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NEARLY OVERCONVERGENT SIEGEL MODULAR FORMS

by Zheng LIU

Abstract. — We introduce a sheaf-theoretic formulation of Shimura's theory of nearly holomorphic Siegel modular forms and differential operators. We use it to define and study nearly overconvergent Siegel modular forms and their p-adic families.

Résumé. — Nous introduisons une formulation faisceau-théorique de la théorie de Shimura des formes modulaires de Siegel quasi holomorphes et des opérateurs différentiels. Nous l'utilisons pour définir et étudier les formes modulaires de Siegel quasi surconvergentes et leurs familles p-adiques

1. Introduction

Shimura developed his theory of nearly holomorphic forms in his study on the algebraicity of special L-values and Klingen Eisenstein series [42, 45]. With the goal of combining this useful tool with Hida and Coleman–Mazur theories for p-adic families of modular forms to study special L-values and Selmer groups by using p-adic congruences and deformations, Urban [49] introduced a sheaf-theoretic formulation of Shimura's theory in the $\mathrm{GL}(2)_{/\mathbb{Q}}$ case. Such a formulation enables him to define and study some basic properties of nearly overconvergent modular forms.

In this article we generalize Urban's work to Siegel modular forms. In the construction of automorphic sheaves over Siegel varieties equipped with integrable connections, we take a different approach from [49] by using a canonical $\mathbf{Q}$-torsor over the Siegel variety and $(\mathfrak{g}, \mathbf{Q})$-modules. Here $\mathfrak{g}$ is the Lie algebra of the algebraic group $\mathbf{G} = \mathrm{GSp}(2n)_{/\mathbb{Z}}$ and $\mathbf{Q}$ is the standard Siegel parabolic subgroup of $\mathbf{G}$. Compared to $\mathbf{G}$-representations,

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$(\mathfrak{g}, \mathbf{Q})$-modules are more adaptive for p-adic deformations. Combining the ideas and techniques in [2] with our sheaf-theoretic formulation of nearly holomorphic Siegel modular forms and differential operators, we introduce the space of nearly overconvergent Siegel modular forms and their p-adic families.

One of the main motivations for considering differential operators, nearly holomorphic forms and their p-adic theory is for arithmetic applications of various integral representations of L-functions or L-values: the algebraicity results on special L-values and Klingen Eisenstein series by the doubling method [20, 21, 44, 45], the construction of p-adic L-functions by evaluating Eisenstein series at CM points [32], by Rankin–Selberg method [24] and by doubling method [14, 15, 37], and the study of p-adic regulators of Heegner cycles by the Waldspurger formula [4], just to name a few.

As we know, the algebraicity of an automorphic representation is mainly related with its archimedean component. When utilizing integral representations to study special L-values, differential operators and nearly holomophic forms naturally show up in the analysis of archimedean zeta integrals. Over the field of complex numbers, roughly speaking, cuspidal nearly holomorphic forms are automorphic forms inside cuspidal automorphic representations whose archimedean components are isomorphic to holomorphic discrete series. The holomorphic forms are those whose archimedean components belong to the lowest K_∞-types of the holomorphic discrete series. The Maass–Shimura differential operators correspond to the action of the Lie algebra on the archimedean components. The theory of nearly holomorphic forms and differential operators aims to introduce nice algebraic or even integral structure to the complex vector space of nearly holomorphic forms and to the action of the Lie algebra. It also provides explicit formulas that help the computation of Fourier coefficients and archimedean zeta integrals. Besides Shimura, the differential operators and nearly holomorphic forms have also been studied in [6, 19, 27, 40, 41] through different approaches.

In Shiumra's theory of nearly holomorphic Siegel modular forms, there are three main ingredients. Let $\mathfrak{h}_n$ be the genus n Siegel upper half space, $\Gamma \subset \mathrm{Sp}(2n, \mathbb{Z})$ be a congruence subgroup, and (ρ, W_ρ) be an algebraic $\mathrm{GL}(n)$-representation of finite rank. Shimura defined

(1) the space $N^r_\rho(\mathfrak{h}_n, \Gamma)$ of $W_\rho(\mathbb{C})$-valued nearly holomorphic forms on $\mathfrak{h}_n$ of level Γ and (non-holomorphy) degree r, together with its algebraic structure defined by using CM points,

(2) the Maass–Shimura differential operator

$$D_{\mathfrak{h}_n,\rho} : N^r_\rho(\mathfrak{h}_n,\Gamma) \to N^{r+1}_{\rho\otimes\tau}(\mathfrak{h}_n,\Gamma),$$

where τ is the symmetric square of the standard representation of $\mathrm{GL}(n)$,

(3) a holomorphic projection $N^r_\kappa(\mathfrak{h}_n,\Gamma) \to N^0_\kappa(\mathfrak{h}_n,\Gamma)$ for a generic weight κ.

Both the differential operators and the holomorphic projection preserve the algebraic structure in (1), and they play important roles in choosing desirable archimedean sections in arithmetic applications of various integral representations of L-functions and L-values.

This paper consists of two parts. In the first part, we construct the automorphic quasi-coherent sheaf $\mathcal{V}_\rho$ over a smooth toroidal compactification X of the Siegel modular variety Y of level Γ defined over $\mathbb{Z}[1/N]$ for some positive integer N. This automorphic sheaf $\mathcal{V}_\rho$ has an increasing filtration $\mathcal{V}^r_\rho$ and we construct a connection

$$\mathcal{V}^r_\rho \longrightarrow \mathcal{V}^{r+1}_\rho \otimes_{\mathcal{O}_X} \Omega^1_X(\log C), \quad C = X - Y. \tag{1.1}$$

Composing this connection with the Kodaira–Spencer isomorphism, we get the differential operator $D_\rho : \mathcal{V}^r_\rho \to \mathcal{V}^{r+1}_{\rho\otimes\tau}$. We show in Section 2.5 that $\mathcal{V}^r_\rho$ together with D_ρ recovers the first two ingredients in Shimura's theory, and there is the commutative diagram

$$\begin{tikzcd} H^0(X^\circ_{\mathbb{C}},\mathcal{V}^r_\rho) \arrow[r, "\sim"] \arrow[d, "D_\rho"] & N^r_\rho(\mathfrak{h}_n,\Gamma) \arrow[r, hook] \arrow[d, "D_{\mathfrak{h}_n,\rho}"] & C^\infty(\Gamma\backslash \mathbf{G}^\circ(\mathbb{R})) \arrow[d, "\mathfrak{q}^+\text{-action}"] \\ H^0(X^\circ_{\mathbb{C}},\mathcal{V}^{r+1}_{\rho\otimes\tau}) \arrow[r, "\sim"] & N^{r+1}_{\rho\otimes\tau}(\mathfrak{h}_n,\Gamma) \arrow[r, hook] & C^\infty(\Gamma\backslash \mathbf{G}^\circ(\mathbb{R})), \end{tikzcd} \tag{1.2}$$

where $X^\circ_{\mathbb{C}}$ is a connected component of the base change of X to $\mathbb{C}$, $\mathbf{G}^\circ = \mathrm{Sp}(2n)$, and $\mathfrak{q}^+ = \left(\begin{smallmatrix} I_n & iI_n \\ iI_n & I_n \end{smallmatrix}\right)(\mathrm{Lie}\,\mathbf{Q})_{\mathbb{C}}\left(\begin{smallmatrix} I_n & iI_n \\ iI_n & I_n \end{smallmatrix}\right)^{-1}$.

Automorphic sheaves are defined over X using algebraic $\mathbf{Q}$-representations free of finite rank and the canonical $\mathbf{Q}$-torsor $T^\times_{\mathcal{H}} = \underline{\mathrm{Isom}}_X(\mathcal{O}^{2n}_X, \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}})$, where $\mathcal{A} \to Y$ is the principally polarized universal abelian scheme, and the isomorphisms are required to respect the Hodge filtration and preserve the symplectic pairing of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ up to similitude. Given an algebraic $\mathbf{Q}$-representation V, the associated automorphic sheaf is defined as the contracted product $\mathcal{V} = T^\times_{\mathcal{H}} \times^{\mathbf{Q}} V$.

If one wants to consider automorphic sheaves further equipped with integrable connections which induce Hecke equivariant maps on global sections, we show in Section 2.2 that the right objects to consider are $(\mathfrak{g},\mathbf{Q})$-modules.

It is the $\mathfrak{g}$-module structure combined with the Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ that gives rise to the desired connection. Then in order to construct the sheaves of nearly holomorphic Siegel modular forms with differential operators, it remains to select suitable $(\mathfrak{g}, \mathbf{Q})$-modules. In Section 2.3 we define, for each algebraic $\mathrm{GL}(n)$-representation ρ free of finite rank, a $(\mathfrak{g}, \mathbf{Q})$-module V_ρ. As a $\mathbf{Q}$-module, V_ρ has an increasing filtration $V_\rho^r, r \geqslant 0$ such that $\mathfrak{g}\cdot V_\rho^r \subset V_\rho^{r+1}$. We define the sheaf of nearly holomorphic forms of weight ρ and (non-holomorphy) degree r as $\mathcal{V}_\rho^r = T_{\mathcal{H}}^\times \times^{\mathbf{Q}} V_\rho^r$. The general construction in Section 2.2 equips $\mathcal{V}_\rho$ with the connection (1.1). The construction of holomorphic projections is postponed to Section 3.7 where it is done in the more general setting of nearly overconvergent families.

In the second part, combining the ideas and techniques in [2] with our construction in the first part, we define and study some basic properties of the space of nearly overconvergent forms and p-adic families of nearly overconvergent forms. When replacing dominant algebraic weights by general p-adic anaylitic weights, it is convenient to construct the corresponding representations of the Lie algebra, which can be viewed as a p-adic deformation of the Lie algebra representations attached to dominant algebraic weights. However, these Lie algebra representations do not integrate to representations of the algebraic group, but only integrate to certain p-adic analytic representations of some rigid analytic subgroup of the rigid analytification of the algebraic group. In order to construct sheaves of p-adic automorphic forms with p-adic analytic weights, one natural approach is to modify the torsor of the algebraic group to a p-adic analytic torsor of its rigid analytic subgroup, and to form the contracted product of the p-adic analytic torsor with the representation of the rigid analytic subgroup.

In [2], for $v, w \geqslant 0$ within a certain range, over the strict neighborhood $\mathcal{X}_{\mathrm{Iw}}(v)$ of the ordinary locus of the compactifed Iwahori-level Siegel variety X_{Iw}, an Iwahori-like space $\mathcal{T}_{\mathcal{F},w}^\times(v)$ inside the $\mathrm{GL}(n)_{\mathrm{an}}$-torsor $T_{\omega,\mathrm{an}}^\times = \underline{\mathrm{Isom}}_X\left(\mathcal{O}_X^n, \omega(\mathcal{A}/Y)^{\mathrm{can}}\right)_{\mathrm{an}}$ is constructed by using canonical subgroups. Here the subscript "an" means the rigid analytification. This $\mathcal{T}_{\mathcal{F},w}^\times(v)$ can be viewed as a torsor of a rigid analytic subgroup $\mathcal{I}_w \subset \mathrm{GL}(n)_{\mathrm{rig}}$, the rigid analytic fibre of the completion of $\mathrm{GL}(n)$ along its special fibre. For a w-analytic weight $\kappa \in \mathrm{Hom}_{\mathrm{cont}}\left((\mathbb{Z}_p^\times)^n, \mathbb{C}_p^\times\right)$, there corresponds a natural representation $W_{\kappa,w}$ of $\mathrm{Lie}(\mathrm{GL}(n))$ which integrates to a representation of $\mathcal{I}_w$. The Banach sheaf $\omega_{\kappa,w}^\dagger$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ of overconvergent modular forms of the w-analytic weight κ is obtained as the contracted product of $\mathcal{T}_{\mathcal{F},w}^\times(v)$ and $W_{\kappa,w}$.

Taking ρ to be the trivial representation and $r = 1$, the construction in Section 2 gives an automorphic coherent sheaf $\mathcal{J} = \mathcal{V}^1_{\mathrm{triv}}$. A quick way to define the Banach sheaf of degree r nearly overconvergent forms of the w-analytic weight κ is to set $\mathcal{V}^{\dagger,r}_{\kappa,w} := \omega^\dagger_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}$ (this is similar to the way of defining $\mathcal{H}^r_k$, $\mathcal{H}^r_{\mathfrak{U}}$ in [49]). For the convenience of defining differential operators and holomorphic projections as in Section 3.6, 3.7, we need a contracted product interpretation for $\mathcal{V}^{\dagger,r}_{\kappa,w}$. Associated to the p-adic analytic weight κ, generalizing the previous V_ρ, there is a natural $\mathfrak{g}$-module $V_{\kappa,w}$ which integrates to a $(\mathfrak{g}, \mathcal{Q}_w)$-module, where $\mathcal{Q}_w \subset \mathbf{Q}_{\mathrm{an}}$ is the rigid analytic group defined as the preimage of $\mathcal{I}_w$ of the projection $\mathbf{Q}_{\mathrm{an}} \to \mathrm{GL}(n)_{\mathrm{an}}$. We define the $\mathcal{Q}_w$-torsor $\mathcal{T}^\times_{\mathcal{H},w}(v)$ as the subspace of $T^\times_{\mathcal{H},\mathrm{an}}$ whose image under the projection $T^\times_{\mathcal{H},\mathrm{an}} \to T^\times_{\omega,\mathrm{an}}$ lies inside $\mathcal{T}^\times_{\omega,w}(v)$. Then $\mathcal{T}^\times_{\mathcal{H},w}(v)$ together with $V_{\kappa,w}$ gives the desired contracted product interpretation for the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$.

Now let $\mathcal{U}$ be an affinoid subdomain of the weight space whose $\mathbb{C}_p$-points are all w-analytic. The construction above works for the universal weight as well and produces the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa^{un},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}$. In Section 3.5 we show that the $\mathcal{A}(\mathcal{U})$-Banach module $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{un},w}(-C))$ is projective. Section 3.9 is devoted to defining the $\mathbb{U}_p$-operators and showing the compactness of the operator $U_p = \mathrm{res} \circ U_{p,n} \circ \cdots \circ U_{p,1}$ acting on $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$. Then the Coleman–Riesz–Serre spectral theory is applied to give the slope decomposition of $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} := \bigcup_{r \geqslant 0} N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ in Section 3.11.

The p-adic theory of nearly holomorphic forms and differential operators has also been considered in [12, 13] (unitary case) and [28] (simplectic case). They define nearly holomorphic forms as global sections of $\left(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}\right)^{\otimes m}$ for some positive integer m, and the differential operators are then the connections induced from the Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$. In order to consider p-adic deformations, their method relies on unit root splitting of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ over the ordinary locus and the q-expansion or Serre–Tate expansion principle, and does not extend to nearly overconvergent forms. We believe that our method here works also for Shimura varieties for unitary groups. In [23], a construction of the Gauss–Manin connections for nearly overconvergent forms is given in the $\mathrm{GL}(2)_{/\mathbb{Q}}$ case, where they consider the action of $\mathrm{GL}(1)$ (the Levi subgroup of the Siegel parabolic of $\mathrm{GL}(2)$) instead of that of $\mathrm{Lie}(\mathrm{GL}(2))$. Note that besides constructing differential operators acting on nearly overconvergent forms of general p-adic analytic weight, there is another problem of taking the differential operator to a p-adic analytic power. This is easy for p-adic forms over the ordinary locus by using the q-expansion principle, but for nearly overconvergent forms there seems no obvious approach. Recently this problem has been addressed for families of nearly overconvergent modular forms in [1]. It is expected that some ideas there extend to our case of nearly overconvergent Siegel modular forms.

Notation. — Let $\mathbf{G}$ be the rank n symplectic similitude group

$$\mathrm{GSp}(2n)_{/\mathbb{Z}} = \left\{g \in \mathrm{GL}(2n)_{/\mathbb{Z}} : {}^{\mathrm{t}}g \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) g = \nu(g) \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)\right\}$$

with the multiplier character $\nu : \mathbf{G} \to \mathbb{G}_m$. Denote by $\mathbf{Q}$ the standard Siegel parabolic subgroup of $\mathbf{G}$ consisting of matrices whose lower left $n \times n$ block is 0 and by $\mathbf{T}$ the maximal torus consisting of diagonal matrices. Write $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{U}$ with $\mathbf{M}$ and $\mathbf{U}$ as its Levi subgroup and unipotent radical. Fix the embedding $\mathrm{GL}(n) \hookrightarrow \mathbf{M}$ sending $a \in \mathrm{GL}(n)$ to $\left(\begin{smallmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1} \end{smallmatrix}\right)$. Let $\mathbf{G}^\circ = \mathrm{Sp}(2n)_{/\mathbb{Z}}$ be the kernel of the multiplier character ν with maximal torus $\mathbf{T}^\circ \cong \mathbb{G}_m^n$ and standard Siegel parabolic subgroup $\mathbf{Q}^\circ = \mathbf{M}^\circ \ltimes \mathbf{U}$. The embedding $\mathrm{GL}(n) \hookrightarrow \mathbf{M}$ gives an isomorphism of $\mathrm{GL}(n)$ onto $\mathbf{M}^\circ$. The maximal torus $\mathbf{T}^\circ$ of $\mathrm{Sp}(2n)$ can also be regarded as a maximal torus of $\mathbf{M}^\circ \cong \mathrm{GL}(n)$. We use $\mathbf{B}$ to denote the Borel subgroup of $\mathbf{M}^\circ$ consisting of upper triangular matrices and $\mathbf{N}$ to denote the unipotent radical of $\mathbf{B}$. For an algebra E, let $\mathrm{Rep}_E\,\mathbf{Q}$ (resp. $\mathrm{Rep}_{E,f}\,\mathrm{GL}(n)$) stand for the category of algebraic representations of the group $\mathbf{Q}$ (resp. $\mathrm{GL}(n)$) base changed to E on locally free E-modules (resp. locally free E-modules of finite rank). The projection $\mathbf{Q} \to \mathrm{GL}(n)$ mapping $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathbf{Q}$ to $a \in \mathrm{GL}(n)$ defines a functor $\mathrm{Rep}_{E,f}\,\mathrm{GL}(n) \to \mathrm{Rep}_E\,\mathbf{Q}$, through which we regard every object in $\mathrm{Rep}_{E,f}\,\mathrm{GL}(n)$ also as a $\mathbf{Q}$-representation. The congruence subgroup $\{\gamma \in \mathbf{G}^\circ(\mathbb{Z}) : \gamma \equiv I_{2n} \mod N\}$ of $\mathbf{G}^\circ(\mathbb{Z})$ is denoted by $\Gamma(N)$.

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2. Nearly holomorphic forms

2.1. Automorphic sheaves over Siegel varieties

Let $N \geqslant 3$ and $Y = Y_{\mathbf{G},\Gamma(N)}$ be the Siegel variety parametrizing principally polarized abelian schemes of relative dimension n with principal level N structure defined over $\mathbb{Z}[1/N]$. Over it there is the universal abelian scheme $\mathbf{p} : \mathcal{A} \to Y$. Take a smooth toroidal compactification X of Y with boundary $C = X - Y$. Then $\mathbf{p} : \mathcal{A} \to Y$ extends to a semi-abelian scheme $\mathbf{p} : \mathcal{G} \to X$. Let $\omega(\mathcal{G}/X)$ be the pullback of $\Omega^1_{\mathcal{G}/X}$ along the zero section of $\mathbf{p}$. According to [35, Proposition 6.9], the locally free sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) = R^1\mathbf{p}_*(\Omega^\bullet_{\mathcal{A}/Y})$ has a canonical extension $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \cong \mathcal{H}^1_{\log\text{-dR}}(\mathcal{G}/X)$ which is a locally free subsheaf of $(Y \to X)_*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)$. This canonical extension $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ is endowed with a symplectic pairing under which $\omega(\mathcal{G}/X)$ is maximally isotropic. The Hodge filtration of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)$ also extends to

$$0 \longrightarrow \omega(\mathcal{G}/X) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \longrightarrow \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X) \longrightarrow 0$$

where ${}^{\mathrm{t}}\mathcal{G}/X$ is the dual semi-abelian scheme of $\mathcal{G}/X$.

There is a standard way to construct, from a representation in $\mathrm{Rep}_{\mathbb{Z}}\,\mathbf{Q}$, a quasi-coherent sheaf over X whose global sections are equipped with Hecke actions. Equip the free sheaf $\mathcal{O}_X^{2n}$ with a two-step filtration with the first n copies as the subsheaf, and a symplectic pairing using the matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Define the right $\mathbf{Q}$-torsor over X

$$T_{\mathcal{H}}^{\times} = \underline{\mathrm{Isom}}_X(\mathcal{O}_X^{2n}, \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}})$$

to be the isomorphisms respecting the filtration and the symplectic pairing up to similitude. The right $\mathbf{Q}$-action is given as

$$(b \cdot \phi)(v) = (\phi \circ b)(v) = \phi(bv)$$

for any open subscheme $U = \mathrm{Spec}(R) \subset X$, $\phi \in T_{\mathcal{H}}^{\times}(U)$, $v \in R^{2n}$ and $b \in \mathbf{Q}(R)$.

With this right $\mathbf{Q}$-torsor, by forming contracted product, one can define the functor

$$\begin{aligned} \mathcal{E} : \mathrm{Rep}_{\mathbb{Z}}\,\mathbf{Q} &\longrightarrow \mathrm{QCoh}(X) \\ V &\longmapsto T_{\mathcal{H}}^{\times} \times^{\mathbf{Q}} V \end{aligned}$$

from the category of algebraic representations of $\mathbf{Q}$ on locally free $\mathbb{Z}$-modules to that of quasi-coherent sheaves over X. Let us give a more detailed description of $\mathcal{E}(V)$ in local affine charts. Let $U = \mathrm{Spec}(R)$ be an affine open subscheme of X such that $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}(U)$ is free over R. We identify elements in $T_{\mathcal{H}}^{\times}(U)$ with ordered basis $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}(U)$, which gives rise to isomorphisms between R^{2n} and $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}(U)$ preserving the Hodge filtration and the symplectic paring up to similitude. Then $\mathcal{E}(V)(U)$ is the set of maps $v : T_{\mathcal{H}}^{\times}(U) \to V \otimes R$ such that $v(\alpha g) = g^{-1} \cdot v(\alpha)$ for all $g \in \mathbf{Q}(R)$ and $\alpha \in T_{\mathcal{H}}^{\times}(U)$.

Moreover, for all $V \in \mathrm{Rep}_{\mathbb{Z}} \mathbf{Q}$, the space of global sections of the associated quasi-coherent sheaf $\mathcal{E}(V)$ comes with a Hecke action constructed via algebraic correspondence (cf. [16, Section VII.3]). Such an $\mathcal{E}(V)$ together with the Hecke action on its global sections is often called an automorphic sheaf. Morphisms between algebraic $\mathbf{Q}$-representations induce Hecke equivariant morphisms between global sections of the corresponding automorphic sheaves. The functor $\mathcal{E}$ is exact and faithful [35, Definition 6.13]. Certainly this functor is not fully faithful (see Example 2.13). Let V_{st} be the standard representation of $\mathbf{G}$ restricted to $\mathbf{Q}$ and W_{st} be the standard representation of $\mathrm{GL}(n)$ regarded as a $\mathbf{Q}$-representation. Then immediately from the definition we see that $\mathcal{E}(V_{\mathrm{st}}) \cong \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ and $\mathcal{E}(W_{\mathrm{st}}) \cong \omega(\mathcal{G}/X)$.

The multiplier character $\nu : \mathbf{G} \to \mathbb{G}_m$ can be seen as an algebraic representation of $\mathbf{Q}$ and we denote its corresponding invertible sheaf over X by $\mathcal{E}(\nu)$. As an invertible sheaf, $\mathcal{E}(\nu)$ is isomorphic to the structure sheaf $\mathcal{O}_X$. However the Hecke action differs by a Tate twist. For $V \in \mathrm{Rep}_{\mathbb{Z}} \mathbf{Q}$ we define $\mathcal{E}(V)(i)$ to be $\mathcal{E}(V \otimes \nu^i) = \mathcal{E}(V) \otimes \mathcal{E}(\nu)^i$.

Remark 2.1. — The Hecke actions are only defined on global sections not on the quasi-coherent sheaves. However, in the following we say, for simplicity, a quasi-coherent sheaf with Hecke actions to mean that Hecke operators act on its global sections, and a Hecke equivariant morphism between quasi-coherent sheaves to mean that the induced map on global sections is Hecke equivariant. Also by a morphism between two automorphic sheaves we mean a Hecke equivariant morphism unless otherwise stated.

2.2. $(\mathfrak{g}, \mathbf{Q})$-modules and Gauss–Manin connection

Let $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}$, $\mathfrak{q} = \mathrm{Lie}\,\mathbf{Q}$ be the Lie algebras of $\mathbf{G}$ and its Siegel parobolic $\mathbf{Q}$.

Definition 2.2. — *Let E be an algebra. A $(\mathfrak{g}, \mathbf{Q})$-module V over E is an algebraic representation of $\mathbf{Q}$ and $\mathfrak{g}$ base changed to E on locally free E-modules, such that the action of $\mathfrak{q} \subset \mathfrak{g}$ on V is the one induced from that of $\mathbf{Q}$ and for any $g \in \mathbf{Q}$, $X \in \mathfrak{g}$, $v \in V$,*

$$g \cdot X \cdot g^{-1} \cdot v = (\mathrm{Ad}(g)X) \cdot v.$$

We denote the category of $(\mathfrak{g}, \mathbf{Q})$-modules over E by $\mathrm{Rep}_E(\mathfrak{g}, \mathbf{Q})$.

It is mentioned on [16, p. 223] that $\mathbf{G}(\mathbb{C})$-equivariant quasi-coherent $\mathcal{D}$-modules over the compact dual $\mathbb{D}^\vee = \mathbf{G}(\mathbb{C})/\mathbf{Q}(\mathbb{C})$ correspond to $(\mathfrak{g}, \mathbf{Q})$-modules. We show below that for an object $V \in \mathrm{Rep}_{\mathbb{Z}}(\mathfrak{g}, \mathbf{Q})$, using the

$\mathfrak{g}$-module structure on V, we can equip its associated automorphic sheaf $\mathcal{E}(V)$ with an integrable connection.

For the locally free sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) = R^1\mathbf{p}_*(\Omega^\bullet_{\mathcal{A}/Y})$ over Y, there is a canonical integrable connection called the Gauss–Manin connection [33]. We record the following result on the extension of the Gauss–Manin connection to toroidal compactification.

Theorem 2.3 ([35, Proposition 6.9]). — *The Gauss–Manin connection $\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) \otimes \Omega^1_Y$ extends to an integrable connection with log poles along the boundary*

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \otimes \Omega^1_X(\log C),$$

which satisfies Griffith transversality and compatible with the symplectic pairing on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$.

Let $U = \mathrm{Spec}(R)$ and basis α be as in our description of the contracted product defining $\mathcal{E}(V)$ in Section 2.1. Given $D \in T_X(U) = \mathrm{Der}_{\mathbb{Z}[1/N]}(R,R)$, a section of the tangent bundle of X over U, by Theorem 2.3 there exists $X(D,\alpha) \in M_{n,n}(R) \cong \mathfrak{g}(R)$ (in fact $\mathfrak{g}(\mathrm{Frac}(R))$ with logarithm poles along the boundary if U intersects with the boundary) such that

$$\nabla(D)(\alpha) = \alpha\, X(D,\alpha). \tag{2.1}$$

For $v \in \mathcal{E}(V)(U)$ we define the operator $\nabla_{\mathcal{E}(V)}(D)$ acting on it as

$$(\nabla_{\mathcal{E}(V)}(D)(v))(\alpha) := D\,v(\alpha) + X(D,\alpha)\cdot v(\alpha). \tag{2.2}$$

Here D acts on $v(\alpha) \in V \otimes R$ through the action of $\mathrm{Der}_{\mathbb{Z}[1/N]}(R,R)$ on R, i.e. by coefficients. The action of $X(D,\alpha)$ on $v(\alpha)$ is the action of the Lie algebra $\mathfrak{g}$ on V.

Proposition 2.4. — *The above defined $\nabla_{\mathcal{E}(V)}(D)(v)$ belongs to $\mathcal{E}(V)(U)$ and the formula* (2.2) *on local sections patches together to an integrable connection with log poles along the boundary*

$$\nabla_{\mathcal{E}(V)} : \mathcal{E}(V) \longrightarrow \mathcal{E}(V) \otimes \Omega^1_X(\log C).$$

Proof. — What we need to show is that for any $g \in \mathbf{Q}(R)$

$$(\nabla_{\mathcal{E}(V)}(D)(v))(\alpha g) = g^{-1}\cdot(\nabla_{\mathcal{E}(V)}(D)(v))(\alpha). \tag{2.3}$$

The Gauss–Manin connection ∇ satisfies that

$$\begin{aligned}\nabla(D)(\alpha g) &= \nabla(D)(\alpha)\cdot g + \alpha\, Dg \\ &= (\alpha g)(g^{-1}X(D,\alpha)g + g^{-1}Dg) \\ &= (\alpha g)(\mathrm{Ad}(g^{-1})X(D,\alpha) + g^{-1}Dg)\end{aligned}$$

i.e.

$$X(D,\alpha g)=\mathrm{Ad}(g^{-1})X(D,\alpha)+g^{-1}Dg.$$

We compute the left hand side of (2.3) by definition,

$$\begin{aligned}\mathrm{LHS}&=D\,v(\alpha g)+X(D,\alpha g)\cdot v(\alpha g)\\&=D(g^{-1}\cdot v(\alpha))+(\mathrm{Ad}(g^{-1})X(D,\alpha)+g^{-1}Dg)\cdot v(\alpha g)\\&=((Dg^{-1})g)\cdot(g^{-1}\cdot v(\alpha))+g^{-1}\cdot(D\,v(\alpha))\\&\qquad+(\mathrm{Ad}(g^{-1})X(D,\alpha)+g^{-1}Dg)\cdot(g^{-1}\cdot v(\alpha))\\&=-(g^{-1}Dg)\cdot(g^{-1}\cdot v(\alpha))+g^{-1}\cdot(D\,v(\alpha))\\&\qquad+(g^{-1}\cdot X(D,\alpha)\cdot g)\cdot(g^{-1}\cdot v(\alpha))+(g^{-1}Dg)\cdot(g^{-1}\cdot v(\alpha))\\&=g^{-1}\cdot(D\,v(\alpha)+X(D,\alpha)\cdot v(\alpha)),\end{aligned}$$

which equals to the right hand side. The compatibility of the action of $\mathfrak{g}$ and $\mathbf{Q}$ is used for the fourth equality. The integrability of the Gauss–Manin connection implies that for $D_1,D_2\in T_X(U)$

$$\begin{aligned}X([D_1,D_2],\alpha)=D_1X(D_2,\alpha)-D_2X(D_1,\alpha)&\\+X(D_1,\alpha)X(D_2,\alpha)-X(D_2,\alpha)X(D_1,\alpha).&\end{aligned}$$

Also,

$$\begin{aligned}&\nabla_{\mathcal{E}(V)}(D_1)\nabla_{\mathcal{E}(V)}(D_2)\\&\quad=D_1D_2v(\alpha)+(D_1X(D_2,\alpha))\cdot v(\alpha)+X(D_2,\alpha)\cdot D_1v(\alpha)\\&\qquad\qquad+X(D_1,\alpha)\cdot D_2v(\alpha)+X(D_1,\alpha)\cdot X(D_2,\alpha)\cdot v(\alpha),\end{aligned}$$

$$\begin{aligned}&\nabla_{\mathcal{E}(V)}(D_2)\nabla_{\mathcal{E}(V)}(D_1)\\&\quad=D_2D_1v(\alpha)+(D_2X(D_1,\alpha))\cdot v(\alpha)+X(D_1,\alpha)\cdot D_2v(\alpha)\\&\qquad\qquad+X(D_2,\alpha)\cdot D_1v(\alpha)+X(D_2,\alpha)\cdot X(D_1,\alpha)\cdot v(\alpha).\end{aligned}$$

Thus

$$\begin{aligned}&\left(\nabla_{\mathcal{E}(V)}(D_1)\nabla_{\mathcal{E}(V)}(D_2)-\nabla_{\mathcal{E}(V)}(D_2)\nabla_{\mathcal{E}(V)}(D_1)\right)(\alpha)\\&\quad=[D_1,D_2]v(\alpha)\\&\qquad+(D_1X(D_2,\alpha)-D_2X(D_1,\alpha)+[X(D_1,\alpha),X(D_2,\alpha)])\cdot v(\alpha)\\&\quad=[D_1,D_2]v(\alpha)+X([D_1,D_2],\alpha)\cdot v(\alpha)=\nabla_{\mathcal{E}(V)}([D_1,D_2])(\alpha),\end{aligned}$$

i.e. the connection $\nabla_{\mathcal{E}(V)}$ is integrable. □

Remark 2.5. — If the $(\mathfrak{g}, \mathbf{Q})$-module V can be constructed from the standard representation V_{st} of $\mathbf{G}$ by taking tensor products, symmetric powers and wedge products, then applying the same operations to the sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} = \mathcal{E}(V_{\mathrm{st}})$ we get the locally free sheaf $\mathcal{E}(V)$ attached to V, so the Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ immediately induces a connection on $\mathcal{E}(V)$. This is the approach adopted in [12]. The point of our construction here is that V does not need to be a representation of $\mathbf{G}$. The construction works for all $(\mathfrak{g}, \mathbf{Q})$-modules and therefore can be easily adapted to deal with p-adic analytic weights and the universal weight (see Sections 3.2, 3.4, 3.6). There is another construction for the connection $\nabla_{\mathcal{E}(V)}$ in [47, Section 3.2] using Grothendieck's sheaves of differentials when V is a finite dimensional $\mathbf{G}$-representation. That approach may be modified to deal with the non-algebraic weight except that there might be some issue with taking duality when infinite dimensional representations are involved.

