k_3)\,\|\psi^i_{k_i}\|_{S^\sharp_{k_i}}, \qquad i=1,2 \tag{8.35}$$

In our case this leads to losses of at most a factor of $1+(k_3-k_1)^2$. Fortunately we are able to prove a stronger bound

$$|I_0| \lesssim \frac{2^j}{|j|}\frac{\langle k_3^-\rangle}{2^{\frac{k_1+k_3}{2}}}\|\psi^1_{k_1}\|_{S_{k_1}}\|\psi^2_{k_2}\|_{S_{k_2}}\|\psi^3_{k_3}\|_{S_{k_3}}\|\psi_j\|_{L^\infty L^2} \tag{8.36}$$

which can absorb these losses and still lead to (8.32). To keep the size of formulae below manageable, we normalize all four norms on the right to 1 in the sequel.

Without restricting the generality of the argument, we restrict our attention to $\sigma(i)=i$, in which case

$$I_0 = \int_0^\infty \psi^1_{k_1}(r)\overline{\psi}^2_{k_2}(r)\psi^3_{k_3}(r)\overline{\psi}_j(r)rdrdt$$

In order to treat $\tilde{N}$, we need to consider also (the others are similar)

$$I_1 = \int_0^\infty \overline{\psi}_j(r)\psi^3_{k_3}(r)\int_r^\infty \frac{1}{s}\psi^1_{k_1}(s)\overline{\psi}^2_{k_2}(s)dsrdrdt$$

$$I_2 = \int_0^\infty \overline{\psi}_j(r)\psi^1_{k_1}(r)\int_r^\infty \frac{1}{s}\psi^2_{k_2}(s)\overline{\psi}^3_{k_3}(s)dsrdrdt$$

Here all factors have the appropriate frequency and modulation localization.

The analysis is similar for all these quadrilinear forms, so we will work with I_0. We switch I_0 to the Fourier space, where it becomes

$$\begin{aligned} I_0 &= \int \psi_{\xi_1}(r)\psi_{\xi_2}(r)\psi_{\xi_3}(r)\psi_\xi(r)rdr\hat{\psi}^1_{k_1}(t,\xi_1)\overline{\hat{\psi}}^2_{k_2}(t,\xi_2)\hat{\psi}^3_{k_3}(t,\xi_3)\overline{\hat{\psi}}_j(t,\xi)d\xi_i d\xi dt \\ &= \int (G_0\chi_D)(\xi,\xi_1,\xi_2,\xi_3)\hat{\psi}^1_{k_1}(t,\xi_1)\overline{\hat{\psi}}^2_{k_2}(t,\xi_2)\hat{\psi}^3_{k_3}(t,\xi_3)\overline{\hat{\psi}}_j(t,\xi)d\xi_i d\xi dt \end{aligned}$$

where G_0 is the quadrilinear form on $\tilde{H}$-generalized eigenfunctions, introduced in Section 4.8, and χ_D is any function which equals 1 in the set D defined in (8.33). Similarly we can write I_1 and I_2 in terms of G_1 and G_2.

In practice we always restrict the support of χ_D to the sets described in cases (i), (ii) above, and we assume it has good regularity. Hence in the support of χ_D we can use the bounds in Proposition 4.15 for G_0, G_1 and G_2. Thus, from (4.55), G_0 satisfies

$$|G_0| \lesssim g_{jk_1k_2k_3} = \frac{2^{\frac{j}{2}}}{|j|}\frac{\langle k_3^-\rangle 2^{-2k_3^+}}{2^{\frac{k_3}{2}}}$$

and is smooth in (ξ_1,ξ_2,ξ_3) on the 2^{k_3} scale, and in ξ on the 2^j scale. We consider the two cases (i) and (ii) described above.

Case 3B(i). Here $|k_1-k_2|, |k_2-k_3| \lesssim 1$ therefore $G_0\chi_D$ has symbol regularity in all variables. Hence separating the variables it suffices to look at G_0 of the form

$$G_0 = g_{jk_1k_2k_3}\chi_{k_1}(\xi_1)\chi_{k_2}(\xi_2)\chi_{k_3}(\xi_3)\chi_j(\xi)$$

where the χ_k's are smooth normalized dyadic bump functions. Then the quadrilinear integral becomes

$$I_0 = g_{jk_1k_2k_3} \int \langle \hat{\chi}_{k_1}, \psi^1_{k_1}\rangle \langle \hat{\chi}_{k_2}, \overline{\psi}^2_{k_2}\rangle \langle \hat{\chi}_{k_3}, \psi^3_{k_3}\rangle \langle \hat{\chi}_j, \overline{\psi}_j\rangle dt$$

Hence using the local energy norm for the first two ψ's, the energy for the last two and (4.11) for the inverse FT of bump functions we obtain

$$|I_0| \lesssim g_{jk_1k_2k_3} 2^{\frac{k_3+j-k_1-k_2}{2}} \lesssim \frac{2^j}{|j|} \frac{\langle k_3^- \rangle}{2^{k_1}}$$

which easily implies (8.36). Since in this case G_1 and G_2 satisfy similar bounds, we also obtain (8.36) for I_1 and I_2.

