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Necessary and Sufficient Conditions for a Subsemigroup of a Cancellative Left Amenable Semigroup to be Left Amenable

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Abstract

Let T be a subsemigroup of a cancellative, left amenable semigroup S. We give conditions which are both necessary and sufficient for the subsemigroup T to be left amenable.

1 Introduction

Let S be a semigroup, and let $B(S)$ denote the set of all bounded, real valued functions on S. For each $f \in B(S)$ and for each $a \in S$, define $f_a \in B(S)$ by $f_a(x) = f(ax)$ for each $x \in S$. We say that S is $left\ amenable$ if there exists a function $\mu : B(S) \to \mathbb{R}$ such that for all $f, g \in B(S)$, for each $a \in S$, and for all $r \in \mathbb{R}$:

1. $sup_{x \in S} f(x) \geq \mu(f) \geq inf_{x \in S} f(x)$
2. $\mu(f_a) = \mu(f)$
3. $\mu(f + g) = \mu(f) + \mu(g)$

4. $\mu(rf) = r\mu(f)$.

The function μ is called a *left invariant mean* on the semigroup S. A group G is called *amenable* if it is left amenable. Examples of amenable groups and semigroups include all commutative semigroups (in particular all abelian groups), all finite groups, and all solvable groups. Examples of nonamenable groups and semigroups include any free group or semigroup of rank two or higher. For more on amenable groups and semigroups, see [1], [2].

In [4], Følner proves the following theorem which states that the amenability of a group G is inherited by all of the subgroups of G.

Theorem 1. *Let G be a group, and let H be a subgroup of G. If G is amenable, then H is amenable.*

In [5], Frey gives an example of a left amenable semigroup S such that S contains a subsemigroup T which is not left amenable, and in [6], Hochster gives an example of an amenable group which contains (and in fact, is generated by) a free subsemigroup on two generators. Thus, in general a subsemigroup T of a left amenable semigroup S need not inherit left amenability. In [5], Frey proves the following theorem.

Theorem 2. *Let S be a cancellative semigroup such that S contains no free subsemigroup on two generators. If S is left amenable, then every subsemigroup of S is left amenable.*

In [3], the author generalizes Theorem 2 and proves the following result.

Theorem 3. *Let S be a cancellative semigroup. Let T be a subsemigroup of S such that T does not contain a free subsemigroup on two generators. If S is left amenable, then T is left amenable.*

One can easily use the example given by Hochster in [6] to show that while the condition given in the statement of Theorem 3 of not containing a free subsemigroup on two generators is sufficient for a subsemigroup T of a cancellative left amenable semigroup S to also be left amenable, it is not necessary.

In this paper, we give conditions which are both necessary and sufficient for a subsemigroup T of a cancellative left amenable semigroup S to also be left amenable. We start in section 2 by stating some results and definitions that we will need later in the proof of the main theorem. We then give the proof of the main theorem in section 3.

2 Background and Basic Definitions

In this section we state some theorems and definitions that we will need in section 3 when proving the main theorem.

We say that a semigroup V has *common right multiples* if for each pair of elements $a, b \in V$, there exist elements $x, y \in V$ such that $ax = by$. A theorem of Ore states that if S is a cancellative semigroup which has common right multiples, then S embeds as a subsemigroup into a group G such that the following three conditions are satisfied:

(i) For each $g \in G$, there exists $x, y \in S$ such that $g = xy^{-1}$.

(ii) If S has semigroup presentation $\langle A \mid R \rangle$, then G is isomorphic to the group defined by the presentation $\langle A \mid R \rangle$.

(iii) The group G is amenable if and only if the semigroup S is left amenable .

The group G is called the *group of right fractions* of S, and S is called the *positive semigroup* of G.

The following lemma is proven in [3].

Lemma 1. *Let S be a cancellative semigroup which has common right multiples. Let T be a subsemigroup of S such that T also has common right multiples. Let H denote the group of right fractions of S, and let D denote the group of right fractions of T. Then T embeds into H, and D is isomorphic to a subgroup of H.*

The following theorem was originally proven for groups by Følner, and was then generalized to cancellative semigroups by Frey [4], [5], [7].

