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NECESSARY CONDITIONS FOR THE CONVERGENCE OF CARDINAL HERMITE SPLINES AS THEIR DEGREE TENDS TO INFINITY

BY

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ABSTRACT. Let $\mathbb{S}_{n,s}$ denote the class of cardinal Hermite splines of degree n having knots of multiplicity s at the integers. In this paper we show that if $f_n \to f$ uniformly on $\mathbf{R}$, where $f_n \in \mathbb{S}_{i_n,s}$, $i_n \to \infty$ as $n \to \infty$, and f is bounded, then f is the restriction to $\mathbf{R}$ of an entire function of exponential type $\leqslant s$. In proving this result, we need to derive some extremal properties of certain splines $\mathcal{E}_{n,s} \in \mathbb{S}_{n,s}$, in particular that $\|\mathcal{E}_{n,s}\|_\infty$ minimises $\|S\|_\infty$ over $S \in \mathbb{S}_{n,s}$ with $\|S^{(n)}\|_\infty = \|\mathcal{E}_{n,s}^{(n)}\|_\infty$.

1. Introduction. For $n = 1,2, \ldots$ and $1 \leqslant s \leqslant n$, let

$$\mathcal{F}_{n,s} = \{f \in C^{n-s}(\mathbf{R}): f|(\nu, \nu + 1) \in C^{n-1}[(\nu, \nu + 1)] \text{ and } f^{(n-1)} \text{ absolutely continuous on } (\nu, \nu + 1), \forall\, \nu \in \mathbf{Z}\}.$$

We let $\mathbb{S}_{n,s}$ denote the set of all cardinal spline functions of degree n in $\mathcal{F}_{n,s}$, i.e.,

$$\mathbb{S}_{n,s} = \{S \in C^{n-s}(\mathbf{R}): S|(\nu, \nu + 1) \in \pi_n, \forall\, \nu \in \mathbf{Z}\},$$

where π_n denotes the set of all polynomials of degree at most n.

Throughout this paper, $\|f\|$ will denote ess $\sup_{x\in\mathbf{R}}|f(x)|$.

In [6] Lipow and Schoenberg have shown that for odd n, $1 \leqslant s \leqslant \frac{1}{2}(n + 1)$, and any function f with $f^{(\nu)}$ of power growth on $\mathbf{R}$, $\nu = 0,1, \ldots, s - 1$, there is a unique $S_{n,s} \in \mathbb{S}_{n,s}$ of power growth such that $S_{n,s}^{(\nu)}$ interpolates $f^{(\nu)}$ at the integers. In [8] Marsden and Riemenschneider have shown that if f is the Fourier-Stieltjes transform of a measure on $(-s\pi, s\pi)$, then $S_{n,s}^{(\nu)} \to f^{(\nu)}$ uniformly on $\mathbf{R}$ as $n \to \infty$, $\nu = 0,1, \ldots, s - 1$. The case $s = 1$ had previously been proved by Schoenberg [10] who established in [11] the partial converse that if f is bounded on $\mathbf{R}$ and $S_{n,1} \to f$ uniformly on $\mathbf{R}$ as $n \to \infty$, then f is the restriction to $\mathbf{R}$ of an entire function of exponential type $\leqslant \pi$.

In §4 of this paper we generalise Schoenberg's result by showing, in particular, that for any $s = 1,2, \ldots,$ if f is bounded on $\mathbf{R}$ and $S_{n,s} \to f$ uniformly on $\mathbf{R}$ as $n \to \infty$, then f is the restriction to $\mathbf{R}$ of an entire function

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of exponential type $\leq s\pi$. To establish this result we need some extremal properties of certain splines $\mathcal{E}_{n,s} \in \mathbb{S}_{n,s}$ which may be regarded as generalisations of the Euler splines employed in **[11]**. For odd s these were defined by Cavaretta in **[1]**. In §2 we define $\mathcal{E}_{n,s}$ for even s and show that for all s, $f \in \mathcal{F}_{n,s}$, $\|f\| \leq 1 = \|\mathcal{E}_{n,s}\|$ and $\|f^{(n)}\|_{\infty} \leq \|\mathcal{E}_{n,s}^{(n)}\|$ implies

$$|f^{(k)}(\nu +)| \leq |\mathcal{E}_{n,s}^{(k)}(\nu +)|, \qquad \forall \ \nu \in \mathbf{Z} \text{ and } k = n - s, \dots, n - 1.$$

In **[1]** Cavaretta shows that for odd s, $S = \mathcal{E}_{n,s}$ minimises $\|S\|$ over all $S \in \mathbb{S}_{n,s}$ with

$$S^{(n)}|(\nu, \nu + 1) = (-1)^{\nu}\|\mathcal{E}_{n,s}^{(n)}\|, \qquad \forall \nu \in \mathbf{Z}.$$

In §3 we show that for all s, $S = \mathcal{E}_{n,s}$ actually minimises $\|S\|$ over all $S \in \mathbb{S}_{n,s}$ with $\|S^{(n)}\| = \|\mathcal{E}_{n,s}^{(n)}\|$.