2.3. The $(\mathfrak{g}, \mathbf{Q})$-module V_κ

Now in order to use the constructions in Sections 2.1 and 2.2 to formulate Shimura's theory of nearly holomorphic forms in a sheaf-theoretic context, what we need is to define a suitable $(\mathfrak{g}, \mathbf{Q})$-module for a given algebraic representation of $\mathrm{GL}(n)$.

Let $(\rho, W_\rho) \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ be an algebraic representation of $\mathrm{GL}(n)$ locally free of finite rank. We define the $(\mathfrak{g}, \mathbf{Q})$-module V_ρ as follows. For any algebra R, set

$$V_\rho(R) := W_\rho(R) \otimes_R R[\underline{Y}] = W_\rho(R) \otimes_R R[Y_{ij}]_{1 \leqslant i \leqslant j \leqslant n}$$

where $\underline{Y} = (Y_{ij})_{1 \leqslant i,j \leqslant n}$ is the symmetric $n \times n$ matrix with the indeterminate $Y_{ij} = Y_{ji}$ as its (i,j) entry. Elements in $V_\rho(R)$ can be regarded as polynomials in the $\frac{n(n+1)}{2}$ variables Y_{ij} with coefficients in $W_\rho(R)$. Define the $\mathbf{Q}$-action on V_ρ by

$$(g \cdot P)(\underline{Y}) = a \cdot P(a^{-1}b + a^{-1}\,\underline{Y}d) \tag{2.4}$$

for $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathbf{Q}(R)$ and $P(\underline{Y}) \in V_\rho(R)$. For the action of $\mathfrak{q} \subset \mathfrak{g}$ on V_ρ, obviously we take the one induced from the $\mathbf{Q}$-action. It remains to define the action of $\mathfrak{u}^-$, the Lie algebra of the unipotent subgroup opposite to $\mathbf{U} \subset \mathbf{G}$, on V_ρ. First, we pick the following basis of $\mathfrak{u}^-$

$$\mu^-_{ii} = E_{i+n,i}, \quad 1 \leqslant i \leqslant n, \quad \mu^-_{ij} = E_{i+n,j} + E_{j+n,i}, \quad 1 \leqslant i < j \leqslant n,$$

where E_{ij} is the $2n \times 2n$ matrix with 1 as the (i,j) entry and 0 elsewhere. We make $\mathfrak{u}^-$ act on V_ρ by the formulas

$$\begin{aligned}(\mu_{ij}^- \cdot P)(\underline{Y}) = &\sum_{1\leqslant k\leqslant n} (Y_{ki}\varepsilon_{kj} + Y_{kj}\varepsilon_{ki}) \cdot P(\underline{Y}) \\ &- \sum_{1\leqslant k\leqslant l\leqslant n} (Y_{ki}Y_{jl} + Y_{kj}Y_{il}) \frac{\partial}{\partial Y_{kl}} P(\underline{Y}), \qquad i \neq j, \\ (\mu_{ii}^- \cdot P)(\underline{Y}) = &\sum_{1\leqslant k\leqslant n} \varepsilon_{ki} \cdot P(\underline{Y}) - \sum_{1\leqslant k\leqslant l\leqslant n} Y_{ki}Y_{il} \frac{\partial}{\partial Y_{kl}} P(\underline{Y}),\end{aligned} \tag{2.5}$$

where $\varepsilon_{ij} \in \mathfrak{gl}(n)$ is the $n \times n$ matrix with 1 as the (i,j) entry and 0 elsewhere, and it acts via the $\mathfrak{gl}(n)$-action on the coefficients of $P(\underline{Y})$. The compatibility of such defined actions of $\mathbf{Q}$ and $\mathfrak{u}^-$ can be shown by direct computation using the formulas. There is also a more conceptual proof. To describe it we construct a representation of the group

$$I_{\mathbf{G}}(\mathbb{Z}_p) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathbf{G}(\mathbb{Z}_p) \,\middle|\, c \equiv 0 \mod p \right\}.$$

Let $\mathbf{Q}^-_{I_{\mathbf{G}}}(\mathbb{Z}_p)$ be the subgroup of $I_{\mathbf{G}}(\mathbb{Z}_p)$ whose elements have 0 as the right upper $n \times n$ corner. we make it act on $W_\rho(\mathbb{Q}_p)$ through its Levi part. Equip $W_\rho(\mathbb{Q}_p)$ with a p-adic norm by choosing a basis of $W_\rho(\mathbb{Q}_p)$, and since it is finite dimensional all norms defined in this way are equivalent. We consider the p-adic analytic induction $\mathrm{Ind}_{\mathbf{Q}^-_{I_{\mathbf{G}}}(\mathbb{Z}_p)}^{I_{\mathbf{G}}(\mathbb{Z}_p)} W_\rho(\mathbb{Q}_p)$. Thanks to the Iwahori decomposition we know

$$\mathrm{Ind}_{\mathbf{Q}^-_{I_{\mathbf{G}}}(\mathbb{Z}_p)}^{I_{\mathbf{G}}(\mathbb{Z}_p)} W_\rho(\mathbb{Q}_p) = W_\rho(\mathbb{Q}_p)\langle Y_{ij}\rangle_{1\leqslant i<j\leqslant n} = W_\rho(\mathbb{Q}_p)\langle \underline{Y}\rangle,$$

with $g \in I_{\mathbf{G}}(\mathbb{Z}_p)$ acting on $P(\underline{Y}) \in W_\rho(\mathbb{Q}_p)\langle \underline{Y}\rangle$ by

$$(g \cdot P)(\underline{Y}) = (a + \underline{Y}c) \cdot P\left((a + \underline{Y}c)^{-1}(b + \underline{Y}d)\right). \tag{2.6}$$

Here $W_\rho(\mathbb{Q}_p)\langle \underline{Y}\rangle$ is the space of strictly convergent power series in $\underline{Y}$ (i.e. convergent on the closed unit ball). Then the formulas (2.4) and (2.5) can be deduced from (2.6), and the compatibility of the actions of $\mathbf{Q}$ and $\mathfrak{u}^-$ on V_ρ follows.

Remark 2.6. — One can check that the formulas (2.4) and (2.5) actually agree with the formulas (2.11) and (2.12) given in [29], so as $\mathfrak{g}(\mathbb{C})$-modules, $V_\rho(\mathbb{C})$ defined here should agree with $\mathcal{O}^f(\mathbf{G}^\circ(\mathbb{R}), K_{\mathbf{G}^\circ(\mathbb{R})}, W_\rho(\mathbb{C}))$ defined in [29].

Remark 2.7. — A $(\mathfrak{g}, \mathbf{Q})$-module of finite rank comes from an algebraic representation of $\mathbf{G}$. However, the $(\mathfrak{g}, \mathbf{Q})$-module defined above is not of finite rank, and it contains a sub-$(\mathfrak{g}, \mathbf{Q})$-module of finite rank. Compared to $\mathbf{G}$-representations, $(\mathfrak{g}, \mathbf{Q})$-modules are more convenient for considering p-adic deformations and p-adic families.

As a $\mathbf{Q}$-representation, V_ρ comes with an increasing filtration

$$\mathrm{Fil}^r V_\rho = V_\rho^r = W_\rho[\underline{Y}]_{\leqslant r}, \tag{2.7}$$

where the subscript $\leqslant r$ means polynomials in $\underline{Y}$ of total degree less or equal to r. Let $\eta_0 = -\sum_{1\leqslant i\leqslant n} E_{i+n,i+n} \in \mathfrak{g}$. Then $\mathrm{Fil}^r V_\rho$ can also be characterized as the sum of generalized η_0-eigenspaces with eigenvalues $\geqslant -r$ [16, p. 230]. The eigenvalues of η_0 are also called F-weights [16]. Viewing the $\mathrm{GL}(n)$-representation W_ρ as a $\mathbf{Q}$-representation we have $V_\rho^0 = W_\rho$. It follows from the definition formulas that

$$\mathfrak{g} \cdot V_\rho^r \subset V_\rho^{r+1}. \tag{2.8}$$

Let V_{triv} be the $(\mathfrak{g}, \mathbf{Q})$-module constructed as above by taking ρ to be the trivial representation. Denote by J the $\mathbf{Q}$-representation V_{triv}^1. We note here the following useful isomorphism of $\mathbf{Q}$-representations

$$V_\rho^r \cong V_\rho^0 \otimes \mathrm{Sym}^r J = W_\rho \otimes \mathrm{Sym}^r J. \tag{2.9}$$

For a dominant weight $\kappa = (k_1, \dots, k_2) \in X(\mathbf{T}^\circ)^+$ of $\mathrm{GL}(n)$ with respect to $\mathbf{B}$. Set $\kappa' = (-k_n, \dots, -k_1)$. We define W_κ to be the algebraic $\mathrm{GL}(n)$-representation

$$\left\{ f : \mathrm{GL}(n) \to \mathbb{A}^1 \;\middle|\; \begin{array}{l} \text{morphism of schemes satisfying} \\ f(gb) = \kappa'(b) f(g) \\ \text{for all } g \in \mathrm{GL}(n) \text{ and } b \in \mathbf{B} \end{array} \right\} \tag{2.10}$$

with $\mathrm{GL}(n)$ acting by left inverse translation. Putting $W_\rho = W_\kappa$ we get the $(\mathfrak{g}, \mathbf{Q})$-module V_κ and $\mathbf{Q}$-representations V_κ^r, $r \geqslant 0$.

Denote by τ the symmetric square of the standard representation of $\mathrm{GL}(n)$. Let $\tau^\vee$ be the dual representation of τ. In the following we mostly consider $\mathrm{GL}(n)$-representations which are tensor products of some κ with symmetric powers of τ and $\tau^\vee$.

Remark 2.8. — We can twist V_ρ by the i-th power of the multiplier character ν and denote the resulting $(\mathfrak{g}, \mathbf{Q})$-module by $V_\rho(i)$. Such a twist will change the F-weights by $-i$ and corresponds to a Tate twist [16, p. 222].

2.4. The sheaf $\mathcal{V}_\kappa^r$ of nearly holomorphic forms

Let κ be a dominant weight of $\mathrm{GL}(n)$. With preparations in previous sections we give the following definitions.

Definition 2.9. — *The locally free sheaf over X of weight κ, (non-holomorphy) degree r nearly holomorphic forms is defined to be $\mathcal{V}^r_\kappa = \mathcal{E}(V^r_\kappa)$.*

When $r = 0$, we also use ω_κ to denote $\mathcal{V}^0_\kappa$ which is the sheaf of weight κ holomorphic forms. More generally for $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ we define the locally free sheaves $\mathcal{V}_\rho = \mathcal{E}(V_\rho)$, $\mathcal{V}^r_\rho = \mathcal{E}(V^r_\rho)$ and denote $\mathcal{V}^0_\rho$ by ω_ρ. The nearly holomorphic forms are defined to be global sections of the sheaf $\mathcal{V}^r_\kappa$.

Definition 2.10. — *Let R be a $\mathbb{Z}[1/N]$-algebra. The space of nearly holomorphic forms (resp. cuspidal nearly holomorphic forms) over R of weight κ, principal level N and (non-holomorphy) degree r is defined to be $N^r_\kappa(\Gamma(N), R) = H^0(X_{/R}, \mathcal{V}^r_\kappa)$ (resp. $N^r_{\kappa,\mathrm{cusp}}(\Gamma(N), R) = H^0(X_{/R}, \mathcal{V}^r_\kappa(-C))$).*

There is the moduli interpretation à la Katz for nearly holomorphic forms. Away from the cusps, a nearly holomorphic form f over R of weight κ, principal level N and degree r is a rule assigning to every quadruple $(A_{/S}, \lambda, \psi_N, \alpha)$ an element $f(A_{/S}, \lambda, \psi_N, \alpha)$ inside $V^r_\kappa(S) = W_\kappa(S)[\underline{Y}]_{\leqslant r}$, where $(A_{/S}, \lambda)$ is a principally polarized dimension n abelian scheme defined over the R-algebra S, ψ_N is a principal level N structure and α is a basis of $\mathcal{H}^1_{\mathrm{dR}}(A/S)$ respecting the Hodge filtration and the symplectic pairing up to similitude. Taking into account the definition of W_κ (2.10), by evaluating $f(A_{/S}, \lambda, \psi_N, \alpha) \in W_\kappa(S)[\underline{Y}]_{\leqslant r}$ at the identity, one may also formulate Katz's interpretation for f as follows. The nearly holomorphic form f is a rule assigning to each quadruple $(A_{/S}, \lambda, \psi_N, \alpha)$ an element $f^{\mathrm{sc}}(A_{/S}, \lambda, \psi_N, \alpha) \in S[\underline{Y}]_{\leqslant r}$ such that for each $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathbf{Q}$ with a belonging to $\mathbf{B}$, we have $f^{\mathrm{sc}}(A_{/S}, \lambda, \psi_N, \alpha g) = \kappa'(a) f^{\mathrm{sc}}(A_{/S}, \lambda, \psi_N, \alpha)$.

It follows directly from Proposition 2.4 and (2.8) that the sheaves $\mathcal{V}_\rho, \mathcal{V}^r_\rho$ are equipped with the integrable connections

$$\nabla_\rho : \mathcal{V}_\rho \longrightarrow \mathcal{V}_\rho \otimes \Omega^1_X(\log C)$$

and

$$\nabla_\rho : \mathcal{V}^r_\rho \longrightarrow \mathcal{V}^{r+1}_\rho \otimes \Omega^1_X(\log C). \tag{2.11}$$

The global sections of the differential sheaf Ω^1_X has a natural Hecke action and the extended Kodaira–Spencer isomorphism [35, Proposition 6.9] says that there is the Hecke-equivariant isomorphism

$$\Omega^1_X(\log C) \cong \mathrm{Sym}^2(\omega(\mathcal{G}/X))(-1) \cong \omega_\tau(-1).$$

There is a canonical isomorphism of locally free sheaves

$$t^+ : \mathcal{V}^{r+1}_{\rho\otimes\tau}(-1) \longrightarrow \mathcal{V}^{r+1}_{\rho\otimes\tau}$$

which is not Hecke equivariant but commutes with Hecke actions up to a twist by the multiplier character. Composing ∇_ρ with t^+ we get the differential operator

$$D_\rho : \mathcal{V}_\rho^r \xrightarrow{\nabla_\rho} \mathcal{V}_\rho^{r+1} \otimes \Omega_X^1(\log C) \xrightarrow{\mathrm{KS}} \mathcal{V}_{\rho\otimes\tau}^{r+1}(-1) \xrightarrow{t^+} \mathcal{V}_{\rho\otimes\tau}^{r+1}.$$

It commutes with Hecke actions up to a multiplier twist (cf. Section 3.10, [49, Section 2.5.2, Proposition 3.3.7]).

Put $\mathcal{J} = \mathcal{E}(J)$ and $\mathcal{J}^\vee$ to be its dual. By (2.9) we have

Proposition 2.11. — *$\mathcal{V}_\rho^r \cong \omega_\rho \otimes \operatorname{Sym}^r \mathcal{J}$ as locally free sheaves over X with Hecke actions.*

Remark 2.12. — In [48, Sections 4.1.2, 4.3.1] Urban defined a locally free sheaf $\mathcal{J}'$ to be the one making the diagram below commutative with bottom row exact,

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \operatorname{Sym}^2(\omega(\mathcal{G}/X))(-1) \arrow[r, hook] \arrow[d] & \omega(\mathcal{G}/X) \otimes \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X)^\vee \arrow[d] \\ \mathcal{J}' \arrow[r, hook] \arrow[d] & \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \otimes \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X)^\vee \arrow[d] \\ \mathcal{O}_X \arrow[r, hook] \arrow[d] & \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X) \otimes \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X)^\vee \arrow[d] \\ 0 & 0, \end{tikzcd}$$

and he defined the sheaf of weight κ degree r nearly holomorphic forms to be $\omega_\kappa \otimes \operatorname{Sym}^r \mathcal{J}'^\vee$. One can show that the sheaf $\mathcal{J}^\vee$ satisfies Urban's condition for defining $\mathcal{J}'$. Hence $\mathcal{J} \cong \mathcal{J}'^\vee$ and our definition of sheaves of nearly holomorphic forms agrees with his.

We end this section with an example showing that the locally free sheaves associated to two non-isomorphic $\mathbf{Q}$-representations can be isomorphic as locally free sheaves without considering the Hecke actions. It also illustrates that the sheaf $\mathcal{J}$ may have splitting that does not come from the $\mathbf{Q}$-representation and such a splitting can give rise to holomorphic but non-Hecke equivariant differential operators.

Example 2.13. — Take $n = 1$, $\mathbf{G} = \mathrm{GL}(2)$ and $\mathbf{G}^\circ = \mathrm{SL}(2)$. We show that the sheaf $\mathcal{J}^\vee = (\mathcal{V}^1_{\mathrm{triv}})^\vee$ and the first jet sheaf $\mathcal{P}^1(\mathcal{O}_X)$ are isomorphic in $\mathrm{QCoh}(X)$ but their corresponding $\mathbf{Q}$-representations are not isomorphic. Let V_1, V_2 be the $\mathbf{Q}$-representations giving rise to $\mathcal{J}^\vee, \mathcal{P}^1(\mathcal{O}_X)$ respectively. Write $Y = Y_{11}$. Then $V_1^\vee = \mathrm{triv} \otimes \mathbb{Z}[Y]_{\leqslant 1}$ with basis $\{Y, 1\}$, and the action of $\mathbf{Q}^\circ$ is given by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot P(Y) = P(a^{-1}b + a^{-2}Y),$$

or in the matrix form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a^{-2} & 0 \\ a^{-1}b & 1 \end{pmatrix}.$$

Clearly V_1 is indecomposable as a $\mathbf{Q}$-representation. On the other hand by [16, Proposition VI 5.1], $V_2^\vee \cong U_1(\mathfrak{g}^\circ) \otimes_{U(\mathfrak{q}^\circ)} \mathrm{triv}$ as a $\mathbf{Q}^\circ$-representation, where $\mathfrak{g}^\circ = \mathrm{Lie}\, \mathbf{G}^\circ = \mathfrak{sl}(2) = \mathrm{Span}\{h, x, y\}$ and $\mathfrak{q}^\circ = \mathrm{Span}\{h, x\}$ with $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. As a basis of $V_2^\vee$ we can take $\{y \otimes 1, 1 \otimes 1\}$, and we have $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ act on them by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (y \otimes 1) = a^{-2} y \otimes 1, \qquad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (1 \otimes 1) = 1 \otimes 1,$$

or in the matrix form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

This is saying that the $\mathbf{Q}$-action on V_2 splits. Hence V_1 and V_2 are not isomorphic as $\mathbf{Q}$-representations.

However as coherent sheaves $\mathcal{J}^\vee$ and $\mathcal{P}^1(\mathcal{O}_X)$ are indeed isomorphic, because the nearly holomorphic form E_2 splits $\mathcal{J}^\vee \cong \omega(\mathcal{G}/X)^{\otimes 2} \otimes \mathcal{J}$ as locally free sheaves [49, Remark 2.3.7]. This non-Hecke equivariant splitting gives rise to Serre's ∂ operator that acts on a modular form f of weight k by

$$\partial f = 12\theta f - kPf,$$

where $\theta = q\frac{\mathrm{d}}{\mathrm{d}q}$ and P is the holomorphic funciton on the upper half plane defined as $P(q) = 1 - 24\sum_{m \geqslant 1} \sigma_1(n) q^n$ with $q = e^{2\pi i z}$ (cf. [30, Section A1.4]). Serre's ∂ operator is a holomorphic differential operator but not Hecke equivariant.

2.5. Equivalence to Shimura's nearly holomorphic forms and differential operators

First recall Shimura's definition of nearly holomorphic forms and Maass–Shimura differential operators. Let $\mathfrak{h}_n = \{z \in M_n(\mathbb{C}) : {}^{\mathrm{t}}z = z,\, \mathrm{Im}\, z > 0\}$ be the genus n Siegel upper half space and $\Gamma \subset \mathbf{G}^\circ(\mathbb{Z}) = \mathrm{Sp}(2n, \mathbb{Z})$ be a congruence subgroup. As usual $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \mathbf{G}(\mathbb{R})$ acts on $\mathfrak{h}_n$ by $\gamma z = (a_\gamma z + b_\gamma)\cdot(c_\gamma z + d_\gamma)^{-1}$. Put $s(z) = (z - \bar{z})^{-1}$ and $\mu(\gamma, z) = c_\gamma z + d_\gamma$.

For an algebraic representation (ρ, W_ρ) of $\mathrm{GL}(n)$ free of finite rank, Shimura defines [45, Section 13.11] the space of $W_\rho(\mathbb{C})$-valued nearly holomorphic forms of degree r, denoted by $N^r_\rho(\mathfrak{h}_n, \Gamma)$, as the set consisting of functions $f \in C^\infty(\mathfrak{h}_n, W_\rho(\mathbb{C}))$ satisfying

(i) $f(z)$ can be written as a degree $\leqslant r$ polynomial in the components of $s(z)$ with coefficients being holomorphic maps from $\mathfrak{h}_n$ to $W_\rho(\mathbb{C})$,
(ii) f transforms under $\gamma \in \Gamma$ by $f(\gamma z) = \rho(\mu(\gamma, z))f(z)$.

When $n = 1$ the function f is also required to satisfy the cusp condition, i.e. for every $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ there exists $a_{in} \in \mathbb{C}$ and $M \in \mathbb{N}$ such that

$$\rho(\mu(\gamma, z))^{-1} f(\gamma z) = \sum_{i=0}^{r} (\pi \mathrm{Im}\, z)^{-i} \sum_{n=0}^{\infty} a_{in} e^{2\pi i z/M}.$$

The Maass–Shimura differential operator $D_{\mathfrak{h}_n,\rho}$ is defined as [45, Section 12.9]

$$\begin{aligned} D_{\mathfrak{h}_n,\rho} : N^r_\rho(\mathfrak{h}_n, \Gamma) &\longrightarrow N^{r+1}_{\rho\otimes\tau}(\mathfrak{h}_n, \Gamma) \\ f(z) &\longmapsto \rho(s(z))(\mathrm{d}_z(\rho(s(z)^{-1})f(z))). \end{aligned} \tag{2.12}$$

Now we show that $N^r_\rho(\mathfrak{h}_n, \Gamma)$, together with the Maass–Shimura differential operator $D_{\mathfrak{h}_n,\rho}$, is nothing but the global sections over $\Gamma\backslash\mathfrak{h}_n$ of the sheaf $\mathcal{V}^r_\rho$ equipped with the differential operator D_ρ defined in the previous sections. Let $Y^\circ_{\mathbb{C}}$ be a connected component of Y base changed to $\mathbb{C}$. Then $Y^\circ_{\mathbb{C}} \cong \Gamma(N)\backslash\mathfrak{h}_n$ as complex manifolds and the universal abelian variety $\mathbf{p} : \mathcal{A}_{\mathbb{C}} \to Y^\circ_{\mathbb{C}}$ is isomorphic to $\mathbf{p} : \Gamma(N)\backslash\mathbb{C}^n \times \mathfrak{h}_n/\mathbb{Z}^{2n} \to \Gamma(N)\backslash\mathfrak{h}_n$. Here $(m_1, m_2) \in \mathbb{Z}^{2n}$ and $\gamma \in \Gamma(N)$ act on $(w, z) \in \mathbb{C}^n \times \mathfrak{h}_n$ by

$$\begin{aligned} (w, z)\cdot(m_1, m_2) &= (w + m_1 z + m_2,\, z), \\ \gamma\cdot(w, z) &= (w\mu(\gamma, z)^{-1},\, \gamma z). \end{aligned}$$

Let $q : \mathfrak{h}_n \to \Gamma(N)\backslash\mathfrak{h}_n$ be the quotient map and $A_{\mathfrak{h}_n} = \mathbb{C}^n \times \mathfrak{h}_n/\mathbb{Z}^{2n} \to \mathfrak{h}_n$ be the pullback of $\mathcal{A}_{\mathbb{C}}$ via q. For each $z = (z_{ij}) \in \mathfrak{h}_n$ the fibre $A_{\mathfrak{h}_n,z} \cong \mathbb{C}^n/\Lambda_z$, where Λ_z is the lattice spanned by e_i, the vector with 1 as the i-th entry and 0 elsewhere, $1 \leqslant i \leqslant n$, and $z_j = {}^{\mathrm{t}}(z_{1j}, z_{2j}, \ldots, z_{nj})$,

$1 \leqslant j \leqslant n$. Let $\lambda_{\mathfrak{h}_n}$ (resp. $\psi_{\mathfrak{h}_n,N}$) be the polarization (principal level N structure) of $A_{\mathfrak{h}_n}$ such that its fibre at z is given by the real Riemann form $E_z : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$, defined as $E_z(w_1, w_2) = -\mathrm{Im}\,({}^{\mathrm{t}}w_1(\mathrm{Im}\, z)^{-1}\overline{(iw_2)})$ (resp. $\frac{1}{N}e_1, \ldots, \frac{1}{N}e_n, \frac{1}{N}z_1, \ldots, \frac{1}{N}z_n$). The $\{e_i, z_j\}_{1\leqslant i,j\leqslant n}$ form a basis of $H_1(A_{\mathfrak{h}_n,z}, \mathbb{Z})$. Over $\mathfrak{h}_n$ we have a global basis $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ for the sheaf $q^*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}_{\mathbb{C}}/Y_{\mathbb{C}}) = \mathcal{H}^1_{\mathrm{dR}}(A_{\mathfrak{h}_n}/\mathfrak{h}_n)$ defined as

$$\alpha_i\left(\sum_{j=1}^n m_{1,j}z_j + m_{2,j}e_j\right) = m_{2,i}, \quad \beta_i\left(\sum_{j=1}^n m_{1,j}z_j + m_{2,j}e_j\right) = m_{1,i}.$$

The basis (α, β) is horizontal with respect to the Gauss–Manin connection, i.e.

$$\nabla(\alpha_i) = \nabla(\beta_i) = 0, \quad 1 \leqslant i \leqslant n.$$

After base changing to $C^\infty(\mathfrak{h}_n, \mathbb{C})$, the Hodge decomposition gives another basis $(\mathrm{d}w, \mathrm{d}\bar{w}) = (\mathrm{d}w_1, \ldots, \mathrm{d}w_n, \mathrm{d}\bar{w}_1, \ldots, \mathrm{d}\bar{w}_n)$ of $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathfrak{h}_n}/\mathfrak{h}_n) \otimes C^\infty(\mathfrak{h}_n, \mathbb{C})$.

Neither $(\mathrm{d}w, \mathrm{d}\bar{w})$ nor (α, β) gives rise to an element of $(q^*T^\times_{\mathcal{H}})(\mathfrak{h}_n) \otimes C^\infty(\mathfrak{h}_n, \mathbb{C})$. The basis $(\mathrm{d}w, \mathrm{d}\bar{w})$ does not satisfy the pairing condition, while (α, β) is not compatible with the Hodge filtration. Nevertheless $(\mathrm{d}w, \beta)$ (resp. $(\mathrm{d}w, -\mathrm{d}\bar{w}\, s)$) does give an element of $(q^*T^\times_{\mathcal{H}})(\mathfrak{h}_n)$ (resp. $(q^*T^\times_{\mathcal{H}})(\mathfrak{h}_n) \otimes C^\infty(\mathfrak{h}_n, \mathbb{C})$), and it is easily checked that

$$(\mathrm{d}w, -\mathrm{d}\bar{w}\, s) = (\mathrm{d}w, \beta)\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

Evaluating global sections of $\mathcal{V}^r_\rho$ over $Y^\circ_{\mathbb{C}}$ at the test object

$$(A_{\mathfrak{h}_n}/\mathfrak{h}_n, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, -\mathrm{d}\bar{w}\, s))$$

defines a map

$$\begin{aligned} \phi : H^0(Y^\circ_{\mathbb{C}}, \mathcal{V}^r_\rho) &\longrightarrow N^r_\rho(\mathfrak{h}_n, \Gamma(N)) \\ f &\longmapsto f(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{N,\mathfrak{h}_n}, (\mathrm{d}w, -\mathrm{d}\bar{w}\, s))|_{\underline{Y}=0}. \end{aligned} \tag{2.14}$$

Proposition 2.14. — *ϕ is well defined and is an isomorphism.*

Proof. — We need to check that the above defined $\phi(f)$ does land inside $N^r_\rho(\mathfrak{h}_n, \Gamma(N))$. First look at the evaluation of f at the test object $(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, \beta))$. Since $(\mathrm{d}w, \beta)$ is holomorphic we have

$$f(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, \beta)) = P_f(\underline{Y}),$$

a polynomial in $\underline{Y}$ of degree $\leqslant r$ with coefficients being holomorphic maps from $\mathfrak{h}_n$ to $W_\rho(\mathbb{C})$. Combining (2.4) and (2.13) we get

$$\begin{aligned}\phi(f) &= f(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, -\mathrm{d}\bar{w}\, s))|_{\underline{Y}=0} \\ &= f\left(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, \beta)\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}\right)\Bigg|_{\underline{Y}=0} \\ &= \begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\cdot f(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, \beta))\Big|_{\underline{Y}=0} \\ &= P_f(\underline{Y}+s)|_{\underline{Y}=0} = P_f(s).\end{aligned}$$

This shows that $\phi(f)$ satisfies condition (i) in the definition of $N^r_\rho(\mathfrak{h}_n, \Gamma(N))$. Under the isomorphism

$$\begin{aligned}\gamma : A_{\mathfrak{h}_n,z} &\longrightarrow A_{\mathfrak{h}_n,\gamma z} \\ w &\longmapsto w\mu(\gamma,z)^{-1}\end{aligned}$$

for $\gamma\in\Gamma(N)$ we have

$$\gamma^*(\mathrm{d}w, -\mathrm{d}\bar{w}\, s) = (\mathrm{d}w, -\mathrm{d}\bar{w}\, s)\begin{pmatrix}\mu(\gamma,z)^{-1} & 0\\ 0 & \mu(\gamma,z)\end{pmatrix},$$

from which we see that $\phi(f)$ also has the transformation property required in condition (ii). Finally the bijectivity of ϕ can be seen from the fact that essentially it sends $P_f(\underline{Y})$ to $P_f(s)$ and we can recover one of them from the other. □

Next we prove the compatibility of D_ρ and $D_{\mathfrak{h}_n,\rho}$ under the map ϕ.