Case 3B(ii). Here $|k_1 - k_2| \lesssim 1, |k_3 - k_2| \gg 1$, and the function $G_0 \chi_D$ has symbol regularity in ξ_3 and ξ, but only on the 2^{k_3} scale in ξ_1 and ξ_2. Furthermore we can use χ_D to localize it in the region $|\xi_1 - \xi_2| \ll 2^{k_3}$. Thus we can separate variables and obtain a decomposition of G_0 of the form

$$G_0(\xi_1, \xi_2, \xi_3, \xi) = g_{jk_1k_2k_3} \sum_{l=1}^{2^{k_1-k_3}} \chi^l_{k_1}(\xi_1)\chi^l_{k_2}(\xi_2)\chi_{k_3}(\xi_3)\chi_j(\xi)$$

where $\chi^l_{k_1}, \chi^l_{k_2}$ have similar 2^{k_3} sized supports. Using an argument similar to the one in Proposition 4.11 we obtain

$$|\hat{\chi}^l_{k_{1,2}}(r)| \lesssim 2^{k_3 + \frac{k_1}{2}} (1 + 2^{k_1} r)^{-\frac{1}{2}} (1 + 2^{k_3} r)^{-N}$$

At this point a direct computation as in case (i) for the corresponding part I_0^l of I_0 gives

$$|I_0^l| \lesssim \frac{2^j}{|j|} \frac{\langle k_3^- \rangle}{2^{k_1}}$$

and, after summation with respect to l,

$$|I_0| \lesssim \frac{2^j}{|j|} \frac{\langle k_3^- \rangle}{2^{k_3}}. \tag{8.37}$$

Unfortunately this bound is not strong enough for (8.36) for either I_0 or I_1. The failure of this argument is that no orthogonality with respect to l is exploited. However, we remark that the bounds for G_2 have an extra factor of $2^{k_3-k_1}$, which is more than enough to prove (8.36) for I_2.

To remedy the above difficulty for I_0 and I_1 we separate the nonlinear expression into two parts: one where r is small, which we can estimate directly, and the other one where r is large, for which we apply the above computation. Given a threshold $m > -k_3^-$ we split $I_0 = I_0^m + J_0^m$ where

$$I_0^m = \int_0^\infty \chi_{\geq m}(r) \psi^1_{k_1}(r) \psi^2_{k_2}(r) \psi^3_{k_3}(r) \psi_j(r) r dr dt$$

The contribution J_0^m of the region $A_{<m}$ is estimated directly via the local energy for the first two factors combined with the pointwise bound (4.24) derived from the energy for the last two factors:

$$|J_0^m| \lesssim \|\chi_{<m}(r) \psi^1_{k_1} \psi^2_{k_2} \psi^3_{k_3} \psi_j\|_{L^1} \lesssim \frac{2^j}{|j|} \frac{\langle m \rangle 2^{(m+k_3)/2}}{2^{k_1}} \tag{8.38}$$

For I_0^m we proceed as in the derivation of (8.37) but with G_0 replaced by its corresponding truncated version G_0^m which by Proposition 4.15 satisfies a better bound than G_0, namely

$$|G_0^m| \lesssim \frac{2^{j/2}}{|j|} \frac{\langle k_3^- \rangle}{2^{k_3/2}} 2^{-N(m+k_3)}$$

This leads to a similar improvement over (8.37), namely

$$|I_0^m| \lesssim \frac{2^j}{|j|} \frac{\langle k_3^- \rangle}{2^{k_3}} 2^{-N(m+k_3)} \tag{8.39}$$

Adding (8.38) and (8.39) gives

$$|I_0| \lesssim \frac{2^j}{|j|} \frac{\langle m \rangle}{2^{k_1}} \left(2^{(m+k_3)/2} + 2^{k_1-k_3} 2^{-N(m+k_3)}\right)$$

Optimizing with respect to m we obtain (8.36) for I_0.

We still need to consider I_1. The bound for I_1^m is identical to the one for I_0^m since (4.58) gives the same bounds for G_0^m and G_1^m. For J_1^m we apply a similar argument but some extra care is required due to the presence of the $[r\partial_r]^{-1}$ operator. Precisely, using the $L^4_{k_1}$ and LE_{k_2} norms for the two factors we obtain

$$2^{k_1} \|\psi^1_{k_1} \psi^2_{k_2}\|_{L^{\frac{4}{3}}(A_{<-k_1})} + 2^{\frac{5k_1-3m}{8}} \sum_{m \geq k_1} \|\psi^1_{k_1} \psi^2_{k_2}\|_{L^{\frac{4}{3}}(A_m)} \lesssim 1$$

and the norm in the LHS above is preserved by the operator $[r\partial_r]^{-1}$ by (1.15). Then using the pointwise bound derived from the energy norm for ψ_j respectively the $L^4_{k_3}$ norm for $\psi^3_{k_3}$ we obtain

$$|J_1^m| \lesssim \frac{2^j}{|j|} \frac{\langle m \rangle 2^{(m+k_3)/4}}{2^{(5k_1+3k_3)/8}}$$

Though slightly weaker than (8.38), this still leads to (8.36) for I_1 when combined with (8.39) for I_1^m. The proof of the proposition is complete.

□

9. The bootstrap estimate for the λ parameter.

In this section we show that the Z_0 regularity of the parameter λ can be bootstrapped, completing the proof of Theorem 8.2. The result in (8.17) is obtained by replacing γ with $\gamma + \gamma_0^2$ in the the following Proposition.