2. The Euler-Chebyshev splines. In **[1]** Cavaretta shows there are functions $\mathcal{E}_{n,s}$ in $\mathbb{S}_{n,s}$ for $n = 1,2, \dots$ and odd $s \leq n$, characterised by the following properties:

$$\mathcal{E}_{n,s}(x + 1) = (-1)^{s}\mathcal{E}_{n,s}(x), \qquad \forall\, x \in \mathbf{R}, \tag{2.1}$$

$\mathcal{E}_{n,s}(x)$ equioscillates between -1 and 1 at points

$$0 \leq \beta_1 < \cdots < \beta_s < 1, \tag{2.2}$$

$$\mathcal{E}_{n,s} \text{ is even or odd about } x = \tfrac{1}{2} \text{ as } n \text{ is even or odd,} \tag{2.3}$$

$$\mathcal{E}_{n,s}^{(n)}(x) > 0 \text{ on } (0, 1). \tag{2.4}$$

We now construct functions $\mathcal{E}_{n,s}$ in $\mathbb{S}_{n,s}$ for $n = 1,2, \dots$ and even $s \leq n$ which are also characterised by properties (2.1)–(2.4).

We shall need the following lemma. Its proof is almost identical to that of Proposition 1 in **[1]** and so will be omitted.

LEMMA 1. *Let* $\{f_1(x), \dots, f_k(x)\}$ *be a Chebyshev system in* $[a, b]$ *and define*

$$g_i(x) = (x - a)(x - b)f_i(x), \qquad i = 1, \dots, k.$$

Let $F(x)$ *be a continuous function on* $[a, b]$ *which vanishes at* a *and* b. *Then there exists a unique linear combination* $\sum_{i=1}^{k} a_i g_i(x)$ *of best approximation in the uniform norm to* $F(x)$. *This best approximation is uniquely characterised by a* $(k + 1)$-*point equioscillation property, i.e. there exist* $k + 1$ *points* $a < x_1 < \cdots < x_{k+1} < b$ *where the error function assumes the value of its norm with alternating signs.*

We first consider the case of odd n. For any p,q, $1 \leq q \leq p$, we define

$$V_{2p+1,2q} = \Big\{ f \in \pi_{2p+1}|\big[0, \tfrac{1}{2}\big] : f^{(2i)}(0) = 0, \quad i = 0, \dots, p - q,$$
$$f^{(2j)}\big(\tfrac{1}{2}\big) = 0, \quad j = 0, \dots, p \Big\}.$$

It follows from the theory of Jerome and Schumaker **[3]** and Lorentz **[7]** that dim $V_{2p+1,2q} = q$ and any f in $V_{2p+1,2q}$ has at most $q+1$ zeros in $[0,\frac{1}{2}]$. Thus if $x(x-\frac{1}{2})f_i(x)$, $i = 1,\ldots,q$, form a basis for $V_{2p+1,2q}$, then $\{f_1(x),\ldots,f_q(x)\}$ form a Chebyshev system on $[0,\frac{1}{2}]$.

Now take any odd n and even s, $4 \leqslant s < n$, and take any f in $V_{n,s}$ with $f^{(n)} > 0$. Let F denote the best approximation to f in the uniform norm in $V_{n-2,s-2}$. Then by Lemma 1, $f - F$ equioscillates at points $0 < \beta_1 < \cdots < \beta_{s/2} < \frac{1}{2}$. Let $G = (f-F)/\|f-F\|$ and define $\mathcal{E}_{n,s}$ in $\mathbb{S}_{n,s}$ by

$$\mathcal{E}_{n,s}(x) = \begin{cases} G(x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ (-1)^n G(1-x), & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

$$\mathcal{E}_{n,s}(x+1) = \mathcal{E}_{n,s}(x), \qquad \forall\, x \in \mathbf{R}. \tag{2.5}$$

For $s = 2$, let G be the element of $V_{n,2}$ with $\|G\| = 1$ and $G^{(n)} > 0$, and again define $\mathcal{E}_{n,s}$ by (2.5). Since $G(0) = G(\frac{1}{2}) = 0$, $\exists\, \beta_1 \in (0,\frac{1}{2})$ with $|G(\beta_1)| = 1$, and so $\mathcal{E}_{n,2}$ equioscillates at β_1 and $\beta_2 = 1 - \beta_1$. Thus for all even s, $\mathcal{E}_{n,s}$ is characterised by properties (2.1) to (2.4).