Proposition 2.15. — $D_{\mathfrak{h}_n,\rho}\circ\phi = \phi\circ D_\rho$

Proof. — Let $\langle\cdot,\cdot\rangle$ be the canonical pairing between the sheaf of differentials $\Omega^1_{\mathfrak{h}_n}$ and the tangent bundle $T_{\mathfrak{h}_n}$. Take $\partial/\partial z_{ij}\in T_{\mathfrak{h}_n}$ and $f\in H^0(Y^\circ_{\mathbb{C}}, \mathcal{V}^r_\rho)$. We show that $\langle D_{\mathfrak{h}_n,\rho}\circ\phi(f),\ \partial/\partial z_{ij}\rangle = \langle\phi\circ D_\rho(f), \partial/\partial z_{ij}\rangle$. Assume $i\neq j$ (the computation for the case $i=j$ is the same and we omit it), the Gauss–Manin connection acts on $(\mathrm{d}w,\beta)$ as

$$\nabla(\partial/\partial z_{ij})(\mathrm{d}w,\beta) = (\mathrm{d}w,\beta)\begin{pmatrix}0 & 0\\ E_{ij}+E_{ji} & 0\end{pmatrix} = (\mathrm{d}w,\beta)\,\mu^-_{ij}. \tag{2.15}$$

Let $P_f(\underline{Y})$ be as in the above proof. According to the definition of D_ρ by (2.2),

$$\begin{aligned}
&\langle (D_\rho f)(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, \beta)),\ \partial/\partial z_{ij}\rangle \\
&\qquad = \frac{\partial}{\partial z_{ij}} P_f(\underline{Y}) + \mu_{ij}^- \cdot P_f(\underline{Y}) \\
&\qquad = \frac{\partial}{\partial z_{ij}} P_f(\underline{Y}) + \sum_{1\leqslant k\leqslant n} (Y_{ki}\varepsilon_{kj} + Y_{kj}\varepsilon_{ki}) \cdot P_f(\underline{Y}) \\
&\qquad\qquad - \sum_{1\leqslant k\leqslant l\leqslant n} (Y_{ki}Y_{jl} + Y_{kj}Y_{il}) \frac{\partial}{\partial Y_{kl}} P_f(\underline{Y}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle \phi \circ D_\rho(f), \partial/\partial z_{ij}\rangle \\
&\qquad = \langle (D_\rho f)(A_{\mathfrak{h}_n}, \lambda_{\mathfrak{h}_n}, \psi_{\mathfrak{h}_n,N}, (\mathrm{d}w, \beta)),\ \partial/\partial z_{ij}\rangle \,|_{\underline{Y}=s} \\
&\qquad = \left.\frac{\partial}{\partial z_{ij}} P_f(\underline{Y})\right|_{\underline{Y}=s} + \sum_{1\leqslant k\leqslant n} (s_{ki}\varepsilon_{kj} + s_{kj}\varepsilon_{ki}) \cdot P_f(s) \\
&\qquad\qquad - \sum_{1\leqslant k\leqslant l\leqslant n} (s_{ki}s_{jl} + s_{kj}s_{il}) \frac{\partial}{\partial s_{kl}} P_f(s).
\end{aligned}$$

Using

$$\frac{\partial s_{kl}}{\partial z_{ij}} = -\left(s\left(\frac{\partial}{\partial z_{ij}}(z-\bar{z})\right)s\right)_{kl} = -(s_{ik}s_{jl} + s_{il}s_{jk}),$$

we get

$$\begin{aligned}
&\text{(2.16)}\quad \langle \phi \circ D_\rho(f), \partial/\partial z_{ij}\rangle \\
&\quad = \left.\frac{\partial}{\partial z_{ij}} P_f(\underline{Y})\right|_{\underline{Y}=s} + \sum_{1\leqslant k\leqslant n} (s_{ki}\varepsilon_{kj} + s_{kj}\varepsilon_{ki}) \cdot P_f(s) + \frac{\partial s_{kl}}{\partial z_{ij}} \frac{\partial}{\partial s_{kl}} P_f(s) \\
&\quad = \frac{\partial}{\partial z_{ij}}(P_f(s)) + \sum_{1\leqslant k\leqslant n} (s_{ki}\varepsilon_{kj} + s_{kj}\varepsilon_{ki}) \cdot P_f(s) \\
&\quad = \frac{\partial}{\partial z_{ij}}\phi(f) + \sum_{1\leqslant k\leqslant n} (s_{ki}\varepsilon_{kj} + s_{kj}\varepsilon_{ki}) \cdot \phi(f).
\end{aligned}$$

On the other hand according to the definition of $D_{\mathfrak{h}_n,\rho}$ (2.12)

$$\begin{aligned}\langle D_{\mathfrak{h}_n,\rho}\circ\phi(f),\,\partial/\partial z_{ij}\rangle\\ &= \langle \rho(s)(d_z(\rho(s^{-1})\phi(f))),\,\partial/\partial z_{ij}\rangle\\ &= \rho(s)\left(\frac{\partial}{\partial z_{ij}}(\rho(s^{-1})\phi(f))\right) = \frac{\partial}{\partial z_{ij}}\phi(f)+\rho\left(s\frac{\partial s^{-1}}{\partial z_{ij}}\right)\phi(f)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f)+\left(\sum_{k=1}^{n}s_{ki}\varepsilon_{kj}+s_{kj}\varepsilon_{ki}\right)\cdot\phi(f).\end{aligned}$$

Comparing with (2.16), we conclude. □

It is also explained in [43, Section 7], [45, Appendix A8] (see also [37, Section 2.4]) that the Maass–Shimura differential operators correspond to the action of the Lie algebra $(\mathrm{Lie}\,\mathbf{G})_{\mathbb{C}}$ on nearly holomorphic forms. Therefore the sheaf-theoretic definition of the differential operators in Section 2.4 can be viewed as a geometric interpretation of the Lie algebra action at the archimedean place, and we have the commutative diagram (1.2).

2.6. Polynomial q-expansions

We first define the Mumford objects. Thanks to the moduli interpretation of $N^r_\kappa(\Gamma(N),R) = H^0\left(X_{/R},\mathcal{V}^r_\kappa\right)$, the evaluation of a nearly holomorphic form at a Mumford object gives rise to its polymonial q-expansion. We also include formulas for the action of differential operators on the polynomial q-expansions.

Following [16, V.1], let $L=\mathbb{Z}^n$ with fixed basis $e_1,\dots,e_n$ and L^* be its dual. Put S_L to be the symmetric quotient of $L\times L$ and $S_{L,\geqslant 0}$ to be the intersection of S_L with the cone dual to the cone inside $S_L^*\otimes_{\mathbb{Z}}\mathbb{R}$ consisting of semi-positive definite forms. Take a basis $s_1,\dots,s_{n(n+1)/2}$ of S_L lying inside $S_{L,\geqslant 0}$, and set $\mathbb{Z}((S_{L,\geqslant 0}))=\mathbb{Z}[[S_{L,\geqslant 0}]][1/s_1s_2\cdots s_{n(n+1)/2}]$. For $\beta\in S_{L,\geqslant 0}$, the corresponding element in $\mathbb{Z}[[S_{L,\geqslant 0}]]$ is sometimes written as q^β.

The natural map $L\to S_L\otimes L^*$ defines a period group $L\subset L^*\otimes \mathbb{G}_{m/\mathbb{Z}((S_{L,\geqslant 0}))}$, principally polarized by the duality between L and L^*. Mumford's construction [16] gives an abelian variety $A_{/\mathbb{Z}((S_{L,\geqslant 0}))}$ with a canonical polarization λ_{can} and a canonical basis $\omega_{\mathrm{can}}=(\omega_{1,\mathrm{can}},\dots,\omega_{n,\mathrm{can}})$ of $\omega(A/\mathbb{Z}((S_{L,\geqslant 0})))$. The exact sequence

$$0\to L^*\otimes\prod_l\varprojlim_m \mu_{l^m}\to\prod_l T_l(A)\to L\otimes\widehat{\mathbb{Z}}\to 0,$$

after base changing to $\mathbb{Z}((N^{-1}S_{L,\geqslant 0}))[\zeta_N, 1/N]$, gives rise to a principal level N structure $\psi_{N,\mathrm{can}}$ for $A_{/\mathbb{Z}((S_{L,\geqslant 0}))}$. Let $D_{ij} \in \mathrm{Der}(\mathbb{Z}((S_{L,\geqslant 0})), \mathbb{Z}((S_{L,\geqslant 0})))$ be the element dual to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $\delta_{i,\mathrm{can}} = \nabla(D_{ii})\,\omega_{i,\mathrm{can}}$. For $\beta \in S_{L,\geqslant 0}$ we have $D_{ij}(q^\beta) = (2-\delta_{ij})\beta_{ij}q^\beta$ with $\delta_{ij} = 0$ if $i \neq j$, and 1 if $i = j$. Then $\delta_{\mathrm{can}} = (\delta_{1,\mathrm{can}}, \dots, \delta_{n,\mathrm{can}})$ together with ω_{can} forms a basis of $\mathcal{H}^1_{\mathrm{dR}}(A/\mathbb{Z}((S_{L,\geqslant 0})))$ respecting both the Hodge filtration and the symplectic pairing.

Evaluating a nearly holomorphic form $f \in N^r_\kappa(\Gamma(N), R)$ at the test object

$$\mathrm{Mum}_N(q) = (A_{/\mathbb{Z}((N^{-1}S_{L,\geqslant 0}))[\zeta_N, 1/Np]}, \lambda_{\mathrm{can}}, \psi_{N,\mathrm{can}}, \omega_{\mathrm{can}}, \delta_{\mathrm{can}})$$

defines its polynomial q-expansion

$$\begin{aligned} N^r_\kappa(\Gamma(N), R) &\longrightarrow \mathbb{Z}[\zeta_N, 1/N][[N^{-1}S_{L,\geqslant 0}]] \otimes W_\kappa(R)[\underline{Y}]_{\leqslant r} \\ f &\longmapsto f(q, \underline{Y}) = f(\mathrm{Mum}_N(q)). \end{aligned} \tag{2.17}$$

Next we compute formulas of differential operators in terms of polynomial q-expansions. Let $\underline{X} = (X_{ij})_{1\leqslant i,j\leqslant n}$ be the symmetric matrix with the indeterminate $X_{ij} = X_{ji}$ as the (i,j) and (j,i) entries for $1 \leqslant i \leqslant j \leqslant n$. The X_{ij}'s form a basis of the $\mathrm{GL}(n)$-representation τ. An element $a \in \mathrm{GL}(n)$ acts on $\underline{X}$ by $a \cdot \underline{X} = {}^{\mathrm{t}}a\,\underline{X}a$. Let $X_{ij}^\vee$ be the basis of $\tau^\vee$ dual to X_{ij}. Then under the trivialization $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$, X_{ij} corresponds to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $X_{ij}^\vee$ corresponds to D_{ij}. From the construction of $\mathrm{Mum}_N(q)$ one can see that

$$\nabla(D_{ij})(\omega_{\mathrm{can}}, \delta_{\mathrm{can}}) = (\omega_{\mathrm{can}}, \delta_{\mathrm{can}})\,\mu^-_{ij}, \qquad X(D_{ij}, (\omega_{\mathrm{can}}, \delta_{\mathrm{can}})) = \mu^-_{ij}.$$

Proposition 2.16. — *Let $f \in N^r_\kappa(\Gamma(N), R)$ be a nearly holomorphic form with polynomial q-expansion $f(q, \underline{Y}) \in \mathbb{Z}[\zeta_N, 1/N][[N^{-1}S_{L,\geqslant 0}]] \otimes W_\kappa(R)[\underline{Y}]_{\leqslant r}$. Then*

$$(D_\kappa f)(q, \underline{Y}) = \sum_{1\leqslant i\leqslant j\leqslant n} \Big(D_{ij} f(q, \underline{Y}) + \mu^-_{ij} \cdot f(q, \underline{Y})\Big) \otimes X_{ij}$$

Example 2.17. — If we apply the above proposition to the case $n = 1$, $\kappa = k \in \mathbb{N}$, we recover the formula given in [49, Proposition 2.4.1] for D_k (denoted δ_k there). In this case the image of the polynomial q-expansion belongs $R[\zeta_N, 1/N][[q^{1/N}]][\underline{Y}]_{\leqslant r}$ and $D_{11} = q\frac{\mathrm{d}}{\mathrm{d}q}$. Write $Y = Y_{11}$. The representations κ and τ are both one-dimensional and we omit writing down their basis.

$$\begin{aligned}(D_\kappa f)(q,Y) &= D_{11}f(q,Y) + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot f(q,Y)\\ &= q\frac{\mathrm{d}}{\mathrm{d}q}f(q,Y) + Y\varepsilon_{11}\cdot f(q,Y) - Y^2\frac{\partial}{\partial Y}f(q,Y)\\ &= \left(q\frac{\mathrm{d}}{\mathrm{d}q} - Y^2\frac{\partial}{\partial q}\right)f(q,Y) + kYf(q,Y).\end{aligned}$$

2.7. Holomorphic differential operators

The purpose of this section is to illurstrate Shimura's construction of holomorphic differential operators in the sheaf-theoretic context. Let $G \hookrightarrow H$ be an embedding of reductive groups over $\mathbb{Q}$, and we assume both $G(\mathbb{R})$ and $H(\mathbb{R})$ have holomorphic discrete series. One of the motivations for studying nearly holomorphic forms is that they help construct differential operators sending holomorphic forms on H of weight κ_0 (often taken to be a scalar weight) to holomorphic forms on G of a specified weight κ. Such holomorphic differential operators have been considered and applied in many works on studying special L-values, e.g. [7, 11, 14, 15, 20, 22, 45].

Let $G = \mathbf{G}^\circ = \mathrm{Sp}(2n)_{/\mathbb{Q}}$ and $H = \mathrm{Sp}(4n)_{/\mathbb{Q}}$ with standard Siegel parabolic subgroups $Q_G = \mathbf{Q}^\circ$ and Q_H. The Shimura variety Y_G (resp. Y_H) of principal level N is defined over $\mathbb{Q}(\zeta_N)$ and is a connected component of $Y = Y_{\mathbf{G},\Gamma(N)}$ (resp. the Siegel variety parametrizing principally polarized abelian schemes of relative dimension $2n$ with a principal level N structure). In the following, sheaves over Y_H and $(\mathrm{Lie}\,H, Q_H)$-modules are denoted with a superscript ^{H}.

Let $\iota : Y_G \times Y_G \hookrightarrow Y_H$ be the embedding corresponding to

$$\begin{array}{ccccc} G & \times & G & \hookrightarrow & H\\ \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix} & \times & \begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix} & \longmapsto & \begin{pmatrix}a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ c_1 & 0 & d_1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix}.\end{array}$$

Denote by $p_1, p_2 : Y_G \times Y_G \to Y_G$ the projection to the first and second factor.

Proposition 2.18. — *Let k be an positive integer (viewed as a scalar weight) and $\kappa \in X(\mathbf{T}^\circ)^+$ be a generic weight such that the holomorphic projection $\mathscr{A}_{k+\kappa} : \mathcal{V}^e_{k+\kappa} \to \omega_{k+\kappa}$ (cf.* [45, Proposition 14.2], *Proposition 3.6,* *Corollary 3.10) exists for* $e = |\kappa| = \sum_{i=1}^{n} \kappa_i$. *Then there exists a nonzero morphism*

$$D_{k,k+\kappa} : \iota^{-1}\omega_k^H \longrightarrow p_1^*\omega_{k+\kappa} \otimes p_2^*\omega_{k+\kappa}.$$

By taking global sections, $D_{k,k+\kappa}$ *induces a holomorphic differential operator sending Siegel modular forms on* $\mathrm{Sp}(4n)$ *of scalar weight* k *to Siegel modular forms on* $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$ *of weight* $(k+\kappa, k+\kappa)$.

Proof. — First, by our construction of differential operators, there is the map

$$D_k^e : \iota^{-1}\omega_k^H \longrightarrow \iota^*\mathcal{V}^{H,e}_{k\otimes\mathrm{Sym}^e\,\tau^H},$$

so we consider the decomposition of the sheaf $\iota^*\mathcal{V}^{H,e}_{k\otimes\mathrm{Sym}^e\,\tau^H}$, especially how $\omega_{k+\kappa}$ appears in the decomposition. Equivalently, we consider the decomposition of $V^{H,e}_{k\otimes\mathrm{Sym}^e\,\tau^H}$ as a $(\mathfrak{g}^\circ \times \mathfrak{g}^\circ, Q_G \times Q_G)$-module.

Write

$$\underline{X}^H = \begin{pmatrix} \underline{X}_1 & \underline{X}_0 \\ {}^{\mathrm{t}}\underline{X}_0 & \underline{X}_2 \end{pmatrix}, \qquad \underline{Y}^H = \begin{pmatrix} \underline{Y}_1 & \underline{Y}_0 \\ {}^{\mathrm{t}}\underline{Y}_0 & \underline{Y}_2 \end{pmatrix}$$

in $n \times n$ blocks. The subspace

$$(\underline{X}_1, \underline{X}_2, \underline{Y}_0)W_k^H[\underline{X}^H, \underline{Y}^H] \cap W_k^H[\underline{X}^H]_e[\underline{Y}^H] \subset W_k^H[\underline{X}^H]_e[\underline{Y}^H]$$

is stable under the action of $\mathfrak{g}^\circ \times \mathfrak{g}^\circ$ and $Q_G \times Q_G$. Here the subscript $_e$ means polynomials of degree equal to e. The quotient of $W_k^H[\underline{X}^H]_e[\underline{Y}^H]$ by this submodule is canonically isomorphic to

$$W_k[\underline{X}_0]_e[\underline{Y}_1, \underline{Y}_2] \tag{2.18}$$

with the induced $(\mathfrak{g}^\circ \times \mathfrak{g}^\circ, Q_G \times Q_G)$-action. Instead of looking at the decomposition of the whole $V^{H,e}_{k\otimes\mathrm{Sym}^e\,\tau^H}|_{(\mathfrak{g}^\circ\times\mathfrak{g}^\circ, Q_G\times Q_G)}$, we consider the decomposition of the quotient (2.18). First it is easy to check that $\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & {}^{\mathrm{t}}a_1^{-1} \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a_2 & b_2 \\ 0 & {}^{\mathrm{t}}a_2^{-1} \end{smallmatrix}\right) \in Q_G \times Q_G$ acts on $P(\underline{X}_0, \underline{Y}_1, \underline{Y}_2) \in W_k[\underline{X}_0]_e[\underline{Y}_1, \underline{Y}_2]$ as

$$\begin{aligned} &\begin{pmatrix} a_1 & b_1 \\ 0 & {}^{\mathrm{t}}a_1^{-1} \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ 0 & {}^{\mathrm{t}}a_2^{-1} \end{pmatrix} \cdot P(\underline{X}_0, \underline{Y}_1, \underline{Y}_2) \\ &\qquad = P\left({}^{\mathrm{t}}a_1\, \underline{X}_0\, a_2,\ a_1^{-1}\, \underline{Y}_1 {}^{\mathrm{t}}a_1^{-1} + a_1^{-1}b_1,\ a_2^{-1}\, \underline{Y}_2 {}^{\mathrm{t}}a_2^{-1} + a_2^{-1}b_2\right). \end{aligned}$$

By [45, Theorem 12.7] we know that as representations of $Q_G \times Q_G$,

$$W_k[\underline{X}_0]_e[\underline{Y}_1, \underline{Y}_2] = \bigoplus_{\kappa\in X(\mathbf{T}^\circ)^+,\ |\kappa|=e} V_{k+\kappa} \boxtimes V_{k+\kappa}. \tag{2.19}$$

By checking the formulas defining the $\mathfrak{g}$-actions, we see that this decomposition actually holds as modules of $(\mathfrak{g}^\circ \times \mathfrak{g}^\circ, Q_G \times Q_G)$. Moreover, for each κ appearing in the decomposition, the highest weight vector inside

$V^0_{k+\kappa} \boxtimes V^0_{k+\kappa}$ is given by $\prod_{i=1}^n \det_i(\underline{X}_0)^{\kappa_i-\kappa_{i+1}}$, where $\det_i$ is the determinant of the upper left $i \times i$ block. Therefore, for $\kappa \in X(\mathbf{T}^\circ)^+$, $|\kappa|$=e, the $(\mathfrak{g}^\circ \times \mathfrak{g}^\circ, Q_G \times Q_G)$-module $V_{k+\kappa} \boxtimes V_{k+\kappa}$ appears as a quotient of $V^H_{k\otimes \mathrm{Sym}^e\,\tau^H}\big|_{(\mathfrak{g}^\circ\times\mathfrak{g}^\circ, Q_G\times Q_G)}$ and one can write down a map

$$V^H_{k\otimes \mathrm{Sym}^e\,\tau'}\Big|_{(\mathfrak{g}^\circ\times\mathfrak{g}^\circ, Q_G\times Q_G} \xrightarrow{\mathrm{mod}\,\underline{X}_1,\,\underline{X}_2,\,\underline{Y}_0} W'_k[\underline{X}_0]_e[\underline{Y}_1,\,\underline{Y}_2] \longrightarrow V_{k+\kappa}\boxtimes V_{k+\kappa},$$

which induces a morphism of sheaves over $Y_G \times Y_G$,

$$\varrho_{k,\kappa} : \iota^*\mathcal{V}^H_{k\otimes \mathrm{Sym}^e\,\tau^H} \longrightarrow p_1^*\mathcal{V}_{k+\kappa} \otimes p_2^*\mathcal{V}_{k+\kappa}.$$

When the holomorphic projection $\mathscr{A}_{k+\kappa} : \mathcal{V}^e_{k+\kappa} \to \omega_{k+\kappa}$ exists, We define the operator $D_{k,k+\kappa}$ as the composition

$$\begin{aligned} D_{k,k+\kappa} : \iota^{-1}\omega^H_k \xrightarrow{D^e_k} \iota^*\mathcal{V}^{H,e}_{k\otimes \mathrm{Sym}^e\,\tau^H} \xrightarrow{\varrho_{k,\kappa}} p_1^*\mathcal{V}^e_{k+\kappa} \otimes p_2^*\mathcal{V}^e_{k+\kappa} \\ \xrightarrow{\mathscr{A}_{k+\kappa}} p_1^*\omega_{k+\kappa} \otimes p_2^*\omega_{k+\kappa}. \end{aligned}$$

It remains to show that such defined $D_{k,k+\kappa}$ is nonzero. This can be done by some computation in local coordinates.

Take an affine open subset $U_H = \mathrm{Spec}(R') \subset Y_H$ such that $U_H \times_{Y_H} (Y_G \times Y_G)$ is of the form $U \times U$ with $U = \mathrm{Spec}(R)$. Also we pick an ordered basis $\alpha = (\alpha_1, \dots, \alpha_{4n})$ of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/U_H)$ respecting both the Hodge filtration and the symplectic pairing such that $\alpha^{(1)} \times \alpha^{(2)} \in T^\times_{\mathcal{H}}(U) \times T^\times_{\mathcal{H}}(U)$ with

$$\begin{aligned} \alpha^{(1)} &= (\iota^*\alpha_1, \dots, \iota^*\alpha_n, \iota^*\alpha_{2n+1}, \dots, \iota^*\alpha_{3n}), \\ \alpha^{(2)} &= (\iota^*\alpha_{n+1}, \dots, \iota^*\alpha_{2n}, \iota^*\alpha_{3n+1}, \dots, \iota^*\alpha_{4n}). \end{aligned}$$

Then $\omega^H_k(U_H) \simeq R'$, $\Omega^1_{Y_H}(U_H) \simeq R'[\underline{X}^H]_1$, $\Omega^1_{Y_G\times Y_G}(U \times U) \simeq R[\underline{X}_1]_1 \otimes R[\underline{X}_2]_1$, and $\Omega^1_{Y_H/Y_G\times Y_G}(U_H) \simeq R'[\underline{X}_0]_1$.

Let $\partial^H_{ij} \in \mathrm{Der}_{\mathbb{Q}(\zeta_N)}(R', R')$, $1 \leqslant i \leqslant j \leqslant 2n$, be the dual basis of X^H_{ij} and write $\partial^H = (\partial^H_{ij})$ in $n\times n$ blocks as $\partial^H = \begin{pmatrix} \partial_1 & \partial_0 \\ {}^{\mathrm{t}}\partial_0 & \partial_2 \end{pmatrix}$. According to (2.2) there is a nonzero degree e homogeneous polynomial $P_\kappa(T_{ij}) \in \mathbb{Q}[T_{ij}]_{1\leqslant i\leqslant j\leqslant n}$ and a polynomial Q in $\underline{X}_0, \partial_0, \underline{Y}_1, \underline{Y}_2$ whose degree in ∂_0 is strictly less than e such that

$$\varrho_{k,\kappa} \circ D^e_k = u_\iota(P_\kappa(X_{0,ij}\partial_{0,ij}) + Q(\underline{X}_0, \partial_0, \underline{Y}_1, \underline{Y}_2)),$$

where $u_\iota : R' \to R\otimes R$ is the quotient map corresponding to the embedding $Y_G \times Y_G \hookrightarrow Y_H$. The holomorphic projection $\mathscr{A}_{k+\kappa}$ is purely defined on $Y_G \times Y_G$, so does not involve any ∂_0. Thus,

$$\begin{aligned} D_{k,k+\kappa} &= \mathscr{A}_{k+\kappa} \circ u_\iota\Big(P_\kappa(X_{0,ij}\partial_{0,ij}) + Q(\underline{X}_0, \partial_0, \underline{Y}_1, \underline{Y}_2)\Big) \\ &= u_\iota\left(P_\kappa(X_{0,ij}\partial_{0,ij}) + \tilde{Q}(\underline{X}_0, \partial_0, \partial_1, \partial_2)\right) \end{aligned}$$

with $\tilde{Q}$ being a polynomial whose degree in ∂_0 is still strictly less than e. This implies that the differential operator $D_{k,k+\kappa} \neq 0$, and the coefficient for the highest weight vector $\prod_{i=1}^n \det_i(\underline{X}_0)$ is $\prod_{i=1}^n \det_i(\partial_0)^{\kappa_i-\kappa_{i+1}} + c(\partial_0, \partial_1, \partial_2)$, where the total degree of the homogeneous polynomial of c is e and every term involves either ∂_1 or ∂_2. □

Remark 2.19. — In the construction of $D_{k,k+\kappa}$, the increase of the weight is contributed by the co-normal differential sheaf $\Omega^1_{Y_H/Y_G\times Y_G}$. We do not consider the part $\Omega^1_{Y_G\times Y_G}$ because its contribution will be killed by the holomophic projection.

Remark 2.20. — Besides Shimura, holomorphic differential operators are also studied by Böcherer [6], Ibukiyama [27] uisng invariant pluri-harmonic polynomials and Harris [19] using Grothendieck's sheaves of differentials. Harris' approach shows the uniqueness (up to scalars) of holomorphic differential operators in many cases (including the case considered above). Therefore, all the holomorphic differential operators constructed in different approaches must be the same up to scalars. On the other hand, different approaches yield different descriptions of the holomorphic differential operators, and have their own advantages in applications.

3. Overconvergent nearly holomorphic forms and their p-adic families

3.1. The weight space

Let p be an odd prime number. The weight space $\mathcal{W}$ is the rigid analytic space defined over $\mathbb{Q}_p$ associated to the noetherian complete algebra $\mathbb{Z}_p[[\mathbf{T}^\circ(\mathbb{Z}_p)]]$. Its $\mathbb{C}_p$-points parametrize continuous homomorphisms from $\mathbf{T}^\circ(\mathbb{Z}_p)$ to $\mathbb{C}_p^\times$, i.e. $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{T}^\circ(\mathbb{Z}_p), \mathbb{C}_p^\times)$. For $\kappa \in \mathcal{W}$ we can write it as $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ with κ_i being a continuous character of $\mathbb{G}_m(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$ such that $\kappa(\mathrm{diag}(a_1, \ldots, a_n)) = \prod_{i=1}^n \kappa_i(a_i)$. If we fix a topological generator of $1 + p\mathbb{Z}_p$, say $1 + p$, then $\mathcal{W}$ can be identified with the disjoint union of n-dimensional open unit balls indexed by $\mathbf{T}^\circ\widehat{(\mathbb{Z}/p\mathbb{Z})}$, the character group of the torsion part $\mathbf{T}^\circ(\mathbb{Z}/p\mathbb{Z})$ of the group $\mathbf{T}^\circ(\mathbb{Z}_p)$. Explicitly we can write the isomorphism as

$$\begin{aligned} \mathcal{W} &\longrightarrow \mathbf{T}^\circ\widehat{(\mathbb{Z}/p\mathbb{Z})} \times \prod_{i=1}^n \mathcal{B}(1, 1^-) \\ \kappa &\longmapsto (\kappa|_{\mathbf{T}^\circ(\mathbb{Z}/p\mathbb{Z})}, \kappa_1(1+p), \kappa_2(1+p), \ldots, \kappa_n(1+p)). \end{aligned}$$

Here $\mathcal{B}(1,1^-)$ is the 1-dimensional rigid open unit ball centered at 1. For an affinoid subdomain $\mathcal{U}\subset\mathcal{W}$, we use $\mathcal{A}(\mathcal{U})$ to denote the affinoid algebra of analytic functions on $\mathcal{U}$ and $\mathcal{A}(\mathcal{U})^\circ$ to denote the subset of $\mathcal{A}(\mathcal{U})$ consisting of power bounded elements.

Given κ we say it is algebraic if $\kappa(\mathrm{diag}(a_1,\dots,a_n)) = a_1^{k_1}a_2^{k_2}\dots a_n^{k_n}$ with k_i, $1\leqslant i\leqslant n$, being integers, and it is arithmetic if it can be written as the product of an algebraic weight and a locally constant character. If κ is arithmetic, we denote by κ_{alg} (resp. κ_{f}) its algebraic part (resp. locally constant part).

Let μ_{p-1} be the group of $(p-1)$-th roots of unity. There is a universal character

$$\kappa^{\mathrm{un}}:\mathcal{W}\times\mathbf{T}^\circ(\mathbb{Z}_p)\longrightarrow\mu_{p-1}\times\mathcal{B}(1,1^-).$$

Take L to be an extension of $\mathbb{Q}_p$ inside $\mathbb{C}_p$, complete with a valuation v such that $v(p)=1$. Denote by m_L the maximal ideal of $\mathcal{O}_L$. For each $w\in v(m_L)$ we can define over L the rigid analytic group $\mathcal{T}^\circ_{1,w}\cong\prod_{i=1}^n\mathcal{B}(1,p^w)$ with $\mathcal{B}(1,p^w)$ being the 1-dimensional closed ball of radius p^w centered at 1 and the rigid analytic group $\mathcal{T}^\circ_w = T^\circ(\mathbb{Z}_p)\mathcal{T}^\circ_{1,w}$. For any affinoid subdomain $\mathcal{U}\subset\mathcal{W}$ there exists some $w\in v(m_L)$ such that the universal character $\kappa^{\mathrm{un}}|_{\mathcal{U}\times\mathbf{T}^\circ(\mathbb{Z}_p)}$ extends to a map between rigid analytic spaces

$$\kappa^{\mathrm{un}}:\mathcal{U}\times\mathcal{T}^\circ_w\longrightarrow\mu_{p-1}\times\mathcal{B}(1,1^-).$$

For such $\mathcal{U}$ and w we say that the universal character κ^{un} over $\mathcal{U}$ is w-analytic. In order to see the existence of such a w, it suffices to look at the case where $\mathcal{U}$ is a closed ball inside the identity connected component $\mathcal{W}^\circ$ of the weight space, i.e. $\mathcal{U}=\mathcal{W}(t)^\circ=\prod_{i=1}^n\mathcal{B}(1,p^t)$ for some $t\in v(m_L)$. Let $Y_1,\dots,Y_n$ (resp. $S_1,\dots,S_n$) be the coordinates of $\mathcal{W}(t)^\circ$ (resp. the neighborhood $a\cdot\prod_{i=1}^n\mathcal{B}(1,p^w)=\prod_{i=1}^n\mathcal{B}(a_i,p^w)$ of $a=\mathrm{diag}(a_1,\dots,a_n)\in\mathbf{T}^\circ(\mathbb{Z}_p)$) with coordinate ring $\mathcal{A}(\mathcal{W}(t)^\circ) = L\langle Y_1,\dots,Y_n\rangle$ (resp. $L\langle S_1,\dots,S_n\rangle$). The universal character can be extended to $\mathcal{W}(t)^\circ\times a\cdot\prod_{i+1}^n\mathcal{B}(1,p^w)$ as long as $(1+p^tY_i)^{a_i}(1+p^tY_i)^{\frac{\log(1+p^wS_i)}{\log(1+p)}}$ belongs to $L\langle Y_i,S_i\rangle$ for all $1\leqslant i\leqslant n$. The factor $(1+p^tY_i)^{a_i}=\sum_{j=1}^\infty\binom{a_i}{j}p^{tj}Y_i^j$ is always inside $1+p^t\mathcal{O}_L\langle Y_i\rangle$, and the factor $(1+p^tY_i)^{\frac{\log(1+p^wS_i)}{\log(1+p)}} = \exp(\log(1+p^tY_i)\cdot\frac{\log(1+p^wS_i)}{\log(1+p)})$ lies inside $L\langle Y_i,S_i\rangle$ if we choose w large enough such that the supreme norm of the function $\log(1+X)$ over $\mathcal{B}(0,p^t)$ satisfies $|\log(1+X)|_{\mathcal{B}(o,p^t)}<p^{w-\frac{1}{p-1}}$. If the universal weight κ^{un} is w-analytic over $\mathcal{U}$, then it is obvious that any point $\kappa\in\mathcal{U}(L)$ is a w-analytic weight.

