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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF NEUTRAL EQUATION

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1. INTRODUCTION

Consider the neutral delay differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+px(t-\tau_0)]+\sum_{i=1}^{n}p_i(t)x(t-\tau_i)=0 \tag{1.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+px(t-\tau_0)]+Q(t)\Big(\sum_{i=1}^{n}a_i x(t-\tau_i)\Big)=0, \tag{1.2}$$

where

(H_1) $p_i \in C([0,\infty),\mathbb{R}^+)$, $p_i(t)$ are periodic functions with common period T and $\tau_0<\min\{\tau_i,\ i=1,2,\dots,n\}$;

(H_2) $Q\in C([0,\infty),\mathbb{R}^+)$, Q is T-periodic and p, $a_i\in\mathbb{R}$;

(H_3) $\tau_i\in\mathbb{R}^+$ and there exists integers n_i such that $\tau_i=n_iT$.

In this paper we obtain the following necessary and sufficient conditions for oscillation of all solutions of (1.1) and (1.2).

Theorem 1. *Suppose that*

$$P_i=\frac{1}{T}\int_0^T p_i(s)\,\mathrm{d}s,\quad i=1,2,\dots,n.$$

If (H_1), (H_3) *hold and* $p\in\mathbb{R}^+$ *then the following conditions are equivalent:*

(a) *Every solution of* (1.1) *oscillates.*

(b) *Every solution of the neutral equation with constant coefficients*

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+px(t-\tau_0)]+\sum_{i=1}^{n}P_ix(t-\tau_i)=0 \tag{1.3}$$

oscillates.

Theorem 2. *Suppose that*

$$P=\frac{1}{T}\int_0^T Q(s)\,\mathrm{d}s\,.$$

If (H_2) *and* (H_3) *hold then the following conditions are equivalent:*

(c) *Every solution of* (1.2) *oscillates.*

(d) *Every solution of the neutral equation with constant coefficients*

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+px(t-\tau_0)]+\sum_{i=1}^{n}Pa_ix(t-\tau_i)=0 \tag{1.4}$$

oscillates.

Theorem 3. *Suppose that*

$$\sigma_i=\int_0^{\tau_i} Q(s)\,\mathrm{d}s,\quad i=0,1,2,\ldots,n.$$

If (H_2) *and* (H_3) *hold then the following conditions are equivalent:*

(e) *Every solution of* (1.2) *oscillates.*

(f) *Every solution of the neutral equation with constant coefficients*

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+px(t-\tau_0)]+\sum_{i=1}^{n}a_ix(t-\sigma_i)=0 \tag{1.5}$$

oscillates.

By a solution of (1.1) (or (1.2)) on $[0,\infty)$, we mean a continuous real valued function x defined on the interval $[-\varrho,\infty)$, where $\varrho=\max\{\tau_i,\ i=0,1,2,\ldots,n\}$ and $\big(x(t)+px(t-\tau_0)\big)$ is differentiable in $[0,\infty)$ and satisfies (1.1) (or (1.2)). As usual, a solution of (1.1) or (1.2) is said to be oscillatory if it has arbitrary large zeros, and nonoscillatory otherwise.

Applications of neutral equations occur in electrical networks containing lossless transmission lines. Such lines arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see [1] and [6]).

2. Proof of Theorem 1

Without any loss of generality, assume that $\tau_{i-1} < \tau_i$, $i = 1, 2, \ldots, n$. For the proof of "(a) implies (b)", assume to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). From a result due to Kulenovic, Ladas and Meimaridou [7], it follows that the characteristic equation

$$F(\lambda) = -\lambda(1 + p\mathrm{e}^{\lambda\tau_0}) + \sum_{i=1}^{n} P_i \mathrm{e}^{\lambda\tau_i} = 0 \tag{2.1}$$

has a real root, say λ_0. For each $\lambda \in \mathbb{R}$ define

$$f_\lambda(t) = \Big(\sum_{i=1}^{n} p_i(t)\mathrm{e}^{\lambda\tau_i}\Big) / (1 + p\mathrm{e}^{\lambda\tau_0}). \tag{2.2}$$