Next consider even n. For any p,q, $0 \leqslant q \leqslant p$, define

$$V_{2p,2q} = \Big\{ f \in \pi_{2p}\big|\big[0,\tfrac{1}{2}\big] : f^{(2i+1)}(0) = 0, \quad i = 0,\ldots,p-q-1, \\ f^{(2j+1)}\big(\tfrac{1}{2}\big) = 0, \quad j = 0,\ldots,p-1 \Big\}.$$

Then dim $V_{2p,2q} = q+1$ and any f in $V_{2p,2q}$ has at most q zeros in $[0,\frac{1}{2}]$. Thus any basis for $V_{2p,2q}$ forms a Chebyshev system.

Now take even n and even s, $2 \leqslant s \leqslant n$, and take any f in $V_{n,s}$ with $f^{(n)} > 0$. Let F denote the best approximation to f in the uniform norm in $V_{n-2,s-2}$. Then $f - F$ equioscillates at points $0 \leqslant \beta_1 < \cdots < \beta_{s/2+1} \leqslant \frac{1}{2}$. Now $f' - F'$ is in $V_{n-1,s}$ and so has at most $\frac{1}{2}s - 1$ zeros in $(0,\frac{1}{2})$. Thus $\beta_1 = 0$ and $\beta_{s/2+1} = \frac{1}{2}$. Let $G = (f-F)/\|f-F\|$ and define $\mathcal{E}_{n,s}$ in $\mathbb{S}_{n,s}$ by (2.5). Then again $\mathcal{E}_{n,s}$ is characterised by properties (2.1)–(2.4).

We note that, for $m = 1,2,\ldots,$

$$\mathcal{E}_{2m-1,1}(x) = (-1)^m \mathcal{E}_{2m-1}(x), \\ \mathcal{E}_{2m,1}(x) = (-1)^m \mathcal{E}_{2m}\big(x - \tfrac{1}{2}\big), \tag{2.6}$$

where $\mathcal{E}_n$ denotes the Euler spline of degree n, see **[11]**.

We also note that, for $n = 1,2,\ldots,$

$$\mathcal{E}_{n,n}(x) = T_n(2x-1), \qquad \forall\, x \in [0,1],$$

where T_n denotes the Chebyshev polynomial of degree n.

It therefore seems appropriate to refer to $\mathcal{E}_{n,s}$ as Euler-Chebyshev splines, or ET-splines, following the similar terminology introduced by Cavaretta in **[1]**. They satisfy the following extremal property which is related to a theorem of Kolmogorov (see **[2]**).

Theorem 1. *Suppose f in $\mathcal{F}_{n,s}$ satisfies*

$$\|f\| < 1 \quad \textit{and} \quad \|f^{(n)}\| < \|\mathcal{E}_{n,s}^{(n)}\|, \tag{2.7}$$

then

$$|f^{(k)}(\nu +)| < |\mathcal{E}_{n,s}^{(k)}(\nu +)|, \qquad \forall\, \nu \in \mathbf{Z}, \quad k = n-s, \ldots, n-1.$$

Proof. We use an elementary and powerful technique introduced by Cavaretta [2].

Without loss of generality we may take $\nu = 0$. Suppose f in $\mathcal{F}_{n,s}$ satisfies (2.7) and is periodic of period an even integer K. We shall assume $|f^{(k)}(0+)| > |\mathcal{E}_{n,s}^{(k)}(0+)|$ for some k, $n - s < k < n - 1$, and reach a contradiction. Choose λ so that $\lambda f^{(k)}(0+) = \mathcal{E}_{n,s}^{(k)}(0+)$ and let $g = \mathcal{E}_{n,s} - \lambda f$, noting that g is also periodic of period K.