Theorem 4. *A cancellative semigroup S is left amenable if and only if for each $\epsilon \in (0, 1)$, and for each finite, nonempty subset $H \subseteq S$, there exists a finite, nonempty subset $E \subseteq S$ such that for each $h \in H$,* $\dfrac{|hE \cap E|}{|E|} > \epsilon$

We call the set E in Theorem 4 a *Følner Set* of the semigroup S. One can use Theorem 4 to prove the following lemma [3].

Lemma 2. *Let S be a cancellative semigroup. If S is left amenable, then S has common right multiples.*

The following lemma of Frey is proven in [5].

Lemma 3. *Let S be a cancellative semigroup. Then S contains no free subsemigroup on two generators if and only if every subsemigroup T of S has common right multiples.*

The following lemma is proven in [1].

Lemma 4. *Let S and Q be semigroups. If S is left amenable, and $f : S \to Q$ is a semigroup homomorphism onto Q, then Q is left amenable.*

3 A Proof of the Main Theorem

In this section, we give conditions which are both necessary and sufficient for a subsemigroup T of a cancellative left amenable semigroup S to also be left amenable. In particular, we prove the following theorem.

Theorem 5. *Let S be a cancellative, left amenable semigroup. Let T be a subsemigroup of S. Then the following are equivalent:*

(i) T is left amenable.

(ii) T has common right multiples.

(iii) For each pair of elements $a, b \in T$, there exist elements $c, d \in T$ such that the subsemigroup of T generated by $\{ac, bd\}$ is not the free semigroup on two generators.

(iv) For each pair of elements $a, b \in T$, there exist elements $u, v \in T$ such that the subsemigroup of T generated by $\{au, bv\}$ is contained in a cyclic subsemigroup of T.

(v) For each pair of elements $a, b \in T$, there exist elements $h, k \in T$ such that the subsemigroup of T generated by $\{ah, bk\}$ is a cyclic subsemigroup of T.

In [5], Frey gives an example of a left amenable semigroup S such that S contains a subsemigroup T which is not left amenable. In particular, Frey defines T to be the free group G_2 on two generators, and then forms S by adjoining an element 0 to T such that $00 = 0$ and such that for each $g \in T$, $0g = g0 = 0$. The semigroup S is left amenable since we can define a left invariant mean μ on S by $\mu(f) = f(0)$ [1]. However, even though the group G_2 has common right multiples, it is still not amenable. Note that in this case, the semigroup S is not cancellative. Thus, the condition in Theorem 5 for S to be cancellative is needed for the theorem to hold.

We note that Theorem 3 follows as a corollary of Theorem 5. In particular, if T is a subsemigroup of a cancellative, left amenable semigroup S such that T has no free subsemigroup on two generators, then it follows by Lemma 3 that T has common right multiples. Thus, by Theorem 5, T is left amenable. Note also that Theorem 2 follows as a corollary of Theorem 3, and therefore as a corollary of Theorem 5 [3]. We finally note that the techniques used to prove the equivalence of *(i)* and *(ii)* are similar to the techniques used in the proof of Theorem 3, and that the techniques used to prove that *(iii)* implies *(ii)* are similar to the techniques use in the proof of Lemma 3.

Proof of Theorem 5. We first prove the equivalence of *(i)* and *(ii)*. First assume that T is left amenable. Since $T \subseteq S$, and since S is cancellative, then T is cancellative. Thus, T is a cancellative, left amenable semigroup, and it follows by Lemma 2 that T has common right multiples.

Conversely, assume that T has common right multiples. Since T is a cancellative semigroup which has common right multiples, then it follows by the theorem of Ore that T embeds into a group of right fractions D. Since S is a cancellative left amenable semigroup, then S also has common right multiples. Thus, S also embeds into a group of right fractions H. It follows by Lemma 1 that D is isomorphic to a subgroup K of H. Since S is left amenable, then it follows by the theorem of Ore that H is amenable. Since K is a subgroup of H and since H is amenable, then it follows by Theorem 1 that K is amenable. Since K is amenable, and since D is isomorphic to K, then it follows by Lemma 4 that D is amenable. Therefore, since D is amenable, then it follows by the theorem of Ore that T is left amenable. Hence, *(i)* and *(ii)* are equivalent.

We next prove that *(ii)* implies *(v)*. Assume that T has common right multiples. Let $a, b \in T$. Since T has common right multiples, then there exist $h, k \in T$ such that $ah = bk$. Since $ah = bk$, then the subsemigroup of T generated by $\{ah, bk\}$ is the cyclic semigroup generated by ah.