For $k = 0, 1, 2 \ldots, n$, we have

$$\begin{aligned} \int_{t-\tau_k}^{t} f_\lambda(s)\,\mathrm{d}s &= \frac{\tau_k}{1 + p\mathrm{e}^{\lambda\tau_0}} \Big\{ \sum_{i=1}^{n} \mathrm{e}^{\lambda\tau_i} \Big(\frac{1}{\tau_k} \int_{t-\tau_k}^{t} p_i(s)\,\mathrm{d}s \Big) \Big\} \\ &= \frac{\tau_k}{1 + p\mathrm{e}^{\lambda\tau_0}} \Big(\sum_{i=1}^{n} \mathrm{e}^{\lambda\tau_i} P_i \Big). \end{aligned} \tag{2.3}$$

Using (2.1) and (2.3) with $\lambda = \lambda_0$ we obtain

$$\int_{t-\tau_k}^{t} f_{\lambda_0}(s)\,\mathrm{d}s = \lambda_0\tau_k, \quad k = 0, 1, 2, \ldots, n. \tag{2.4}$$

Set

$$x(t) = \exp\Big(-\int_0^t f_{\lambda_0}(s)\,\mathrm{d}s \Big).$$

By (2.4) we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + px(t-\tau_0)] + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) \\ &= \Big(-f_{\lambda_0}(t) - pf_{\lambda_0}(t-\tau_0)\mathrm{e}^{\lambda_0\tau_0} + \sum_{i=1}^{n} p_i(t)\mathrm{e}^{\lambda_0\tau_i} \Big) \exp\Big(-\int_0^t f_{\lambda_0}(s)\,\mathrm{d}s \Big). \end{aligned} \tag{2.5}$$

Clearly, if $f_{\lambda_0}(t)$ is a T-periodic function then $f_{\lambda_0}(t) = f_{\lambda_0}(t-\tau_0)$. Using (2.2) in (2.5) for $\lambda = \lambda_0$ we see that $x(t)$ is a nonoscillatory solution of (1.1), a contradiction to our assumption. Hence "(a) implies (b)" has been proved.

Now we prove that (b) implies (a). If possible, suppose that $y(t)$ is a nonoscillatory solution of (1.1). Without any loss of generality, assume that $y(t) > 0$ for $t \geqslant t_0$. To complete the proof of this theorem we have to obtain a contradiction. Whenever we write an inequality, we mean that it holds for large t.

It is known from [7], that all solutions of (1.3) are oscillatory if and only if the characteristic equation $F(\lambda) = 0$ has no real roots. Since $F(0) > 0$, we have the following lemmas.

Lemma 1. *There exists a positive number m_0 such that*

$$-\mu(1 + p\mathrm{e}^{\mu\tau_0}) + \sum_{i=1}^{n} P_i \mathrm{e}^{\mu\tau_i} \geqslant m_0 \quad \text{for } \mu \in \mathbb{R}.$$

Lemma 2. *If $y(t)$ is a solution of* (1.1) *then $x(t) = y(t) + py(t - \tau_0)$ is also a solution of* (1.1).

Their proofs are very straightforward and hence are omitted. Now set

$$\begin{aligned} x(t) &= y(t) + py(t - \tau_0), \\ z(t) &= x(t) + px(t - \tau_0), \\ z_1(t) &= z(t) + pz(t - \tau_0) \end{aligned}$$

and

$$z_m(t) = z_{m-1}(t) + pz_{m-1}(t - \tau_0), \quad m = 2, 3, \ldots$$

It follows from Lemma 2 that $x(t)$, $z(t)$ and $z_m(t)$, $m = 1, 2 \ldots$ are solutions of (1.1). Clearly, $x'(t) = -\sum\limits_{i=1}^{n} p_i(t) y(t - \tau_i)$ implies that $x'(t) < 0$ eventually. Similarly, $z'(t) < 0$, $z_m'(t) < 0$, $m = 1, 2, \ldots$ eventually. Let

$$\Lambda_m = \{\lambda > 0 \mid z_m'(t) + f_\lambda(t) z_m(t) \leqslant 0 \text{ for large } t\}.$$

Now we prove the following two lemmas for Λ_m.