Since $\|\lambda f\| < \|\mathcal{E}_{n,s}\|$ and because of the equioscillation of $\mathcal{E}_{n,s}$, g has at least Ks distinct zeros per period. Thus, by repeated application of Rolle's theorem, $g^{(n-s)}$ has at least Ks distinct zeros per period. If $k = n - s$, then $g^{(n-s)}(0) = 0$ and so $g^{(n-s+1)}$ has at least $K(s-1)+1$ zeros per period which are not at integers. If $k > n - s$, then $g^{(n-s+1)}$ has at least $K(s-1)$ zeros per period which are not at integers, and so $g^{(k)}$ has at least $K(n-k)$ zeros per period which are not at integers. But $g^{(k)}(0+) = 0$ and so $g^{(k+1)}$ has at least $K(n-k-1)+1$ changes of sign per period which are not at integers. Thus for all k, $g^{(n)}$ has at least one change of sign per period which is not at an integer. But this contradicts $|\lambda f^{(n)}(x)| < \mathcal{E}_{n,s}^{(n)}(x)|$ in every interval $(\nu, \nu+1)$, $\nu \in \mathbf{Z}$.

We may extend to nonperiodic f in precisely the same manner as in [2]. □

3. An extremal property of ET-splines. For $n = 1, 2, \ldots,$ $1 < s < n$, and numbers $\alpha_1, \ldots, \alpha_s, \lambda$, we define

$$\Pi_n(\alpha_1, \ldots, \alpha_s; \lambda)$$

$$= \begin{vmatrix}
1 & \cdots & 1 & (1-\lambda) & 0 & 0 & \cdots & & 0 \\
\alpha_1 & & \alpha_s & 1 & (1-\lambda) & 0 & & & \cdot \\
\alpha_1^2 & \cdots & \alpha_s^2 & 1 & \binom{2}{1} & (1-\lambda) & & & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & 0 \\
\alpha_1^{n-s} & \cdots & \alpha_s^{n-s} & 1 & \binom{n-s}{1} & \binom{n-s}{2} & \cdots & \binom{n-s}{n-s-1} & (1-\lambda) \\
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\alpha_1^n & \cdots & \alpha_s^n & 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-s-1} & \binom{1}{n-s}
\end{vmatrix}$$

This determinant has the following properties, which follow from the work of Micchelli [9] or by using the method of Lee and Sharma [5].

For fixed $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s < 1$, $\Pi_n(\lambda) \equiv \Pi_n(\alpha_1, \ldots, \alpha_s; \lambda)$ is a polynomial in λ with real distinct roots of sign $(-1)^s$. If $\alpha_1 > 0$, $\Pi_n(\lambda) = a\lambda^{n-s+1} + \ldots$, where sign $a = (-1)^{(s+1)(n+s+1)}$. If $\alpha_1 = 0$, $\Pi_n(\lambda) = a\lambda^{n-s} + \ldots$, where sign $a = (-1)^{(s+1)(n+s)}$. If the nonzero α_i, $i = 1, \ldots, s$, are symmetric about $\frac{1}{2}$, then $\Pi_n(\lambda)$ is reciprocal.

Now fix $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s < 1$ and take r, $1 \leq r \leq s$. For $x \in [0, 1]$ we define

$$\begin{aligned}\Pi(x, \lambda) &= \Pi_n(\alpha_1, \ldots, \alpha_{r-1}, x, \alpha_{r+1}, \ldots \alpha_s; \lambda)\\ &= p_0(x)\lambda^{n-s+1} + p_1(x)\lambda^{n-s} + \cdots + p_{n-s+1}(x).\end{aligned}$$

Then it is easy to show that

$$\frac{\partial^j}{\partial x^j}\Pi(1, \lambda) = \lambda \frac{\partial^j}{\partial x^j}\Pi(0, \lambda), \qquad j = 0, \ldots, n - s, \tag{3.1}$$

and

$$\Pi(\alpha_i, \lambda) = 0, \qquad i \neq r. \tag{3.2}$$

We now define the 'B-spline'

$$B_r(x) = \begin{cases} p_\nu(x - \nu), & x \in [\nu, \nu + 1), \quad \nu = 0, \ldots, n - s + 1,\\ 0, & x < 0 \text{ and } x \geq n - s + 2.\end{cases}$$

From (3.1) we see that $B_r \in \mathbb{S}_{n,s}$ and from (3.2) we have $B_r(\alpha_i + \nu) = 0$ for all $\nu \in \mathbf{Z}$ and $i \neq r$. Also

$$\sum_{\nu=-\infty}^{\infty} B_r(x + \nu)t^\nu = t^{n-s+1}\Pi(x, t^{-1}), \qquad x \in [0, 1). \tag{3.3}$$