Let $\mathfrak{T}^\circ_{1,w}$ be the formal group defined by

$$\mathfrak{T}^\circ_{1,w}(R)=\mathrm{Ker}\left(\mathbf{T}^\circ(R)\longrightarrow\mathbf{T}^\circ(R/p^wR)\right)\tag{3.1}$$

for all flat, p-adically complete $\mathcal{O}_L$-algebras R. As a formal scheme $\mathfrak{T}^\circ_{1,w}$ is isomorphic to $\mathrm{Spf}(\mathcal{O}_L\langle S_1,\dots,S_n\rangle)$. The identity component $\mathcal{W}(t)^\circ$ of $\mathcal{W}(t)$ has a natural formal model $\mathfrak{W}(t)^\circ$ isomorphic to $\mathrm{Spf}(\mathcal{O}_L\langle Y_1,\dots,Y_n\rangle)$. Given an affinoid subdomain $\mathcal{U}\subset\mathcal{W}(t)^\circ$ and an open formal subscheme $\mathfrak{U}$ of an admissible blow-up of $\mathfrak{W}(t)^\circ$ such that $\mathcal{U}$ is the rigid fibre of $\mathfrak{U}$, the above discussion shows that for $w\in v(m_L)$ big enough the formal universal character

$$\kappa^{\mathrm{un}}:\mathfrak{U}\times\mathfrak{T}^\circ_{1,w}\longrightarrow\widehat{\mathbb{G}}_m$$

can be defined and it specializes to a formal character $\kappa:\mathfrak{T}^\circ_{1,w}\to\widehat{\mathbb{G}}_m$ for each $\kappa\in\mathcal{U}(L)$.

3.2. The analytic $(\mathfrak{g},\mathcal{Q}_w)$-modules $V_{\kappa,w}$ and $V_{\kappa^{\mathrm{un}},w}$

This section is an analogue of Section 2.3 in the p-adic analytic and formal setting. Fix the p-adic field L and $w\in v(m_L)$ as in the previous section. Let $\mathfrak{A}_L$ be the category of L-affinoid algebras and $\mathbf{Adm}_{\mathcal{O}_L}$ be the category of admissible $\mathcal{O}_L$-algebras, i.e. the flat $\mathcal{O}_L$-algebras that are quotients of $\mathcal{O}_L\langle X_1,\dots,X_s\rangle$ for some $s\in\mathbb{N}$. First we define several rigid analytic groups and formal groups. Like the formal torus $\mathfrak{T}^\circ_{1,w}$ we define the formal groups $\mathfrak{M}^\circ_{1,w}$ and $\mathfrak{B}_{1,w}$ over $\mathcal{O}_L$ by

$$\begin{aligned}\mathfrak{M}^\circ_{1,w}(R)&=\mathrm{Ker}\,(\mathrm{GL}(n,R)\longrightarrow\mathrm{GL}(n,R/p^wR)),\\ \mathfrak{B}_{1,w}(R)&=\mathrm{Ker}\,(\mathbf{B}(R)\longrightarrow\mathbf{B}(R/p^wR))\end{aligned}$$

for all $R\in\mathbf{Adm}_{\mathcal{O}_L}$. Define $\mathfrak{N}_{1,w}$ to be the unipotent part of $\mathfrak{B}_{1,w}$. Let $\mathrm{GL}(n)_{\mathrm{an}}$, $\mathbf{B}_{\mathrm{an}}$, $\mathbf{N}_{\mathrm{an}}$, $\mathbf{T}^\circ_{\mathrm{an}}$, $\mathbf{Q}_{\mathrm{an}}$, $\mathbf{U}_{\mathrm{an}}$ be the rigid analytic groups associated to the groups schemes $\mathrm{GL}(n)$, $\mathbf{B}$, $\mathbf{N}$, $\mathbf{T}^\circ$, $\mathbf{Q}$, $\mathbf{U}$, and $\mathrm{GL}(n)_{\mathrm{rig}}$, $\mathbf{B}_{\mathrm{rig}}$, $\mathbf{T}^\circ_{\mathrm{rig}}$, $\mathbf{Q}_{\mathrm{rig}}$ be the generic fibre of the formal completion of $\mathrm{GL}(n)$, $\mathbf{B}$, $\mathbf{T}^\circ$, $\mathbf{Q}$ along p. The rigid fibre $\mathcal{M}^\circ_{1,w}$, $\mathcal{B}_{1,w}$, $\mathcal{T}^\circ_{1,w}$ of the formal groups $\mathfrak{M}^\circ_{1,w}$, $\mathfrak{B}_{1,w}$, $\mathfrak{T}^\circ_{1,w}$ can be naturally regarded as rigid analytic subgroups of $\mathrm{GL}(n)_{\mathrm{rig}}$, $\mathbf{B}_{\mathrm{rig}}$, $\mathbf{T}^\circ_{\mathrm{rig}}$. Set $I(\mathbb{Z}_p)=\{g\in\mathrm{GL}(n,\mathbb{Z}_p):g\mod p\in\mathbf{B}(\mathbb{Z}/p\mathbb{Z})\}$ to be the Iwahori subgroup of $\mathrm{GL}(n,\mathbb{Z}_p)$ and $N_I^-(\mathbb{Z}_p)$ to be the unipotent subgroup of $I(\mathbb{Z}_p)$ consisting of lower triangular matrices with 1 on the diagonal. There is the Iwahori decomposition $I(\mathbb{Z}_p)=N_I^-(\mathbb{Z}_p)\,\mathbf{B}(\mathbb{Z}_p)$. We define the rigid analytic subgroup $\mathcal{I}_w$ of $\mathrm{GL}(n)_{\mathrm{rig}}$ by $\mathcal{I}_w=I(\mathbb{Z}_p)\cdot\mathcal{M}^\circ_{1,w}$. Fixing a set S of representatives in $I(\mathbb{Z}_p)$ of $I(\mathbb{Z}/p^{[w]}\mathbb{Z})$, the group $\mathcal{I}_w$ can be written as the disjoint union $\coprod_{\gamma\in S}\gamma\cdot\mathcal{M}^\circ_{1,w}$. Similarly we define $\mathcal{B}_w=\mathbf{B}(\mathbb{Z}_p)\cdot\mathcal{B}_{1,w}\subset\mathbf{B}_{\mathrm{rig}}$. The group $\mathcal{T}^\circ_w=\mathbf{T}^\circ(\mathbb{Z}_p)\cdot\mathcal{T}^\circ_{1,w}\subset\mathbf{T}^\circ_{\mathrm{rig}}$ is already defined in last section.

There is a projection $\pi : \mathbf{Q}_{\mathrm{an}} \to \mathrm{GL}(n)_{\mathrm{an}}$ sending $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to a. We define the rigid analytic subgroup $\mathcal{Q}_w \subset \mathbf{Q}_{an}$ as

$$\mathcal{Q}_w = \pi^{-1}(\mathcal{I}_w) \tag{3.2}$$

Note that $\mathcal{Q}_w$ is not contained inside $\mathbf{Q}_{\mathrm{rig}}$.

Now take $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathcal{W}(\mathbb{C}_p)$ to be a w-analytic weight and set $\kappa' = (-\kappa_n, \dots, -\kappa_1)$ which is also w-analytic. Extend κ' to a character of $\mathcal{B}_w$ through the quotient map $\mathcal{B}_w \to \mathcal{T}_w^\circ$. Define the w-analytic left $\mathcal{I}_w$-module $W_{\kappa,w}$ by

$$W_{\kappa,w}(R) = \left\{ f : \mathcal{I}_w(R) \to R \text{ analytic} \;\middle|\; \begin{array}{l} f(xb) = \kappa'(b) f(x) \\ \text{for all } b \in \mathcal{B}_w(R),\, x \in \mathcal{I}_w(R) \end{array} \right\}$$

for all $R \in \mathfrak{A}_L$, with $\mathcal{I}_w$ acting through the left inverse translation. Because of the Iwahori decomposition, $W_{\kappa,w}$ consists of analytic functions on

$$N_I^-(\mathbb{Z}/p^{[w]}\mathbb{Z}) \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mathcal{B}(0,p^w) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}(0,p^w) & \mathcal{B}(0,p^w) & \cdots & 1 \end{pmatrix}.$$

Therefore as a module over R we see $W_{\kappa,w}(R) = \bigoplus_{N_I^-(\mathbb{Z}/p^{[w]}\mathbb{Z})} R\langle T_{ij}\rangle_{1\leqslant j<i\leqslant n}$, i.e. $|N_I^-(\mathbb{Z}/p^{[w]}\mathbb{Z})|$ copies of strictly convergent power series in $n(n-1)/2$ variables.

From this description we see that there is a natural formal model of $W_{\kappa,w}$, whose R-points are $\oplus_{N_I^-(\mathbb{Z}/p^{[w]}\mathbb{Z})} R\langle T_{ij}\rangle_{1\leqslant j<i\leqslant n}$ for $R \in \mathbf{Adm}_{\mathcal{O}_L}$, equipped with a functorial action of $I(\mathbb{Z}_p)$ and $\mathfrak{M}_{1,w}^\circ$. We denote the formal model still by $W_{\kappa,w}$.

With $W_{\kappa,w}$ we define the w-analytic $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa,w}$ in the same way as we define the $(\mathfrak{g}, \mathbf{Q})$-module V_κ from the algebraic representation W_κ of $\mathrm{GL}(n)$ in Section 2.3. For all $R \in \mathfrak{A}_L$

$$V_{\kappa,w}(R) = W_{\kappa,w}(R) \otimes_R R[\underline{Y}] = W_{\kappa,w}(R) \otimes_R R[Y_{ij}]_{1\leqslant i\leqslant j\leqslant n}.$$

The action of $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathcal{Q}_w$ and $\mu_{ij}^- \in \mathfrak{u}^-$ on $P(Y) \in V_{\kappa,w}$ is given by the formulas

$$(g \cdot P)(\underline{Y}) = a \cdot P(a^{-1}b + a^{-1}\underline{Y}d), \tag{3.3}$$

$$
\begin{aligned}
(\mu_{ij}^{-}\cdot P)(\underline{Y}) &= \sum_{1\leqslant k\leqslant n}(Y_{ki}\varepsilon_{kj}+Y_{kj}\varepsilon_{ki})\cdot P(Y)\\
&\qquad - \sum_{1\leqslant k\leqslant l\leqslant n}(Y_{ki}Y_{jl}+Y_{kj}Y_{il})\frac{\partial}{\partial Y_{kl}}P(Y),\qquad i\neq j,\\
(\mu_{ii}^{-}\cdot P)(\underline{Y}) &= \sum_{1\leqslant k\leqslant n}Y_{ki}\varepsilon_{ki}\cdot P(\underline{Y}) - \sum_{1\leqslant k\leqslant l\leqslant n}Y_{ki}Y_{il}\frac{\partial}{\partial Y_{kl}}P(\underline{Y}).
\end{aligned}
\tag{3.4}
$$

The compatibility is checked in the same way as in Section 2.3. As $\mathcal{Q}_w$-representations we have the filtration

$$
\operatorname{Fil}^r V_{\kappa,w}(R) = V_{\kappa,w}^r(R) = W_{\kappa,w}(R)\otimes_R R[\underline{Y}]_{\leqslant r}
$$

satisfying $\mathfrak{g}\cdot V_{\kappa,w}^r \subset V_{\kappa,w}^{r+1}$. By definition, if we regard the $\mathcal{I}_w$-representation $W_{\kappa,w}$ as a $\mathcal{Q}_w$-representation via the projection $\mathcal{Q}_w \to \mathcal{I}_w$, then $V_{\kappa,w}^0 = W_{\kappa,w}$ as $\mathcal{Q}_w$-representations. For $i\in\mathbb{Z}$ we can twist $V_{\kappa,w}$ by the i-th power of the multiplier ν and get the w-analytic $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa,w}(i)$.

Recall that J is defined to be the algebraic representation V_{triv}^1 of $\mathbf{Q}$. It restricts to an analytic $\mathcal{Q}_w$-representation. Parallel to (2.9) we have

$$
V_{\kappa,w}^r \cong V_{\kappa,w}^0\otimes \operatorname{Sym}^r J = W_{\kappa,w}\otimes\operatorname{Sym}^r J
$$

as analytic $\mathcal{Q}_w$-representations.

More generally, given $(\rho, W_\rho)\in \operatorname{Rep}_{\mathbb{Z},f}\mathrm{GL}(n)$, an algebraic representation of $\mathrm{GL}(n)$ free of finite rank, the tensor product $W_{\kappa\otimes\rho,w} = W_{\kappa,w}\otimes W_\rho$ is an analytic $\mathcal{I}_w$-representation, and we can define the corresponding analytic $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa\otimes\rho,w}$ and $\mathcal{Q}_w$-representation $V_{\kappa\otimes\rho,w}^r$ for $r\geqslant 0$.

All of the above constructions carry over to the universal w-analytic weight κ^{un} over an affinoid subdomain $\mathcal{U}\subset\mathcal{W}$.

3.3. The Andreatta–Iovita–Pilloni construction

We briefly recall the constructions in [2, Chapter 3,4,5]. Let σ be the Frobenious endomorphism of $\mathcal{O}_L/p\mathcal{O}_L$. For any finite group scheme H over $\mathcal{O}_L$, we denote by H^D its Cartier dual and ω_H its sheaf of invariant differentials. Given a Barsotti–Tate group G over $\mathcal{O}_L$ of dimension n, the Hasse invariant $\mathrm{Ha}(G)\in\det(\omega_{G[p]^D})^{\otimes p-1}$ is defined to be the determinant of the σ-linear endomorphism on $\omega_{G[p]^D}$ induced by the relative Frobenius. The Hodge height $\mathrm{Hdg}(G)\in[0,1]$ is defined as the truncated valuation of $\mathrm{Ha}(G)$.

Let $\mathbf{NAdm}_{\mathcal{O}_L}$ be subcategory of $\mathbf{Adm}_{\mathcal{O}_L}$ consisting of those objects that are normal. Fix $R\in\mathbf{NAdm}_{\mathcal{O}_L}$ and suppose that G is a rank n

semi-abelian scheme over $S = \mathrm{Spec}(R)$ whose restriction to an open dense subscheme of S is abelian. Take a positive integer $m \in \mathbb{N}_{>0}$ and $v < \frac{1}{2p^{m-1}}$ (resp. $v < \frac{1}{3p^{m-1}}$ if $p = 3$) such that for any $x \in S_{\mathrm{rig}}$ the Hodge height $\mathrm{Hdg}(x) := \mathrm{Hdg}(G_x[p^\infty]) \leqslant v$. Write R_w to denote $R/p^w R$. We summarize, in the following theorem, some results about canonical subgroups in families used in [2].

Theorem 3.1 ([2, Proposition 4.1.3, Proposition 4.3.1]). — *There is a finite flat canonical subgroup $H_m \subset G[p^m]$ of level m over S, which, at each point $x \in S_{\mathrm{rig}}$, specializes to the canonical subgroup $H_{m,x} \subset G_x[p^m]$ as constructed in* [17, Theorem 6]. *Moreover, assuming $H_m^D(R[1/p]) \simeq (\mathbb{Z}/p^m\mathbb{Z})^n$, then there is a free sub-sheaf of R-modules $\mathcal{F} \subset \omega_G$ of rank n containing $p^{\frac{v}{p-1}}\omega_G$, equipped with an isomorphism*

$$\mathrm{HT}_w : H_m^D(R[1/p]) \otimes_{\mathbb{Z}} R_w \xrightarrow{\sim} \mathcal{F} \otimes_R R_w,$$

induced from the Hodge–Tate map on H_m^D, for all $w \in (0, m - v\frac{p^m}{p-1}] \cap v_p(\mathcal{O}_L)$.

Fix $N \geqslant 3$ prime to p. Let K be a finite extension of $\mathbb{Q}_p$ with valuation v such that $v(p) = 1$ and a uniformizer ϖ. Denote by Y the Siegel variety defined over $\mathcal{O}_K$ parametrizing principally polarized abelian schemes of dimension n with principal level N structure. Let X be a smooth toroidal compactification. The universal abelian scheme $A \to Y$ extends to a semi-abelian scheme $G \to X$. Set $\mathfrak{X}$ to be the formal scheme obtained by completing X along its special fibre. On the associated rigid analytic space $X_{\mathrm{rig}} = X_{\mathrm{an}}$, we have the Hodge height function $\mathrm{Hdg} : X_{\mathrm{rig}} \to [0,1]$. For $v \in v(\mathcal{O}_K)$ we define the open subset $\mathcal{X}(v) = \{x \in X_{\mathrm{rig}} : \mathrm{Hdg}(x) \leqslant v\}$. Let $\tilde{\mathfrak{X}}(v)$ be the admissible blow-up of $\mathfrak{X}$ along the ideal (Ha, p^v), and $\mathfrak{X}(v)$ be the p-adic completion of the normalization of the largest open formal sub-scheme of $\tilde{\mathfrak{X}}(v)$ where the ideal (Ha, p^v) is generated by Ha. This $\mathfrak{X}(v)$ is a formal model of $\mathcal{X}(v)$. By construction the semi-abelian scheme $G \to X$ gives rise to semi-abelian schemes over $\mathcal{X}(v)$ and $\mathfrak{X}(v)$, which we still denote by G. For $m \in \mathbb{N}_{>0}$ and $v < \frac{1}{2p^{m-1}}$ (resp. $v < \frac{1}{3p^{m-1}}$ if $p = 3$), there is the level m canonical subgroup $H_m \subset G[p^m]$. Define $\mathcal{X}_1(p^m)(v) = \underline{\mathrm{Isom}}_{\mathcal{X}(v)}((\mathbb{Z}/p^m\mathbb{Z})^n, H_m^D)$ to be the finite étale cover of $\mathcal{X}(v)$ parametrizing the trivializations ψ of the Cartier dual of H_m. The group $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{X}_1(p)(v)$. The quotient $\mathcal{X}_{\mathrm{Iw}}(v) = \mathcal{X}_1(p)(v))/\mathbf{B}(\mathbb{Z}/p\mathbb{Z})$ is also finite étale over $\mathcal{X}(v)$. As formal models of $\mathcal{X}_1(p^m)(v)$, $\mathcal{X}_{\mathrm{Iw}}(v)$, we take $\mathfrak{X}_1(p^m)(v)$, $\mathfrak{X}_{\mathrm{Iw}}(v)$ to be the normalizations of $\mathfrak{X}(v)$ inside the corresponding rigid spaces. There is the chain of formal schemes

$$\mathfrak{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathfrak{X}_{\mathrm{Iw}}(v) \xrightarrow{\pi_0} \mathfrak{X}(v).$$

Let $\mathfrak{Y}, \mathfrak{Y}(v), \mathfrak{Y}_1(p^m)(v), \mathfrak{Y}_{\mathrm{Iw}}(v)$ be the open formal subschemes of $\mathfrak{X}, \mathfrak{X}(v)$, $\mathfrak{X}_1(p^m)(v)$, $\mathfrak{X}_{\mathrm{Iw}}(v)$ that are the complements of the boundary C. Although $\mathfrak{Y}(v)$, $\mathfrak{Y}_1(p^m)(v)$, $\mathfrak{Y}_{\mathrm{Iw}}(v)$ are not moduli spaces, they admit modular interpretations for $R \in \mathbf{NAdm}$ (cf. [2, Proposition 5.2.1.1]). Let Y_{an} be the analytification of Y with the natural open immersion $Y_{\mathrm{an}} \hookrightarrow X_{\mathrm{an}}$. Set $\mathcal{Y}(v), \mathcal{Y}_1(p^m)(v), \mathcal{Y}_{\mathrm{Iw}}(v)$ to be the fibre products of $\mathcal{X}(v), \mathcal{X}_1(p^m)(v), \mathcal{X}_{\mathrm{Iw}}(v)$ with Y_{an} over X_{an}.

By the construction of $\mathfrak{X}_1(p^m)(v)$, we can apply Theorem 3.1 to construct a locally free sub-sheaf $\mathcal{F} \subset \omega(\mathcal{G}/\mathfrak{X}_1(p^m)(v))$ of rank n, equipped with the isomorphism

$$\mathrm{HT}_w \circ \psi : (\mathbb{Z}/p^m\mathbb{Z})^n \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}_1(p^m)(v),w} \xrightarrow{\sim} \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w} \tag{3.5}$$

for $w \in (0, m - v\frac{p^m}{p-1}] \cap v(\mathcal{O}_K)$.

From now on we assume $w \in (m-1+\frac{v}{p-1}, m - \frac{p^m v}{p-1}] \cap v(\mathcal{O}_K)$, so m is determined by w. Define the $\mathfrak{M}^\circ_{1,w}$-torsor $\mathfrak{T}^\times_{\mathcal{F},w}(v)$ over $\mathfrak{X}_1(p^m)(v)$ by

$$\mathfrak{T}^\times_{\mathcal{F},w}(v) = \underline{\mathrm{Isom}}_{\mathfrak{X}_1(p^m)(v),\psi,w}(\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}, \mathcal{F}),$$

where the subscript ψ, w means that we require the isomorphism to be w-compatible with (3.5) as explained below. We always fix the canonical global basis of the n copies of the structure sheaf $\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}$ and the canonical basis of the $\mathbb{Z}/p^m\mathbb{Z}$-module $(\mathbb{Z}/p^m\mathbb{Z})^n$. Then locally over $\mathfrak{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(p^m)(v)$, an isomorphism α from R^n to $\mathcal{F}(\mathfrak{U})$ corresponds to an ordered basis $\alpha_1, \dots \alpha_n$ of the free R-module $\mathcal{F}(\mathfrak{U})$ and ψ gives rise to an ordered basis $x_1, \dots, x_n$ of $H^D_m(R[1/p])$. We say that α is w-compatible with (3.5) if $\alpha_i \equiv \mathrm{HT}_w(x_i) \mod p^w R$ for all $1 \leqslant i \leqslant n$. An element $a \in \mathfrak{M}^\circ_{1,w}(R)$ acts on α by sending it to $\alpha \circ a$, or equivalently sending the corresponding basis $(\alpha_1, \dots, \alpha_n)$ to $(\alpha_1, \dots, \alpha_n)\, a$. This action makes $\mathfrak{T}^\times_{\mathcal{F},w}(v)$ a $\mathfrak{M}^\circ_{1,w}$-torsor over $\mathfrak{X}_1(p^m)(v)$.

For a w-analytic weight $\kappa \in \mathcal{W}(K)$ we can form the contracted product and get a locally free formal sheaf

$$\tilde{\mathfrak{w}}^\dagger_{\kappa,w} := \mathfrak{T}^\times_{\mathcal{F},w}(v) \times^{\mathfrak{M}^\circ_{1,w}} W_{\kappa,w}$$

over $\mathfrak{X}_1(p^m)(v)$. In particular this $\tilde{\mathfrak{w}}^\dagger_{\kappa,w}$ is a flat formal Banach sheaf in the sense of [2, Definition A.1.1.1]. Therefore we can apply the procedure worked out in [2, A.2.2] to get the associated Banach sheaf $\tilde{\omega}^\dagger_{\kappa,w}$ over the rigid analytic fibre $\mathcal{X}_1(p^m)(v)$ (see [2, Definition A.2.1.2] for the definition of a Banach sheaf). For any affinoid subdomain $\mathcal{U} \subset \mathcal{X}_1(p^m)(v)$ and an admissible blow-up $h : \mathfrak{X}' \to \mathfrak{X}_1(p^m)(v)$ such that $\mathcal{U}$ is the rigid fibre of an open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}'$, the local sections of $\tilde{\omega}^{\dagger}_{\kappa,w}$ over $\mathcal{U}$ are

$$\tilde{\omega}^{\dagger}_{\kappa,w}(\mathcal{U}) = h^{*}\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K,$$

which is naturally equipped with a complete norm (independent of h up to equivalence) with $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}(\mathfrak{U})$ being the unit ball.

The group $I(\mathbb{Z}/p^m\mathbb{Z})$ acts on $\mathfrak{X}_1(p^m)(v)$ with $\mathfrak{X}_{\mathrm{Iw}}(v)$ as the quotient. Under this action the sheaf $\tilde{\omega}^{\dagger}_{\kappa,w}$ is $I(\mathbb{Z}/p^m\mathbb{Z})$-equivariant. Since $I(\mathbb{Z}/p^m\mathbb{Z})$ is a finite group, the $I(\mathbb{Z}/p^m\mathbb{Z})$-invariant of the pushforward $\pi_{1,*}\tilde{\omega}^{\dagger}_{\kappa,w}$ is a Banach sheaf over $\mathcal{X}_{\mathrm{Iw}}(v)$.

Definition 3.2. — *The Banach sheaf of w-analytic, v-overconvergent, weight κ Siegel modular forms of principal level N is defined as*

$$\omega^{\dagger}_{\kappa,w} := (\pi_{1,*}\tilde{\omega}^{\dagger}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})}.$$

We also want to associate to the Banach sheaf $\omega^{\dagger}_{\kappa,w}$ a contracted product interpretation, which will bring us some convenience when defining certain morphisms. By taking the rigid fibre of the $\mathfrak{M}^{\circ}_{1,w}$-torsor $\mathfrak{T}^{\times}_{\mathcal{F},w}(v)$ over $\mathfrak{X}_1(p^m)(v)$, we get

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v) \xrightarrow{\pi_2} \mathcal{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathcal{X}_{\mathrm{Iw}}(v).$$

The rigid analytic space $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ is a $\mathcal{M}^{\circ}_{1,w}$-torsor over $\mathcal{X}_1(p^m)(v)$ and the cover $\pi_1 : \mathcal{X}_1(p^m)(v) \to \mathcal{X}_{\mathrm{Iw}}(v)$ is finite étale. The group $I(\mathbb{Z}_p)$ acts on $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$. The $I(\mathbb{Z}_p)$-action together with the $\mathcal{M}^{\circ}_{1,w}$-torsor structure on $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ makes it an $\mathcal{I}_w$-torsor over $\mathcal{X}_{\mathrm{Iw}}(v)$. Let $\mathcal{S}$ be the category whose objects are affinoid subdomains of $\mathcal{X}_{\mathrm{Iw}}(v)$ admitting local sections of the projection $\pi_1 \circ \pi_2$ with inclusions as morphisms. We can define a presheaf on $\mathcal{S}$ by the contracted product

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}. \tag{3.6}$$

It is isomorphic to the restriction of the sheaf $\omega^{\dagger}_{\kappa,w}$ to $\mathcal{S}$. We call (3.6) a contracted product interpretation of $\omega^{\dagger}_{\kappa,w}$. Since the objects of $\mathcal{S}$ form a basis of the Grothendieck topology on $\mathcal{X}_1(p^m)(v)$, in order to construct morphisms between the sheaves over $\mathcal{X}_{\mathrm{Iw}}(v)$, it suffices to construct morphisms between their restrictions to $\mathcal{S}$ by using the contracted product interpretation. By abuse of notation we write $\omega^{\dagger}_{\kappa,w} = \mathcal{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}$.

Define $\mathcal{O}_K$ schemes $T_\omega = \underline{\mathrm{Hom}}_X(\mathcal{O}^n_X, \omega(\mathcal{G}/X))$, $T^{\times}_\omega = \underline{\mathrm{Isom}}_X(\mathcal{O}^n_X, \omega(\mathcal{G}/X))$ over X. Let $T_{\omega,\mathrm{an}}, T^{\times}_{\omega,an}$ be their rigid analytifications, and $\mathfrak{T}_\omega, \mathfrak{T}^{\times}_\omega$ be their formal completions along the special fibres. Also take $T_{\omega,\mathrm{rig}}, T^{\times}_{\omega,\mathrm{rig}}$ to be the rigid fibre of $\mathfrak{T}_\omega$, $\mathfrak{T}^{\times}_\omega$. Set $\mathcal{T}_{\omega,\mathrm{an}}(v), \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v), \mathcal{T}_{\omega,\mathrm{rig}}(v), \mathcal{T}^{\times}_{\omega,\mathrm{rig}}(v)$ to be the corresponding base changes to $\mathcal{X}_{\mathrm{Iw}}(v)$. Due to the requirement

$w \in (m-1+\frac{v}{p-1}, m-v\frac{p^m}{p-1}] \cap v(\mathcal{O}_K)$, the argument of [2, Proposition 5.3.1] shows that there is a natural open immersion

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v) \hookrightarrow \mathcal{T}_{\omega,\mathrm{rig}}(v) \cap \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v).$$

Local sections of the projection $\mathcal{T}^{\times}_{\mathcal{F},w}(v) \to \mathcal{X}_{\mathrm{Iw}}(v)$ correspond to local basis of the sheaf $\omega(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))$ satisfying w-compatibility conditions defined by the Hodge–Tate map HT_w. Note that $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ does not lie inside $\mathcal{T}^{\times}_{\omega,\mathrm{rig}}(v)$. When κ is algebraic this open immersion induces a canonical inclusion of $\omega_\kappa|_{\mathcal{X}_{\mathrm{Iw}}(v)}$ into $\omega^{\dagger}_{\kappa,w}$.

In [2] another two formal schemes are introduced. They are defined as

$$\mathfrak{IW}_w(v) = \mathfrak{T}^{\times}_{\mathcal{F},w}/\mathfrak{B}_{1,w}, \qquad \text{and} \qquad \mathfrak{IW}^+_w(v) = \mathfrak{T}^{\times}_{\mathcal{F},w}/\mathfrak{N}_{1,w},$$

with maps

$$\mathfrak{IW}^+_w(v) \xrightarrow{g} \mathfrak{IW}_w(v) \xrightarrow{\pi_3} \mathfrak{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathfrak{X}_{\mathrm{Iw}}(v).$$

The group $\mathfrak{T}^{\circ}_{1,w}$ acts on $\mathfrak{IW}^+_w(v)$ over $\mathfrak{IW}_w(v)$, and so on the pushforward of the structure sheaf $g_*\mathcal{O}_{\mathfrak{IW}^+_w(v)}$. Define the invertible sheaf $\mathfrak{L}_\kappa = g_*\mathcal{O}_{\mathfrak{IW}^+_w(v)}[\kappa']$ to be the κ'-invariant of the $\mathfrak{T}^{\circ}_{1,w}$-action on $g_*\mathcal{O}_{\mathfrak{IW}^+_w(v)}$. Take the rigid fibres $\mathcal{IW}^+_w(v)$, $\mathcal{IW}^+_w(v)$, $\mathcal{L}_\kappa$. There is a $\mathbf{B}(\mathbb{Z}_p)$-action on $\mathcal{IW}^+_w(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ which, together with κ, makes $\pi_{3,*}\mathcal{L}_\kappa$ a $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$-equivariant Banach sheaf with respect to the natural $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$-action on $\mathcal{X}_1(p^m)(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$. In [2] the invariant $(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{B(\mathbb{Z}/p^m\mathbb{Z})}$ is defined to be the Banach sheaf of w-analytic, v-overconvergent, weight κ Siegel modular forms. It is easy to see that the map $W_{\kappa,w} \to \mathbb{A}^1_K$ by evaluation at identity induces an isomorphism between $(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})}$ and the sheaf $\omega^{\dagger}_{\kappa,w}$ in Definition 3.2.

All the above constructions run parallelly for the w-analytic universal weight κ^{un} corresponding to $\mathcal{U} \subset \mathcal{W}$, so that we can define the Banach sheaf $\omega^{\dagger}_{\kappa^{\mathrm{un}},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}$ and the flat formal Banach sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}$ over $\mathfrak{X}_1(p^m)(v) \times \mathfrak{U}$.

