Nearly Gorenstein rings arising from finite graphs

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Abstract

The classification of complete multipartite graphs whose edge rings are nearly Gorenstein as well as that of finite perfect graphs whose stable set rings are nearly Gorenstein is achieved.

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Gorenstein graded algebras associated to combinatorial objects like graphs or simplicial complexes have attracted a lot of interest. See, e.g., [5], [16], [2]. Recently several extensions of the class of Gorenstein rings (inside the class of Cohen–Macaulay rings) have been discussed in, e.g., [6], [7], hence it is natural to search for the combinatorial counterpart.

According to [7], when R is a Cohen–Macaulay graded $\mathbb{K}$-algebra over the field $\mathbb{K}$ with canonical module ω_R, it is called *nearly Gorenstein* if the canonical trace ideal $\mathrm{tr}(\omega_R)$ contains the maximal graded ideal $\mathfrak{m}_R$ of R. Here $\mathrm{tr}(\omega_R)$ is the ideal generated by the image of ω_R through all homomorphism of R-modules into R. As $\mathrm{tr}(\omega_R)$ describes the

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non-Gorenstein locus of R ([7, Lemma 2.1]), one has $\mathrm{tr}(\omega_R) = R$ if and only if R is a Gorenstein ring.

In the present paper we initiate the study of nearly Gorenstein rings belonging to two classes of algebras associated to graphs. Throughout, $\mathbb{K}$ is any field. Assume G is a simple graph (it possesses no loops or multiple edges) with vertex set $V(G) = [d] := \{1, \dots, d\}$.

The *edge ring* $\mathbb{K}[G]$ is the $\mathbb{K}$-subalgebra of the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$ generated by the monomials $x_i x_j$ for all edges $\{i, j\} \in E(G)$. When $V(G)$ can be partitioned $V(G) = \sqcup_{k=1}^{n} V_k$ with $n \geqslant 2$ and $|V_k| = r_k$ for $k = 1, \dots, n$ such that $E(G)$ consists of all the pairs $\{i, j\}$ with $i \in V_a$ and $j \in V_b$ for $1 \leqslant a < b \leqslant n$, we say that G is a *complete multipartite graph* of type $r_1, \dots, r_n$ which is denoted $K_{r_1,\dots,r_n}$. Related algebraic properties for these graphs have been recently studied in [10] and [11]. In Proposition 5 and in Theorem 6 we prove the following result.

Theorem A. Assume $G = K_{r_1,\dots,r_n}$. Set $R = \mathbb{K}[G]$. Then

1. if $n = 2$ and $1 \leqslant r_1 \leqslant r_2$, the ring R is nearly Gorenstein if and only if $r_1 = 1$, or $r_2 \in \{r_1, r_1 + 1\}$.
2. if $n \geqslant 3$ the ring R is nearly Gorenstein if and only if R is Gorenstein.

Since Ohsugi and Hibi in [14] have explicitly listed the complete multipartite graphs whose edge ring is Gorenstein (see Theorem 1 below), Theorem A offers a full description for the nearly Gorenstein property, as well.

The other class of algebras we consider deals with the stable sets in G. A nonempty set W of vertices is called *stable* (or *independent*) if there is no edge $\{i, j\}$ in G with $i, j \in W$. The *stable set ring* of G denoted $\mathrm{Stab}_{\mathbb{K}}(G)$ is the $\mathbb{K}$-subalgebra in the polynomial ring $\mathbb{K}[x_1, \dots, x_d, t]$ generated by those monomials $(\prod_{i \in W} x_i) \cdot t$ with W any stable set in G. When G is a perfect graph, it is known [15] that $\mathrm{Stab}_{\mathbb{K}}(G)$ is Cohen–Macaulay, and that it is Gorenstein if and only if all maximal cliques of G have the same cardinality [16]. Recall that a set $C \subset V(G)$ is called a clique if the subgraph induced by C is a complete graph.

The size of the maximal cliques in G is also relevant to describe in Theorem 13 for which perfect graphs the algebra $\mathrm{Stab}_{\mathbb{K}}(G)$ is nearly Gorenstein. We prove the following.