Now assume

$$\Pi_n(\alpha_1, \ldots, \alpha_s; (-1)^s) \neq 0. \tag{3.4}$$

Then following the method of Schoenberg [11], we may write

$$\left\{ \sum_{\nu=-\infty}^{\infty} B_r(\nu + \alpha_r)t^\nu \right\}^{-1} = \sum_{\nu=-\infty}^{\infty} \omega_\nu t^\nu, \tag{3.5}$$

where the series is convergent on some annulus about $|t| = 1$ and $|\omega_\nu| = O(\beta^\nu)$ as $\nu \to \pm\infty$ for some $0 < \beta < 1$.

We now define the 'fundamental spline'

$$L_r(x) = \sum_{\nu=-\infty}^{\infty} \omega_\nu B_r(x - \nu).$$

Then

$$\begin{aligned}L_r(k + \alpha_r) &= \sum_{\nu=-\infty}^{\infty} \omega_\nu B_r(k + \alpha_r - \nu)\\ &= \delta_{k0}, \qquad \forall\, k \in \mathbf{Z}, \quad \text{by (3.5).}\end{aligned}$$

It follows from the theory of [9] that if $S \in \mathbb{S}_{n,s}$ is of power growth, then

$$S(x) = \sum_{r=1}^{s} \sum_{k=-\infty}^{\infty} S(k + \alpha_r) L_r(x - k). \tag{3.6}$$

Now take x in (0, 1). Then

$$\frac{\partial^n}{\partial x^n} \Pi(x, \lambda) = (-1)^{n+r+1} n! \Pi_{n-1}(\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots \alpha_s; \lambda).$$

So, by (3.3),

$$\sum_{\nu=-\infty}^{\infty} B_r^{(n)}(\nu + x) t^\nu$$

$$= (-1)^{n+r+1} n! t^{n-s+1} \Pi_{n-1}\left(\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_s; t^{-1}\right) \tag{3.7}$$

Now

$$L_r^{(n)}(k + x) = \sum_{\nu=-\infty}^{\infty} \omega_\nu B_r^{(n)}(k + x - \nu)$$

and so

$$\sum_{k=-\infty}^{\infty} L_r^{(n)}(k + x) t^k = \left(\sum_{i=-\infty}^{\infty} \omega_i t^i \right) \left(\sum_{j=-\infty}^{\infty} B_r^{(n)}(j + x) t^j \right).$$

So by (3.7), (3.5) and (3.3),

$$\sum_{k=-\infty}^{\infty} L_r^{(n)}(k + x) t^k$$

$$= \frac{(-1)^{n+r+1} n! \Pi_{n-1}\left(\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_s; t^{-1}\right)}{\Pi_n\left(\alpha_1, \ldots, \alpha_s; t^{-1}\right)}. \tag{3.8}$$

Then from (3.8) and the properties of $\Pi_n(\lambda)$, we have the following result.

$$\sum_{k=-\infty}^{\infty} L_r^{(n)}(k + x) t^k = \frac{(-1)^{r+s} K \prod_{i=1}^{n-s+1} (1 + \mu_i t)}{\prod_{j=1}^{n-s+1} (1 - \lambda_j t)} \quad \text{if } \alpha_1 > 0,$$

$$= \frac{(-1)^{r+s+1} K \prod_{i=1}^{n-s} (1 + \mu_i t)}{\prod_{j=1}^{n-s} (1 - \lambda_j t)} \quad \text{if } \alpha_1 = 0, \quad r > 1,$$

$$= \frac{K \prod_{i=1}^{n-s+1} (1 + \mu_i t)}{t \prod_{j=1}^{n-s} (1 - \lambda_j t)} \quad \text{if } \alpha_1 = 0, \quad r = 1,$$

where K, μ_i, λ_j are constants (depending on $r, n, \alpha_1, \ldots, \alpha_s$) with $K > 0$ and sign μ_i = sign $\lambda_j = (-1)^s$, $\forall\, i,j$.

We therefore have (see [4, p. 395]),

$$\operatorname{sign} L_r^{(n)}(k + x) = \begin{cases} (-1)^{q+r+k} & s \text{ odd}, \\ (-1)^{q+r+1}, & s \text{ even}, \end{cases}$$

where

$$q = \begin{cases} 1, & \text{if } \alpha_1 > 0, \\ 0, & \text{if } \alpha_1 = 0. \end{cases} \tag{3.9}$$

We are now in a position to prove our result.