3.4. Nearly overconvergent Siegel modular forms

3.4.1. The Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ and its global sections

Recall that in Section 2.4 we defined the locally free sheaf of finite rank $\mathcal{J}$ over X, and for $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ we have $\mathcal{V}^r_\rho \cong \omega_\rho \otimes \mathrm{Sym}^r\,\mathcal{J}$ as locally free sheaves with Hecke actions. Take the rigid analytification of $\mathcal{J}$ and pull it back to $\mathcal{X}_{\mathrm{Iw}}(v)$. We denote the resulting coherent sheaf over $\mathcal{X}_{\mathrm{Iw}}(v)$

still by $\mathcal{J}$. Similarly, let $\mathfrak{J}$ be the locally free formal sheaf of finite rank over $\mathfrak{X}_{\mathrm{Iw}}(v)$ obtained by completing $\mathcal{J}$ along the special fibre of X and pulling it back. Then $\mathrm{Sym}^r \mathcal{J}$ is the rigid fibre of $\mathrm{Sym}^r \mathfrak{J}$. Since $\mathrm{Sym}^r \mathcal{J}$ is locally free of finite rank and $\mathcal{X}_{\mathrm{Iw}}(v)$ is quasi-compact, it can be equipped with a Banach sheaf structure by choosing a cover and local basis. All such structures are equivalent to the one given by the formal model $\mathrm{Sym}^r \mathfrak{J}$. The tensor product of $\mathrm{Sym}^r \mathcal{J}$ with a Banach sheaf is still a Banach sheaf under the tensor product semi-norm. The flatness of $\mathrm{Sym}^r \mathcal{J}$ guarantees that the sheaf conditions of the Banach sheaf are preserved under the operation of tensoring with $\mathrm{Sym}^r \mathcal{J}$. Also, the spaces of local sections of the tensor product sheaf is complete with respect to the tensor product semi-norm.

Definition 3.3. — *The Banach sheaf of w-analytic, v-overconvergent nearly holomorphic forms of principal level N, weight κ (resp. universal weight κ^{un} over $\mathcal{U} \subset \mathcal{W}$) and (non-holomorphy) degree r is defined as*

$$\mathcal{V}_{\kappa,w}^{\dagger,r} := \omega_{\kappa,w}^{\dagger} \otimes \mathrm{Sym}^r \mathcal{J} \qquad (\textit{resp. } \mathcal{V}_{\kappa^{\mathrm{un}},w}^{\dagger,r} := \omega_{\kappa^{\mathrm{un}},w}^{\dagger} \otimes \mathrm{Sym}^r \mathcal{J}).$$

The space of global sections of a Banach sheaf over a quasi-compact rigid analytic space can be equipped with a norm by choosing a suitable admissible covering by affinoids. All such norms are equivalent and the space of global sections are complete under these norms.

Definition 3.4. — *The K-Banach space (resp. $\mathcal{A}(\mathcal{U})$-Banach module) of w-analytic, v-overconvergent nearly holomorphic forms of principal level N, weight κ (resp. universal weight κ^{un} over $\mathcal{U} \subset \mathcal{W}$) and (non-holomorphy) degree r is defined as*

$$N_{\kappa,w,v}^{\dagger,r} := H^0(\mathcal{X}_{\mathrm{Iw}}(v), \mathcal{V}_{\kappa,w}^{\dagger,r}) \quad (\textit{resp. } N_{\mathcal{U},w,v}^{\dagger,r} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}_{\kappa^{\mathrm{un}},w}^{\dagger,r}),$$

and the corresponding cuspidal part is defined as

$$N_{\kappa,w,v,\mathrm{cusp}}^{\dagger,r} := H^0(\mathcal{X}_{\mathrm{Iw}}(v), \mathcal{V}_{\kappa,w}^{\dagger,r}(-C))$$
$$(\textit{resp. } N_{\mathcal{U},w,v,\mathrm{cusp}}^{\dagger,r} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}_{\kappa^{\mathrm{un}},w}^{\dagger,r}(-C)).$$

Following [49] we also call overconvergent nearly holomorphic forms nearly overconvergent forms.

For later use we also define a locally free formal Banach sheaf $\tilde{\mathfrak{V}}_{\kappa,w}^{\dagger,r}$ over $\mathfrak{X}_1(p^m)(v)$ as

$$\tilde{\mathfrak{V}}_{\kappa,w}^{\dagger,r} := \tilde{\mathfrak{w}}_{\kappa,w}^{\dagger} \otimes \mathrm{Sym}^r \mathfrak{J}.$$

Let $\tilde{\mathcal{V}}_{\kappa,w}^{\dagger,r}$ be its rigid fibre which is an $I(\mathbb{Z}/p^m\mathbb{Z})$-equivariant Banach sheaf. Then we have $\mathcal{V}_{\kappa,w}^{\dagger,r} = (\pi_{1,*}\tilde{\mathcal{V}}_{\kappa,w}^{\dagger,r})^{I(\mathbb{Z}/p^m\mathbb{Z})}$.

3.4.2. The $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ and contracted product interpretation of $\mathcal{V}^{\dagger,r}_{\kappa,w}$

The definition of the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ as $\omega^{\dagger}_{\kappa,w}\otimes\operatorname{Sym}^r\mathcal{J}$ is already convenient for constructing unramified Hecke operators and $\mathbb{U}_p$-operators. However, for the construction of differential operators and holomorphic projections, it is preferable to have a contracted product interpretation involving a $\mathcal{Q}_w$-torsor and the $\mathcal{Q}_w$-submodule $V^r_{\kappa,w}$ of the $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa,w}$ defined in Section 3.2.

The $\mathcal{O}_K$-scheme $T^{\times}_{\mathcal{H}}=\underline{\mathrm{Isom}}_X(\mathcal{O}^{2n}_X,\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}})$ is defined as in Section 2.1. Let $T^{\times}_{\mathcal{H},\mathrm{an}}$ be its analytification and $\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)$ be the base change to $\mathcal{X}_{\mathrm{Iw}}(v)$. There is a natural projection

$$\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)\longrightarrow\mathcal{T}^{\times}_{\omega,\mathrm{an}}(v).$$

We define the $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ as

$$\mathcal{T}^{\times}_{\mathcal{H},w}(v):=\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)\times_{\mathcal{T}^{\times}_{\omega,\mathrm{an}}(v)}\mathcal{T}^{\times}_{\mathcal{F},w}(v).$$

It is not difficult to see that the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ admits the following contracted product interpretation

$$\mathcal{V}^{\dagger,r}_{\kappa,w}=\mathcal{T}^{\times}_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w}V^r_{\kappa,w}.$$

3.4.3. Summary

We record below several interpretations of the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ and its global sections, which we will use later for convenience according to different purposes.

(i) $\mathcal{V}^{\dagger,r}_{\kappa,w}=\omega^{\dagger}_{\kappa,w}\otimes\operatorname{Sym}^r\mathcal{J}$,

(ii) $\mathcal{V}^{\dagger,r}_{\kappa,w}=(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})}\otimes\operatorname{Sym}^r\mathcal{J}$, and for global sections

$$N^{\dagger,r}_{\kappa,w,v}=H^0(\mathcal{IW}_w(v),\mathcal{L}_\kappa\otimes(\pi_1\circ\pi_3)^*\operatorname{Sym}^r\mathcal{J})^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})},$$

(iii) $\mathcal{V}^{\dagger,r}_{\kappa,w}=(\pi_{1,*}\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})}$, and for global sections

$$\begin{aligned}N^{\dagger,r}_{\kappa,w,v}&=H^0(\mathcal{X}_1(p^m)(v),\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})}\\&=(H^0(\mathfrak{X}_1(p^m)(v),\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}\otimes\pi_1^*\operatorname{Sym}^r\mathfrak{J})[1/p])^{I(\mathbb{Z}/p^m\mathbb{Z})},\end{aligned}$$

(iv) $\mathcal{V}^{\dagger,r}_{\kappa,w}=\mathcal{T}^{\times}_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w}V^r_{\kappa,w}$.

It is easy to see that in all the above constructions we can replace κ by the w-analytic universal weight κ^{un} corresponding to $\mathcal{U}\subset\mathcal{W}$, and consider the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U}$ as well as the $\mathcal{A}(\mathcal{U})$-Banach module $N^{\dagger,r}_{\mathcal{U},w,v}:=H^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U},\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w})$.

In the following we need also to consider the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} := \omega^{\dagger}_{\kappa,w}\otimes\omega_\rho\otimes\operatorname{Sym}^r\mathcal{J}$ and its global sections $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ for some $(\rho,W_\rho)\in\operatorname{Rep}_{\mathbb{Z},f}\operatorname{GL}(n)$. Here ω_ρ is the base change to $\mathcal{X}_{\mathrm{Iw}}(v)$ of the analytification of the automorphic sheaf $\mathcal{E}(W_\rho)$. From $\mathcal{E}(W_\rho)$ one also gets the locally free formal sheaf of finite rank $\mathfrak{w}_\rho$ over $\mathfrak{X}_{\mathrm{Iw}}(v)$ whose rigid fibre is ω_ρ. When working with $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w}$ and $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$, we can replace $\operatorname{Sym}^r\mathcal{J}$ and $\operatorname{Sym}^r\mathfrak{J}$ in (ii), (iii) by $\omega_\rho\otimes\operatorname{Sym}^r\mathcal{J}$, $\mathfrak{w}_\rho\otimes\operatorname{Sym}^r\mathfrak{J}$, and $V^r_{\kappa,w}$ in (iv) by $V^r_{\kappa\otimes\rho,w}$.

3.5. The Banach $\mathcal{A}(\mathcal{U})$-module $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is projective

The goal of this section is to prove the proposition below following the arguments in [2, Section 8].

Proposition 3.5. — *$N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is a projective Banach $\mathcal{A}(\mathcal{U})$-module. For every $\kappa\in\mathcal{U}$ with the corresponding maximal ideal $\mathfrak{m}_\kappa\subset\mathcal{A}(\mathcal{U})$, we have $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}\otimes\mathcal{A}(\mathcal{U})/\mathfrak{m}_\kappa\xrightarrow{\sim}N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}}$.*

Proof. — We use the interpretation (iii) in Section 3.4.3 and the same proof works if we replace κ^{un} by $\kappa^{\mathrm{un}}\otimes\rho$ with $\rho\in\operatorname{Rep}_{\mathbb{Z},f}\operatorname{GL}(n)$. Our case differs very little from that in [2, Section 8]. Instead of repeating the whole proof here, we just point out the main ingredients there and explain that their arguments for the formal Banach sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ are applicable to $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)=\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}\otimes\pi_1^*\operatorname{Sym}^r\mathfrak{J}(-C)$. Below for simplicity we write $\pi_1^*\operatorname{Sym}^r\mathfrak{J}$ as $\operatorname{Sym}^r\mathfrak{J}$.

We use the notation in [2, Section 8.2]. Let $X^\star$ be the minimal compactification of Y. There is a proper morphism $X\to X^\star$. Like $\mathfrak{X}(v)$ one can define $\mathfrak{X}^\star(v)$ to be the p-adic completion of the normalization of the largest open formal subscheme of the blow-up of $\mathfrak{X}^\star$ along the ideal (Ha,p^v) where it is generated by Ha. We have the projection $\eta:\mathfrak{X}_1(p^m)(v)\to\mathfrak{X}^\star(v)$. We may assume that $\mathcal{U}$ lies inside the identity component $\mathcal{W}^\circ$ and take $\mathfrak{U}$ to be the open formal subscheme of an admissible blow-up of $\mathfrak{W}^\circ$ whose rigid fibre is $\mathcal{U}$. We use the subscript l to mean reduction modulo ϖ^l. [2, Corollary 8.1.6.2] shows that $\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa^{\mathrm{un}},w}$ is a small formal Banach sheaf over $\mathfrak{X}_1(p^m)(v)$ with $\operatorname{Sym}^r\mathfrak{J}_1$ as the required coherent sheaf in the definition of small formal Banach sheaves (cf. [2, Definition A.1.2.1]).

First we claim that the proposition follows from the following base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$. For all $l \in \mathbb{N}$, considering the diagram

$$\begin{tikzcd} X_1(p^m)(v)_l \times \mathfrak{U}_l \arrow[r, "i"] \arrow[d, "\eta_l \times 1"] & X_1(p^m)(v)_{l+1} \times \mathfrak{U}_{l+1} \arrow[d, "\eta_{l+1} \times 1"] \\ X^{\star}(v)_l \times \mathfrak{U}_l \arrow[r, "i'"] & X^{\star}(v)_{l+1} \times \mathfrak{U}_{l+1} \end{tikzcd},$$

the base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ is

$$i'^{*}(\eta_{l+1} \times 1)_{*}\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w,l+1}(-C) = (\eta_l \times 1)_{*}\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w,l}(-C). \tag{3.7}$$

Once this base change property is proved, we deduce that $(\eta \times 1)_{*}\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ is a small formal Banach sheaf with $(\eta \times 1)_{*} \operatorname{Sym}^r \mathfrak{J}_1$ as the required coherent sheaf. Then applying [2, Theorem A.1.2.2] and the arguments in [2, Corollary 8.2.3.1, 8.2.3.2], we conclude that the module $H^0(\mathcal{X}_1(p^m)(v) \times \mathcal{U}, \tilde{\mathcal{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C))$ is a projective Banach $\mathcal{A}(\mathcal{U})$-module and the map

$$H^0(\mathcal{X}_1(p^m)(v)\times\mathcal{U}, \tilde{\mathcal{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C))\otimes \mathcal{A}(\mathcal{U})/\mathfrak{m}_{\kappa} \longrightarrow H^0(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger}_{\kappa,w}(-C))$$

is an isomorphism. The statement of the proposition follows by taking the invariant of the finite group $I_n(\mathbb{Z}/p^m\mathbb{Z})$.

We are left to show the base change property (3.7). Like in [2, Section 8], we look at the projection η_l in a formal neighborhood of a geometric point in the minimal compactification and reduce the base change property to the vanishing of certain locally free sheaves over abelian schemes (3.11). In [2, Section 8], only the vanishing for ρ being trivial is needed.

Let $V' \subset V = \mathbb{Z}^{\oplus 2n}$ be an isotropic direct factor of rank r'. We start by recalling the description given in [2, Section 8.2] of the localization of the projection from the toroidal compactification to the minimal compactification at a geometric point $\bar{x} \in X^{\star}(v)_l$ belonging to the stratum $Y_{V',l} \subset X^{\star}_l$. Let $A_{V'} \to Y_{V'}$ be the universal abelian scheme, and $\mathcal{B}_{V'} \to Y_{V'}$ be the abelian scheme parametrizing the extensions of $A_{V'}$ by $V' \otimes \mathbb{G}_m$ which is isogenous to $A^{r'}_{V'}$. Over $\mathcal{B}_{V'}$ lies $\mathcal{M}_{V'}$ which is a torsor under the torus with character group $S^{\vee}_{V'}$, isogenous to $\mathrm{Hom}(\operatorname{Sym}^2 V/V'^{\perp}, \mathbb{G}_m)$. Let $\mathcal{M}_{V'} \to \mathcal{M}_{V',\mathcal{S}}$ be the torus embedding associated to a polyhedral decomposition $\mathcal{S}$ of the cone $C(V/V'^{\perp})$ of symmetric semi-definite bilinear forms on $V/V'^{\perp}$. Like in Section 3.3 one defines $\mathfrak{Y}_{V'}(v), \mathfrak{Y}_1(p^m)_{V'}(v), \mathfrak{B}_{V'}(v), \mathfrak{M}_{V',\mathcal{S}}(v)$. Put $\mathfrak{B}_1(p^m)_{V'}(v) = \mathfrak{B}_{V'}(v) \times_{\mathfrak{A}^r_{V'}} (\mathfrak{A}_{V'}/H_{m,V'})^r$ and $\mathfrak{M}_1(p^m)_{V',\mathcal{S}}(v) = \mathfrak{M}_{V',\mathcal{S}}(v) \times_{\mathfrak{B}_{V'}(v)} \mathfrak{B}_1(p^m)_{V'}(v)$. The completion $\widehat{X_1(p^m)(v)}_{l,\bar{x}}$ at a geometric point $\bar{x}$ inside $Y_{V'}(v)_l \subset X^{\star}(v)_l$ is isomorphic to a disjoint union of spaces $\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{l,\bar{y}}/\Gamma_1(p^m)_{V'}$ with some geometric point $\bar{y} \in Y_1(p^m)_{V'}(v)_l$. The spaces fit into the diagram

$$\begin{tikzcd}
\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{l,\bar{y}} \arrow[r, "h_2"] \arrow[d, "h_1"] & \mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{l,\bar{y}}/\Gamma_1(p^m)_{V'} \arrow[r] \arrow[d] & X_1\widehat{(p^m)}(v)_{l,\bar{x}}. \\
\mathcal{B}_1(\widehat{p^m)_{V'}}(v)_{l,\bar{y}} \arrow[r] & Y_1(\widehat{p^m)_{V'}}(v)_{l,\bar{y}}
\end{tikzcd} \tag{3.8}$$

Because of the exact sequence

$$\begin{aligned}0 \longrightarrow \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,1} \otimes \operatorname{Sym}^r \mathfrak{J}_1(-C) \xrightarrow{\varpi^{l-1}} \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,l} \otimes \operatorname{Sym}^r \mathfrak{J}_l(-C) \\ \longrightarrow \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,l-1} \otimes \operatorname{Sym}^r \mathfrak{J}_{l-1}(-C) \to 0,\end{aligned}$$

the base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ will follow from the vanishing result

$$H^1(\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}, \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,1} \otimes \operatorname{Sym}^r \mathfrak{J}_1(-C)) = 0 \tag{3.9}$$

for all $\kappa \in \mathcal{U}$. The coherent $\operatorname{Sym}^r \mathfrak{J}$ has a filtration with graded pieces being automorphic sheaves attached to algebraic $\mathrm{GL}(n)$-representations that are free of finite rank and the sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,1}$ is an inductive limit of iterated extensions of the trivial sheaf [2, Corollary 8.1.6.2]. Therefore, (3.9) will follow from the general vanishing result: For all $\rho \in \operatorname{Rep}_{\mathbb{Z},f} \mathrm{GL}(n)$ and $i > 0$,

$$H^i(\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}, \mathfrak{w}_{\rho,1}(-C)) = 0, \tag{3.10}$$

where $\mathfrak{w}_{\rho,1}$ is the pullback to $\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}$ of the automorphic sheaf ω_ρ on X. The proof of (3.10) is an adaption of [36, Section 8.2] in the situation (3.8). It is enough to show that

$$H^i\Big(\Gamma_1(p^m)_{V'}, H^j(\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}, h_2^*\mathfrak{w}_{\rho,1}(-C))\Big) = 0, \quad \text{if } i+j>0.$$

Over $\mathcal{B}_{V'}$ there is the universal semi-abelian scheme

$$0 \longrightarrow V' \otimes \mathbb{G}_m \longrightarrow G_{V'} \longrightarrow A_{V'} \longrightarrow 0.$$

Using the $\mathrm{GL}(n)$-torsor $\underline{\mathrm{Isom}}_{\mathcal{B}_{V'}}(\mathcal{O}^n_{\mathcal{B}_{V'}}, \omega(G_{V'}/\mathcal{B}_{V'}))$ one constructs a locally free sheaf of finite rank $\omega_{V',\rho}$ over $\mathcal{B}_{V'}$, whose pullback $\mathfrak{w}_{V',\rho,1}$ to $\mathcal{B}_1(\widehat{p^m)_{V'}}(v)_{l,\bar{y}}$ satisfies

$$h_1^*\mathfrak{w}_{V',\rho,1} = h_2^*\mathfrak{w}_{\rho,1}.$$

The action of $\Gamma_1(p^m)_{V'}$ on $S_{V'}$ factors through a quotient which acts freely on $\{\lambda \in S_{V'} \cap C(V/V'^{\perp})^\vee : \lambda > 0\}$. Take S_0 to be a set of representatives of the orbits. Applying [36, Lemma 8.2.3.12], [16, Theorem V.2.7] we get

$$\begin{aligned}
&H^i\Big(\Gamma_1(p^m)_{V'}, H^j(\mathcal{M}_1\widehat{(p^m)_{V'}}_{,\mathcal{S}}(v)_{1,\bar{y}}, h_2^*\mathfrak{w}_{\rho,1}(-C))\Big)\\
&= H^i\left(\Gamma_1(p^m)_{V'}, \prod_{\substack{\lambda\in S_{V'}\cap C(V/V'^{\perp})^\vee,\\ \lambda>0}} H^j(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes\mathfrak{w}_{V',\rho,1})\right)\\
&= \begin{cases}\prod_{\lambda\in S_0} H^j(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes\mathfrak{w}_{V',\rho,1}) & i=0\\ 0 & i>0,\end{cases}
\end{aligned}$$

where $\mathcal{L}(\lambda)$ is an ample invertible sheaf over the abelian scheme $\mathfrak{B}_1(p^m)_{V'}(v)$ for $\lambda \in S_0$ [16, p. 143]. We reduce to show

$$H^j(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes\mathfrak{w}_{V',\rho,1}) = 0, \quad \text{if } j>0. \tag{3.11}$$

Over $\mathcal{B}_{V'}$, the sheaf of invariant differentials of the torus part and the quotient abelian part of the semi-abelian scheme $G_{V'}$ can be trivialized. Hence the sheaf $\omega_{V',\rho}$ can be constructed by using a torsor of a unipotent subgroup $N_{V'} \subset \mathrm{GL}(n)$ with the $N_{V'}$-representation $\rho|_{N_{V'}}$. Then [36, Lemma 8.2.4.16] says that $\rho|_{N_{V'}}$ admits a filtration with $N_{V'}$ acting trivially on each graded piece. Thus $\omega_{V',\rho}$ is an iterated extension of the trivial sheaf, and (3.11) follows from the vanishing results for $H^j(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda))$, $j>0$ [39, Section III.16]. □

3.6. The differential operators

Let $\Omega^1_{\mathcal{X}_{\mathrm{Iw}}(v)}$ be the sheaf of differentials on $\mathcal{X}_{\mathrm{Iw}}(v)$ defined as in [18, Ex. 4.4.1]. Over $\mathcal{X}_{\mathrm{Iw}}(v)$ we have the integrable Gauss–Manin connection

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))^{\mathrm{can}} \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))^{\mathrm{can}} \otimes \Omega^1_{\mathcal{X}_{\mathrm{Iw}}(v)}(\log C).$$

For a w-analytic weight $\kappa \in \mathcal{W}(\mathbb{C}_p)$ and $\rho \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$, we defined in Section 3.2 the $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa\otimes\rho,w}$. The Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} = \omega^\dagger_{\kappa,w}\otimes\omega_\rho\otimes\mathrm{Sym}^r\mathcal{J}$ on $\mathcal{X}_{\mathrm{Iw}}(v)$ has the contracted product interpretation $\mathcal{T}^\times_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w} V_{\kappa\otimes\rho,w}$. Using this contracted product interpretation and the construction in Section 2.2, we obtain a connection

$$\nabla_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \longrightarrow \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}\otimes\Omega_{\mathcal{X}_{\mathrm{Iw}}(v)}(\log C) \cong \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}(-1).$$

Recall that τ is the symmetric square of the standard representation of $\mathrm{GL}(n)$. Composing it with $t^+ : \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}(-1) \to \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}$ we get the following differential operator which can be thought of as a p-adic analytic version of the Maass–Shimura differential operators

$$D_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \longrightarrow \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho\otimes\tau,w}.$$

Besides, there is the Shimura's E-operator [45, Section 12.9]

$$E_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \xrightarrow{\epsilon_{\kappa\otimes\rho,w}} \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}(1) \xrightarrow{t^-} \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w},$$

whose construction relies only on the fact that we have the morphism of $\mathcal{Q}_w$-representations

$$V^r_{\kappa\otimes\rho,w}/V^0_{\kappa\otimes\rho,w} \longrightarrow V^{r-1}_{\kappa\otimes\rho,w} \otimes V^0_{\tau^\vee}(1) = V^{r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w}(1).$$

We can also iterate the operators and obtain

$$D^e_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\rho,w} \longrightarrow \mathcal{V}^{\dagger,r+e}_{\kappa\otimes\rho\otimes\mathrm{Sym}^e\,\tau,w},$$
$$E^e_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\rho,w} \longrightarrow \mathcal{V}^{\dagger,r-e}_{\kappa\otimes\rho\otimes\mathrm{Sym}^e\,\tau^\vee,w},$$

for $e \in \mathbb{N}$. A section of the sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w}$ lies inside $\mathcal{V}^{\dagger,r'}_{\kappa\otimes\rho,w}$ for $0 \leqslant r' < r$ if and only is it is annihilated by $E^{r'+1}_{\kappa\otimes\rho,w}$.

Also, note that because there is a natural isomorphism $V^r_{\kappa\otimes\rho,w}/V^{r-1}_{\kappa\otimes\rho,w} \xrightarrow{\sim} V^0_{\kappa\otimes\rho\otimes\mathrm{Sym}^r\,\tau^\vee,w}(r)$, we have

$$\frac{1}{r!}E^r_{\kappa\otimes\rho,w} : \mathcal{V}^{\dagger,r}_{\rho,w} \xrightarrow{\sim} \mathcal{V}^{\dagger,0}_{\kappa\otimes\rho\otimes\mathrm{Sym}^r\,\tau^\vee,w}. \tag{3.12}$$

3.7. The holomorphic projection

Besides the definition of the space of nearly holomorphic forms, its algebraic structure and the Maass–Shimura differential operators, another main ingredient in Shimura's theory of nearly holomorphic forms is the holomorphic projection. Shimura's construction [45, Proposition 14.2] can be adapted to our p-adic analytic context.

Define the functions $\mathrm{Log}_1, \dots, \mathrm{Log}_n$ on the weight space $\mathcal{W}$ by

$$\mathrm{Log}_i(\kappa) := \frac{\log_p(\kappa_i(1+p)^t)}{\log_p((1+p)^t)},$$

for $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathcal{W}$ and some $t \in \mathbb{N}$ sufficiently large. Let $K(\mathrm{Log}_1, \dots, \mathrm{Log}_n)$ be the fraction field of $K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$. For an affinoid subdomain $\mathcal{U} \subset \mathcal{W}$ such that $\kappa^{\mathrm{un}}|_{\mathcal{U}}$ is w-analytic, we prove in this section the following proposition.

Proposition 3.6. — *There is an $\mathcal{A}(\mathcal{U})$-linear continuous map*

$$\mathscr{A} : N^{\dagger,r}_{\mathcal{U},w,v} \longrightarrow N^{\dagger,0}_{\mathcal{U},w,v} \otimes_K K(\mathrm{Log}_1, \ldots, \mathrm{Log}_n)$$

whose restriction to $N^{\dagger,0}_{\mathcal{U},w,v}$ is the identity.

In order to simplify notation for the rest of this section we omit all the subscripts from the differential operators and E-operators as well as the subscript w from $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau\otimes\mathrm{Sym}^{e'}\tau^\vee,w}$.

Suppose that $\mathrm{Spm}(R) \subset \mathcal{X}_{\mathrm{Iw}}(v)$ is an affinoid subdomain such that there exists a section $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$, and we regard α as a basis $(\alpha_1, \ldots, \alpha_{2n})$ of $\mathcal{H}^1_{\mathrm{dR}}(A/R)$ satisfying certain conditions. Given $D \in \mathrm{Der}_K(R, R)$, define $X(D, \alpha) \in \mathfrak{g} \otimes R$ by

$$\nabla(D)\alpha = \alpha \cdot X(D, \alpha),$$

and denote by $\overline{X(D, \alpha)}$ its image in the quotient $\mathfrak{g}/\mathfrak{q} \cong \mathfrak{u}^-$. The Levi subgroup $\mathbf{M}$ acts on $\mathfrak{u}^-$ by conjugation, i.e. $a \in \mathrm{GL}(n, R)$ acts on $\overline{X(D, \alpha)}$ by sending it to ${}^{\mathrm{t}}a^{-1}\overline{X(D, \alpha)}a^{-1}$. This $\mathrm{GL}(n)$-action is isomorphic to $\tau^\vee$. Given $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$ and a basis $\{e_i\}_{1\leqslant i\leqslant n(n+1)/2}$ of the $\mathrm{GL}(n)$-representation τ, the dual basis $\{e_i^\vee\}_{1\leqslant i\leqslant n(n+1)/2}$ gives rise to a basis $\{D_{e_i^\vee,\alpha}\}_{1\leqslant i\leqslant n(n+1)/2}$ of the tangent space $\mathrm{Der}_K(R, R)$. One can check by definition that the element $\overline{X(D_{e_i^\vee,\alpha}, \alpha)}$ inside $\mathfrak{u}^- \otimes R$ is independent of the choice of $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$, and we abbreviate it as $\overline{X(e_i^\vee)}$.

Lemma 3.7. — $\left\{\overline{X(D_{e_i^\vee})}\right\}_{1\leqslant i\leqslant n(n+1)/2}$ *form a basis of* $\mathfrak{u}^- \otimes R \cong \tau^\vee(R)$, *which is dual to the basis* $\{e_i\}_{1\leqslant i\leqslant n(n+1)/2}$.

Proof. — The statement is an equality statement and does not depend on the choice of $\{e_i\}_{1\leqslant i\leqslant n(n+1)/2}$. Hence it suffices to prove it for the Siegel variety Y, and we can further reduce to the Siegel upper half space $\mathfrak{h}_n$ and take α to be the holomorphic basis $(\mathrm{d}w, \beta)$ of $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathfrak{h}_n}/\mathfrak{h}_n)$ constructed in Section 2.5. Denote by KS the Kodaira–Spencer map. Explicit computation using (2.15) shows that

$$\mathrm{KS}(\mathrm{d}w_i \mathrm{d}w_j) = 2\pi i \cdot \mathrm{d}z_{ij} \quad 1 \leqslant i \leqslant j \leqslant n. \tag{3.13}$$

Put $\underline{X} = (X_{ij})$ as in Section 2.6. Then $(X_{ij})_{1\leqslant i\leqslant j\leqslant n}$ can be regarded as a basis spanning the representation τ. It is dual to the basis μ^-_{ij} of $\mathfrak{u}^-$. (3.13) shows that $\mathrm{d}z_{ij}$ corresponds to X_{ij} under the basis $(\mathrm{d}w, \beta)$ so $\partial/\partial z_{ij} = D_{X^\vee_{ij},(\mathrm{d}w,\beta)}$. By (2.15) we have $\overline{X(X^\vee_{ij})} = \mu^-_{ij}$ and the statement is proved. $\square$

The morphism $\tau \otimes \tau^\vee \to \mathrm{triv}$ of $\mathrm{GL}(n)$-representations induces the contraction operator

$$\theta^e : \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau \otimes \mathrm{Sym}^e \tau^\vee} \longrightarrow \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}}.$$

Lemma 3.8. — *The composition*

$$E^e \theta^e D^e : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee} \xrightarrow{D^e} \mathcal{V}^{\dagger,e}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee \otimes \mathrm{Sym}^e \tau} \xrightarrow{\theta^e} \mathcal{V}^{\dagger,e}_{\kappa^{\mathrm{un}}} \xrightarrow{E^e} \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee}$$

is an $\mathcal{O}_{\mathcal{X}_{\mathrm{Iw}(v)}} \times \mathcal{U}$-linear morphism of Banach sheaves over $\mathcal{X}_{\mathrm{Iw}(v)} \times \mathcal{U}$ induced by an endomorphism of the $\mathcal{Q}_w$-representation $V^0_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee}$.