Theorem B. Let G be a perfect graph and $G_1, \dots, G_s$ its connected components. Let δ_i denote the maximal cardinality of cliques of G_i. Then $\mathrm{Stab}_{\mathbb{K}}(G)$ is nearly Gorenstein if and only if for each G_i its maximal cliques have the same cardinality and $|\delta_i - \delta_j| \leqslant 1$ for $1 \leqslant i < j \leqslant s$.

To prove Theorems A and B we observe that the algebras R which occur are Cohen–Macaulay domains, so ω_R can be identified with an ideal in R. By [7, Lemma 1.1], its trace can be computed as

$$\begin{aligned} \mathrm{tr}(\omega_R) &= \omega_R \cdot \omega_R^{-1}, \text{ where} \\ \omega_R^{-1} &= \{x \in Q(R) : x \cdot \omega_R \subseteq R\} \end{aligned}$$

is the anti-canonical ideal of R and $Q(R)$ denotes the field of fractions of R.

We refer the reader to [1] and [2] for the undefined graph or algebraic notions.

1 Edge rings

In this section unless stated otherwise $G = K_{r_1,\dots r_n}$ is the complete multipartite graph on $[d]$ with vertices partitioned $V(G) = V_1 \sqcup \dots \sqcup V_n$, $n \geqslant 2$, $|V_k| = r_k$ for all k. In this context $d = \sum_{k=1}^{n} r_k$ and without loss of generality, we will always assume that $1 \leqslant r_1 \leqslant \ldots \leqslant r_n$.

The graph G satisfies the so called *odd cycle condition*, i.e. for any two odd cycles in G which have no common vertex there is a bridge between them. Indeed, when $n = 2$ there is no odd cycle and anything to prove. Assume $n \geqslant 3$, and C_1 and C_2 be two disjoint odd cycles in G. Since G is multipartite, each of these contains vertices from at least two of the components $V_1, \dots, V_n$, so one finds $v \in C_1 \cap V_a$ and $w \in C_2 \cap V_b$ with $a \neq b$. Then vw is an edge in G and a bridge between C_1 and C_2. Consequently, by [13] the edge ring

$$R = \mathbb{K}[G] = \mathbb{K}[x_i x_j : i \in V_a, j \in V_b, 1 \leqslant a < b \leqslant n] \subset \mathbb{K}[x_1, \dots, x_d]$$

is normal, hence a Cohen–Macaulay domain ([12]). Before we address the nearly Gorenstein property, we recall that Ohsugi and Hibi [14] classified the complete multipartite edge rings which are Gorenstein. With notation as above, their result is the following.

Theorem 1. *(Ohsugi, Hibi [14, Remark 2.8]) The edge ring of the complete multipartite graph $K_{r_1,\dots,r_n}$ is Gorenstein if and only if*

1. $n = 2$ *and* $(r_1, r_2) \in \{(1, m), (m, m) : m \geqslant 1\}$*, or*
2. $n = 3$ *and* $1 \leqslant r_1 \leqslant r_2 \leqslant r_2 \leqslant 2$*, or*
3. $n = 4$ *and* $r_1 = r_2 = r_3 = r_4 = 1$*.*

For some complete multipartite graphs the edge ring fits into classes of algebras for which the nearly Gorenstein property is already understood.

Example 2. When $r_1 = \cdots = r_n = 1$, the edge ring R is the squarefree Veronese subalgebra of degree 2 in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, and according to [7, Theorem 4.14], R is nearly Gorenstein if and only if it is Gorenstein. The latter property holds if and only if $n \leqslant 4$, by using work of De Negri and Hibi [5], or Bruns, Vasconcelos and Villarreal [3].

Example 3. According to Higashitani and Matsushita [10, Proposition 2.2], when $n = 2$, or when $n = 3$ and $r_1 = 1$, the corresponding edge ring is isomorphic to a Hibi ring, and for the latter the nearly Gorenstein property is described in [7]. We refer to [9] for background on Hibi rings.