THEOREM 2. *If $S \in \mathbb{S}_{n,s}$ satisfies $\|S\| < 1$, then $\|S^{(n)}\| < \|\mathcal{E}_{n,s}^{(n)}\|$.*

PROOF. Take $\beta_1, \ldots, \beta_s$ as in (2.2). By (2.3) we know the nonzero β_i, $i = 1, \ldots, s$, are symmetric about $\frac{1}{2}$ and so $\Pi_n(\beta_1, \ldots, \beta_s; \lambda)$ is a reciprocal polynomial in λ. If n and s are both even or both odd, then $\beta_1 = 0$. Otherwise $\beta_1 > 0$. Thus in all cases, $\Pi_n(\beta_1, \ldots, \beta_s; \lambda)$ is a polynomial in λ of even degree and so

$$\Pi_n(\beta_1, \ldots, \beta_s; (-1)^s) \neq 0.$$

Since (3.4) is satisfied, we may define the 'fundamental spline' L_r for $r = 1, \ldots, s$. Then for any $S \in \mathbb{S}_{n,s}$ satisfying $\|S\| < 1$, we have from (3.6),

$$\begin{aligned} |S^{(n)}(x)| &= \left| \sum_{r=1}^{s} \sum_{k=-\infty}^{\infty} S(k + \beta_r) L_r^{(n)}(x - k) \right| \\ &< \sum_{r=1}^{s} \sum_{k=-\infty}^{\infty} |L_r^{(n)}(x - k)|, \qquad \forall\, x \in \mathbf{R}. \end{aligned} \tag{3.10}$$

But it follows from (3.9) and (2.2) that equality is attained in (3.10) for $S = \mathcal{E}_{n,s}$. $\square$

For $s = 1$ this result was proved by Schoenberg **[11]**, and for $s = n$ the result follows immediately from the properties of Chebyshev polynomials.

It is clear from the proof of Theorem 2 that the condition $\|S\| < 1$ in the statement of the theorem can be replaced by the weaker condition

$$|S(k + \beta_i)| < 1, \qquad \forall\, k \in \mathbf{Z}, \quad i = 1, \ldots, s.$$

4. Limits of cardinal splines. We need a further property of ET-splines.

LEMMA 2. *For $s = 1,2, \ldots,$ there are constants K_s such that $\|\mathcal{E}_{n,s}^{(\nu)}\| < K_s(s\pi)^n$ for all $n > s$ and $\nu = 0, \ldots, n$.*

PROOF. First suppose s is odd, $s = 2t - 1$. It follows from the work of **[1]** that for any $n > s$,

$$\mathcal{E}_{n,s} = \mathcal{E}_{n,1} + \mu_1 \mathcal{E}_{n-2,1} + \cdots + \mu_{t-1} \mathcal{E}_{n-2t+2,1}, \tag{4.1}$$

where $\mu_1, \ldots, \mu_{t-1}$ are chosen to minimise $\|\mathcal{E}_{n,s}\|$.

We first consider odd $n > s$. Then it follows from (4.1) and (2.6) that we may write

$$\mathcal{E}_{n,s} = (-1)^{(n+1)/2} \phi_n / \|\phi_n\|,$$

where

$$\phi_n(x) = \sum_{r=1}^{\infty} \frac{\cos(2r-1)\pi x}{(2r-1)^{n+1}} + \lambda_1^{(n)} \sum_{r=1}^{\infty} \frac{\cos(2r-1)\pi x}{(2r-1)^{n-1}}$$

$$+ \cdots + \lambda_{t-1}^{(n)} \sum_{r=1}^{\infty} \frac{\cos(2r-1)\pi x}{(2r-1)^{n-2t+3}}$$

$$= \sum_{r=1}^{\infty} \frac{\cos(2r-1)\pi x}{(2r-1)^{n+1}} \{1 + \lambda_1^{(n)}(2r-1)^2 + \cdots + \lambda_{t-1}^{(n)}(2r-1)^{2t-2}\},$$

and $\lambda_1^{(n)}, \ldots, \lambda_{t-1}^{(n)}$ are chosen to minimise $\|\phi_n\|$.