Proposition 9.1. *Let $T \in (0, \infty]$. Consider $\tilde{\lambda}$ which satisfies the bound*

$$\|\tilde{\lambda} - 1\|_{Z[0,T]} \ll 1. \tag{9.1}$$

and a function ψ satisfying

$$\|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T]} \leq \gamma \ll 1 \tag{9.2}$$

Then the functions λ, α defined by

$$\psi_2(t,1) = i e^{i\alpha(t)} h_1^{\lambda(t)}(1) \tag{9.3}$$

satisfy the bounds

$$\|\lambda - 1\|_{Z_0[0,T]} + \|\alpha\|_{Z_0[0,T]} \lesssim \gamma \tag{9.4}$$

Proof. From the fixed time analysis in Proposition 7.3 we have the uniform bound in $[0,T]$

$$\|\psi_2(1)-1\|_{L_t^\infty L_r^\infty[\frac{1}{2},2]} \lesssim \gamma$$

which leads to the uniform bound

$$\|\lambda - 1\|_{L^\infty} + \|\alpha\|_{L^\infty} \lesssim \gamma$$

To continue we recall the equation for ψ_2 from (3.31):

$$\partial_r \psi_2 = iA_2\psi - \frac{A_2\psi_2}{r}$$

Approximating the second A_2 with h_3^λ we rewrite this in the form

$$L_\lambda \psi_2 = iA_2\psi - \frac{(A_2 - h_3^\lambda)\psi_2}{r} := i\psi + N$$

A solution to the homogeneous equation is h_1^λ. Then by the same reasoning as in Proposition 7.3 we must have

$$\psi_2(r) = i\lambda h_1^\lambda(r) + L_\lambda^{-1}(i\psi + N)$$

where the factor λ in the first term on the right is dictated by the requirement that $\psi_2 - ih_1 \in X$ which plays the role of the boundary condition at infinity. Also L_λ^{-1} is the $LX \to X$ inverse of L_λ. For functions $f \in L^1(rdr)$ we have the integral formula

$$L_\lambda^{-1} f(r) = -h_1^\lambda(r)\int_r^\infty \frac{f(s)}{h_1^\lambda(s)}ds$$

We will be able to apply this formula for N above, but not for $i\psi$. However, in the case of $i\psi$ there is another computation that allows us to replace L_λ^{-1} by L^{-1}. Precisely,

$$L_\lambda^{-1} f(r) = \frac{\lambda h_1^\lambda(r)}{h_1(r)} L^{-1} f(r) - \lambda h_1^\lambda(r)\int_r^\infty \left(\frac{1}{\lambda h_1^\lambda(s)} - \frac{1}{h_1(s)}\right) f(s)ds$$

which holds for all $f \in LX$. The integral converges even for all $f \in L^2$.

Then we rewrite ψ_2 in the form

$$\psi_2(r) = i\lambda h_1^\lambda(r)(1 + (h_1(r))^{-1}A(r) - B(r))$$

where

$$A(r) = L^{-1}\psi, \qquad B(r) = \int_r^\infty \psi(s)\left(\frac{1}{\lambda h_1^\lambda(s)} - \frac{1}{h_1(s)}\right) - i\frac{N(s)}{\lambda h_1^\lambda(s)}ds$$

Set $r=1$ and recall that $\psi_2(1) = ie^{i\theta}h_1^\lambda(1)$. Then we obtain

$$e^{i\theta} = \lambda(1 - A(1) - B(1))$$

Since Z_0 is an algebra, it suffices to estimate $A(1)$ and $B(1)$ in Z_0. For $A(1)$ we will establish a linear Z bound,

$$\|A(1)\|_{Z[0,T]} \lesssim \gamma \tag{9.5}$$

which is facilitated by the fact that λ does not appear in the expression for A. On the other hand λ does appear in the B expression; however, there this does not matter since for $B(1)$ we establish a stronger bound

$$\|B(1)\|_{L_t^2[0,T]} \lesssim \gamma \tag{9.6}$$

using only the pointwise bound for λ. Together (9.5) and (9.6) imply (9.4). We consider these two bounds separately:

1. The estimate (9.5) for the linear term.
By the local energy decay estimate for ψ we can replace $A(1)$ with a local average

$$\tilde{A} = \int A(r)\chi(r)dr$$

where χ is a bump function supported near 1 with the normalization

$$\int \chi(r)h_1(r)dr = 1$$

The difference admits good L^2 and L^∞ bounds,

$$\|A(1) - \tilde{A}\|_{L^\infty} \lesssim \|\psi\|_{L^\infty L^2}, \qquad \|A(1) - \tilde{A}\|_{L^2} \lesssim \|\psi\|_{LE}$$

Hence it remains to show that

$$\|\tilde{A}\|_{Z_0[0,T]} \lesssim \|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T]} \tag{9.7}$$

Using the Fourier expansion for ψ in the $\tilde{H}$ frame, respectively for $L^{-1}\psi$ in the H frame, we can represent $L^{-1}\psi$ in the form

$$L^{-1}\psi(r) = \int \xi^{-1}\phi_\xi(r)\hat{\psi}(\xi)d\xi$$

Then for $\tilde{A}$ we have

$$\tilde{A} = \int h(\xi)\hat{\psi}(\xi)d\xi, \qquad h(\xi) = \xi^{-1}\int \phi_\xi(r)\chi(r)dr.$$

Given the representation of ϕ_ξ in Section 4, it follows that h is a smooth function in $(0,\infty)$ which has symbol type regularity, rapid decay at infinity and whose size near $\xi = 0$ is given by

$$h(\xi) \approx \frac{\xi^{-\frac{3}{2}}}{|\log \xi|}, \qquad \xi \ll 1.$$

The bound (9.2) for ψ in the proposition is given in terms of the λ frame, which is inconvenient as it makes it difficult to track the different modulations. Fortunately, we are able to use the bound (6.30) to transfer enough of the $WS^\sharp[\tilde{\lambda}]$ norm to the fixed frame. It remains to show that

$$\|\tilde{A}\|_{Z[0,T]} \lesssim \|\psi\|_{WS^r[1][0,T]} \tag{9.8}$$

At this stage we can replace ψ by any admissible $WS^r[1]$ extension to the real line, and show that the above bound holds globally in time.