Theorem 4 ([7, Theorem 5.4], [9]). *Let P be a finite poset. Then the Hibi ring R of the distributive lattice of the order ideals in P is nearly Gorenstein if and only if P is the disjoint union of pure connected posets $P_1, \dots, P_q$ such that $|\operatorname{rank}(P_i) - \operatorname{rank}(P_j)| \leqslant 1$ for $1 \leqslant i < j \leqslant q$.*

In particular, R is a Gorenstein ring if and only if P is pure.

Based on that, when G is a complete bipartite graph we obtain the following classification.

Proposition 5. *Let $G = K_{r_1,r_2}$ be the complete bipartite graph with $1 \leqslant r_1 \leqslant r_2$. Then the edge ring $\mathbb{K}[G]$ is nearly Gorenstein if and only if $r_1 = 1$, or $r_1 \geqslant 2$ and $r_2 \in \{r_1, r_1 + 1\}$.*

When $2 \leqslant r_1 = r_2 - 1$, the ring $\mathbb{K}[G]$ is nearly Gorenstein and not Gorenstein.

Proof. By [10, Proposition 2.2], $\mathbb{K}[G]$ is isomorphic to the Hibi ring associated to the distributive lattice of order ideals in the poset P which consists of two disjoint chains with $r_1 - 1$ and $r_2 - 1$ elements, respectively. By Theorem 4, $\mathbb{K}[G]$ is nearly Gorenstein if and only if $r_1 = 1$, or $r_1 \geqslant 2$ and $r_2 \in \{r_1, r_1 + 1\}$. □

For non-bipartite graphs we prove the following result.

Theorem 6. *Let R be the edge ring of a complete multipartite graph $K_{r_1,\dots,r_n}$ with $n \geqslant 3$. The following statements are equivalent:*

(i) R is a Gorenstein ring;

(ii) R is a nearly Gorenstein ring.

Proof. Clearly, $(i) \Rightarrow (ii)$. We'll prove the converse.

When $n = 3$ and $r_1 = 1 \leqslant r_2 \leqslant r_3$, by [10, Proposition 2.2] the ring R is isomorphic to the Hibi ring associated to the distributive lattice of order ideals in a poset Q with maximal chains $q_1 < \cdots < q_{r_1}$, $q_{r_1+1} < \cdots < q_{r_1+r_2}$ and $q_1 < q_{r_1+r_2}$. The poset Q is connected, hence R is nearly Gorenstein if and only if it is Gorenstein, i.e. $1 = r_1 \leqslant r_2 \leqslant r_3 \leqslant 2$.

We now consider the remaining cases: either $n = 3$ and $r_1 \geqslant 2$, or $n \geqslant 4$. Assume, by contradiction that R is nearly Gorenstein and not Gorenstein, i.e.

$$\operatorname{tr}(\omega_R) = \mathfrak{m}_R. \tag{1}$$

The monomials in R and ω_R have a nice combinatorial description as feasable integer solutions to some systems of inequalities. This can be described as follows. We denote $H = \sum_{\{i,j\}\in E(G)} \mathbb{N}(\mathbf{e}_i + \mathbf{e}_j) \subset \mathbb{N}^d$ the affine semigroup generated by the columns of the vertex-edge incidence matrix for G, and $\mathcal{C} = \mathbb{R}_+ H$ the rational cone over H.

For $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{N}^d$, it follows from [13] and [18, Proposition 3.4] that $\mathbf{u} \in H$ (equivalently, $\mathbf{x}^{\mathbf{u}} \in R$) if and only if

$$\sum_{i=1}^{d} u_i \equiv 0 \bmod 2, \tag{2}$$

$$u_1, \dots, u_d \geqslant 0, \quad \text{and}$$

$$\sum_{i \notin V_k} u_i \geqslant \sum_{j \in V_k} u_j \text{ for all } k = 1, \dots, n.$$

The latter inequalities are equivalent to

$$\sum_{i=1}^{d} u_i \geqslant 2 \sum_{j \in V_k} u_j, \text{ for } k = 1, \dots, n. \tag{3}$$