Let $\lambda_1, \ldots, \lambda_{t-1}$ be the unique solution of the equations

$$1 + (2r-1)^2\lambda_1 + \cdots + (2r-1)^{2t-2}\lambda_{t-1} = 0, \qquad r = 1, \ldots, t-1.$$

Let

$$\psi_n(x) = \sum_{r=t}^{\infty} \frac{\cos(2r-1)\pi x}{(2r-1)^{n+1}} \{1 + \lambda_1(2r-1)^2 + \cdots + \lambda_{t-1}(2r-1)^{2t-2}\}.$$

Then $\|(2t-3)^{n+1}\psi_n\| \to 0$ as $n \to \infty$. Since $\|\phi_n\| < \|\psi_n\|$, $\|(2t-3)^{n+1}\phi_n\| \to 0$ as $n \to \infty$ and so for $r = 1, \ldots, t-1$,

$$\left(\frac{2t-3}{2r-1}\right)^{n+1} \{1 + \lambda_1^{(n)}(2r-1)^2 + \cdots + \lambda_{t-1}^{(n)}(2r-1)^{2t-2}\} \to 0 \quad \text{as } n \to \infty.$$

So $\lambda_i^{(n)} \to \lambda_i$ as $n \to \infty$, $i = 1, \ldots, t-1$. Thus

$$(2t-1)^{n+1}\phi_n(x) = f_n(x) + a_n \cos(2t-1)\pi x + O\left(\left[\frac{2t-1}{2t+1}\right]^n\right)$$

where $f_n(x)$ is of the form $\sum_{r=1}^{t-1} b_r \cos(2r-1)\pi x$ and

$$a_n \to a = 1 + (2t-1)^2\lambda_1 + \cdots + (2t-1)^{2t-2}\lambda_{t-1} \neq 0 \quad \text{as } n \to \infty.$$

Now for each n, there is an integer j, $1 \leqslant j \leqslant 2t-1$, such that

$$f_n\left(\frac{j}{2t-1}\right) a_n \cos j\pi > 0,$$

and so

$$(2t-1)^{n+1}\left|\phi_n\left(\frac{j}{2t-1}\right)\right| > |a_n| + O\left(\left[\frac{2t-1}{2t+1}\right]^n\right).$$

So $\exists\ \delta > 0$ such that

$$s^{n+1}\|\phi_n\| > \delta, \qquad \forall\ n \geqslant s. \tag{4.2}$$

Writing

$$g_n(x) = \sum_{r=1}^{\infty} \frac{\cos(2r-1)\pi x}{(2r-1)^{n+1}},$$

we have

$$\|g_n^{(\nu)}\| < \pi^\nu\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) < 2\pi^\nu$$

for $n = 1,2,\ldots$ and $\nu < n$. Also $\|g_n^{(n)}\| = 2\|g_n^{(n-1)}\| < 4\pi^{n-1}$. So

$$\|\phi_n^{(\nu)}\| < 4\pi^{n-1}\{1 + |\lambda_1^{(n)}| + \cdots + |\lambda_{t-1}^{(n)}|\}, \qquad \nu < n,$$

and so there is a constant K such that

$$\|\phi_n^{(\nu)}\| < K\pi^n \quad \text{for all } n > s \text{ and } \nu < n. \tag{4.3}$$

Thus

$$\|\mathcal{E}_{n,s}^{(\nu)}\| = \|\phi_n^{(\nu)}\|/\|\phi_n\| < \frac{Ks}{\delta}(s\pi)^n, \qquad \forall\, n > s, \quad \nu < n,$$

by (4.2) and (4.3).

The result for even n follows similarly.

Next suppose s is even, $s = 2t$. We first note that

$$\mathcal{E}_{n,2}^{(n-1)}(x)/\|\mathcal{E}_{n,2}^{(n)}\| = x - \tfrac{1}{2}, \qquad \forall\, x \in (0, 1).$$

So

$$\mathcal{E}_{n,2} = (-1)^{[n/2]}h/\|h\|,$$

where

$$h(x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n}\cos 2k\pi\left(x - \tfrac{1}{2}\right) + \sum_{k=1}^{\infty} \frac{1}{(2k)^n} & \text{if } n \text{ even}, \\ \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n}\sin 2k\pi\left(x - \tfrac{1}{2}\right) & \text{if } n \text{ odd}. \end{cases}$$

It follows that for even n,

$$\mathcal{E}_{n,s} = (-1)^{n/2}\phi_n/\|\phi_n\|,$$

where

$$\phi_n(x) = \mu + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n}\cos 2k\pi\left(x - \tfrac{1}{2}\right) + \lambda_1^{(n)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{n-2}}\cos 2k\pi\left(x - \tfrac{1}{2}\right)$$
$$+ \cdots + \lambda_{t-1}^{(n)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{n-2t+2}}\cos 2k\pi\left(x - \tfrac{1}{2}\right),$$

and $\mu, \lambda_1^{(n)}, \ldots, \lambda_{t-1}^{(n)}$ are chosen to minimise $\|\phi_n\|$.