The fact that *(v)* implies *(iv)* follows immediately.

We next prove that *(iv)* implies *(ii)*. Assume *(iv)*. Let $a, b \in T$. Let $u, v \in T$ be such that the subsemigroup Y of T generated by $\{au, bv\}$ is contained in a cyclic subsemigroup V of T. Since V is cyclic, then V is commutative. Since $Y \subseteq V$, then Y is also commutative. Thus, we see that $a(ubv) = b(vau)$. Since a and b were arbitrary, then it follows that T has common right multiples.

Finally, we prove the equivalence of *(ii)* and *(iii)*. First, assume that T has common right multiples. Let $a, b \in T$. It follows by the arguments given above proving the equivalence of *(ii)*, *(iv)*, and *(v)*, that there exist elements $c, d \in T$ such that the subsemigroup W of T generated by $\{ac, bd\}$ is a cyclic subsemigroup of T. Since W is cyclic then it is not a free semigroup on two generators.

Conversely, assume *(iii)*. Let $a, b \in T$. Let $c, d \in T$ be such that the subsemigroup M of T generated by $\{ac, bd\}$ is not the free semigroup on two generators. Let $z_1 = ac$, and let $z_2 = bd$. Since M is not the free semigroup on two generators, then there exist distinct words $u_1u_2u_3\cdots u_n$ and $v_1v_2v_3\cdots v_m$ over $\{z_1, z_2\}$ such that $u_1u_2u_3\cdots u_n = v_1v_2v_3\cdots v_m$. Since $u_1u_2u_3\cdots u_n$ and $v_1v_2v_3\cdots v_m$ are distinct words, then either $n \neq m$ or else both $n = m$ and there exists $i \in \{1, \ldots, n\}$ such that $u_i \neq v_i$. We may assume that $n \leq m$.

First assume that there exists exists $i \in \{1, \ldots, n\}$ such that $u_i \neq v_i$. Let t be the smallest integer in $\{1, \ldots, n\}$ such that $u_t \neq v_t$. We may assume that $u_t = z_1$ and $v_t = z_2$. Assume that $t = 1$. In this case, we see that $acu_2u_3\cdots u_n = z_1u_2u_3\cdots u_n = z_2v_2v_3\cdots v_m = bdv_2v_3\cdots v_m$. Thus, if $t = 1$, then a and b have common right multiples.

Now assume that $t \geq 2$. In this case, we see that $u_1\cdots u_{t-1} = v_1\cdots v_{t-1} = \alpha$, and moreover that $\alpha u_t\cdots u_n = u_1u_2u_3\cdots u_n = v_1v_2v_3\cdots v_m = \alpha v_t\cdots v_m$. Since T is cancellative, then it follows that $ac\cdots u_n = z_1\cdots u_n = u_t\cdots u_n = v_t\cdots v_m = z_2\cdots v_m = bd\cdots v_m$. Thus, if $t \geq 2$, then a and b have common right multiples.

Next, assume that for each $i \in \{1, \ldots, n\}$, that we have that $u_i = v_i$. Since the words $u_1u_2u_3\cdots u_n$ and $v_1v_2v_3\cdots v_m$ are distinct, then it must be the case that $n < m$. Let $\beta = u_1u_2u_3\cdots u_n = v_1v_2v_3\cdots v_n$. Since $v_1v_2v_3\cdots v_m$ is a word over $\{z_1, z_2\}$, then either $v_{n+1} = z_1$ or else $v_{n+1} = z_2$. If $v_{n+1} = z_1$, then define u_{n+1} to be z_2. If $v_{n+1} = z_2$, then define u_{n+1} to be z_1. In either case, u_{n+1} and v_{n+1} are defined to be elements of $\{z_1, z_2\}$ such that $u_{n+1} \neq v_{n+1}$. We may assume that $u_{n+1} = z_1$ and $v_{n+1} = z_2$. Thus, we have that $\beta u_{n+1} = u_1u_2u_3\cdots u_nu_{n+1} = v_1v_2v_3\cdots v_mu_{n+1} = \beta v_{n+1}\cdots v_mu_{n+1}$. Again, by cancellativity in the semigroup T, we have that $ac = z_1 = u_{n+1} = v_{n+1}\cdots v_mu_{n+1} = z_2\cdots v_mu_{n+1} = bd\cdots v_mu_{n+1}$. Hence, it follows in any case that a and b have common right multiples. $\square$

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