Since R is normal, by [4], [17] (see also [2, Theorem 6.3.5(b)]), a $\mathbb{K}$-basis for ω_R is given by the monomials $\mathbf{x}^{\mathbf{u}}$ where $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{Z}^d$ satisfies

$$\sum_{i=1}^{d} u_i \equiv 0 \bmod 2, \tag{4}$$

$$u_1, \ldots, u_d \geqslant 1, \text{and} \tag{5}$$

$$\sum_{i=1}^{d} u_i \geqslant 2 + 2 \sum_{j \in V_k} u_j, \text{ for } k = 1, \ldots, n. \tag{6}$$

From the equations above it is easy to see that if the monomial $\mathbf{x}^{\mathbf{u}}$ is in R or in ω_R, we can permute the exponents x_i and x_j whenever $i, j \in V_k$ for some k, and we obtain another monomial in R, or in ω_R, respectively.

In what follows $\mathbf{u} = (u_1, \ldots, u_d)$ and $\mathbf{v} = (v_1, \ldots, v_d)$.

For a monomial $\mathbf{x}^{\mathbf{u}} \in \omega_R$ and $1 \leqslant k \leqslant n$ we say that V_k (or simply, k) is a *heavy component* in $\mathbf{u}$ if

$$\sum_{i=1}^{d} u_i = 2 + 2 \sum_{j \in V_k} u_j. \tag{7}$$

Claim 7. For any $\mathbf{x}^{\mathbf{u}} \in \omega_R$ there exist at most two heavy components in $\mathbf{u}$. In particular, there is at least one non-heavy component in $\mathbf{u}$.

Proof. Indeed, if $k_1 < k_2 < k_3$ are heavy components in $\mathbf{u}$, then by adding the equations (7) for these indices we get

$$3 \sum_{i=1}^{d} u_i = 6 + \sum_{j \in V_{k_1} \cup V_{k_2} \cup V_{k_3}} 2u_j,$$

If $n = 3$, then $\sum_{i=1}^{d} u_i = 6$. Since $u_i \geqslant r_i \geqslant 2$ for all i, we infer that $r_1 = r_2 = r_3 = 2$, and $\mathbb{K}[G]$ is a Gorenstein ring (by Theorem 1), which is not the case.

If $n \geqslant 4$, then $\sum_{i=1}^{d} u_i < 6$. As $\sum_{i=1}^{d} u_i$ is even, we get that $n = 4$ and $r_1 = r_2 = r_3 = r_4 = 1$. Example 2 implies that R is a Gorenstein ring, which is false. □

Claim 8. For any $1 \leqslant i \leqslant d$ there exists a monomial $\mathbf{x}^{\mathbf{u}} \in \omega_R$ such that $u_i = 1$.

Proof. We fix i and we denote $a_i = \min\{u_i : \prod x_i^{u_i} \in \omega_R\}$. By (5), $a_i \geqslant 1$. Assume $a_i \geqslant 2$, and say $i \in V_k$.

If $r_k > 1$, we may pick $j \in V_k$, $j \neq i$. Then it is easy to check that the monomial $m = \frac{\mathbf{x}^{\mathbf{u}}}{x_i} x_j \in \omega_R$ and $\deg_{x_i}(m) = a_i - 1$, a contradiction.

When $r_k = 1$, then $n \geqslant 4$ and by the previous claim there is at least one non-heavy component V_{k_1} in $\mathbf{u}$ which is different from V_k. We pick $j \in V_{k_1}$ and since the monomial $m = \frac{\mathbf{x}^{\mathbf{u}}}{x_i} x_j \in \omega_R$ and $\deg_{x_i}(m) = a_i - 1$ we obtain a contradiction. □

It follows at once that

$$\gcd(\mathbf{x}^{\mathbf{u}} : \mathbf{x}^{\mathbf{u}} \in \omega_R) = \prod_{i=1}^{d} x_i,$$

where the greatest common divisor is computed in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_d]$.