Proof. — There exists a contraction map

$$\tilde{\theta}^e : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee \otimes \mathrm{Sym}^e \tau \otimes \mathrm{Sym}^e \tau^\vee} \to \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee}$$

induced from a morphism of the corresponding representations such that

$$E^e \theta^e D^e = \tilde{\theta}^e E^e D^e.$$

Therefore it is enough to show that the map

$$E^e D^e : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee} \longrightarrow \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee \otimes \mathrm{Sym}^e \tau \otimes \mathrm{Sym}^e \tau^\vee}$$

is induced from a morphism of $\mathcal{I}_w$-representations. Still take $\underline{X} = (X_{ij})$ as a basis of τ and write $V_{\kappa^{\mathrm{un}},w} = W_{\kappa^{\mathrm{un}},w}[\underline{Y}]$ with $\underline{Y} = (Y_{ij})_{1 \leqslant i \leqslant j \leqslant n}$ as in Section 3.2. Locally over $\mathrm{Spm}(R) \subset \mathcal{X}_{\mathrm{Iw}}(v)$, we fix a section $\alpha \in \mathcal{T}^\times_{\mathcal{H},w}(v)(R)$ and let $D_{X^\vee_{ij},\alpha}$ be the basis of $\mathrm{Der}_K(R,R)$ associated to $X^\vee_{ij}$ and α. With these choices of local coordinates, the map $E^e D^e$ can be written as

$$\begin{aligned} &\mathcal{T}^\times_{\mathcal{H},w}(v)(R) \times^{\mathcal{Q}_w(R)} V^0_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee}(R) \\ &\qquad\qquad \xrightarrow{E^e D^e} \mathcal{T}^\times_{\mathcal{H},w}(v)(R) \times^{\mathcal{Q}_w(R)} V^0_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee \otimes \mathrm{Sym}^e \tau \otimes \mathrm{Sym}^e \tau^\vee}(R) \\ &\qquad (\alpha, u) \longmapsto (\alpha, P_{\alpha,u,e}(\underline{X}, \underline{Y})), \end{aligned}$$

with $P_{\alpha,u,e}(\underline{X}, \underline{Y})$ being a homogenous polynomial of degree e in $\underline{X}$ and degree e in $\underline{Y}$ whose coefficients lie in $V^0_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee}(R)$. The claim that $E^e D^e$ is induced from a morphism of $\mathcal{I}_w$-representations is equivalent to the equality

$$a \cdot (P_{\alpha,u,e}(\underline{X}, \underline{Y})) = P_{\alpha \cdot a,u,e}(\underline{X}, \underline{Y}), \tag{3.14}$$

for all $a \in \mathcal{I}_w(R)$ and $u \in V^0_{\kappa^{\mathrm{un}} \otimes \mathrm{Sym}^e \tau^\vee}(R)$. By (2.2) the operator E^e annihilates all terms in $D^e((\alpha, u))$ involving derivations of the base ring R or the action of $\mathfrak{q} \subset \mathfrak{g}$, because they do not increase the degree in $\underline{Y}$. Thus,

$$P_{\alpha,u,e}(\underline{X}, \underline{Y}) = \sum_{1 \leqslant i \leqslant j \leqslant n} \left(\overline{X(X^\vee_{ij})} \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y}) \right) X_{ij},$$

where $\overline{X(X_{ij}^\vee)}$ is regarded as an element of $\mathfrak{u}^-$ through $\mathfrak{u}^- \cong \mathfrak{g}/\mathfrak{q}$. We show (3.14) by induction. The $e = 0$ case is true by definition of the contracted product. Assuming it is true for $e-1$, then

$$\begin{aligned}
&a \cdot P_{\alpha,u,e}(\underline{X}, \underline{Y}) \\
&\quad = a \cdot \sum_{1\leqslant i\leqslant j\leqslant n} \left(\overline{X(X_{ij}^\vee)} \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y})\right) X_{ij} \\
&\quad = \sum_{1\leqslant i\leqslant j\leqslant n} \left(\left({}^{\mathrm{t}}a^{-1}\overline{X(X_{ij}^\vee)}a^{-1}\right) \cdot \left(a \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y})\right)\right) \left(a \cdot X_{ij}\right) \\
&\quad = \sum_{1\leqslant i\leqslant j\leqslant n} \left(\overline{X(X_{ij}^\vee)} \cdot \left(a \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y})\right)\right) X_{ij} \\
&\quad = \sum_{1\leqslant i\leqslant j\leqslant n} \left(\overline{X(X_{ij}^\vee)} \cdot P_{\alpha\cdot a,u,e-1}(\underline{X}, \underline{Y})\right) X_{ij} \\
&\quad = P_{\alpha\cdot a,u,e}(\underline{X}, \underline{Y}).
\end{aligned}$$

The second equality uses the compatibility of the action of $\mathfrak{g}$ and $\mathcal{I}_w$ and the third equality follows from Lemma 3.7. $\square$

Denote by $\varphi(\kappa^{\mathrm{un}}, e)$ the endomorphism of $W_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee} = V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee}$ giving rise to $E^e\theta^e D^e$.

Lemma 3.9. — *There exists an element* $\widetilde{\varphi} \in \mathrm{End}(W_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee,w})$ *and a nonzero* $\eta \in K[\mathrm{Log}_1, \ldots, \mathrm{Log}_n]$ *such that* $\widetilde{\varphi}\circ\varphi(\kappa^{\mathrm{un}}, e) = \varphi(\kappa^{\mathrm{un}}, e)\circ\widetilde{\varphi} = \eta$.

Proof. — As an $\mathcal{A}(\mathcal{U})$-Banach module, we have

$$W_{\kappa^{\mathrm{un}},w} \cong \oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}\mathcal{A}(\mathcal{U})\langle \underline{T}\rangle,$$

the direct sum of $|N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})|$ copies of strictly convergent power series in $\underline{T} = (T_{ij})_{1\leqslant i<j\leqslant n}$. Let $W^0 = \mathcal{A}(\mathcal{U})[\underline{T}]$ be the polynomial part of one copy. Fix a basis $\underline{Z} = (Z_{ij})_{1\leqslant i,j\leqslant n}$, $Z_{ij} = Z_{ji}$ of $\tau^\vee$ with $a \in \mathrm{GL}(n)$ acting by $a \cdot \underline{Z} = {}^{\mathrm{t}}a^{-1}\,\underline{Z}a^{-1}$. Then

$$W_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau^\vee,w} \cong \oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}\mathcal{A}(\mathcal{U})[\underline{Z}]_e\langle \underline{T}\rangle,$$

where the subscript e means homogenous polynomials of degree e. Like W^0, set $W^0_e = W^0\otimes\mathrm{Sym}^e\tau^\vee = \mathcal{A}(\mathcal{U})[\underline{Z}]_e[\underline{T}]$. Both W^0 and W^0_e are closed under the action of $\mathfrak{gl}(n)$, and $\varphi(\kappa^{\mathrm{un}}, e)$ restricts to an endomorphism of the $\mathfrak{gl}(n)$-module W^0_e. We can write W^0_e as a direct sum of its weight spaces $W^0_e = \oplus_\lambda W^0_{e,\lambda}$ with each $W^0_{e,\lambda}$ free of finite rank generated by some monomials of the form $\prod_{1\leqslant i<j\leqslant n} T_{ij}^{s_{ij}} \cdot \prod_{1\leqslant k\leqslant l\leqslant n} Z_{kl}^{t_{kl}}$, $s_{ij}, t_{kl} \geqslant 0$, $\sum t_{kl} = e$. The endomorphism $\varphi(\kappa^{\mathrm{un}}, e)$ restricts to an $\mathcal{A}(\mathcal{U})$-linear map $\varphi_\lambda : W^0_{e,\lambda} \to W^0_{e,\lambda}$

for each λ and the corresponding matrix, with respect to the basis consisting of monomials, has entries in $\mathcal{O}_K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$.

The first claim is that the determinant of φ_λ in non-zero. For $\kappa \in \mathcal{U}$ write $\varphi_{\lambda,\kappa}$ to denote the specialization of φ_λ at κ. Fix an arbitrary $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathcal{U}$ and consider $\kappa + k = (\kappa_1 + k, \dots, \kappa_n + k)$ with k varying in $\mathbb{N}$. Set $Q(k)$ to be the determinant of $\varphi_{\lambda,\kappa+k}$. It is a polynomial in k and is non-zero as observed in [45, (14.3)]. Hence the determinant of φ_λ cannot be zero.

Then in order to show the existence of the $\tilde{\varphi}$, it suffices to show that there exists $\eta \in \mathcal{O}_K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$ such that the minimal polynomial P_λ of φ_λ divides η in $\mathcal{O}_K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$ for all λ. Let L be the algebraic closure of the field $K(\mathrm{Log}_1, \dots, \mathrm{Log}_n)$. For a generic $\kappa \in \mathcal{U}$, the specialization W^0_κ of W^0 at κ is isomorphic to the irreducible Verma module with highest weight κ. According to [3, Lemma 5], for generic κ, the $\mathfrak{gl}(n)$-module $W^0_{e,\kappa} = W^0_\kappa \otimes \mathrm{Sym}^e \tau^\vee$ admits a Jordan–Hölder series of finite length with graded pieces being irreducible Verma modules, and the length is independent of κ. Let l be this length. It follows that the subset of L consisting all the eigenvalues of φ_λ for all λ is finite. Also, for each $u \in W^0_{e,\lambda,\kappa}$ with κ generic, the dimension of the space $\mathrm{Span}\{\varphi^m_{\lambda,\kappa}(u) : m \in \mathbb{N}\}$ is bounded by l. Therefore as λ varies, the degree of the minimal polynomial P_λ is uniformly bounded and all the roots are contained in a finite set. This implies the existence of the desired $\eta \in \mathcal{O}_K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$. □

Proof of Proposition 3.6. — Let $\tilde{\varphi}, \eta$ be as in the previous lemma for $e = r$. Then $\eta^{-1}\tilde{\varphi}$ induces the morphism

$$\Phi_r : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^r \tau^\vee, w} \longrightarrow \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^r \tau^\vee, w} \otimes_K K(\mathrm{Log}_1, \dots, \mathrm{Log}_n),$$

which is the inverse of $E^r\theta^r D^r$. Set $\mathscr{A}_r = 1 - \theta^r D^r \Phi_r E^r$. Then

$$E^r \mathscr{A}_r = E^r(1 - \theta^r D^r \Phi_r E^r) = E^r - (E^r\theta^r D^r \Phi_r)E^r = E^r - E^r = 0,$$

showing that $\mathscr{A}_r$ sends $N^{\dagger,r}_{\mathcal{U},w,v}$ into $N^{\dagger,r-1}_{\mathcal{U},w,v} \otimes_K K(\mathrm{Log}_1, \dots, \mathrm{Log}_n)$. Meanwhile $\mathscr{A}_r$ is identity on $N^{\dagger,r-1}_{\mathcal{U},w,v}$ because E^r annihilates $N^{\dagger,r-1}_{\mathcal{U},w,v}$. By induction we obtain the desired $\mathscr{A} = \mathscr{A}_1 \circ \mathscr{A}_2 \circ \cdots \circ \mathscr{A}_r$. □

Corollary 3.10. — *There exists a nonzero $\eta \in K[\mathrm{Log}_1, \dots, \mathrm{Log}_n]$ such that each $F \in N^{\dagger,r}_{\mathcal{U},w,v}$ can be written as*

$$\eta F = F_0 + \theta D F_1 + \cdots + \theta^r D^r F_r$$

with $F_i \in N^{\dagger,0}_{\mathcal{U}\otimes \mathrm{Sym}^i \tau^\vee, w,v}$.

3.8. Unramified Hecke operators

Let ℓ be a prime integer with $(\ell, Np) = 1$. For an element γ_ℓ of the double coset $\mathrm{GSp}(2n,\mathbb{Z}_\ell)\backslash \mathrm{GSp}(2n,\mathbb{Q}_\ell)/\mathrm{GSp}(2n,\mathbb{Z}_\ell)$, the action of the Hecke operator T_{γ_ℓ} on $N^{\dagger,r}_{\kappa,w,v}$ can be defined in the standard way by using algebraic correspondence of ℓ-(quasi-)isogenies of type γ_ℓ. Let $Y_{\mathrm{Iw},K}$ be the moduli scheme over K parametrizing principally polarized abelian schemes (A,λ) with a principal level N structure and a self-dual full flag $\mathrm{Fil}_\bullet A[p]$. Define $C_{\gamma_\ell} \subset Y_{\mathrm{Iw},K} \times Y_{\mathrm{Iw},K}$ to be the moduli space, whose R-points $C_{\gamma_\ell}(R)$ for any K-algebra R consists of (quasi-)isogenies

$$\pi : (A_1, \lambda_1, \psi_{N,1}, \mathrm{Fil}_\bullet A_1[p]) \to (A_2, \lambda_2, \psi_{N,2}, \mathrm{Fil}_\bullet A_2[p])$$

of type γ_ℓ. Here for $i = 1,2$, $\lambda_i, \psi_{N,i}$ and $\mathrm{Fil}_\bullet A_i[p]$ need to satisfy $\pi^*\lambda_2 = \nu(\gamma_\ell)\lambda_1$, $\pi \circ \psi_{N,1} = \psi_{N,2}$, $\pi \circ \mathrm{Fil}_\bullet A_1[p] = \mathrm{Fil}_\bullet A_2[p]$. Being of type γ_ℓ means that under certain $\mathbb{Z}_\ell$-basis of the Tate modules $T_\ell(A_i)$, the matrix of the morphism induced by π on Tate modules is γ_ℓ. Denote by p_1 (resp. p_2) the projection of C_{γ_ℓ} to the first (resp. second) factor. Put $\mathcal{C}_{\gamma_\ell}(v) = C_{\gamma_\ell,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},K,\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. Then we have

$$\begin{tikzcd} & \mathcal{C}_{\gamma_\ell}(v) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(v). \end{tikzcd} \tag{3.15}$$

Write $p_i^*\mathcal{T}^\times_{\mathcal{H},w}(v) = \mathcal{C}_{\gamma_\ell}(v) \times_{p_i,\mathcal{Y}_{\mathrm{Iw}}(v)} \mathcal{T}^\times_{\mathcal{H},w}(v)$. Due to the functoriality of the Hodge–Tate map and the canonical subgroups, the (quasi-)isogeny π induces an isomorphism $\pi^* : p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \to p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$ (cf. [2, Lemma 6.1.1]). Applying π^* to the first factor of the contracted product $p_i^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,w}$, we obtain

$$\pi^* : p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w} \xrightarrow{\sim} p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}.$$

The Hecke operator T_{γ_ℓ} is defined as the composition

$$\begin{aligned} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{p_{2*}} H^0(\mathcal{C}_{\gamma_\ell}(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{\pi^*} H^0(\mathcal{C}_{\gamma_\ell}(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \\ \xrightarrow{\mathrm{Tr}\, p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}). \end{aligned}$$

Such defined T_{γ_ℓ} maps bounded functions to bounded functions and restricts to an action on $N^{\dagger,r}_{\kappa,w,v}$ by the discussion of [2, Section 5.5]. The action also preserves the cuspidal part (see Remark 3.15).

3.9. The $\mathbb{U}_p$-operators

Let $T^+ \subset \mathbf{T}(\mathbb{Q})$ be the set

$$T^+ = \{\mathrm{diag}(p^{a_1}, \dots, p^{a_n}, p^{a_0-a_1}, \dots, p^{a_0-a_n}) : a_1 \leqslant \cdots \leqslant a_n, a_0 \geqslant 2a_n\}.$$

We want to attach a Hecke operator to each element of T^+. All such operators will be called $\mathbb{U}_p$-operators. Let

$$\gamma_{p,i} = \begin{pmatrix} I_i & 0 & 0 & 0 \\ 0 & I_{n-i} & 0 & 0 \\ 0 & 0 & p^2 I_i & 0 \\ 0 & 0 & 0 & I_{n-i} \end{pmatrix} \quad 1 \leqslant i \leqslant n-1, \tag{3.16}$$

$$\text{and} \quad \gamma_{p,n} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

An element $\gamma_p \in T^+$ can be uniquely written as $\gamma_p = p^{s_0} \prod_{j=1}^n \gamma_{p,j}^{s_j}$ with $s_0 \in \mathbb{Z}$, $s_1 \dots, s_n \in \mathbb{N}$. We make the scalar p act on $Y_{\mathrm{Iw},K}$ by sending $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ to $(A, \lambda, \psi_N \circ p, \mathrm{Fil}_\bullet A[p])$. This action is invertible and induces a map on the global sections of the sheaf $\mathcal{V}_{\kappa,w}^{\dagger,r}$, which we take as the Hecke operator corresponding to $p \in T^+$ and denote by $\langle p \rangle$. We define the Hecke operator attached to p^{s_0} as $\langle p \rangle^{s_0}$ for all $s_0 \in \mathbb{Z}$. It remains to define the operators $U_{p,i}$ associated to $\gamma_{p,i}$ for $1 \leqslant i \leqslant n$.

3.9.1. The operator $U_{p,n}$

Let $C_n \subset Y_{\mathrm{Iw},K} \times Y_{\mathrm{Iw},K}$ be the moduli space parametrizing the quintuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$, with $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ being the moduli problem defining $Y_{\mathrm{Iw},K}$ and $L \subset A[p]$ satisfying $L \oplus \mathrm{Fil}_n A[p] = A[p]$. Denote by $\pi : A \to A/L$ the universal isogeny. There are two projections p_1, p_2 from C_n to $Y_{\mathrm{Iw},K}$. One is by forgetting L, and the other sends $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ to $(A/L, \lambda', \pi \circ \psi_N, \mathrm{Fil}_\bullet A/L[p])$, with λ' defined by $\pi^*\lambda' = p\lambda$ and $\mathrm{Fil}_i A/L[p] = \pi \circ \mathrm{Fil}_i A[p]$, $1 \leqslant i \leqslant n$. Let

$$\mathcal{C}_n(v) = C_{n,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},K}} \mathcal{Y}_{\mathrm{Iw}}(v) \subset \mathcal{Y}_{\mathrm{Iw}}(v) \times \mathcal{Y}_{\mathrm{Iw}}(v).$$

Then by [17, Theorem 8], there is the diagram

$$\begin{tikzcd} & \mathcal{C}_n(v) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(p)(\tfrac{v}{p}). \end{tikzcd} \tag{3.17}$$

The universal isogeny π induces an isomorphism $\pi^* : p_2^*\mathcal{T}^{\times}_{\mathcal{H},w}(\frac{v}{p}) \to p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ (cf. [2, Lemma 6.2.1.2]) that gives rise to $\pi^* : p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w} \xrightarrow{\sim} p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}$. The operator $U_{p,n}$ is defined as the composition

$$
\begin{aligned}
(3.18)\quad H^0\left(\mathcal{Y}_{\mathrm{Iw}}(p)\left(\frac{v}{p}\right), \mathcal{V}^{\dagger,r}_{\kappa,w}\right) &\xrightarrow{p_{2*}} H^0(\mathcal{C}_n(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \\
&\xrightarrow{\pi^*} H^0(\mathcal{C}_n(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{p^{-n(n+1)/2}\,\mathrm{Tr}\,p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}).
\end{aligned}
$$

See Section 3.9.5 for the normalizer $p^{-n(n+1)/2}$.

3.9.2. The operators $U_{p,i}$, $i = 1,\dots,n-1$

In order to define the operators $U_{p,i}$, we first generalize the notion of w-analyticity to $\underline{w}$-analyticity for $\underline{w} = (w_{jk})_{1\leqslant k<j\leqslant n}$ satisfying

(i) $w_{jk} = w$ or $w-1$ for some w as before,
(ii) $w_{j+1,k} \geqslant w_{j,k}$, and $w_{j,k-1} \geqslant w_{jk}$.

Recall $N_I^-(\mathbb{Z}_p) \subset I(\mathbb{Z}_p)$ is the subgroup of lower triangular elements with 1 as diagonal entries. Let $\mathcal{N}^-_{\underline{w}}$ be the rigid analytic group

$$
\begin{aligned}
N^-(\mathbb{Z}_p)\cdot &\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mathcal{B}(0,p^{w_{21}}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}(0,p^{w_{n1}}) & \mathcal{B}(0,p^{w_{n2}}) & \cdots & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p\mathbb{Z}_p + \mathcal{B}(0,p^{w_{21}}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p\mathbb{Z}_p + \mathcal{B}(0,p^{w_{n1}}) & p\mathbb{Z}_p + \mathcal{B}(0,p^{w_{n2}}) & \cdots & 1 \end{pmatrix}.
\end{aligned}
$$

Then $\mathcal{I}'_{\underline{w}} = \mathcal{N}^-_{\underline{w}}\mathcal{T}^{\circ}_{w-1}\,\mathbf{N}_{an}$ is a rigid analytic space with the group $\mathcal{T}^{\circ}_{w-1}\,\mathbf{N}_{\mathrm{an}}$ acting by the right multiplication. Due to the requirement (i) (ii) on $\underline{w}$, the space $\mathcal{I}'_{\underline{w}}$ is also stable under the left multiplication by the group $\mathcal{I}_w$. Like in Section 3.2 we define the $\mathcal{I}_w$-module $W_{\kappa,\underline{w}}$ by

$$
W_{\kappa,\underline{w}}(R) = \left\{ \begin{array}{l} f : \mathcal{I}'_{\underline{w}}(R) \to R,\ f|_{\mathcal{N}^-_{\underline{w}}} \text{ is analytic and } f(xtn) = \kappa'(t)f(x) \\ \text{for all } x \in \mathcal{I}'_{\underline{w}}(R), t \in \mathcal{T}^{\circ}_{w-1}(R), n \in \mathbf{N}_{an}(R) \end{array} \right\}
$$

for all $R \in \mathfrak{A}_L$. The group $\mathcal{I}_{\underline{w}}$ acts on it through the left inverse translation. We write $\underline{w}' \leqslant \underline{w}$ if $w'_{jk} \leqslant w_{jk}$ for all $1 \leqslant k < j \leqslant n$. If $\underline{w}' \leqslant \underline{w}$, then

$W_{\kappa,\underline{w}'} \subset W_{\kappa,\underline{w}}$ and elements in $W_{\kappa,\underline{w}'}$ satisfy stronger analyticity condition. We define $V_{\kappa,\underline{w}}$, by the same formulas as (3.3) (3.4), and Banach sheaves

$$\omega^{\dagger}_{\kappa,\underline{w}} = \mathcal{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,\underline{w}} \subset \omega^{\dagger}_{\kappa,w}, \quad \mathcal{V}^{\dagger,r}_{\kappa,\underline{w}} = \mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V_{\kappa,\underline{w}} \subset \mathcal{V}^{\dagger,r}_{\kappa,w}.$$

Next we extend the action of $\mathcal{I}_w$ on $W_{\kappa,\underline{w}}$ to $\Delta^-_{I,w} = \mathcal{I}_w T^{\circ -}\mathcal{I}_w$, where $T^{\circ -} = \{\mathrm{diag}(p^{b_1},\dots,p^{b_n}) \in \mathrm{GL}(n,\mathbb{Q}) : b_1 \geqslant \dots \geqslant b_n\}$. With this extension the $\mathcal{Q}_w$-action on $V^r_{\kappa,\underline{w}}$ extends to $\Delta^-_{Q,w} = \mathcal{Q}_w T^- \mathcal{Q}_w$ with $T^- \subset \mathbf{T}(\mathbb{Q})$ consisting of $\mathrm{diag}(p^{b_1},\dots,p^{b_n},p^{b_0-b_1},\dots,p^{b_0-b_n})$, $b_1 \geqslant \dots \geqslant b_n, b_0 \geqslant 2b_1$. Given $h = h' t_h h''$ with $h', h'' \in \mathcal{I}_w$ and $t_h \in T^{\circ -}$, we make it act on $f \in W_{\kappa,\underline{w}}$ by

$$(f \cdot h)(x) = f(h^{-1} x t_h). \tag{3.19}$$

One can check that this action is well defined and has norm less or equal to 1 with respect to the supreme norm on $W_{\kappa,\underline{w}}$. If $t_h = \mathrm{diag}(p^{b_1},\dots,p^{b_n})$, then h sends $W_{\kappa,\underline{w}}$ into $W_{\kappa,\underline{w}'}$, with $w'_{jk} = \max_{k \leqslant t < s \leqslant j}\{w_{st} + b_s - b_t, w_{jk} - 1\} \leqslant w_{jk}$, and increases the analyticity.

Now fix $1 \leqslant i \leqslant n-1$ and consider the moduli scheme C_i over K parametrizing $(A,\lambda,\psi_N,\mathrm{Fil}_\bullet A[p], L)$, where $(A,\lambda,\psi_N,\mathrm{Fil}_\bullet A[p])$ is the moduli problem defining Y_{Iw}, and $L \subset A[p^2]$ is a Lagrangian subgroup such that $L[p] \oplus \mathrm{Fil}_i A[p] = A[p]$. Denote by $\pi : A \to A/L$ the universal isogeny. Define the projection $p_1 : C_i \to Y_{\mathrm{Iw},K}$ by forgetting L, and $p_2 : C_i \to Y_{\mathrm{Iw},K}$ by sending $(A,\lambda,\psi_N,\mathrm{Fil}_\bullet A[p], L)$ to $(A/L, \lambda', \pi\circ\psi_N, \mathrm{Fil}_\bullet A/L[p])$. Here the polarization λ' is defined by $\pi^*\lambda' = p^2\lambda$ and $\mathrm{Fil}_\bullet A/L[p]$ is defined as

$$\mathrm{Fil}_j A/L[p] = \begin{cases} \pi(\mathrm{Fil}_j A[p]), & \text{if } 1 \leqslant j \leqslant i, \\ \pi(\mathrm{Fil}_j A[p] + p^{-1}(\mathrm{Fil}_j A[p] \cap L)), & \text{if } i < j \leqslant n, \\ (\mathrm{Fil}_{2n-j} A/L[p])^{\perp}, & \text{if } n+1 \leqslant j \leqslant 2n. \end{cases}$$

For example if $x_1,\dots,x_n,x_{n+1},\dots,x_{2n}$ is a basis of $A[p^2]$ compatible with $\mathrm{Fil}_\bullet A[p]$ and the Weil pairing, then we can take

$$L = \langle px_{i+1},\dots,px_n,px_{n+1},\dots,px_{2n-i},x_{2n-i+1},\dots,x_{2n}\rangle,$$

and correspondingly we have

$$\begin{aligned}\mathrm{Fil}_\bullet A/L[p] = \langle p\bar{x}_1\rangle \subset \dots \subset \langle p\bar{x}_1,\dots,p\bar{x}_i\rangle \subset \langle p\bar{x}_1,\dots,p\bar{x}_i,\bar{x}_{i+1}\rangle \subset \cdots \\ \subset \langle p\bar{x}_1,\dots,p\bar{x}_i,\bar{x}_{i+1},\dots,\bar{x}_n\rangle \subset \cdots\end{aligned}$$

where $\bar{x}_j$ stands for $x_j \mod L$.

Set $\mathcal{C}_i(v) = C_{i,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$.

Proposition 3.11 ([2, Proposition 6.2.2.1]). — *If* $\mathrm{Hdg}(A[p^\infty]) < \frac{p-2}{p(2p-2)}$ *and* $\mathrm{Fil}_n A[p]$ *is the canonical subgroup of level* 1*, then* $\mathrm{Hdg}(A[p^\infty]/L) \leqslant \mathrm{Hdg}(A[p^\infty])$ *and the* $\mathrm{Fil}_n A/L[p]$ *defined above is the canonical subgroup of level* 1 *of* A/L*.*

Now we have the diagram

$$\begin{tikzcd} & \mathcal{C}_i(v) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(v). \end{tikzcd} \tag{3.20}$$

The pullback $\pi^* : p_2^* \mathcal{T}^\times_{\mathcal{H},\mathrm{an}}(v) \xrightarrow{\sim} p_1^* \mathcal{T}^\times_{\mathcal{H},\mathrm{an}}(v)$ does not send $p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v)$ into $p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$, but to

$$p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v) \circ \begin{pmatrix} pI_{n-i} & 0 & 0 & 0 \\ 0 & I_i & 0 & 0 \\ 0 & 0 & pI_{n-i} & 0 \\ 0 & 0 & 0 & p^2 I_i \end{pmatrix} \mathcal{Q}_w \subset p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v) \circ \Delta^-_{Q,w}.$$

Given local section (α, u) of the contracted product $p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}}$, there is a $\gamma_\alpha \in \Delta^-_{Q,w}$ such that $(\pi^*\alpha)\circ\gamma_\alpha^{-1}$ lies inside $p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$, and we can define

$$\begin{aligned} \widetilde{\pi}^* : p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}} &\longrightarrow p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w'}} \\ (\alpha, u) &\longmapsto ((\pi^*\alpha)\circ\gamma_\alpha^{-1}, \gamma_\alpha \cdot u), \end{aligned} \tag{3.21}$$

with

$$w'_{jk} = \begin{cases} \max\{w_{jk}-1, w-1\}, & \text{if } 1 \leqslant k \leqslant n-i < j \leqslant n, \\ w_{jk}, & \text{otherwise.} \end{cases}$$

It is easy to see that the right hand side of (3.21) does not depend on the choice of γ_α and $\widetilde{\pi}^*$ is well defined.

The operator $U_{p,i}$ is defined as the composition

$$\begin{aligned} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}) \xrightarrow{p_{2*}} H^0(\mathcal{C}_i(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}) &\xrightarrow{\widetilde{\pi}^*} H^0(\mathcal{C}_i(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,\underline{w'}}) \\ &\xrightarrow{p^{-i(n+1)}\,\mathrm{Tr}\,p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,\underline{w'}}). \end{aligned} \tag{3.22}$$

The normalizer $p^{-i(n+1)}$ is justified in Section 3.9.5.

3.9.3. A compact operator U_p

From (3.18) and (3.22) we see that the composition $U_{p,n}\circ U_{p,n-1}\circ\cdots\circ U_{p,1}$ maps $N^{\dagger,r}_{\kappa,w,v}$ continuously into $N^{\dagger,r}_{\kappa,w-1,pv}$. Let $\mathrm{res}: N^{\dagger,r}_{\kappa,w-1,pv}\to N^{\dagger,r}_{\kappa,w,v}$ be the natural restriction map. Define the operator U_p as

$$U_p = \mathrm{res}\circ U_{p,n}\circ U_{p,n-1}\circ\cdots\circ U_{p,1} : N^{\dagger,r}_{\kappa,w,v}\longrightarrow N^{\dagger,r}_{\kappa,w,v}.$$

In the following we show that the map $\mathrm{res}: N^{\dagger,r}_{\kappa,w-1,pv}\to N^{\dagger,r}_{\kappa,w,v}$ is a compact morphism between two K-Banach modules. To this end it will be convenient to use the interpretation (ii) of $N^{\dagger,r}_{\kappa,w,v}$ in Section 3.4.3, i.e.

$$N^{\dagger,r}_{\kappa,w,v} = H^0(\mathcal{IW}_w(v),\mathcal{L}_\kappa\otimes(\pi_1\circ\pi_3)^*\operatorname{Sym}^r\mathcal{J})^{\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})}.$$

Since the group $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$ is finite, there is a continuous projection from $H^0(\mathcal{IW}_w(v),\mathcal{L}_\kappa\otimes(\pi_1\circ\pi_3)^*\operatorname{Sym}^r\mathcal{J})$ to its $\mathbf{B}(\mathbb{Z}/p^m\mathbb{Z})$-invariant part. Thus it is enough to show the compactness of the restriction

$$H^0(\mathcal{IW}_{w-1}(pv),\mathcal{L}_\kappa\otimes(\pi_1\circ\pi_3)^*\operatorname{Sym}^r\mathcal{J}) \longrightarrow H^0(\mathcal{IW}_w(v),\mathcal{L}_\kappa\otimes(\pi_1\circ\pi_3)^*\operatorname{Sym}^r\mathcal{J}).$$

Since the sheaf $\mathcal{L}_\kappa\otimes(\pi_1\circ\pi_3)^*\operatorname{Sym}^r\mathcal{J}$ is coherent, by [34, Proposition 2.4.1] we reduce to prove that $\mathcal{IW}_w(v)$ is relatively compact inside $\mathcal{IW}_{w-1}(pv)$ (relative to $\operatorname{Spm}(K)$).

According to [34, Definition 2.1.1], given a quasi-compact rigid analytic space $\mathcal{Z}$ and an admissible open quasi-compact subset $\mathcal{V}\subset\mathcal{Z}$, $\mathcal{V}$ is called relatively compact inside $\mathcal{Z}$ (relative to $\operatorname{Spm}(K)$), written as $\mathcal{V}\Subset\mathcal{Z}$, if there exists a formal model $\mathfrak{Z}$ of $\mathcal{Z}$ together with an open sub-formal scheme $\mathfrak{V}\subset\mathfrak{Z}$ with rigid fibre $\mathfrak{V}_{\mathrm{rig}}=\mathcal{V}$, such that the closure $\overline{\mathfrak{V}}_0$ of the reduction $\mathfrak{V}_0\subset\mathfrak{Z}_0$ is proper (over $\operatorname{Spec}(k)$, $k=\mathcal{O}_K/\varpi$).