We decompose ψ in frequency with respect to the fixed $\lambda = 1$ frame,

$$\psi = \sum \psi_k, \qquad \psi_k = P_k\psi$$

Correspondingly $\tilde{A} = \sum A_k$, where

$$A_k = \langle g_k, \psi_k\rangle, \qquad \hat{g}_k(\xi) = \tilde{\chi}_k(\xi)h(\xi)$$

By Proposition 4.11, the functions g_k satisfy the pointwise bounds

$$|g_k(r)| \lesssim \frac{\log(1+r^2)}{\langle k^- \rangle^2}(1+2^k r)^{-N} 2^{-Nk^+}$$

The contribution of high frequencies $k \geq 0$ is easily estimated in $L^\infty \cap L^2$,

$$\|A_k\|_{L^\infty} \lesssim 2^{-Nk} \|\psi_k\|_{L^\infty L^2}, \qquad \|A_k\|_{L^2} \lesssim 2^{-Nk} \|\psi_k\|_{LE_k}$$

It suffices to show that for the low frequencies we have

$$\|A_k\|_Z \lesssim \frac{2^{-k}}{|k|} \|\psi_k\|_{S_k^r}, \qquad k < 0 \tag{9.9}$$

On one hand we can use local energy decay to obtain an L^2 bound,

$$\|A_k\|_{L^2} \lesssim \frac{2^{-2k}}{|k|} \|\psi_k\|_{LE_k}$$

which suffices at low modulations,

$$\|Q_{\leq 2k} A_k\|_Z \lesssim \|Q_{\leq 2k} A_\mu\|_{\dot{H}^{\frac{1}{2}}} \lesssim 2^k \|A_k\|_{L^2} \lesssim \frac{2^{-k}}{|k|} \|\psi_k\|_{LE_k}.$$

On the other hand for high modulations we can use the high modulation component of the S_k^r norm:

$$\|Q_{>2k} A_k\|_Z \lesssim \|Q_{>2k} A_k\|_Z \lesssim \|g_k\|_{L^2} \|Q_{>2k} \psi_k\|_{Z L^2} \lesssim \frac{2^{-k}}{|k|} \|\psi_k\|_{S_k^r}$$

This concludes the proof of (9.9) and thus the estimate for $A(1)$.

2. The estimate (9.6) for the nonlinear term.

The analysis in this case is identical whether we work in a compact interval or on the real line. It suffices to place the integrand in the formula of $B(r)$ in $L_t^2 L_r^1[1, \infty)$. For the first term we note that

$$\left| \frac{1}{\lambda h_1^\lambda} - \frac{1}{h_1} \right| \lesssim r^{-1}.$$

Then we can use the local energy bound (6.6):

$$\|r^{-1} \psi\|_{L_t^2 L_r^1([1,\infty];dr)} = \|r^{-2} \psi\|_{L_t^2 L_r^1(A_{>0})} \lesssim \|\psi\|_{LE} \lesssim \|\psi\|_{WS^\sharp[\lambda]} \lesssim \gamma$$

For the second term in B we need to estimate

$$\|rN\|_{L_t^2 L_r^1([1,\infty];dr)} = \|N\|_{L_t^2 L^1(A_{>0})}$$

where

$$N = (A_2 - 1)\psi + \frac{1}{r}(A_2 - h_3^{\tilde{\lambda}})\psi_2$$

Since for $r > 1$ we have $|A_2 - 1| \lesssim |\psi_2|^2$, for the first term we use the local energy decay (6.6) and the uniform in time bounds (4.17) and (4.16) to write

$$\|(A_2 - 1)\psi\|_{L^2 L^1} \lesssim \|\langle r \rangle^{-\frac{1}{2} - \delta} \psi\|_{L^2} \|\langle r \rangle^{-\delta} \psi_2\|_{L^\infty L^2} \|\langle r \rangle^{2\delta} \psi_2\|_{L^\infty} \lesssim \gamma$$

for some $0 < \delta < \frac{1}{4}$.

Finally, for the second term in N we use the local energy decay (7.55) and the uniform in time bound (4.17):

$$\|\langle r\rangle^{-1}(A_2 - h_3^{\tilde{\lambda}})\psi_2\|_{L^2L^1} \lesssim \|\langle r\rangle^{-\frac{1}{2}-\delta}(A_2 - h_3^{\tilde{\lambda}})\|_{L^2}\|\langle r\rangle^{-\frac{1}{2}+\delta}\psi_2\|_{L^\infty L^2} \lesssim \gamma$$

□

Remark 1. *It is likely that λ actually belongs to Z. However this is not needed in the present paper, and proving it would require a much more involved analysis of the nonlinear contribution B above.*

10. The bootstrap argument

In this section we prove Theorem 1.3, i.e. our main global well-posedness result. It is convenient to state the result in a more precise form:

Theorem 10.1. *a) Let $m = 1$ and $\epsilon \leq 1, \gamma \ll 1$. Then for each 1-equivariant initial data u_0 satisfying*

$$\|u_0 - \mathcal{Q}^1\|_{\dot{H}^1} \leq \epsilon\gamma, \qquad \|\bar{u}_0 - \bar{Q}\|_X \leq \gamma \tag{10.1}$$

there exists a unique global solution u so that $\bar{u} - \bar{Q} \in C(\mathbb{R}; X)$ and

$$\|\bar{u} - \bar{Q}\|_{C(\mathbb{R};X)} \lesssim \gamma \tag{10.2}$$

Furthermore, this solution has a Lipschitz dependence on the initial data in X, uniformly on compact time intervals.