Lemma 3. $\Lambda_m \neq \emptyset$ *for each $m \in I^+$, $I^+ = \{1, 2, \ldots\}$.*

Proof. First we prove that $\Lambda_1 \neq \emptyset$. Since $z'(t) < 0$ eventually, we have

$$\begin{aligned} \frac{z_1(t)}{z(t - \tau_i)} &= \frac{z_1(t - \tau_i + \tau_0)}{z(t - \tau_i)} \cdot \frac{z_1(t)}{z_1(t - \tau_i + \tau_0)} \\ &= \frac{z(t - \tau_i + \tau_0) + pz(t - \tau_i)}{z(t - \tau_i)} \cdot \frac{z_1(t)}{z_1(t - \tau_i + \tau_0)} \\ &\leqslant (1 + p) \frac{z_1(t)}{z_1(t - \tau_i + \tau_0)}. \end{aligned} \tag{2.6}$$

Let $\tau_i - \tau_0 = \alpha_i T$, where $\alpha_i \in I^+$. From (2.6) we have

$$\frac{z_1(t)}{z(t-\tau_i)} \leqslant (1+p)\frac{z_1(t)}{z_1(t-\alpha_i T)}. \tag{2.7}$$

From (2.7) we have

$$\frac{z_1(t)}{1+p} \leqslant z(t-\tau_i). \tag{2.8}$$

Using (2.8) and the fact that $z(t)$ is a solution of (1.1) we have

$$z_1'(t) + \frac{1}{1+p}\Big(\sum_{i=1}^{n} p_i(t)\Big) z_1(t) \leqslant z_1'(t) + \sum_{i=1}^{n} p_i(t) z(t-\tau_i) = 0. \tag{2.9}$$

Consequently, from (2.9) we obtain

$$\frac{z_1'(t)}{z_1(t)} \leqslant -\Big(\sum_{i=1}^{n} p_i(t)\Big)/(1+p). \tag{2.10}$$

Now using (2.10) we have

$$\begin{aligned} \frac{z_1(t-\alpha_i T)}{z_1(t)} &= \exp\left(-\int_{t-\alpha_i T}^{t} \frac{z_1'(s)}{z_1(s)}\,\mathrm{d}s\right) \\ &\geqslant \exp\left\{\frac{1}{1+p}\int_{t-\alpha_i T}^{t}\Big(\sum_{i=1}^{n} p_i(s)\Big)\,\mathrm{d}s\right\} \\ &= \exp\left\{\frac{\alpha_i T}{1+p}\Big(\frac{1}{\alpha_i T}\int_{t-\alpha_i T}^{t}\Big(\sum_{i=1}^{n} p_i(s)\Big)\,\mathrm{d}s\Big)\right\} \\ &= \exp\left\{\frac{\alpha_i T}{1+p}\Big(\sum_{i=1}^{n} P_i\Big)\right\}. \end{aligned} \tag{2.11}$$

From (2.7) and (2.11) we have

$$\frac{z(t-\tau_i)}{z_1(t)} \geqslant \frac{1}{1+p}\exp\left\{\frac{\alpha_i T}{1+p}\Big(\sum_{i=1}^{n} P_i\Big)\right\}. \tag{2.12}$$

Hence

$$\sum_{i=1}^{n} p_i(t)\frac{z(t-\tau_i)}{z_1(t)} \geqslant \sum_{i=1}^{n} p_i(t) K_i, \tag{2.13}$$

where

$$K_i = \frac{1}{1+p} \exp\left\{\frac{\alpha_i T}{1+p}\Big(\sum_{i=1}^{n} P_i\Big)\right\}.$$

Consider the functions

$$g_i(\theta) = \frac{\mathrm{e}^{\theta\tau_i}}{1+p\mathrm{e}^{\theta\tau_0}}, \quad i = 1,2,\ldots,n.$$

Clearly, $g_i'(\theta) > 0$ and $g_i(\theta) \to \infty$ monotonically as $\theta \to \infty$. Since $g_i(0) = 1/(1+p)$ and $K_i > 1/(1+p)$, there exists a constant $\mu_i > 0$ such that $g_i(\mu_i) = K_i$, $i = 1,2,\ldots,n$. Let $\mu = \min\{\mu_1,\mu_2,\ldots,\mu_n\}$. Then $\mu > 0$ and $g_i(\mu_i) \geqslant g_i(\mu)$. Consequently, from (2.13) we have

$$\sum_{i=1}^{n} p_i(t)\frac{z(t-\tau_i)}{z_1(t)} \geqslant \sum_{i=1}^{n} p_i(t)K_i = \sum_{i=1}^{n} p_i(t)g_i(\mu_i) \geqslant \sum_{i=1}^{n} p_i(t)g_i(\mu),$$

that is,

$$\sum_{i=1}^{n} p_i(t)z(t-\tau_i) \geqslant \Big(\sum_{i=1}^{n} p_i(t)g_i(\mu)\Big)z_1(t) = f_\mu(t)z_1(t).$$

Hence by the above inequality we have

$$z_1'(t) + f_\mu(t)z_1(t) \leqslant z_1'(t) + \sum_{i=1}^{n} p_i(t)z(t-\tau_i) = 0.$$