Lemma 3.12. — *$\mathcal{X}_1(p^m)(v)$ is relatively compact inside $\mathcal{X}_1(p^m)(pv)$.*

Proof. — First $\mathcal{X}(pv)\Subset\mathcal{X}$ because X is proper. Then using [34, Proposiion 2.3.1] we get $\mathcal{X}(v)\Subset\mathcal{X}(pv)$. Both of the projections $\mathcal{X}_1(p^m)(v)\to\mathcal{X}(v)$ and $\mathcal{X}_1(p^m)(pv)\to\mathcal{X}(pv)$ are finite. The statement follows from [34, Lemma 2.1.8]. □

Proposition 3.13. — *$\mathcal{IW}_w(v)$ is relatively compact inside $\mathcal{IW}_{w-1}(pv)$.*

Proof. — By construction we have the formal model $f:\mathfrak{IW}_{w-1}(pv)\to\mathfrak{X}_1(p^m)(pv)$. By the previous lemma we can take an admissible formal blow-up $\mathfrak{X}_1(p^m)(pv)'\to\mathfrak{X}_1(p^m)(pv)$ with an open formal subscheme $\mathfrak{X}_1(p^m)(v)'\subset\mathfrak{X}_1(p^m)(pv)'$, such that $\mathfrak{X}_1(p^m)(v)'_{\mathrm{rig}}=\mathcal{X}_1(p^m)(v)$ and the closure $\overline{\mathfrak{X}_1(p^m)(v)'_0}$ inside $\mathfrak{X}_1(p^m)(pv)'_0$ is proper. Base changing f via the blow-up we get

$$\begin{tikzcd}
\mathfrak{IW}_{w-1}(v)' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{IW}_{w-1}(pv)' \arrow[d] \\
\mathfrak{X}_1(p^m)(v)' \arrow[r, hook] & \mathfrak{X}_1(p^m)(pv)'
\end{tikzcd}.$$

There is an open covering of $\mathfrak{X}_1(p^m)(pv)'$ by affine open subschemes such that over each member $\mathrm{Spf}(R)$ of it, $\mathfrak{IW}_{w-1}(pv)' \times_{\mathfrak{X}_1(p^m)(pv)'} \mathrm{Spf}(R)$ is isomorphic to

$$\mathrm{Spf}(R) \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p^{w-1}\mathfrak{B}(0,1) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^{w-1}\mathfrak{B}(0,1) & p^{w-1}\mathfrak{B}(0,1) & \cdots & 1 \end{pmatrix}$$
$$\cong \mathrm{Spf}(R) \times \mathfrak{B}(0,1)^{n(n-1)/2} \cong \mathrm{Spf}(R\langle T_{ij}\rangle_{1\leqslant j<i\leqslant n}).$$

Over $\mathfrak{IW}_{w-1}(pv)' \times_{\mathfrak{X}_1(p^m)(pv)'} \mathrm{Spf}(R)$ one can define the ideal sheaf attached to the ideal generated by p and T_{ij}, $1 \leqslant j < i \leqslant n$, which is independent of the choice of the coordinate T_{ij}. Such locally defined ideal sheaves glue together to an ideal sheaf $\mathscr{I}$ over $\mathfrak{IW}_{w-1}(pv)'$. Let $\mathfrak{IW}_{w-1}(pv)''$ be the blow-up of $\mathfrak{IW}_{w-1}(pv)'$ along $\mathscr{I}$. Take $\mathfrak{IW}_w(pv)''$ to be its open sub-formal scheme where the ideal sheaf $\mathscr{I}$ is generated by p. From the local description of $\mathscr{I}$, we know that the closure $\overline{\mathfrak{IW}_w(pv)''_0}$ of $\mathfrak{IW}_w(pv)''_0$ inside $\mathfrak{IW}_{w-1}(pv)''_0$ is proper over the base $\mathfrak{X}_1(p^m)(pv)'_0$. Take $\mathfrak{IW}_w(v)''$ to be the inverse image of $\mathfrak{X}_1(p^m)(v)'$ under the projection $\mathfrak{IW}_w(pv)'' \to \mathfrak{X}_1(p^m)(pv)'$. Then $\mathfrak{IW}_w(v)''$ is an open sub-formal scheme of $\mathfrak{IW}_{w-1}(pv)''$ with rigid fibre equal to $\mathcal{IW}_w(v)$. Now we have the picture

$$\begin{tikzcd}
\mathfrak{IW}_w(v)''_0 \arrow[r, hook] \arrow[d] & \mathfrak{Z}_0 \arrow[r, hook] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \overline{\mathfrak{IW}_w(pv)''_0} \arrow[r, hook] \arrow[d, "h"] & \mathfrak{IW}_{w-1}(pv)''_0 \arrow[dl] \\
\mathfrak{X}_1(p^m)(v)'_0 \arrow[r, hook] & \overline{\mathfrak{X}_1(p^m)(v)'_0} \arrow[r, hook] & \mathfrak{X}_1(p^m)(pv)'_0 &
\end{tikzcd}$$

with the vertical map h being proper. Due to the properness of the scheme $\overline{\mathfrak{X}_1(p^m)(v)'_0}$ and the map g (implied by that of h), the scheme $\mathfrak{Z}_0$ is proper. Then the closure of $\mathfrak{IW}_w(v)''_0$ inside $\mathfrak{IW}_{w-1}(pv)''_0$ must be proper since it is contained in $\mathfrak{Z}$. $\square$

All the arguments apply to the universal weight case by working relatively over $\mathcal{U} \subset \mathcal{W}$, as well as the cuspidal case by replacing $\mathrm{Sym}^r \mathcal{J}$ with $\mathrm{Sym}^r \mathcal{J}(-C)$. We record the following corollary.

Corollary 3.14. — *The operators*

$$U_p : N^{\dagger,r}_{\kappa,w,v} \to N^{\dagger,r}_{\kappa,w,v}, \qquad U_p : N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}} \to N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}}$$

$$(\textit{resp. } U_p : N^{\dagger,r}_{\mathcal{U},w,v} \to N^{\dagger,r}_{\mathcal{U},w,v}, \qquad U_p : N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} \to N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}})$$

are compact operators of K-Banach spaces (resp. $\mathcal{A}(\mathcal{U})$-Banach modules).

3.9.4. Tensoring with $\tau, \tau^\vee$

We consider the algebraic $\mathrm{GL}(n)$-representations $(\rho_{\mathrm{alg}}, W_{\rho_{\mathrm{alg}}})$ that are obtained by taking tensor products of symmetric powers $\mathrm{Sym}^{e_1}\tau_{\mathrm{alg}}$ and $\mathrm{Sym}^{e_2}\tau^\vee_{\mathrm{alg}}$ with $e_1, e_2 \in \mathbb{N}$. Here we add the subscript $_{\mathrm{alg}}$ to indicate that the action of $\Delta^-_{I,w}$ is the one given by the algebraic action of $\mathrm{GL}(n)$. The notation $\rho, \tau, \tau^\vee$ will be saved for the $\Delta^-_{I,w}$-modules which are obtained from the algebraic ones by a renormalization explained below.

First we define two characters χ_1, χ_2 on the semi-group $\Delta^-_{I,w}$. Given $h = h' t_h h''$ with $h', h'' \in \mathcal{I}_w$ and $t_h = \mathrm{diag}(p^{b_1}, \ldots, p^{b_n}) \in T^{\circ -}$, put

$$\chi_1(h) = p^{-2b_n}, \qquad \chi_2(h) = p^{2b_1}.$$

We define the $\Delta^-_{I,w}$-modules $\tau, \tau^\vee$ as

$$\tau := \tau_{\mathrm{alg}} \otimes \chi_1, \qquad \tau^\vee := \tau^\vee_{\mathrm{alg}} \otimes \chi_2.$$

By taking tensor products of $\tau, \tau^\vee$, we associate to each ρ_{alg} the renormalized $\Delta^-_{I,w}$-module ρ. The reason we consider this renormalization of ρ_{alg} is that it makes the action of $\Delta^-_{Q,w}$ on V^r_ρ integral.

Then all the $\mathbb{U}_p$-operators can be constructed for $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ in exactly the same way as when ρ is trivial and Corollary 3.14 holds for the action of U_p on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ and $N^{\dagger,r}_{\kappa\otimes\rho,w,v,\mathrm{cusp}}$.

3.9.5. The normalizations of the $\mathbb{U}_p$-operators

We show that by our choice of the normalizations of the $\mathbb{U}_p$-operators, all the eigenvalues of the compactor operator U_p acting on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ are p-adically integral. Since $\mathcal{V}^r_{\kappa\otimes\rho,w}$ has a filtration with $\mathcal{V}^0_{\kappa\otimes\rho\otimes\mathrm{Sym}^e\tau^\vee,w}$ as graded pieces, it is enough to consider the case $r = 0$.

For an integer $l \geqslant 0$, let Y_l be the Siegel variety modulo p^l and $Y_l[1/\mathrm{Ha}]$ be its ordinary locus. Denote by $S(p^m)_l$ the finite étale cover of $Y_l[1/\mathrm{Ha}]$ parametrizing, the quintuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\leqslant j\leqslant n})$, where the abelian scheme A over an $\mathcal{O}_K/p^l$-algebra is ordinary, and $\mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ is a complete flag of the free $\mathbb{Z}/p^m\mathbb{Z}$-module ${}^{\mathrm{t}}A[p^m]^{\text{ét}}$ with trivializations of graded pieces $\phi_j : \mathbb{Z}/p^m\mathbb{Z} \simeq \mathrm{Fil}_j / \mathrm{Fil}_{j+1} {}^{\mathrm{t}}A[p^m]^{\text{ét}}$. Put $\mathfrak{S}(p^\infty) = \varinjlim_l \varprojlim_m S(p^m)_l$. The Hodge–Tate map gives rise to the embedding

$$\begin{tikzcd} \mathfrak{S}(p^\infty) \arrow[r, hook] \arrow[dr] & \mathfrak{IW}_w^+(v) \arrow[d] \\ & \mathfrak{X}_{\mathrm{Iw}}(v), \end{tikzcd} \tag{3.23}$$

which induces an injective map

$$\mathrm{res} : N^{\dagger,r}_{\kappa\otimes\rho,w,v} \to H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho)[1/p], \tag{3.24}$$

where $\mathfrak{V}^r_\rho$ is the pullback to $\mathfrak{S}(p^\infty)$ of the locally free sheaf $\mathcal{V}^r_\rho$ of finite rank over X. We can define an $\mathbb{U}_p$-action on $H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho)$ such that (3.24) is $\mathbb{U}_p$-equivariant. Then the integrality of the U_p-eigenvalues on $N^{\dagger,r}_{\kappa\otimes\rho,w,v}$ follows. We deal with the case of the operator $U_{p,i}$ for $1 \leqslant i \leqslant n-1$. Other cases are basically the same.

First we construct the correspondence analogous to (3.20)

$$\begin{tikzcd} & C_{i,m,l}(0) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S(p^m)_l & & S(p^{m-1})_l, \end{tikzcd}$$

where $C_{i,m,l}(0)$ parametrizes sextuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\leqslant j\leqslant n}, L)$ whose first five components form the quintuple defining $S(p^m)_l$. The flag $\mathrm{Fil}_\bullet {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ gives a self-dual flag of $\mathrm{Fil}_\bullet A[p]$, and $L \subset A[p^2]$ is the one used in defining C_i. The projection p_1 is forgetting L. The projection p_2 sends $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\leqslant j\leqslant n}, L)$ to $(A', \lambda', \pi \circ \psi_N, \mathrm{Fil}_\bullet {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}}, (\phi'_j)_{1\leqslant j\leqslant n})$. Here $A' = A/L$, and the universal isogeny $\pi : A \to A'$ induces a map ${}^{\mathrm{t}}\pi : {}^{\mathrm{t}}A'[p^m]^{\text{ét}} \to {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ and a well-defined map $p\cdot {}^{\mathrm{t}}\pi^{-1} : {}^{\mathrm{t}}A[p^m]^{\text{ét}} \to {}^{\mathrm{t}}A'[p^m]^{\text{ét}}$. Set $\mathrm{Fil}_j {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}} = p \cdot {}^{\mathrm{t}}\pi^{-1}(\mathrm{Fil}_j {}^{\mathrm{t}}A[p^m]^{\text{ét}}) \cap {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}}$ and

$$\phi'_j = \begin{cases} p^2 \cdot {}^{\mathrm{t}}\pi^{-1} \circ \phi_j, & \text{if } 1 \leqslant j \leqslant n-i, \\ p \cdot {}^{\mathrm{t}}\pi^{-1} \circ \phi_j, & \text{if } n-i+1 \leqslant j \leqslant n. \end{cases}$$

Taking the inverse limit with respect to m followed by the direct limit with respect to l, we get $\mathfrak{C}_{i,\infty}(0) = \varinjlim_l \varprojlim_m C_{i,m,l}(0)$ and the correspondence

$$\begin{tikzcd} & \mathfrak{C}_{i,\infty}(0) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathfrak{S}(p^\infty) & & \mathfrak{S}(p^\infty). \end{tikzcd}$$

By our normalization of the $\Delta^-_{I,w}$-action on V^r_ρ, the group $I(\mathbb{Z}_p)T^{\circ -}I(\mathbb{Z}_p)$ acts on it integrally. This guarantees that the map $\widetilde{\pi}^* : p_2^*\mathfrak{V}_\rho \to p_1^*\mathfrak{V}_\rho$ can be defined in a manner similar to (3.21). Once we have checked that $\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho)$, we can define the operator $U_{p,i}$ as

$$\begin{aligned} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho) \xrightarrow{p_2*} H^0(\mathfrak{C}_{i,\infty}(0), p_2^*\mathfrak{V}^r_\rho) \xrightarrow{\widetilde{\pi}^*} H^0(\mathfrak{C}_{i,\infty}(0), p_1^*\mathfrak{V}^r_\rho) \\ \xrightarrow{p^{-i(n+1)}\,\mathrm{Tr}\, p_1} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho). \end{aligned}$$

It is not difficult to check that such defined $\mathbb{U}_p$-operators on $H^0(\mathfrak{S}(p^\infty), \mathfrak{w}_\rho)$ make (3.24) $\mathbb{U}_p$-equivariant.

We are left to show the inclusion

$$\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho).$$

Essentially this containment reflects the fact that the projection p_1 is ramified and $p^{i(n+1)}$ is its pure inseparability degree. Thanks to the projection formula we have

$$p_{1,*}p_1^*\mathfrak{V}^r_\rho = p_{1,*}\mathcal{O}_{\mathfrak{C}_{i,\infty}(0)} \otimes \mathfrak{V}^r_\rho.$$

Therefore it suffices to show

$$\mathrm{Tr}\, p_1(p_{1,*}\mathcal{O}_{\mathfrak{C}_{i,\infty}(0)}) \subset p^{i(n+1)}\mathcal{O}_{\mathfrak{S}(p^\infty)}. \tag{3.25}$$

Let $S(p^\infty)_0$ be the reduction of $\mathfrak{S}(p^\infty)$. Take $y_0 \in S(p^\infty)_0$, $y_0' \in p_2(p_1^{-1}(y_0))$. We show (3.25) in the formal neighborhoods $\widehat{\mathfrak{S}(p^\infty)}_{y_0}, \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')}$. We explicate the projection p_1 using the Serre–Tate coordinates [26, Sections 8.2, 8.3]. The formal neighborhood $\widehat{\mathfrak{S}(p^\infty)}_{y_0}$ is isomorphic to $\mathrm{Hom}_{\mathrm{sym}}(T_pA^{\text{ét}}_{y_0} \times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}, \widehat{\mathbb{G}}_m)$. A point $z \in \widehat{\mathfrak{S}(p^\infty)}_{y_0}$ corresponds to a bilinear map $q : T_pA^{\text{ét}}_{y_0} \times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0} \to \widehat{\mathbb{G}}_m$ which is symmetric if we identify ${}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ with $A^{\text{ét}}_{y_0}$ via the polarization. Given any basis $x_1, \ldots, x_n$ of $T_pA^{\text{ét}}_{y_0}$, let ${}^{\mathrm{t}}x_1, \ldots, {}^{\mathrm{t}}x_n$ the basis of ${}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ which are obtained as the image of $x_1, \ldots, x_n$ under the polarization. Write $q(x_i, {}^{\mathrm{t}}x_j) = 1 + T_{jk}$, $1 \leqslant j, k \leqslant n$. We have

$T_{jk}=T_{kj}$. The $\{T_{jk}\}_{1\leqslant j\leqslant k\leqslant n}$ is a Serre–Tate coordinate of $\widehat{\mathfrak{S}(p^\infty)}_{y_0}$. Similarly, for $\widehat{\mathfrak{S}(p^\infty)}_{y_0'}$ with a given basis $x_1',\dots,x_n'$ of $T_pA^{\text{ét}}_{y_0'}$, we get a corresponding Serre–Tate coordinate $\{T'_{jk}\}_{1\leqslant j\leqslant k\leqslant n}$. The isogeny $\pi: A_{y_0}\to A_{y_0'}$ induces a map on the Tate modules. Now fix basis $x_1,\dots,x_n$ and $x_1',\dots,x_n'$ of $T_pA^{\text{ét}}_{y_0}$ and $T_pA^{\text{ét}}_{y_0'}$, such that with respect to them the matrix for the map $\pi: T_pA^{\text{ét}}_{y_0}\to T_pA^{\text{ét}}_{y_0'}$ is given by $\left(\begin{smallmatrix} I_{n-i} & 0\\ 0 & p^2I_i\end{smallmatrix}\right)$. Then under the basis ${}^{\mathrm{t}}x_1,\dots,{}^{\mathrm{t}}x_n$ and ${}^{\mathrm{t}}x_1',\dots,{}^{\mathrm{t}}x_n'$ of $T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ and $T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0'}$, the matrix for ${}^{\mathrm{t}}\pi: T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0'}\to T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}$ is given by $\left(\begin{smallmatrix} I_{n-i} & 0\\ 0 & I_i\end{smallmatrix}\right)$. For each $(z,z')\in \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')}\subset \widehat{\mathfrak{S}(p^\infty)}_{y_0}\times\widehat{\mathfrak{S}(p^\infty)}_{y_0'}$, let $q: T_pA^{\text{ét}}_{y_0}\times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0}\to\widehat{\mathbb{G}}_m$ (resp. $q': T_pA^{\text{ét}}_{y_0'}\times T_p{}^{\mathrm{t}}A^{\text{ét}}_{y_0'}\to\widehat{\mathbb{G}}_m$) be the corresponding bilinear map for z (resp. z'). We have $q(x_j,{}^{\mathrm{t}}\pi(x_k'))=q'(\pi(x_j),x_k')$. In terms of the coordinates T_{jk} and T'_{jk}, we see that T'_{jk} can be taken to be the local coordinates of $\widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')}$, and the projection $p_1:\widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')}\to\widehat{\mathfrak{S}(p^\infty)}_{y_0}$ is given by

$$\begin{aligned}\mathcal{O}_K[[T_{jk}]]&\longrightarrow \mathcal{O}_K[[T'_{jk}]]\\ T_{jk}&\longmapsto\begin{cases} T'_{jk} & \text{if } 1\leqslant j\leqslant k\leqslant n-i,\\ (T'_{jk}+1)^p-1 & \text{if } 1\leqslant j\leqslant n-i<k\leqslant n,\\ (T'_{jk}+1)^{p^2}-1 & \text{if } n-i+1\leqslant j\leqslant k\leqslant n.\end{cases}\end{aligned}$$

An easy computation shows that the pure inseparability degree of p_1 is $p^{i(n+1)}$ and $\mathrm{Im}\,(\mathrm{Tr}\,p_1)\subset p^{i(n+1)}\mathcal{O}_K[[T_{jk}]]$.

Before ending this section we include the following remark concerning the Hecke actions preserving the cuspidality.

Remark 3.15. — The injection (3.24) is equivariant under the action of both unramified Hecke operators and $\mathbb{U}_p$-operators. It is also easy to check that

$$N^{\dagger,r}_{\kappa\otimes\rho,w,v,\mathrm{cusp}}=N^{\dagger,r}_{\kappa\otimes\rho,w,v}\cap H^0(\mathfrak{S}(p^\infty),\mathfrak{V}^r_\rho(-C))[1/p].$$

Hence it is enough to notice that the space $H^0(\mathfrak{S}(p^\infty),\mathfrak{V}^r_\rho(-C))$ is preserved under those operators. This follows from the fact that classical cuspidal nearly homomorphic forms are stable under Hecke actions, and that the classical cuspidal nearly homomorphic forms are dense inside $H^0(\mathfrak{S}(p^\infty),\mathfrak{V}^r_\rho(-C))$.

3.10. Interchanging the Hecke and differential operators

Let ρ be as in Section 3.9.4. In this section we discuss the commutator of Hecke operators and differential operators. Recall that the operators

$D_{\kappa\otimes\rho,w}$ and $E_{\kappa\otimes\rho,w}$ are defined as the compositions

$$\begin{aligned} D_{\kappa\otimes\rho,w} &: \mathcal{V}^r_{\kappa\otimes\rho,w,v} \xrightarrow{\nabla_{\kappa\otimes\rho,w}} \mathcal{V}^{r+1}_{\kappa\otimes\rho\otimes\tau_{\mathrm{alg}},w,v}(-1) \xrightarrow{t^+} \mathcal{V}^{r+1}_{\kappa\otimes\rho\otimes\tau,w,v}, \\ E_{\kappa\otimes\rho,w} &: \mathcal{V}^r_{\kappa\otimes\rho,w,v} \xrightarrow{\epsilon_{\kappa\otimes\rho,w}} \mathcal{V}^{r-1}_{\kappa\otimes\rho\otimes\tau^\vee_{\mathrm{alg}},w,v}(1) \xrightarrow{t^-} \mathcal{V}^{r-1}_{\kappa\otimes\rho\otimes\tau^\vee,w,v}. \end{aligned} \tag{3.26}$$

We first show that Hecke operators commute with the connection $\nabla_{\kappa\otimes\rho,w}$ and the operator $\epsilon_{\kappa\otimes\rho,w}$.

Lemma 3.16. — *The $\mathbb{U}_p$-operators and unramified Hecke operators commute with the connection $\nabla_{\kappa\otimes\rho,w}$ and the operator $\epsilon_{\kappa\otimes\rho,w}$.*

Proof. — The $\mathbf{Q}$-representation J admits a filtration $0 \to \mathrm{triv} \to J \to \tau^\vee_{\mathrm{alg}}(1) \to 0$. The operator $\epsilon_{\kappa\otimes\rho,w}$ by definition is induced from the quotient morphism $J \to \tau^\vee_{\mathrm{alg}}(1)$, and is easily seen to commute with all $\mathbb{U}_p$-operators as well as unramified Hecke operators.

The commutativity of the connection $\nabla_{\kappa\otimes\rho,w}$ with Hecke operators is a result of the functoriality of the Gauss–Manin connection, which says that for any map of abelian schemes

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A' \arrow[d] \\ S \arrow[r, "f"] & R \end{tikzcd},$$

we have

$$\begin{tikzcd} f^*\mathcal{H}^1_{\mathrm{dR}}(A'/R) \arrow[r, "f^*\nabla"] \arrow[d, "\varphi^*"] & f^*\mathcal{H}^1_{\mathrm{dR}}(A'/R)\otimes f^*\Omega^1_R \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & f^*\mathcal{H}^1_{\mathrm{dR}}(A'/R)\otimes \Omega^1_S \arrow[d, "\varphi^*\otimes 1"] \\ \mathcal{H}^1_{\mathrm{dR}}(A/S) \arrow[rr, "\nabla"] & & \mathcal{H}^1_{\mathrm{dR}}(A/S)\otimes\Omega^1_S \end{tikzcd}.$$

Let π be the universal isogeny $A \to A' = A/L$ over $\mathcal{C}_i(v)$. The commutativity of $\mathbb{U}_{p,i}$, $1 \leqslant i \leqslant n-1$, with $\nabla_{\kappa\otimes\rho,w}$ will follow from the commutativity if the following diagram

$$\begin{tikzcd} p_2^*\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \arrow[r, "\tilde{\pi}^*"] \arrow[d, "p_2^*\nabla_{\kappa\otimes\rho,w}"] & p_1^*\mathcal{V}^{\dagger,r}_{\kappa\otimes\rho,w} \arrow[d, "p_1^*\nabla_{\kappa\otimes\rho,w}"] \\ p_2^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}\otimes\Omega^1_{\mathcal{C}_i(v)} \arrow[r, "\tilde{\pi}^*\otimes 1"] & p_1^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w}\otimes\Omega^1_{\mathcal{C}_i(v)} \end{tikzcd}.$$

Write a local section of $p_2^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\rho,w} = p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa\otimes\rho,w}$ as (α,u). For any $g \in \mathcal{Q}_w$, $(\alpha,u) = (\alpha\circ g, g^{-1}\cdot u)$. Take $\gamma \in \Delta^-_{Q,w}$ such that $\pi^*\alpha\circ\gamma^{-1} \in$

$p_1^*\mathcal{T}_{\mathcal{H},w}^{\times}(v)$. If D is a local section of the tangent bundle of $\mathcal{C}_i(v)$ then

$$\begin{aligned}(p_1^*\nabla_{\rho,w}(D)\circ\tilde{\pi}^*)(\alpha,v) &= p_1^*\nabla_{\rho,w}(D)((\pi^*\alpha\circ\gamma^{-1},\,\gamma\cdot v))\\ &= (\pi^*\alpha\circ\gamma^{-1},\,D(\gamma\cdot v)+X(D,\pi^*\alpha\circ\gamma^{-1})\cdot\gamma\cdot v)\\ &= \tilde{\pi}^*(\alpha,\,\gamma^{-1}\cdot D(\gamma\cdot v)+\gamma^{-1}\cdot X(D,\alpha\circ\gamma^{-1})\cdot\gamma\cdot v)\\ &= \tilde{\pi}^*(\alpha,\,Dv+X(D,\alpha)\cdot v)\\ &= (\tilde{\pi}^*\circ p_2^*\nabla_{\rho,w}(D))(\alpha,\,v),\end{aligned}$$

where the third equality follows from the functoriality of the Gauss–Manin connection. The commutativity of $\nabla_{\kappa\otimes\rho,w}$ with other Hecke operators can be shown similarly. □

Next we show that interchanging the order of $\mathbb{U}_p$-operators and t^+,t^- in (3.26) leads to powers of p. Define the following two characters

$$\begin{aligned}\nu_{p,D},\,\nu_{p,E}:T^+&\longrightarrow\mathbb{Q}^\times\\ t=\mathrm{diag}(p^{a_1},\dots,p^{a_n},p^{a_0-a_1},\dots,p^{a_0-a_n})&\longmapsto\begin{cases}\nu_{p,D}(t)=p^{a_0-2a_1},\\ \nu_{p,E}(t)=p^{a_0-2a_n}.\end{cases}\end{aligned}$$

Both $\nu_{p,D}$ and $\nu_{p,E}$ are trivial on scalar matrices, and $\nu_{p,D}(\gamma_{p,i})=p^2$, $\nu_{p,E}(\gamma_{p,i})=1$ for $1\leqslant i\leqslant n-1$, and $\nu_{p,D}(\gamma_{p,n})=\nu_{p,E}(\gamma_{p,n})=p$. For ℓ coprime to Np, define

$$\begin{aligned}\nu_\ell:\mathrm{GSp}(2n,\mathbb{Z}_\ell)\backslash\mathrm{GSp}(2n,\mathbb{Q}_\ell)/\mathrm{GSp}(2n,\mathbb{Z}_\ell)&\longrightarrow\mathbb{Q}^\times\\ \gamma_\ell&\longmapsto|\nu(\gamma_\ell)|_\ell^{-1},\end{aligned}$$

where ν is the multiplier character.

Lemma 3.17.

(1) $\nu_{p,D}(\gamma_p)\cdot t^+U_{\gamma_p}=U_{\gamma_p}t^+,\quad t^-U_{\gamma_p}=\nu_{p,E}(\gamma_p)\cdot U_{\gamma_p}t^-$,
(2) $\nu_\ell(\gamma_\ell)\cdot t^+T_{\gamma_\ell}=T_{\gamma_\ell}t^+,\quad t^-T_{\gamma_\ell}=\nu_\ell(\gamma_\ell)\cdot T_{\gamma_\ell}t^-$.

Proof. — (2) is obvious since the corresponding representations differ by a twist of the multiplier character. (1) is basically the same, but when defining the $\mathbb{U}_p$-operators, we renormalized the algebraic representations τ_{alg}, $\tau_{\mathrm{alg}}^\vee$ to the $\Delta_{I,w}^-$-modules τ, $\tau^\vee$ by twisting the characters χ_1, χ_2. Therefore, for $\gamma_p=\mathrm{diag}(p^{a_1},\dots,p^{a_n},p^{a_0-a_1},\dots,p^{a_0-a_n})\in T^+$, the commutators of U_{γ_p} with t^+, t^- are $\nu_p(\gamma_p)\cdot\chi_1(\gamma_p^\circ)=\nu_{p,D}(\gamma_p)$, and $\nu_p(\gamma_p)\cdot\chi_2(\gamma_p^\circ)^{-1}=\nu_{p,E}(\gamma_p)$. □

Corollary 3.18.

(1) $\nu_{p,D}(\gamma_p)\cdot D_{\kappa\otimes\rho}U_{\gamma_p}=U_{\gamma_p}D_{\kappa\otimes\rho},\quad E_{\kappa\otimes\rho}U_{\gamma_p}=\nu_{p,E}(\gamma_p)\cdot U_{\gamma_p}E_{\kappa\otimes\rho}$,
(2) $\nu_\ell(\gamma_\ell)\cdot D_{\kappa\otimes\rho}T_{\gamma_\ell}=T_{\gamma_\ell}D_{\kappa\otimes\rho},\quad E_{\kappa\otimes\rho}T_{\gamma_\ell}=\nu_\ell(\gamma_\ell)\cdot T_{\gamma_\ell}E_{\kappa\otimes\rho}$.

In particular, for the compact operator U_p we have

$$p^{2n-1}\cdot D_{\kappa\otimes\rho}U_p = U_pD_{\kappa\otimes\rho},\quad E_{\kappa\otimes\rho}U_p = p\cdot U_pE_{\kappa\otimes\rho}.$$

3.11. The slope decomposition

We consider the slope decomposition of the operator U_p acting on the union of projective $\mathcal{A}(\mathcal{U})$-Banach modules $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} := \bigcup_{r\geqslant 0} N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$. On each $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ the action of U_p is compact. Applying the Coleman–Riesz–Serre theory developed in [9] on the spectrum of compact operators, one can define the Fredohlm determinant $P_r(T)=\det(1-TU_p|_{N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}})$, which belongs to $\mathcal{A}(\mathcal{U})\{\{T\}\}$, the $\mathcal{A}(\mathcal{U})$-algebra of power series with convergence radius being infinity. Because of the integrality of the operator U_p, all the coefficients of $P_r(T)$ are power bounded, i.e. $P_r(T)\in\mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$.

Proposition 3.19. — *The sequence*

$$0\longrightarrow N^{\dagger,r-1}_{\kappa,w,v,\mathrm{cusp}}\longrightarrow N^{\dagger,r}_{\kappa,w,v\,\mathrm{cusp}}\xrightarrow{\frac{1}{r!}E^r_{\kappa,w}} N^{\dagger,0}_{\kappa\otimes\mathrm{Sym}^r\tau^\vee,w,v\,\mathrm{cusp}}\longrightarrow 0 \tag{3.27}$$

is exact.

Proof. — Let $\eta:\mathfrak{X}_1(p^m)(v)\to\mathfrak{X}^\star(v)$ be as in Section 3.5. Combining the vanishing result (3.9) there and (3.12), we get the exact sequence of small formal Banach sheaves over $\mathfrak{X}^\star(v)$

$$0\longrightarrow \eta_*\tilde{\mathfrak{V}}^{\dagger,r-1}_{\kappa,w}(-C)\longrightarrow\eta_*\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa,w}(-C)\xrightarrow{\frac{1}{r!}E^r_{\kappa,w}}\eta_*\tilde{\mathfrak{V}}^{\dagger,0}_{\kappa\otimes\mathrm{Sym}^r\tau^\vee,w}(-C)\longrightarrow 0.$$

Due to the smallness we know that the augmented Cěch complexes of the above sheaves are exact after inverting p [2, Theorem A.1.2.2]. Thus we deduce the exactness of the sequence

$$\begin{aligned}0\longrightarrow H^0(\mathcal{X}_1(p^m)(v),\tilde{\mathcal{V}}^{\dagger,r-1}_{\kappa,w}(-C))\longrightarrow H^0(\mathcal{X}_1(p^m)(v),\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w}(-C))\\ \xrightarrow{\frac{1}{r!}E^r_{\kappa,w}} H^0(\mathcal{X}_1(p^m)(v),\tilde{\mathcal{V}}^{\dagger,0}_{\kappa\otimes\mathrm{Sym}^r\tau^\vee,w}(-C))\longrightarrow 0.\end{aligned}$$

The proposition follows by taking the invariants of $I(\mathbb{Z}/p^m\mathbb{Z})$. □

Combining (3.27) and the equality

$$E^r_{\kappa,w}U_p = p^rU_pE^r_{\kappa,w},$$

we see that there exist $C_r(T)\in\mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$ such that

$$P_r(T)=P_{r-1}(T)C_r(p^rT).$$

Therefore we can define $P_\infty(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$ as the limit

$$P_\infty(T) := \lim_{r\to\infty} P_r(T).$$

Given $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ dividing $P_\infty(T)$, one checks by definition [10, p. 434–435] that for sufficiently large r, the resultant $\mathrm{Res}(Q(T), P_\infty(T)/P_r(T))$ is a unit in $\mathcal{A}(\mathcal{U})$, so $Q(T)$ divides $P_r(T)$.