b) Let ψ be the reduced field associated to the solution u above and (α, λ) defined by (7.51). Then the following estimates are valid:

$$\|\psi\|_{l^2S^\sharp} \lesssim \epsilon\gamma, \qquad \|W\psi\|_{l^2N^\sharp} \lesssim \epsilon\gamma^2, \qquad \|\lambda - 1\|_{Z_0} + \|\alpha\|_{Z_0} \lesssim \gamma \tag{10.3}$$

In addition, for any function $\tilde{\lambda}$ satisfying (8.15) we have

$$\|\psi\|_{WS^\sharp[\tilde{\lambda}]} \lesssim \gamma, \qquad \|W_{\tilde{\lambda}}\psi\|_{WN^\sharp[\tilde{\lambda}]} \lesssim \epsilon\langle\log\epsilon\rangle^2\gamma^2 \tag{10.4}$$

We prove the theorem in two stages. First we use a bootstrap argument establish global well-posedness and bounds for regular solutions, i.e. solutions with initial data in $Q + H^2$. Then we use a density argument to extend this result to initial data in $Q + X$.

10.1. **Regular solutions.** a) Given an initial data u_0 so that $u_0 - Q \in H^2$, by Theorem 1.1 we know that there exists a unique short time solution u so that $u(t) - Q \in C(0, T; H^2)$ for some small T. In addition, we assume that u_0 satisfies (10.1).

We will use a bootstrap argument to extend the time interval T for which the solution exists, with some suitable bounds. We begin by describing the bounds we will bootstrap. These are all expressed in terms of the reduced field ψ and the soliton parameter λ. We remark that in view of (1.4), (2.2), and Theorem 7.1(a), the bounds (10.1) imply that the initial data $\psi(0)$ satisfies

$$\|\psi(0)\|_{L^2} \lesssim \epsilon\gamma, \qquad \|\psi(0)\|_{LX} \lesssim \gamma \tag{10.5}$$

We choose

$$\gamma < \gamma_0 = C\gamma \ll 1$$

where C is chosen larger than the constants used in defining $\lesssim$ in all estimates in Theorem 8.2. Suppose we have a solution u as above in some interval $[0, T]$. With λ defined as in (3.34), the first bootstrap bound will be

$$\|\lambda - 1\|_{Z_0[0,T]} \leq \gamma_0 \tag{10.6}$$

The second bootstrap bound is concerned with the size of ψ as an L^2 solution for a Schrödinger equation,

$$\|\psi\|_{l^2 S^\sharp[0,T]} \leq \epsilon\gamma_0 \tag{10.7}$$

while the third bootstrap bound keeps track of the norm of ψ in $WS^\sharp[\lambda]$ type spaces,

$$\|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T]} \leq \gamma_0 \tag{10.8}$$

for some function $\tilde{\lambda}$ which has the property that

$$\|\tilde{\lambda} - 1\|_{Z[0,T]} \lesssim \gamma_0, \qquad \|\tilde{\lambda} - \lambda\|_{(L^2 \cap L^\infty)[0,T]} \lesssim \gamma_0 \tag{10.9}$$

We denote by

$$\begin{aligned} \mathcal{A} = \{\ & T_0 \geq 0;\ \text{the Schrödinger map equation (1.1) admits a solution } u \in C(0, T_0; H^2(\mathbb{R}^2)) \\ & \text{so that its reduced field } \psi \text{ and soliton parameter } \lambda \text{ satisfy (10.6), (10.7) and (10.8)} \\ & \text{for all } T \leq T_0\} \end{aligned}$$

Our goal is to prove that $\mathcal{A} = [0, \infty)$. Once this is done, it follows that we have a global solution satisfying (10.6), (10.7) and (10.8). The bounds on $W\psi$ and $W_{\tilde{\lambda}}\psi$ in (10.3), respectively (10.4) follow from Propositions 8.3, 8.4,8.5. The estimate of (10.2) follows from Theorem 7.1(b); this is where the qualitative property $u - Q \in L^2$ is is used in order to uniquely identify u as the map associated to ψ via Theorem 7.1(b).

By definition $\mathcal{A}$ is an interval containing 0. Thus it suffices to prove the following two properties:

(i) $\mathcal{A}$ is open in $[0, \infty)$.
(ii) $\mathcal{A}$ is closed in $[0, \infty)$.

(i) $\mathcal{A}$ is open. Let $T_0 \in \mathcal{A}$. Then $u(T_0) - Q \in H^2$, therefore by Theorem 1.1 we have a local solution $u - Q \in C([T_0, T_0 + \delta_0]; H^2)$. From (10.6), (10.7) and (10.8) at T_0, applying Theorem 8.2, we obtain the bounds

$$\|\lambda - 1\|_{Z_0[0,T_0]} \lesssim \gamma + \gamma_0^2, \quad \|\psi\|_{l^2 S^\sharp[0,T_0]} \lesssim \epsilon(\gamma + \gamma_0^2), \quad \|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T_0]} \lesssim \gamma + \gamma_0^2 \tag{10.10}$$

which are stronger since $\gamma \ll \gamma_0 \ll 1$. It remains to show that the above norms cannot grow much when replacing T_0 by $T_0 + \delta$ with small δ. From the first norm we obtain $|\lambda(T_0) - 1| \lesssim \gamma + \gamma_0^2$. Since λ is a continuous function of time, it follows that for small δ we have

$$\|1_{[T_0, T_0+\delta]}(\lambda - 1)\|_{L^2 \cap L^\infty} \lesssim \gamma + \gamma_0^2$$

This concludes the bootstrap for (10.6).