This implies that $\mu \in \Lambda_1$. To show that Λ_n is nonempty we adopt exactly the same procedure where $z_{n-1}(t)$ serves the purpose of $z(t)$. Hence the proof of this lemma is completed. □

Lemma 4. *There exists a positive real number λ^* such that λ^* is the upper bound of Λ_m for all $m \in I^+$.*

Proof. Clearly, for $k = 1,2,\ldots,n$,

$$\frac{z_{m-1}(t-\tau_k)}{z_m(t)} \leqslant \frac{z_{m-1}(t-\tau_k)}{pz_{m-1}(t-\tau_0)} \tag{2.14}$$

and

$$\begin{aligned}\frac{1}{z_{m-1}(t-\tau_k+\tau_0)}\Big(\sum_{i=1}^{n} p_i(t)z_{m-2}(t-\tau_i)\Big) &\geqslant p_k(t)\frac{z_{m-2}(t-\tau_k)}{z_{m-1}(t-\tau_k+\tau_0)} \\ &= p_k(t)\frac{z_{m-2}(t-\tau_k)}{z_{m-2}(t-\tau_k+\tau_0)+pz_{m-2}(t-\tau_k)} \\ &\geqslant \frac{p_k(t)}{1+p}.\end{aligned} \tag{2.15}$$

Since $p_i(t) > 0$, $i = 1,2,\dots,n$ are periodic functions, there exists a constant $\beta > 0$ such that $p_i(t) > \beta$ for all $t \in [0,\infty)$ and all $i = 1,2,\dots,n$. Now using (2.15) and the fact that $z_{m-2}(t)$ is a solution of (1.1) we have

$$(2.16)\quad z'_{m-1}(t) + \frac{\beta}{1+p} z_{m-1}\big(t-(\tau_k-\tau_0)\big) \leqslant z'_{m-1}(t) + \sum_{i=1}^{n} p_i(t) z_{m-2}(t-\tau_i) = 0.$$

Applying a lemma of Ladas, Sficas and Stavroulakis [10] to (2.16) we have

$$(2.17)\qquad z_{m-1}(t) \geqslant \Big(\frac{\beta(\tau_k-\tau_0)}{2(1+p)}\Big)^2 z_{m-1}\big(t-(\tau_k-\tau_0)\big), \quad k = 1,2,\dots,n.$$

Replacing t by $t-\tau_0$ in (2.17) and using the resulting inequality along with the fact that $z_m(t) \geqslant p z_{m-1}(t-\tau_0)$ we obtain

$$(2.18)\quad z'_m(t) + \Big\{\sum_{i=1}^{n} p_i(t) \frac{4(1+p)^2}{p(\beta(\tau_i-\tau_0))^2}\Big\} z_m(t) \geqslant z'_m(t) + \sum_{i=1}^{n} p_i(t) z_{m-1}(t-\tau_i) = 0.$$

Since $g_i(\theta) \to 0$ as $\theta \to -\infty$, there exists a real number μ_i such that

$$g_i(\mu_i) = \frac{4(1+p)^2}{p(\beta(\tau_i-\tau_0))^2}, \quad i = 1,2,\dots,n.$$

Take $\lambda^* = \max\{\mu_1,\mu_2,\dots,\mu_n\}$. Since $g'_i(t) > 0$, we have $g_i(\lambda^*) \geqslant g_i(\mu_i)$, $i = 1,2,\dots,n$. Hence from (2.18) we have

$$\begin{aligned}(2.19)\quad z'_m(t) + f_{\lambda^*}(t) z_m(t) &= z'_m(t) + \Big(\sum_{i=1}^{n} p_i(t) g_i(\lambda^*)\Big) z_m(t)\\ &> z'_m(t) + \Big(\sum_{i=1}^{n} p_i(t) g_i(\mu_i)\Big) z_m(t)\\ &= z'_m(t) + \Big(\sum_{i=1}^{n} p_i(t) \frac{4(1+p)^2}{p(\beta(\tau_i-\tau_0))^2}\Big) z_m(t) \geqslant 0.\end{aligned}$$

Hence λ^* is the upper bound of Λ_m, because if $\lambda^* < \lambda \in \Lambda_m$ then

$$f_{\lambda^*}(t) = \sum_{i=1}^{n} p_i(t) g_i(\lambda^*) < \sum_{i=1}^{n} p_i(t) g_i(\lambda) = f_\lambda(t).$$