Now take $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ whose constant term is 1 and the leading coefficient is a unit of $\mathcal{A}(U)$, such that $P_\infty(T) = Q(T)S(T)$ with $S(T)$ relatively prime to $Q(T)$. We call such a $Q(T)$ admissible for $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}}$. Applying [9, Theorem.3.3] to get the slope decomposition

$$N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} = N^r_{Q,\mathcal{U},\mathrm{cusp}} \oplus F^r_{Q,\mathcal{U}}, \tag{3.28}$$

satisfying

(1) the direct summand $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ is a projective $\mathcal{A}(\mathcal{U})$-Banach module of finite rank, and we have $\det(1 - TU_p|_{N^r_{Q,\mathcal{U},\mathrm{cusp}}}) = Q(T)$,
(2) the operator $Q^*(U_p)$ is invertible on $F^r_{Q,\mathcal{U}}$, where $Q^*(T) = T^{\deg Q}Q(1/T)$.

Since $Q(T)$ is of finite degree and is picked such that $\mathrm{Res}(Q(T), P_\infty(T)/P_r(T))$ is a unit in $\mathcal{A}(\mathcal{U})$ for $r \gg 0$, the module $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ stops increasing after r is sufficiently large. We define $N_{Q,\mathcal{U},\mathrm{cusp}}$ as $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ for $r \gg 0$. The subscripts w, v are omitted. Since all eigenvalues of U_p acting on $N_{Q,\mathcal{U},\mathrm{cusp}}$ are nonzero, and U_p increases analyticity and overconvergence, the module does not depend on w, v. Elements in the finite rank projective $\mathcal{A}(\mathcal{U})$-Banach module $N_{Q,\mathcal{U},\mathrm{cusp}}$ are Q-finite slope families of cuspidal nearly overconvergnent forms, and we have the Q-finite slope projection

$$e_{Q,\mathcal{U}} : N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} \longrightarrow N_{Q,\mathcal{U},\mathrm{cusp}}.$$

Remark 3.20. — With the finite rank projective $\mathcal{A}(\mathcal{U})$-Banach module $N_{Q,\mathcal{U},\mathrm{cusp}}$, one can apply the machinery developed in [9] to construct the eigenvariety for nearly overconvergent Siegel modular forms. We do not attempt this here because from the point of view of automorphic representations, if π is an irreducible cuspidal automorphic representation generated by a cuspdial nearly holomorphic form, then π_∞ has a lowest K_∞-type and forms inside the lowest K_∞-types are holomorphic, i.e. cuspidal nearly holomorphic forms do not provide new interesting Hecke eigensystems than holomorphic forms.

3.12. p-adic splitting of $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over ordinary locus

Let Y, X, $\mathfrak{X}$, $\mathfrak{X}(v)$, $\mathfrak{X}_{\mathrm{Iw}}(v)$, $\mathcal{X} = X_{\mathrm{rig}}$, $\mathcal{X}(v)$, $\mathcal{X}_{\mathrm{Iw}}(v)$ be defined as in Section 3.3. Over X (resp. Y) there is the semi-abelian scheme $\mathbf{p} : \mathcal{G} \to X$ (resp. the universal abelian scheme $\mathbf{p} : \mathcal{A} \to Y$). Denote by $\mathbf{p} : G_0 \to X_0$ (resp. $\mathbf{p} : A_0 \to Y_0$) the reduction modulo ϖ. Set $X_{0,\mathrm{ord}}$, $Y_{0,\mathrm{ord}}$ to be the ordinary locus of X_0, Y_0. Fix a lift $\sigma : \mathcal{O}_K \to \mathcal{O}_K$ of the Frobenius of the residue field $k = \mathcal{O}_K/\varpi$. Let $F : X_{0,\mathrm{ord}} \to X_{0,\mathrm{ord}}$ be the absolute Frobenius and consider the commutative diagram

$$\begin{tikzcd} X_{0,\mathrm{ord}} \arrow[r, hook] \arrow[d, "F"] & \mathfrak{X}(0) \arrow[r] \arrow[d, "u"] & \mathrm{Spf}(\mathcal{O}_K) \arrow[d, "\sigma"] \\ X_{0,\mathrm{ord}} \arrow[r, hook] & \mathfrak{X}(0) \arrow[r] & \mathrm{Spf}(\mathcal{O}_K) \end{tikzcd},$$

where u is the lift of the absolute Frobenius defined by sending an ordinary semi-abelian scheme $\mathcal{G}$ to its quotient by the connected part of $\mathcal{G}[p]$, and composing with the base change by σ. The isogeny $\mathcal{A} \to \mathcal{A}/\mathcal{A}[p]^\circ$ induces a morphism

$$\Phi : u^*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}}$$

of formal coherent sheaves over $\mathfrak{X}(0)$. By [31, Theorem 4.1], the locally free formal sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}}$ of rank $2n$ has a unique Φ-stable locally free formal sub-sheaf $\mathfrak{U}_{\mathcal{H}}$ of rank n, over which Φ restricts to an isomorphism. This $\mathfrak{U}_{\mathcal{H}}$ gives rise to a splitting, called the unit-root splitting, of the Hodge filtration:

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}} = \omega(\mathcal{G}/\mathfrak{X}(0)) \oplus \mathfrak{U}_{\mathcal{H}}.$$

Moreover $\mathfrak{U}_{\mathcal{H}}$ is stable under the Gauss–Manin connection. The unit-root splitting pulls back to $\mathfrak{X}_{\mathrm{Iw}}(p)(0)$, and induces a projection $\mathfrak{J} \to \mathcal{O}_{\mathfrak{X}_{\mathrm{Iw}}(p)(0)}$. Taking the generic fibre we get the projection

$$\begin{aligned} (3.29)\quad & H^0(\mathcal{X}_{\mathrm{Iw}}(0), \mathcal{V}^{\dagger,r}_{\kappa,w}) \\ & = H^0(\mathcal{X}_{\mathrm{Iw}}(0), \omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}) \longrightarrow H^0(\mathcal{X}_{\mathrm{Iw}}(0), \omega^{\dagger}_{\kappa,w}). \end{aligned}$$

The Igusa tower $\mathfrak{S}(p^\infty)$ defined in Section 3.9.5 is étale over $\mathfrak{X}_{\mathrm{Iw}}(0)$ with the group $\mathbf{T}^\circ(\mathbb{Z}_p)$ acting on it. The space of weight κ p-adic forms consists of functions on $\mathfrak{S}(p^\infty)$ that are κ'-invariant under the action of $\mathbf{T}^\circ(\mathbb{Z}_p)$, i.e.

$$M^{p\text{-adic}}_{\kappa} = H^0(\mathfrak{S}(p^\infty), \mathcal{O}_{\mathfrak{S}(p^\infty)})[\kappa'].$$

Composing (3.24) with $r = 0$ and (3.29) we obtain the map

$$\xi_p : N^{\dagger,r}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_{\kappa}[1/p],$$

sending nearly overconvergent forms to p-adic forms.

Let $\kappa \in \mathcal{W}(K)$ be an arithmetic weight with algebraic part κ_{alg} and finite order part κ_{f}. Set

$$\Gamma_1(N,p^m) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N) : c \equiv 0 \bmod p^m,\ a \bmod p^m \in \mathbf{N}(\mathbb{Z}/p^m\mathbb{Z}) \right\}.$$

Denote by $N^r_\kappa(\Gamma_1(N,p^m),K)$ the space of weight κ_{alg}, degree r classical nearly holomorphic Siegel modular forms of level $\Gamma_1(N,p^m)$ with nebentypus κ_{f} at p.

Proposition 3.21. — *The following restriction of ξ_p to classical nearly holomorphic Siegel modular forms*

$$\xi_{p,\mathrm{cl}} : N^r_\kappa(\Gamma_1(N,p^m),K) \hookrightarrow N^{\dagger,r}_{\kappa,w,v} \xrightarrow{\xi_p} M^{p\text{-adic}}_\kappa[1/p]$$

is injective.

Proof. — Take $f \in \operatorname{Ker}\xi_{p,\mathrm{cl}}$. Under the map $\phi : N^r_\kappa(\Gamma_1(N,p^m),K) \otimes_K \mathbb{C} \to N^r_\kappa(\mathfrak{h}_n, \Gamma_1(N,p^m))$ defined as (2.14), the image $\phi(f)$ of f is a polynomial in $(\operatorname{Im} z)^{-1}$ with coefficients being holomorphic maps from $\mathfrak{h}_n$ to $W_{\kappa_{\mathrm{alg}}}(\mathbb{C})$. By definition ϕ is equivalent to the projection from $\mathcal{V}^r_\kappa$ to $\mathcal{V}^0_\kappa$ through the C^∞ splitting given by the Hodge decomposition of $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathfrak{h}_n}/\mathfrak{h}_n) \otimes C^\infty(\mathfrak{h}_n,\mathbb{C})$. Let $S \subset \mathfrak{h}_n$ be the subset consisting of ordinary CM points. It is analytically dense in $\mathfrak{h}_n$. At each point of S, the unit-root splitting agrees with the C^∞ splitting [32, Lemma 5.1.27]. Therefore $f \in \operatorname{Ker}\xi_{p,\mathrm{cl}}$ implies that $\phi(f) = 0$ and f=0. $\square$

In general it is conjectured that for all w-analytic weight κ, the map ξ_p is injective. The injectivity is proved in the $n = 1$ case [49, Proposition 3.2.4].

3.13. Polynomial q-expansions and p-adic q-expansions

The embedding (3.23) induces, by restriction, the injective map

$$N^{\dagger,r}_{\kappa,w,v} \longrightarrow H^0(\mathfrak{S}(p^\infty), \operatorname{Sym}^r \mathfrak{J})[1/p]. \tag{3.30}$$

For each geometrically connected component $\mathfrak{S}(p^\infty)^\circ$, with the Mumford object constructed in Section 2.6, one can define a map

$$\iota : \operatorname{Spf}(\mathcal{O}_K[1/t][[N^{-1}S_{L,\geqslant 0}]]) \longrightarrow \mathfrak{S}(p^\infty)^\circ.$$

The canonical basis $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$ induces an isomorphism

$$\iota^* \operatorname{Sym}^r \mathfrak{J} \simeq \mathcal{O}_K[1/t][[N^{-1}S_{L,\geqslant 0}]][\underline{Y}]_{\leqslant r},$$

which, together with (3.30), defines a p-adic polymonial q-expansion map

$$\epsilon_{\iota,q,\mathrm{poly}} : N^{\dagger,r}_{\kappa,w,v} \longrightarrow \mathcal{O}_K[[N^{-1}S_{L,\geqslant 0}]][\underline{Y}]_{\leqslant r}[1/p].$$

Remark 3.22. — Note that the image of $\epsilon_{\iota,q,\mathrm{poly}}$ are polynomials in $\underline{Y}$ with scalar coefficients, while the polynomial q-expansion $f(q,\underline{Y})$, defined as (2.17) for a classical nearly holomorphic form f of an arithmetic weight κ, is a polynomial in $\underline{Y}$ with coefficients inside the representation W_κ. To obtain the polynomial q-expansion here from the polymonial q-expansion in (2.17), one simply applies the canonical map $\mathfrak{e}_{\mathrm{can}} : W_{\kappa_{\mathrm{alg}}} \to \mathbb{A}^1$, defined as the evaluation at the identity.

If c is the number of geometrically connected components of $\mathfrak{Y}_1(p^\infty)(0)$, we can choose $\iota_1, \dots, \iota_c$ such that ι_j maps $\mathrm{Mum}_N(q)$ to the j-th component. We define the polynomial q-expansion map $\varepsilon_{q,\mathrm{poly}}$ as $\bigoplus_{j=1}^{c} \varepsilon_{\iota_j,q,\mathrm{poly}}$. Then it follows from the irreduciblity of the Igusa tower $\mathfrak{S}(p^\infty)$ [26, Corollary 8.17] that the map $\varepsilon_{q,\mathrm{poly}}$ is injective. Similarly, we can define the polynomial q-expansion map for families of nearly overconvergent forms.

Proposition 3.23. — *The polynomial q-expansion maps*

$$\varepsilon_{q,\mathrm{poly}} : N^{\dagger,\infty}_{\kappa,w,v} \longrightarrow (\mathcal{O}_K[[N^{-1}S_{L,\geqslant 0}]][\underline{Y}][1/p])^{\oplus c},$$
$$\varepsilon_{q,\mathrm{poly}} : N^{\dagger,\infty}_{\mathcal{U},w,v} \longrightarrow (\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L,\geqslant 0}]][\underline{Y}][1/p])^{\oplus c}$$

are injective.

In Section 3.12 we defined a map $\xi_p : N^{\dagger,r}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_\kappa[1/p]$ using the unit root splitting. Composing ξ_p with the q-expansion map for p-adic forms, we get the map

$$\varepsilon_{q,p\text{-adic}} : N^{\dagger,\infty}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_\kappa[1/p] \longrightarrow (\mathcal{O}_K[[N^{-1}S_{L,\geqslant 0}]][1/p])^{\oplus c},$$

and we call it the p-adic q-expansion of nearly overconvergent forms. Similarly, we define the p-adic q-expansion for $N^{\dagger,\infty}_{\mathcal{U},w,v}$. Since δ_{can} is exactly the unit-root part, $\varepsilon_{q,p\text{-adic}}$ is nothing but $\varepsilon_{q,\mathrm{poly}}|_{\underline{Y}=0}$. In the case when the map ξ_p is injective, the p-adic q-expansion $\varepsilon_{q,p\text{-adic}}$ will also be injective. For families we define the p-adic q-expansion simply as $\varepsilon_{q,\mathrm{poly}}|_{\underline{Y}=0}$.

Proposition 3.24. — *Suppose that the subdomain $\mathcal{U} \subset \mathcal{W}$ is a closed ball centered at an arithmetic point and $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ is admissible for $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}}$. Then*

$$\varepsilon_{q,p\text{-adic}} : N_{Q,\mathcal{U},\mathrm{cusp}} \longrightarrow (\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L,\geqslant 0}]][1/p])^{\oplus c}$$

is injective.

Proof. — Take $F \in N_{Q,\mathcal{U},\mathrm{cusp}}$ with $\varepsilon_{q,p\text{-adic}}(F) = 0$. Then Proposition 3.21 implies that for each $\kappa \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ such that the specialization F_κ is a classical nearly holomorphic form, we have $F_\kappa = 0$. We reduce to show that the subset of $\mathcal{U}(\overline{\mathbb{Q}}_p)$ consisting of points κ with F_κ being classical is Zariski dense inside $\mathcal{U}$. By the construction of $N_{Q,\mathcal{U},\mathrm{cusp}}$, we know that $F \in N^r_{Q,\mathcal{U},\mathrm{cusp}}$ for some $r \in \mathbb{N}$. Then F can be written as (Corollary 3.10)

$$\eta F = F_0 + \theta D F_1 + \cdots + \theta^r D^r F_r$$

with $F_i \in N^{\dagger,0}_{\mathcal{U}\otimes \mathrm{Sym}^i \tau^\vee, w, v}$ and $\eta \in K[\mathrm{Log}_1, \ldots, \mathrm{Log}_n]$ nonzero. By Corollary 3.18, the slopes of $F_0, F_1, \ldots, F_n$ are bounded in terms of Q and r. Therefore, if an arithmetic weight $\kappa \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ is away from the zeroes of η with κ_{alg} sufficiently regular with respect to the bound on slopes, then the classicity of $F_{0,\kappa}, \ldots, F_{n,\kappa}$ can be deduced from [2, Proposition 7.3.1] and [5], from which the classicity of F_κ follows. Such arithmetic points are Zariski dense in $\mathcal{U}$ as $\mathcal{U}$ is a closed ball centered at an arithmetic point. □

3.14. Families by q-expansions

Keep the assumption on $\mathcal{U}, Q$ as in Proposition 3.24. Let $\Sigma \subset \mathcal{U}(\overline{\mathbb{Q}}_p)$ be a Zariski dense subset consisting of arithmetic points. Define

$$N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}} \subset (\mathcal{A}^\circ(\mathcal{U})[[N^{-1}S_{L,\geqslant 0}]][\underline{Y}][1/p])^{\oplus c}$$
$$(\text{resp. } N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}} \subset (\mathcal{A}^\circ(\mathcal{U})[[N^{-1}S_{L,\geqslant 0}]][1/p])^{\oplus c})$$

as the sub-$\mathcal{A}(\mathcal{U})$-module consisting of elements whose specializations at almost all $\kappa \in \Sigma$ are the polynomial q-expansions (resp. p-adic q-expansions) of forms in $N_{Q_\kappa,\kappa,\mathrm{cusp}}$.

Proposition 3.25. — *With $\mathcal{U}, Q$ as in Proposition 3.24, the polynomial q-expansion map induces an isomorphism from $N_{Q,\mathcal{U},\mathrm{cusp}}$ to $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$.*

Proof. — We follow the argument of [50, Theorem 1.2.2], [25, Theorem 7.3.1]. Abbreviate $\mathcal{A}(\mathcal{U})$, $N_{Q,\mathcal{U}\,\mathrm{cusp}}$, $N_{Q_\kappa,\kappa,\mathrm{cusp}}$, $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$, as $\mathcal{A}$, N, N_κ, $N^{\Sigma,\mathrm{poly}}$. Let I be the set consisting of monomials $q^{\beta_i}\prod Y_{jk}^{a_{jk}}$, where $a_{jk} \in \mathbb{N}$, $1 \leqslant j \leqslant k \leqslant n$, and $\beta_i \in N^{-1}S_{L,\geqslant 0}$ with the subscript $1 \leqslant i \leqslant c$ meaning the i-th connected component. By taking coefficients, there is a natural embedding

$$(\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L,\geqslant 0}]][\underline{Y}][1/p])^{\oplus c} \hookrightarrow \mathcal{A}^I.$$

Denote by $K(\mathcal{A})$ the fraction field of $\mathcal{A}$. The $\mathcal{A}$-module N is finite projective. Let $d = \mathrm{rank}_{\mathcal{A}}(N) = \dim_{K(\mathcal{A})}(N \otimes K(\mathcal{A})) < \infty$, and pick $F_1, \ldots, F_d \in N$ such that they span $N \otimes K(\mathcal{A})$ over $K(\mathcal{A})$. Write their images inside $\mathcal{A}^I$ under the polynomial q-expansion map as $(a(F_j, i))_{i\in I}$, $1 \leqslant j \leqslant d$.

Thanks to the injectivity of the map $\varepsilon_{q,\mathrm{poly}}$, we can choose $i_1,\dots,i_d$ such that $D=\det(a(F_j,i_t))_{1\leqslant j,t\leqslant d}\neq 0$. We claim that

$$DN^{\Sigma,\mathrm{poly}}\subset \varepsilon_{q,\mathrm{poly}}(N).$$

Otherwise, there exists $G=(a(G,i))_{i\in I}\in DN^{\Sigma,\mathrm{poly}}\backslash\varepsilon_{q,\mathrm{poly}}(N)$. Subtracting from G a linear combination of the $\varepsilon_{q,\mathrm{poly}}(F_j)$'s, we get a nonzero $G'\in N^{\Sigma,\mathrm{poly}}$ with $a(G,i_t)=0$ for all $1\leqslant t\leqslant d$. The Zariski density of Σ implies that there exists $\kappa\in\Sigma$ such that $\varepsilon_{q,\mathrm{poly}}(F_1)_\kappa,\dots,\varepsilon_{q,\mathrm{poly}}(F_d)_\kappa,G'_\kappa$ are $\overline{\mathbb{Q}}_p$-linearly independent and $G'_\kappa=\varepsilon_{q,\mathrm{poly}}(f)$ for some $f\in N_\kappa$. Thus, $F_{1,\kappa},\dots,F_{d,\kappa},f$ are linearly independent inside N_κ, but N_κ is of dimension d. The claim is proved. Therefore,

$$N^{\Sigma,\mathrm{poly}}=\varepsilon_{q,\mathrm{poly}}(N)\otimes K(\mathcal{A})\cap\mathcal{A}^I,$$

and we also deduce that $N^{\Sigma,\mathrm{poly}}$ is a finitely generated $\mathcal{A}$-module because $\mathcal{A}$ is noetherian. In fact $\mathcal{A}$ is a noetherian UFD and a Jacobson ring [8, Section 5.2.6, Theorem 1, 3].

Now take an arbitrary $G''\in\varepsilon_{q,\mathrm{poly}}(N)\otimes K(\mathcal{A})\cap\mathcal{A}^I$, we want to prove that G'' actually lies inside $\varepsilon_{q,\mathrm{poly}}(N)$. Since $\mathcal{A}$ is a UFD we can take some $\eta\in\mathcal{A}$ such that $\eta G''\in\varepsilon_{q,\mathrm{poly}}(N)$ and $\eta' G''\notin\varepsilon_{q,\mathrm{poly}}(N)$ for any η' strictly divides η. Take $F\in N$ such that $\eta G''=\varepsilon_{q,\mathrm{poly}}(F)$. If $\mathfrak{m}\subset\mathcal{A}$ is a maximal ideal containing η, then $\varepsilon_{q,\mathrm{poly}}(F_{\kappa_\mathfrak{m}})=\eta(\kappa_\mathfrak{m})G''_{\kappa_\mathfrak{m}}=0$. The injectivity of $\varepsilon_{q,\mathrm{poly}}$ at weight κ implies that $F_{\kappa_\mathfrak{m}}=0$, and $F\in\mathfrak{m}N$ by Proposition 3.5. This shows that $F\in\bigcap_{\eta\in\mathfrak{m}}\mathfrak{m}N$. The $\mathcal{A}$-module N is finite projective, so there exists $a_1,\dots,a_l\in\mathcal{A}$ such that $\mathcal{A}=\sum a_i\mathcal{A}$ and each localization N_{a_i} is free of finite rank over A_{a_i}. Each $\mathcal{A}_{a_i}$ is still a noetherian UFD [38, Lemma (19.B)] and a Jacobson ring [46, Tag 00G6]. Let $\eta_1,\dots,\eta_b$ be all the prime factors of η. Then $\eta_j A_{a_i}$ is a prime ideal that is the intersection of all maximal ideals containing η_j in A_{a_i}. It follows that $\sqrt{\eta}A_{a_i}=\bigcap_j\eta_jA_{a_i}=\bigcap_{\eta\in\mathfrak{m},\mathfrak{m}\in\mathrm{Max}(A_{a_i})}\mathfrak{m}A_{a_i}$ and $\sqrt{\eta}N_{a_i}=\bigcap_{\eta\in\mathfrak{m},\mathfrak{m}\in\mathrm{Max}(A_{a_i})}\mathfrak{m}N_{a_i}$. Then since $F\in\bigcap_{\eta\in\mathfrak{m}}\mathfrak{m}N$, we have $F\in\sqrt{\eta}N_{a_i}$ for all i, and hence $F\in\sqrt{\eta}N$. By our choice of η this implies that η is a unit in $\mathcal{A}$. □

If we apply the same argument to $N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}}$, due to the lack of injectivity of the map $\varepsilon_{q,p\text{-adic}}$ at all points in $\mathcal{U}$, we only get a weaker result.

Proposition 3.26. — *With $\mathcal{U},Q$ as in Proposition 3.24, there exists a nonzero $\eta\in\mathcal{A}(\mathcal{U})$ such that $\eta N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}}$ belongs to $\varepsilon_{q,p\text{-adic}}(N_{Q,\mathcal{U},\mathrm{cusp}})$.*

BIBLIOGRAPHY

[1] F. Andreatta & A. Iovita, "Triple product p-adic L-functions associated to finite slope p-adic families of modular forms", with an appendix by Eric Urban, `https://arxiv.org/abs/1708.02785`, 2017.

[2] F. Andreatta, A. Iovita & V. Pilloni, “p-adic families of Siegel modular cuspforms”, *Ann. Math.* **181** (2015), no. 2, p. 623-697.

[3] I. N. Bernshtein, I. M. Gel′fand & S. I. Gel′fand, “Structure of representations that are generated by vectors of highest weight”, *Funkts. Anal. Prilozh.* **5** (1971), no. 1, p. 1-9.

[4] M. Bertolini, H. Darmon & K. Prasanna, “Generalized Heegner cycles and p-adic Rankin L-series”, *Duke Math. J.* **162** (2013), no. 6, p. 1033-1148, With an appendix by Brian Conrad.

[5] S. Bijakowski, V. Pilloni & B. Stroh, “Classicité de formes modulaires surconvergentes”, *Ann. Math.* **183** (2016), no. 3, p. 975-1014.

[6] S. Böcherer, “Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II”, *Math. Z.* **189** (1985), no. 1, p. 81-110.

[7] S. Böcherer & C.-G. Schmidt, “p-adic measures attached to Siegel modular forms”, *Ann. Inst. Fourier* **50** (2000), no. 5, p. 1375-1443.

[8] S. Bosch, U. Güntzer & R. Remmert, *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 261, Springer, 1984, xii+436 pages.

[9] K. Buzzard, “Eigenvarieties”, in *L-functions and Galois representations*, London Mathematical Society Lecture Note Series, vol. 320, Cambridge University Press, 2007, p. 59-120.

[10] R. F. Coleman, “p-adic Banach spaces and families of modular forms”, *Invent. Math.* **127** (1997), no. 3, p. 417-479.

[11] M. Courtieu & A. Panchishkin, *Non-Archimedean L-functions and arithmetical Siegel modular forms*, second ed., Lecture Notes in Mathematics, vol. 1471, Springer, 2004, viii+196 pages.

[12] E. Eischen, “p-adic differential operators on automorphic forms on unitary groups”, *Ann. Inst. Fourier* **62** (2012), no. 1, p. 177-243.

[13] E. Eischen, J. Fintzen, E. Mantovan & I. Varma, “Differential operators and families of automorphic forms on unitary groups of arbitrary signature”, *Doc. Math.* **23** (2018), p. 445-495.

[14] E. Eischen, M. Harris, J. Li & C. Skinner, “p-adic L-functions for unitary groups, part II: zeta-integral calculations”, `https://arxiv.org/abs/1602.01776`, 2016.

[15] E. Eischen & X. Wan, “p-adic Eisenstein series and L-functions of certain cusp forms on definite unitary groups”, *J. Inst. Math. Jussieu* **15** (2016), no. 3, p. 471-510.

[16] G. Faltings & C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 22, Springer, 1990, With an appendix by David Mumford, xii+316 pages.

[17] L. Fargues, “La filtration canonique des points de torsion des groupes p-divisibles”, *Ann. Sci. Éc. Norm. Supér.* **44** (2011), no. 6, p. 905-961, With collaboration of Yichao Tian.

[18] J. Fresnel & M. van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser, 2004, xii+296 pages.

[19] M. Harris, “Arithmetic vector bundles and automorphic forms on Shimura varieties. II”, *Compos. Math.* **60** (1986), no. 3, p. 323-378.

[20] ———, “L-functions and periods of polarized regular motives”, *J. Reine Angew. Math.* **483** (1997), p. 75-161.

[21] ———, “Cohomological automorphic forms on unitary groups. II. Period relations and values of L-functions”, in *Harmonic analysis, group representations, automorphic forms and invariant theory*, Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore, vol. 12, World Scientific, 2007, p. 89-149.

[22] ———, "A simple proof of rationality of Siegel-Weil Eisenstein series", in *Eisenstein series and applications*, Progress in Mathematics, vol. 258, Birkhäuser, 2008, p. 149-185.

[23] R. Harron & L. Xiao, "Gauss-Manin connections for p-adic families of nearly overconvergent modular forms", *Ann. Inst. Fourier* **64** (2014), no. 6, p. 2449-2464.

[24] H. Hida, "A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II", *Ann. Inst. Fourier* **38** (1988), no. 3, p. 1-83.

[25] ———, *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, 1993.

[26] ———, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer, 2004, xii+390 pages.

[27] T. Ibukiyama, "On differential operators on automorphic forms and invariant pluriharmonic polynomials", *Comment. Math. Univ. St. Pauli* **48** (1999), no. 1, p. 103-118.

[28] T. Ichikawa, "Integrality of nearly (holomorphic) Siegel modular forms", `https://arxiv.org/abs/1508.03138`, 2015.

[29] H. P. Jakobsen & M. Vergne, "Restrictions and expansions of holomorphic representations", *J. Funct. Anal.* **34** (1979), no. 1, p. 29-53.

[30] N. M. Katz, "p-adic properties of modular schemes and modular forms", in *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Mathematics, vol. 350, Springer, 1973, p. 69-190.

[31] ———, "Travaux de Dwork", in *Séminaire Bourbaki (1971/1972)*, Lecture Notes in Mathematics, vol. 317, Springer, 1973, p. 167-200.

[32] ———, "p-adic L-functions for CM fields", *Invent. Math.* **49** (1978), no. 3, p. 199-297.

[33] N. M. Katz & T. Oda, "On the differentiation of de Rham cohomology classes with respect to parameters", *J. Math. Kyoto Univ.* **8** (1968), p. 199-213.

[34] M. Kisin & K. F. Lai, "Overconvergent Hilbert modular forms", *Am. J. Math.* **127** (2005), no. 4, p. 735-783.

[35] K.-W. Lan, "Toroidal compactifications of PEL-type Kuga families", *Algebra Number Theory* **6** (2012), no. 5, p. 885-966.

[36] ———, *Compactifications of PEL-type Shimura varieties and Kuga families with ordinary loci*, World Scientific, 2017.

[37] Z. Liu, "p-adic L-functions for ordinary families of symplectic groups", To appear in *J. Inst. Math. Jussieu*, 2016.

[38] H. Matsumura, *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., 1980, xv+313 pages.

[39] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Oxford University Press, 1970, viii+242 pages.

[40] M. A. Nappari, "Holomorphic Forms Canonically Attached to Nearly Holomorphic Automorphic forms", PhD Thesis, Brandeis University (USA), 1992.

[41] A. Pitale, A. Saha & R. Schmidt, "Lowest weight modules of $\mathrm{Sp}_4(\mathbb{R})$ and nearly holomorphic Siegel modular forms", `https://arxiv.org/abs/1501.00524`, 2015.

[42] G. Shimura, "The special values of the zeta functions associated with cusp forms", *Commun. Pure Appl. Math.* **29** (1976), no. 6, p. 783-804.

[43] ———, "Invariant differential operators on Hermitian symmetric spaces", *Ann. Math.* **132** (1990), no. 2, p. 237-272.

[44] ———, *Euler products and Eisenstein series*, Regional Conference Series in Mathematics, vol. 93, American Mathematical Society, 1997, xx+259 pages.

[45] ———, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, vol. 82, American Mathematical Society, 2000, x+302 pages.

[46] The Stacks Project Authors, "Stacks Project", `https://stacks.math.columbia.edu`, 2015.

[47] J. Tilouine, "Companion forms and classicality in the $\mathbf{GL}_2(\mathbb{Q})$-case", in *Number theory*, Ramanujan Mathematical Society Lecture Notes Series, vol. 15, Ramanujan Mathematical Society, 2011, p. 119-141.

[48] E. Urban, "On the rank of Selmer groups for elliptic curves over $\mathbb{Q}$", in *Automorphic representations and L-functions*, Studies in Mathematics. Tata Institute of Fundamental Research, vol. 22, Tata Institute of Fundamental Research, 2013, p. 651-680.

[49] ———, "Nearly overconvergent modular forms", in *Iwasawa theory 2012*, Contributions in Mathematical and Computational Sciences, vol. 7, Springer, 2014, p. 401-441.

[50] A. J. Wiles, "On ordinary λ-adic representations associated to modular forms", *Invent. Math.* **94** (1988), no. 3, p. 529-573.

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