Next we consider the second norm in (10.10). For this we use the equation (8.1) for ψ. The linear part is well-posed in L^2, therefore it suffices to obtain a good bound for $W\psi$ in $[T_0, T_0 + \delta]$,

$$\|W\psi\|_{L^1([T_0, T_0+\delta]; L^2)} \lesssim \delta$$

where, here and below, the implicit constant is allowed to depend on the uniform H^2 bound for $u-Q$ in $[T_0, T_0+\delta_0]$. Indeed, from this and the result of Theorem 8.2, we obtain

$$\|\psi\|_{l^2 S^\sharp[0,T_0+\delta]} \lesssim \|\psi(0)\|_{L^2} + \|1_{[0,T_0]} W\psi\|_{l^2 N^\sharp} + \|1_{[T_0,T_0+\delta]} W\psi\|_{l^2 N^\sharp} \lesssim (\gamma+\gamma_0^2)\epsilon + \delta$$

Similarly, for the third norm in (10.10) we use the equation (8.11) for ψ. The linear part is well-posed in LX, therefore it suffices to obtain a good bound for $W_{\tilde{\lambda}}\psi$ in $[T_0, T_0+\delta]$,

$$\|W_{\tilde{\lambda}}\psi\|_{L^1([T_0,T_0+\delta];LX)} \lesssim \delta$$

with respect to a new[2] function $\tilde{\lambda}$ associated to λ on the interval $[0, T+\delta]$. Indeed, in view of the result in Theorem 8.2 and (6.29),

$$\begin{aligned} \|\psi\|_{WS^\sharp[\tilde{\lambda}][0,T_0+\delta]} &\lesssim \|\psi_0\|_{LX} + \|1_{[0,T_0]} W_{\tilde{\lambda}}\psi\|_{WN^\sharp[\tilde{\lambda}]} + \|1_{[T_0,T_0+\delta]} W_{\tilde{\lambda}}\psi\|_{WN^\sharp[\tilde{\lambda}]} \\ &\lesssim (\gamma+\gamma_0^2) + \delta \end{aligned}$$

By choosing δ small enough, we can then use the result of part b) in Theorem 8.2 to bootstrap the bounds above and claim the last two bounds in (10.10) for $T_0+\delta$.

Using also the embedding (4.25), it suffices to show the fixed time bound

$$\|W\psi\|_{L^2} \lesssim 1, \qquad \|W_{\tilde{\lambda}}\psi\|_{L^1\cap L^2} \lesssim 1 \tag{10.11}$$

For this we use the H^2 regularity for $u-Q$ and its consequences in Corollary 3.4. By Theorem 1.1, this regularity persists up to time T_0 and also for a short time past T_0. We consider each term in W or $W_{\tilde{\lambda}}$. For the cubic term we simply use Sobolev embeddings to get $\|\psi\|_{L^2\cap L^\infty} \lesssim 1$. For the ψ_2 term we use the same, plus the L^2 bound for ψ_2/r. Finally, we split the δA_2 term into two. For large r we have the r^{-2} decay factor so we only need to use the pointwise boundedness of δA_2. However, for small r we need to cancel that factor. This is easily done since for $r \ll 1$ we have

$$|\delta A_2| \lesssim r^2 + |\psi_2|^2 \lesssim r^2 + |u_1|^2 + |u_2|^2 \lesssim r^2$$

where in the last step we have used Sobolev embeddings for u_1 and u_2, which vanish at the origin.

(ii) $\mathcal{A}$ is closed. Suppose that (10.6), (10.7) and (10.8) hold for all $T < T_0$. Then we have a Schrödinger map $u - Q \in C([0,T_0); H^2)$. Passing to the limit in (10.6) we obtain (10.6) for $T = T_0$. In particular this shows that λ stays close to 1 up to $T = T_0$. Then, by Theorem 1.1, it follows that the H^2 bounds persist up to (and beyond) $T = T_0$. Once we have $u - Q \in C([0,T_0]; H^2)$ we repeat the above argument using Theorem 8.2 in $[0, T-\delta]$ and then the bounds (10.11) in $[T-\delta, T]$ with small $\delta > 0$ in order to prove (10.7) and (10.8) for $T = T_0$.

b) The linear bounds in (10.3) and (10.4) have been established above and the nonlinear bounds follow from (8.18), (8.20) and (8.21).

[2]The exact choice of $\tilde{\lambda}$ does not matter here; to fix things one could simply take $\tilde{\lambda} = 1 + Q_{\leq 1}(1_{[0,T+\delta]}(\lambda-1))$.

10.2. **Rough solutions.** Given an initial data u_0 which satisfies

$$\|\bar{u}_0 - \bar{Q}\|_X \lesssim \gamma,$$

we approximate it in the above topologies with a sequence of more regular initial data $u_0^{(n)} \in Q + H^2$. Such approximations can be obtained for instance by removing both the low and the high frequencies in the H frame,

$$\bar{u}_0^{(n)} = \Pi_{\mathbb{S}^2}(\bar{Q} + P^H_{[-n,n]}(\bar{u}_0 - \bar{Q}))$$

where $\Pi_{\mathbb{S}^2}$ represents the radial projection onto the sphere. Since the projections $P^H_{[-n,n]}(\bar{u}_0 - \bar{Q})$ stay pointwise small, the convergence of $\bar{u}_0^{(n)}$ to $\bar{u}_0$ in $\bar{Q} + X$ follows from the algebra property of X and the bound (4.27) in Proposition 4.6.