Consequently,

$$z'_m(t) + f_{\lambda^*}(t) z_m(t) \leqslant z'_m(t) + f_\lambda(t) z_m(t) = 0,$$

a contradiction to (2.19). This completes the proof of this lemma. □

Now we proceed to find our final contradiction. Set $\Lambda = \bigcup \Lambda_i \subset \mathbb{R}$ and $\gamma = m/(1+pe^{\tau_0\lambda^*})$. By Lemma 4, the set Λ is bounded. Hence it possesses a finite supremum λ_*. Let $\lambda \in \Lambda$ be such that $\lambda_* - \gamma/2 < \lambda < \lambda_*$. Then there exists an integer m such that $\lambda \in \Lambda_m$. Set

$$\varphi_\lambda(t) = z_m(t)\exp\left(\int_0^t f_\lambda(s)\,\mathrm{d}s\right). \tag{2.20}$$

Clearly, $\varphi'_\lambda(t) \leqslant 0$ eventually and

$$\begin{aligned}
&z'_{m+1}(t) + f_{\lambda+\gamma}(t)z_{m+1}(t)\\
&= \big(z_m(t)+pz_m(t-\tau_0)\big)' + f_{\lambda+\gamma}(t)\big(z_m(t)+pz_m(t-\tau_0)\big)\\
&= -\Big(\sum_{i=1}^n p_i(t)z_m(t-\tau_i)\Big) + f_{\lambda+\gamma}(t)\big(z_m(t)+pz_m(t-\tau_0)\big)\\
&= -\Big(\sum_{i=1}^n p_i(t)\varphi_\lambda(t-\tau_i)\Big)\exp\left(-\int_0^{t-\tau_i} f_\lambda(s)\,\mathrm{d}s\right)\\
&\quad + f_{\lambda+\gamma}(t)\left\{\varphi_\lambda(t)\exp\left(-\int_0^t f_\lambda(s)\,\mathrm{d}s\right)\right.\\
&\quad \left.+ p\varphi_\lambda(t-\tau_0)\exp\left(-\int_0^{t-\tau_0} f_\lambda(s)\,\mathrm{d}s\right)\right\}\\
&\leqslant \varphi_\lambda(t-\tau_0)\exp\left(-\int_0^t f_\lambda(s)\,\mathrm{d}s\right)\\
&\quad \times\left\{-\sum_{i=1}^n p_i(t)\exp\left(\int_{t-\tau_i}^t f_\lambda(s)\,\mathrm{d}s\right)\right.\\
&\quad \left.+ f_{\lambda+\gamma}(t)\left(1+p\exp\left(\int_{t-\tau_0}^t f_\lambda(s)\,\mathrm{d}s\right)\right)\right\}.
\end{aligned} \tag{2.21}$$

Now using (2.3) in (2.21) we obtain

$$\begin{aligned}
&z'_{m+1}(t) + f_{\lambda+\gamma}(t)z_{m+1}(t)\\
&\leqslant \frac{\varphi_\lambda(t-\tau_0)\exp(-\int_0^t f_\lambda(s)\,\mathrm{d}s)}{(1+pe^{N\tau_0})^{-1}}\left\{-\frac{\sum_{i=1}^n p_i(t)e^{N\tau_i}}{1+pe^{N\tau_0}} + f_{\lambda+\gamma}\right\},
\end{aligned} \tag{2.22}$$

where

$$N = \frac{1}{1+pe^{\lambda\tau_0}}\left\{\sum_{i=1}^n P_i e^{\lambda\tau_i}\right\}.$$

Lemma 1 implies

$$N = \frac{1}{1+p\mathrm{e}^{\lambda\tau_0}}\Big\{\sum_{i=1}^{n} P_i \mathrm{e}^{\lambda\tau_i}\Big\} \geqslant \lambda + \frac{m_0}{1+p\mathrm{e}^{\lambda\tau_0}} \geqslant \lambda + \frac{m_0}{1+p\mathrm{e}^{\lambda^*\tau_0}} = \lambda+\gamma.$$

Hence

$$f_N(t) \geqslant f_{\lambda+\gamma}(t). \tag{2.23}$$

By using (2.23) in (2.22) we obtain

$$z'_{m+1}(t) + f_{\lambda+\gamma}(t) z_{m+1}(t) \leqslant 0.$$

This implies $\lambda+\gamma \in \Lambda_{m+1}$. Consequently, $\lambda+\gamma\in\Lambda$. But

$$\lambda+\gamma > \lambda_* + \frac{\gamma}{2} > \lambda_*,$$

a contradiction to our assumption. Hence the proof is completed.