For odd n,

$$\mathcal{E}_{n,s} = (-1)^{(n-1)/2}\phi_n/\|\phi_n\|,$$

where

$$\phi_n(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \sin 2k\pi\left(x - \tfrac{1}{2}\right) + \lambda_1^{(n)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \sin 2k\pi\left(x - \tfrac{1}{2}\right)$$

$$+ \cdots + \lambda_{t-1}^{(n)} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{n-2t+2}} \sin 2k\pi\left(x - \tfrac{1}{2}\right),$$

and $\lambda_1^{(n)}, \ldots, \lambda_{t-1}^{(n)}$ are chosen to minimise $\|\phi_n\|$.

The result now follows by the same method as for odd s. □

We now apply Lemma 2 and Theorems 1 and 2 in proving the following:

Lemma 3. *For $s = 1,2,\ldots$, there are constants L_s such that if S in $\mathbb{S}_{n,s}$ satisfies $\|S\| < 1$, then $\|S^{(k)}\| < L_s(s\pi)^k$, for all $n > s$ and $k < n - s$.*

Proof. Take S in $\mathbb{S}_{n,s}$ with $\|S\| < 1$. Then by Theorem 2, $\|S^{(n)}\| < \|\mathcal{E}_{n,s}^{(n)}\|$. So by Theorem 1,

$$|S^{(k)}(\nu +)| \leqslant |\mathcal{E}_{n,s}^{(k)}(\nu +)|, \qquad \forall\, \nu \in \mathbf{Z}, \quad k = n - s + 1, \ldots, n - 1.$$

So by Lemma 2,

$$\|S^{(n)}\| < K_s(s\pi)^n \tag{4.4}$$

and

$$|S^{(k)}(\nu +)| < K_s(s\pi)^n, \qquad \forall\, \nu \in \mathbf{Z}, \quad k = n - s + 1, \ldots, n - 1. \tag{4.5}$$

It follows from (4.4) and (4.5) for $k = n - 1$ that $\|S^{(n-1)}\| < 2K_s(s\pi)^n$. Proceeding in this manner we deduce that

$$\|S^{(n-s+1)}\| < sK_s(s\pi)^n. \tag{4.6}$$

Let $T(x) = S(Mx)$, where $M = [\frac{1}{2}K_s s^{n+1}\pi^s]^{-1/(n-s+1)}$. Then

$$|T^{(n-s+1)}(x)| = M^{n-s+1}|S^{(n-s+1)}(x)|$$
$$< \left[\tfrac{1}{2}K_s s^{n+1}\pi^s\right]^{-1} sK_s(s\pi)^n \qquad \text{(by (4.6))}$$
$$= 2\pi^{n-s} < \|\mathcal{E}_{n-s+1}^{(n-s+1)}\|.$$

So by a theorem of Kolmogorov (see [2]), for $k < n - s$,

$$\|T^{(k)}\| < \|\mathcal{E}_{n-s+1}^{(k)}\| < 2\pi^k \qquad (\text{see } [\mathbf{11}]). \tag{4.7}$$

So

$$\|S^{(k)}\| = M^{-k}\|T^{(k)}\| < M^{-k}2\pi^k \qquad \text{(by (4.7))}$$
$$= 2\left[\tfrac{1}{2}K_s(s\pi)^s\right]^{k/(n-s+1)}(s\pi)^k < L_s(s\pi)^k,$$

where $L_s = \max\{2, K_s(s\pi)^s\}$. □

By the method of Schoenberg [**11**], we may deduce from Lemma 3 our final result.

THEOREM 3. *For a given natural number s, suppose* $f_n \in \mathbb{S}_{i_n,s}$, *where* $i_n \to \infty$ *as* $n \to \infty$. *If* $f_n \to f$ *uniformly on* $\mathbf{R}$ *and* f *is bounded, then* f *is the restriction to* $\mathbf{R}$ *of an entire function of exponential type* $\leq s$.

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