According to the previous step in the proof, for the initial data $u_0^{(n)}$ we obtain global solutions $u^{(n)}$ with $\bar{u}^{(n)} - \bar{Q} \in C(\mathbb{R}; X)$, so that the bounds (10.10) hold uniformly for the corresponding functions $\lambda^{(n)}$ and $\psi^{(n)}$. In particular we have a uniform bound

$$\|\psi^{(n)}\|_{L^\infty LX} \lesssim \gamma \tag{10.12}$$

By the first part of Theorem 7.1, the X convergence of $\bar{u}_0^{(n)}$ to $\bar{u}_0$ implies that $\psi_0^{(n)}$ converges to ψ_0 in LX. By the short time result in Theorem 8.1, it follows that the sequence $\psi^{(n)}$ converges in $WS^\sharp[1][0,1]$ to some solution ψ to (8.1) with initial data ψ_0. In view of the uniform bound (10.12) we can reiterate and obtain a global solution ψ to (8.1), so that for all $T > 0$

$$\psi^{(n)} \to \psi \qquad \text{in } WS^\sharp[1][0,T]$$

Furthermore, ψ satisfies (10.10) globally in time. We note however that above we do not obtain uniform convergence with respect to T.

Finally, given ψ we apply the second part of Theorem 7.1 to construct a global Schrödinger map u so that

$$\bar{u}^{(n)} - \bar{Q} \to \bar{u} - \bar{Q} \qquad \text{in } L^\infty X$$

The local in time Lipschitz dependence of the solution $\bar{u}$ in $\bar{Q} + L^\infty X$ on the initial data $\bar{u}_0 \in \bar{Q} + X$ follows also by iterated application of Theorem 8.1, with the transition back and forth between u and ψ done via Theorem 7.1.

11. The $\dot{H}^1$ instability result

In this section we prove the instability result in Theorem 1.4. Let $\epsilon, \gamma \ll 1$ and α_0, λ_0 as in (1.10), i.e. so that $|\alpha_0| + |\lambda_0 - 1| \approx \gamma$. We interpret ϵ as a frequency parameter, and choose the initial data u_0 so that $\bar{u}_0$ takes values in the $(\vec{i}, \vec{k})$ plane and

$$\bar{u}(r) = \begin{cases} \bar{Q}_{\alpha_0,\lambda_0}(r), & r \ll \epsilon^{-1} \\ \bar{Q}(r), & r \gg \epsilon^{-1} \end{cases}$$

with a smooth transition on the ϵ^{-1} scale between the two regions, so that

$$|r(r\partial_r)^\alpha(\bar{u}_0 - \bar{Q})| \lesssim_\alpha \gamma, \qquad r \approx \epsilon^{-1} \text{ and } \alpha \geq 1. \tag{11.1}$$

Using this and the form of the energy from (1.3), a direct computation shows that the bound (1.11) holds,

$$\|u_0 - Q_{\alpha_0,\lambda_0}\|_{\dot{H}^1} \lesssim \epsilon\gamma.$$

To study the evolution of u we switch to the ψ variable. From the above information we characterize ψ at time $t = 0$. The construction of the Coulomb gauge associated to $u(0)$ is trivial in this case. Since $\bar{u}$ stays in the $(\vec{i}, \vec{k})$ plane and $\bar{v}(\infty) = \vec{k}$, from the form of the ODE (3.15) it follows that $\bar{v}$ stays in the same plane. This implies that

$$\bar{w}(r) = \vec{j}, \qquad \bar{v}(r) = \bar{u}(r) \times \vec{j}$$

Recalling that

$$\psi(0) = \partial_r u \cdot v + i \partial_r u \cdot w$$

and using the above characterizations for u, v, w we obtain the following characterization for ψ:

$$\psi(0) = 0, \qquad r \ll \epsilon^{-1} \text{ and } r \gg \epsilon^{-1}$$
$$|r^2 (r\partial_r)^\alpha \psi| \lesssim_\alpha \gamma, \qquad r \approx \epsilon^{-1}$$

In other words, $\psi(0)$ is a bump of size $\gamma\epsilon^2$ localized in the annulus $r \approx \epsilon^{-1}$. Hence it satisfies the L^2 bound

$$\|\psi(0)\|_{L^2} \lesssim \gamma\epsilon \tag{11.2}$$

Using the above estimates on ψ and the characterization of the $\psi_\xi(r)$ from section 4.2, it also follows that the Fourier transform of ψ satisfies

$$|(\xi\partial_\xi)^\alpha \mathcal{F}_{\tilde{H}} \psi(0,\xi)| \lesssim_{\alpha,N} \gamma \frac{\langle \ln \epsilon \rangle}{\langle \ln \xi \rangle} \xi^{\frac{1}{2}} \langle \xi \epsilon^{-1} \rangle^{-N}, \qquad \alpha, N \in \mathbb{N} \tag{11.3}$$

which directly leads to an LX bound

$$\|\psi(0)\|_{LX} \lesssim \gamma \tag{11.4}$$

This places us in the framework of the rest of the paper. Precisely, we obtain a global solution u as in Theorem 10.1, which also satisfies the bounds (10.3), (10.4).