Since ω_R is generated by monomials, one gets that ω_R^{-1} is also generated by monomials in $\mathbb{K}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$. If $f = \mathbf{x}^{\mathbf{u}}/\mathbf{x}^{\mathbf{v}} \in \omega_R^{-1}$ with $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$ coprime monomials in S, then $\mathbf{x}^{\mathbf{v}}$ divides the greatest common divisor of the monomials in ω_R. Hence, in order to determine a system of generators for the R-module ω_R^{-1} it is enough to scan among the (non-reduced) fractions $f = \mathbf{x}^{\mathbf{u}}/(x_1 \dots x_d)$, where $\mathbf{x}^{\mathbf{u}}$ is in the set

$$\mathcal{B} = \left\{ \mathbf{x}^{\mathbf{u}} \in S : \sum_{i=1}^{d} u_i \equiv 0 \bmod 2, \quad \mathbf{x}^{\mathbf{u}} \cdot \omega_R \subseteq x_1 \dots x_d R \right\}.$$

A monomial $\mathbf{x}^{\mathbf{u}}$ is in $\mathcal{B}$ if and only if $\sum_{i=1}^{d} u_i \equiv 0 \bmod 2$ and

$$x_1^{u_1+v_1-1} \cdots x_d^{u_d+v_d-1} \in R$$

for all $x_1^{v_1} \cdots x_d^{v_d}$ in ω_R. That is equivalent, via (2), (4), (3), to the fact that

$$\sum_{i=1}^{d} u_i \equiv d \bmod 2, \text{ and} \tag{8}$$

$$\sum_{i=1}^{d} u_i + \sum_{i=1}^{d} v_i \geqslant d - r_k + 2\sum_{j \in V_k} u_j + 2\sum_{j \in V_k} v_j, \tag{9}$$

$$\text{for } k = 1, \dots, d, \text{ and any } \mathbf{x}^{\mathbf{v}} \in \omega_R.$$

For $k = 1, \dots, n$ we set

$$E_k = \min\left\{ \sum_{i=1}^{d} v_i - 2\sum_{j \in V_k} v_k : \mathbf{x}^{\mathbf{v}} \in \omega_R \right\}.$$

Therefore, (9) is equivalent to

$$\sum_{i=1}^{d} u_i \geqslant d - r_k - E_k + 2\sum_{j \in V_k} u_j \text{ for } k = 1, \dots, n. \tag{10}$$

Before computing E_k we make a simple observation regarding d and the r_i's.

Claim 9. *$2r_i + 2 \leqslant d$ for all $i = 1, \dots, n-1$.*

Proof. Indeed, if that were not the case, then $2r_n + 2 \geqslant 2r_{n-1} + 2 > d$, hence $2r_n \geqslant 2r_{n-1} \geqslant d-1$. This implies $r_n + r_{n-1} \geqslant d-1$, equivalently that $1 = \sum_{i=1}^{n-2} r_i$, which is not possible in our setup. □

Next we show that E_k does not depend on k.

Claim 10. $E_k = 2$ for $k = 1, \dots, n$.

Proof. We fix $1 \leqslant k \leqslant n$. Clearly, $E_k \geqslant 2$, by (6). Then $E_k = 2$ once we find

$$\mathbf{x}^{\mathbf{v}} \in \omega_R \text{ such that } \sum_{i=1}^{d} v_i = 2 + 2\sum_{j \in V_k} v_j. \tag{11}$$

Using Eqs. (4), (5), (6), and translating $v_i = r_i + s_i$ for $i = 1, \dots, n$, we observe that finding $\mathbf{v}$ as in (11) is equivalent to finding integers $s_1, \dots, s_n$ such that

$$s_1, \dots, s_n \geqslant 0, \tag{12}$$

$$\sum_{i=1}^{n} s_i \geqslant 2s_\ell + 2r_\ell + 2 - d, \text{ for } 1 \leqslant \ell \leqslant n, \ell \neq k, \text{ and} \tag{13}$$

$$\sum_{i=1}^{n} s_i = 2s_k + 2 + 2r_k - d. \tag{14}$$

The s_ℓ represents the sum of the components of $\mathbf{v}$ from V_ℓ, each decreased by one. Note that (14) already implies that $\sum_{i=1}^{n} s_i \equiv d \bmod 2$.

We have two cases to consider.