3. Proof of Theorems 2 and 3

Suppose that every solution of (1.2) oscillates. If possible, suppose that $y(t)$ is a nonoscillatory solution of (1.4). Set

$$\begin{aligned} s(t) &= \Big(\int_0^t Q(\theta)\,\mathrm{d}\theta\Big)\Big/\Big(\frac{1}{T}\int_0^T Q(\theta)\,\mathrm{d}\theta\Big) \\ &= \frac{1}{P}\Big(\int_0^t Q(\theta)\,\mathrm{d}\theta\Big) \text{ and } u(t) = y\big(s(t)\big). \end{aligned} \tag{3.1}$$

Clearly, $s(t)\to\infty$ as $t\to\infty$ and $s(t)$ is increasing on $[0,\infty)$. Since Q is T-periodic we have

(3.2)

$$\int_{t-\tau_i}^{t} Q(\theta)\,\mathrm{d}\theta = \int_0^{\tau_i} Q(\theta)\,\mathrm{d}\theta = \tau_i\Big(\frac{1}{\tau_i}\int_0^{\tau_i} Q(\theta)\,\mathrm{d}\theta\Big) = \tau_i P, \quad i=0,1,2,\ldots,n.$$

Further, by using (3.2) we have

$$\begin{aligned} u(t-\tau_i) &= y\Big(\frac{1}{P}\int_0^{t-\tau_i} Q(\theta)\,\mathrm{d}\theta\Big) \\ &= y\Big(s - \frac{1}{P}\int_{t-\tau_i}^{t} Q(\theta)\,\mathrm{d}\theta\Big) = y(s-\tau_i), \quad i=0,1,2,\ldots,n. \end{aligned} \tag{3.3}$$

From (3.2) and the fact that $\frac{ds}{dt} = Q(t)/P$ we have

$$\begin{aligned}
&\frac{d}{dt}[u(t) + pu(t-\tau_0)] + Q(t)\Big(\sum_{i=1}^{n} a_i u(t-\tau_i)\Big) \\
&= \frac{d}{ds}[y(s) + py(s-\tau_0)]\frac{ds}{dt} + Q(t)\Big(\sum_{i=1}^{n} a_i y(s-\tau_i)\Big) \\
&= \frac{1}{P}\Big\{\frac{d}{}dsy(s) + py(s-\tau_0)] + P\Big(\sum_{i=1}^{n} a_i y(s-\tau_i)\Big)\Big\}Q(t) = 0.
\end{aligned}$$

Hence $u(t)$ is a nonoscillatory solution of (1.2), a contradiction to our assumption. Thus "(c) implies (d)" has been proved.

The proof of "(d) implies (c)" can be easily done by using this variable transformation in the reverse way.

This completes the proof of Theorem 2.

Now suppose that (e) holds. If possible, suppose that $y(t)$ is a nonoscillatory solution of (1.5). Set

$$s = \int_0^t Q(\theta)\,d\theta\,.$$

Proceeding along the lines of Theorem 2, it can be shown that $u(t) = y\big(\int_0^t Q(\theta)\,d\theta\big)$ is a nonoscillatory solution of (1.2), a contradiction to our assumption. Hence "(e) implies (f)" holds. The proof of "(f) implies (e)" can be done by using this transformation in the reverse way.

Remark. (i) Theorem 2 and Theorem 3 generalize Theorem 1 due to Philos [11] and Theorem 1 due to Ladas Philos and Sficas [8], respectively.

(ii) We may see that the condition

$$\tau_0 < \min\{\tau_i,\ i = 1,2,3,\dots,n\}$$

assumed in Theorem 1 is only useful to show that (b) implies (a).

(iii) When $n = 1$, it is not necessary to assume the above condition explicitly because the same follows from the assumption that the characteristic equation

$$F(\lambda) = -\lambda(1 + pe^{\lambda\tau_0}) + P_1 e^{\lambda\tau_1} = 0$$

has no real roots and this holds if and only if all solutions of (1.3) are oscillatory. Indeed, $\tau_0 \geqslant \tau_1$ implies that $F(\lambda) < 0$ for large value of λ. Further, $F(0) > 0$ implies that $F(\lambda) = 0$ has a real root.

(iv) When $p = 0$, Theorem 1 reduces to the main result due to Philos [12]. If $p < 0$ the problem remains open.

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