By (3.37), the desired bound (1.12) would follow from the estimate

$$|\psi_2(1,t) - i| \lesssim \gamma |\log \epsilon|^{-1} \quad |t| > t_0 \tag{11.5}$$

We will in effect prove a stronger bound

$$\|\psi_2(t) - i h_1\|_{\dot{H}^1} \lesssim \gamma |\log \epsilon|^{-1} \quad |t| > t_0 \tag{11.6}$$

By Theorem 10.1, we propagate the bounds (11.2) and (11.4) along the flow:

$$\|\psi(t)\|_{L^2} \lesssim \gamma\epsilon, \qquad \|\psi(t)\|_{LX} \lesssim \gamma \tag{11.7}$$

By (10.4) we have a good[3] bound for the nonlinearity in the ψ equation,

$$\|(i\partial_t - \tilde{H}_{\tilde{\lambda}})\psi\|_{WS^\sharp[\tilde{\lambda}]} \lesssim \gamma |\log \epsilon|^{-1} \tag{11.8}$$

where $\tilde{\lambda}$ is as in (8.15). This shows that we can approximate ψ in LX by the solution to the corresponding linear equation, with $\gamma|\log\epsilon|^{-1}$ errors. Then, by Proposition 6.7, we can compare solutions to the linear $\tilde{H}$ equation with solutions to the linear $\tilde{H}_{\tilde{\lambda}}$ equation,

$$\|\psi(t) - e^{it\tilde{H}}\psi(0)\|_{LX} \lesssim \gamma |\log \epsilon|^{-1} \tag{11.9}$$

Thus by Proposition 7.3 it suffices to look at $\tilde{\psi}(t) = e^{it\tilde{H}}\psi(0)$ and show that the corresponding $\tilde{\psi}_2$ associated to it satisfies (11.6).

[3]We actually get a stronger $\gamma\epsilon|\log\epsilon|^2$ bound here.

From (7.20) it follows that

$$\|\tilde{\psi}_2(t) - ih_1\|_{\dot{H}^1} \lesssim \|L^{-1}\tilde{\psi}(t)\|_{\dot{H}^1} \tag{11.10}$$

Denoting $g(t) = L^{-1}\tilde{\psi}(t)$, we will prove that

$$\|g(t)\|_{\dot{H}^1} \lesssim \gamma(\epsilon + \frac{|\log \epsilon|}{|\log t|}). \tag{11.11}$$

Together with (11.10) this establishes (11.6) and concludes the proof of the theorem. Therefore we are left with proving (11.11).

The function g has the Fourier expansion

$$g(r,t) = \int_0^\infty \xi^{-1}\mathcal{F}_{\tilde{H}}\tilde{\psi}(\xi,t)\phi_\xi(r)d\xi = \int_0^\infty \xi^{-1}e^{it\xi^2}\mathcal{F}_{\tilde{H}}\psi_0(\xi)\phi_\xi(r)d\xi$$

where $\psi_0 = \psi(0)$. We denote by $g_k, \tilde{\psi}_k$ the dyadic pieces of g respectively $\tilde{\psi}$ in the H, respectively $\tilde{H}$ calculus. We have the following

Lemma 11.1. *Let q be as in* (4.1). *Then we have*

$$\|g_k\|_{\dot{H}^1} \lesssim \|\tilde{\psi}_k(t)\|_{L^2} + \left|\int \xi^{-1}q(\xi)\mathcal{F}_{\tilde{H}}\psi_k(\xi,t)d\xi\right| \tag{11.12}$$

Proof. For g_k we have the straightforward L^2 relations

$$\|g_k(t)\|_{L^2} \lesssim 2^{-k}\|\tilde{\psi}_k(t)\|_{L^2}, \qquad \|(\partial_r + \frac{h_3}{r})g_k(t)\|_{L^2} = \|\tilde{\psi}_k(t)\|_{L^2}$$

Combining them we obtain

$$\|\chi_{r\gtrsim 2^{-k}}g_k(t)\|_{\dot{H}^1} \lesssim \|\tilde{\psi}_k(t)\|_{L^2}$$

It remains to estimate the part of g_k in the region $\{r \lesssim 2^{-k}\}$. We consider the more difficult case $k<0$. A similar but simpler argument applies in the case $k \geq 0$. In the above region we use (4.1) to obtain:

$$g_k(r,t) = \int_0^\infty \xi^{-1}q(\xi)\mathcal{F}_{\tilde{H}}\tilde{\psi}_k(\xi,t)(h_1 + O(\xi^2 r\log r))d\xi$$

respectively

$$\partial_r g_k(r) = \int_0^\infty \xi^{-1}q(\xi)\mathcal{F}_{\tilde{H}}\tilde{\psi}_k(\xi,t)(h_1' + O(\xi^2\log r))d\xi$$

The O term in both formulas admits the same bound as before. The contribution of the principal part corresponds to the second term on the right-hand side of (11.12). □

Applying the above lemma gives

$$\|g(t)\|_{\dot{H}^1} \lesssim \sum_k \|\psi_k(0)\|_{L^2} + \sum_k \left|\int \xi^{-1}q(\xi)e^{it\xi^2}\mathcal{F}_{\tilde{H}}\psi_k(0,\xi)d\xi\right|$$

The first term is easily estimated by $\gamma\epsilon$ using (11.3). For the integrals we use stationary phase together with (4.3) and (11.3) to obtain

$$\left|\int \xi^{-1}e^{it\xi^2}\mathcal{F}_{\tilde{H}}\psi_k(0,\xi)\frac{1}{\xi^{\frac{1}{2}}\log\xi}d\xi\right| \lesssim_N \gamma\frac{|\ln\epsilon|}{|k|^2}\langle 2^k\epsilon^{-1}\rangle^{-N}(1+2^{2k}|t|)^{-N}$$

Hence for large t we obtain the bound

$$\|g(t)\|_{\dot{H}^1} \lesssim \gamma \left(\epsilon + \frac{\ln \epsilon}{|\ln t|} \right)$$

which concludes the proof of (11.11).

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