Case $k = n$:

We let $s_\ell = \lfloor d/2 \rfloor - r_\ell - 1$ for $\ell = 1, \dots, n-1$. For (14) to hold, we must let

$$\begin{aligned} s_n &= \sum_{i=1}^{n-1} s_i - 2 - 2r_n + d = (n-1)\lfloor d/2 \rfloor - d + r_n - (n-1) - 2 - 2r_n + d \\ &= (n-1)(\lfloor d/2 \rfloor - 1) - r_n - 1 \geqslant 2(\lfloor d/2 \rfloor - 1) - r_n - 1 \geqslant d - r_n - 2 \geqslant 0. \end{aligned}$$

For $\ell < n$, one has $s_\ell \geqslant 0$ by the previous Claim. Also, $2s_\ell + 2 + 2r_\ell - d$ is either 0 or 1, depending on d being even or odd. Therefore, (13) and (12) are all verified.

Case $1 \leqslant k \leqslant n-1$:

We let $s_n = 0$ and $s_\ell = \lfloor d/2 \rfloor - r_\ell - 1$ for $\ell = 1, \dots, n-1$ where $\ell \neq k$. For (14) to hold, we must let

$$s_k = \Big(\sum_{1 \leqslant i \leqslant n-1, i \neq k} s_i\Big) + s_n - 2 - 2r_k + d. \tag{15}$$

Clearly, $s_k \geqslant 0$ since $d \geqslant 2r_k + 2$. Arguing as in the other case, for $k \neq \ell < n$ one has $s_\ell \geqslant 0$ and (13) holds. We are left to verify that

$$\sum_{i=1}^{n} s_i \geqslant 2s_n + 2r_n + 2 - d. \tag{16}$$

Substituting (14) into the left hand side term above, (16) is equivalent to

$$s_k + r_k \geqslant s_n + r_n.$$

Using (15) we get that

$$\begin{aligned} s_k + r_k &= \Big(\sum_{1 \leqslant i \leqslant n-1, i \neq k} s_i\Big) + s_n + d - r_k - 2 \\ &= \Big(\sum_{1 \leqslant i \leqslant n-1, i \neq k} s_i\Big) + s_n + r_n + (d - r_k - r_n - 2) \geqslant s_n + r_n, \end{aligned}$$

where for the latter inequality we used the observation that $d \geqslant r_k + r_n + 2$ in our setup. Consequently, $s_1, \dots, s_n$ fulfil (12), (13), (14), and $E_k = 2$. □

We can now finish the proof of Theorem 6.

Let $m = x_1^{a_1} \dots x_d^{a_d}$ be a monomial generator for ω_R. Then $\deg m = \sum_{i=1}^d a_i \geqslant 2 + 2\sum_{j \in V_k} a_j$ for all $k = 1, \dots, n$. In particular, $\deg m \geqslant 2r_n + 2$.

Let $f = \mathbf{x}^{\mathbf{u}}/(x_1 \cdots x_d)$ be a monomial in ω_R^{-1}, with $\mathbf{x}^{\mathbf{u}} \in \mathcal{B}$. By (10),

$$\deg \mathbf{x}^{\mathbf{u}} = \sum_{i=1}^d u_i \geqslant d - r_k - 2 + 2\sum_{j \in V_k} u_j \text{ for all } k = 1, \dots, n.$$

Since $d > r_n + 2$ in our setup, we find a component k_1 such that $\sum_{j \in V_{k_1}} u_j > 0$.

The product $m \cdot f$ is a monomial in R of degree at least

$$(2r_n + 2) + (d - r_{k_1} - 2 + 2\sum_{j \in V_{k_1}} u_j) - d \geqslant 2r_n - r_{k_1} + 2 \geqslant 3.$$

Consequently, $\operatorname{tr}(\omega_R) = \omega_R \cdot \omega_R^{-1} \subsetneq \mathfrak{m}_R$, a contradiction with (1). □

2 Stable set rings

In this section we consider an algebra generated by the monomials coming from the stable sets of a graph.

Let G be a finite simple graph on $[n]$ and $E(G)$ is the set of edges of G. A subset $C \subset [n]$ is a *clique* of G if $\{i, j\} \in E(G)$ for all $i, j \in C$ with $i \neq j$. A subset $W \subset [n]$ is *stable* in G if $\{i, j\} \notin E(G)$ for all $i, j \in W$ with $i \neq j$. In particular, the empty set as well as each $\{i\} \subset [n]$ is both a clique of G and a stable subset of G. Let $\Delta(G)$ denote the *clique complex* of G which is the simplicial complex on $[n]$ consisting of all cliques of G. Let δ denote the maximal cardinality of cliques of G. Thus $\dim \Delta(G) = \delta - 1$. We say that G is *pure* if $\Delta(G)$ is a pure simplicial complex, i.e. the cardinality of each maximal clique of G is δ. The *chromatic number* of a graph is the smallest number of colors that can be used for its vertices such that no adjacent vertices have the same color. The graph

G is called *perfect* if for all induced subgraphs H of G, including G itself, the chromatic number is equal to the maximal cardinality of cliques contained in H, see [1, p. 165].

Let $\mathbb{K}[x_1, \ldots, x_n, t]$ denote the polynomial ring in $n+1$ variables over the field $\mathbb{K}$. If, in general, $W \subset [n]$, then $x^W t$ stands for the squarefree monomial

$$x^W t = \big(\prod_{i \in W} x_i\big) \cdot t \in \mathbb{K}[x_1, \ldots, x_n, t].$$

Let $\mathrm{Stab}_{\mathbb{K}}(G)$ denote the subalgebra of $\mathbb{K}[x_1, \ldots, x_n]$ which is generated by those $x^W t$ for which W is a stable set of G. Letting $\deg(x^W t) = 1$ for any stable set W, the algebra $\mathrm{Stab}_{\mathbb{K}}(G)$ becomes standard graded. We call $\mathrm{Stab}_{\mathbb{K}}(G)$ the *stable set ring* of G.

It is known [15, Example 1.3 (c)] that $\mathrm{Stab}_{\mathbb{K}}(G)$ is normal if G is perfect. It follows that, when G is perfect, $\mathrm{Stab}_{\mathbb{K}}(G)$ is spanned over $\mathbb{K}$ by those monomials $(\prod_{i=1}^{n} x_i^{a_i}) t^q$ with $\sum_{i \in C} a_i \leqslant q$ for each maximal clique C of G. Furthermore, the canonical module $\omega_{\mathrm{Stab}_{\mathbb{K}}(G)}$ of $\mathrm{Stab}_{\mathbb{K}}(G)$ is spanned over $\mathbb{K}$ by those monomials $(\prod_{i=1}^{n} x_i^{a_i}) t^q$ with each $a_i > 0$ and with $\sum_{i \in C} a_i < q$ for each maximal clique C of G. Thus [16, Theorem 2.1 (b)] implies that $\mathrm{Stab}_{\mathbb{K}}(G)$ is Gorenstein if and only if G is pure.

The following lemma captures a sufficient combinatorial condition for $\mathrm{Stab}_{\mathbb{K}}(G)$ to be nearly Gorenstein.

Lemma 11. *Let G be a finite simple perfect graph such that $\mathrm{Stab}_{\mathbb{K}}(G)$ is nearly Gorenstein. Then every connected component of G is pure.*

Proof. Assume $V(G) = [n]$. Denote $R = \mathrm{Stab}_{\mathbb{K}}(G)$. Since each $x_i t$ as well as t belongs to R, the quotient field of R is the rational function field $\mathbb{K}(x_1, \ldots, x_n, t)$ over $\mathbb{K}$.

Suppose G_1 is a connected component of G which is not pure. Let δ and δ_1 denote the maximal cardinality of cliques of G and of G_1, respectively. Then there is an edge $\{i_0, j_0\} \in E(G_1)$ for which i_0 belongs to a clique C of G with $|C| = \delta_1$ and for which j_0 belongs to *no* clique C of G with $|C| = \delta_1$.

Let $z = \prod_{i=1}^{n} x_i^{a'_i} t^{q'} \in \omega_R^{-1}$. Set $v_1 = x_1 \cdots x_n t^{\delta+1}$. It is easy to check that $v_1 \in \omega_R$ and that each monomial belonging to ω_R is divisible (in $\mathbb{K}[x_1, \ldots, x_n, t]$) by v_1. Hence $a_i \geqslant -1$ for all i. Clearly, $x_{j_0} v_1 \in \omega_R$ and $1 \neq x_{j_0} v_1 z \in R$, hence $q' \geqslant -\delta$.

Since G is not pure, R is not a Gorenstein ring and thus

$$\mathrm{tr}(\omega_R) = \omega_R \cdot \omega_R^{-1} = \mathfrak{m}_R.$$

Let $w' = \prod_{i=1}^{n} x_i^{a'_i} t^{q'} \in \omega_R^{-1}$ and $w = \prod_{i=1}^{n} x_i^{a_i} t^q \in \omega_R$ with $w'w = x_{i_0} t$. Since $q' \geqslant -\delta$ and $q \geqslant \delta + 1$, one has $q' = -\delta$ and $q = \delta + 1$.

Let $v = x_1 x_2 \cdots x_n t^{\delta+1} \cdot x_{i_0}^{\delta - \delta_1}$. One has $v \in \omega_R$ and $x_{j_0} v \in \omega_R$. We claim that $w' \cdot x_{j_0} v \in \mathfrak{m}_R$ is divisible by $x_{i_0} x_{j_0} t$, but it is not divisible by t^2. This is clear when $\delta > \delta_1$. In case $\delta = \delta_1$, since i_0 belongs to a clique C of G with $|C| = \delta$, one has $a_{i_0} = 1$. Thus $a'_{i_0} = 0$ and the claim is verified.

Thus $w' \cdot x_{j_0} v$ must be of the form $x^W t$, where W is a stable set of G, which contradicts $\{i_0, j_0\} \in E(G)$. Hence $\mathfrak{m}_R \subsetneq \mathrm{tr}(\omega_R)$, as desired. □

Recall that the a-invariant of any graded algebra R with canonical module ω_R is defined as $a(R) = -\min\{i : (\omega_R)_i \neq 0\}$.

Corollary 12. *If G is a perfect graph then $a(\mathrm{Stab}_{\mathbb{K}}(G)) = -\dim \Delta(G) - 2$.*

Proof. Let δ be the maximal size of a clique in G. From the proof of the Lemma 11, $v = x_1 \cdots x_n t^{\delta+1}$ is in $(\omega_{\mathrm{Stab}_{\mathbb{K}}(G)})_{\delta+1}$ and v divides every monomial in $\omega_{\mathrm{Stab}_{\mathbb{K}}(G)}$. This gives the conclusion. □

Theorem 13. *Let G be a finite simple graph with $G_1, \ldots, G_s$ its connected components and suppose that G is perfect. Let δ_i denote the maximal cardinality of cliques of G_i. Then $\mathrm{Stab}_{\mathbb{K}}(G)$ is nearly Gorenstein if and only if each G_i is pure and $|\delta_i - \delta_j| \leqslant 1$ for $1 \leqslant i < j \leqslant s$.*

Proof. Suppose that $\mathrm{Stab}_{\mathbb{K}}(G)$ is nearly Gorenstein. It follows from Lemma 11 that each G_i is pure and each $\mathrm{Stab}_{\mathbb{K}}(G_i)$ is Gorenstein. Since $\mathrm{Stab}_{\mathbb{K}}(G)$ is the Segre product of $\mathrm{Stab}_{\mathbb{K}}(G_1), \ldots, \mathrm{Stab}_{\mathbb{K}}(G_s)$, it follows from [7, Corollary 4.16] and [8, Corollary 2.8] that

$$|a(\mathrm{Stab}_{\mathbb{K}}(G_i)) - a(\mathrm{Stab}_{\mathbb{K}}(G_j))| \leqslant 1 \text{ for all } i, j.$$

Corollary 12 yields $|\delta_i - \delta_j| \leqslant 1$ for $1 \leqslant i < j \leqslant s$. Furthermore, the "If" part also follows from [7, Corollary 4.16] and [8, Corollary 2.8]. □

Corollary 14. *Let G be a finite simple graph which is pefect and connected. Then the ring $\mathrm{Stab}_{\mathbb{K}}(G)$ is nearly Gorenstein if and only if it is Gorenstein.*

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