Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 269-283

NECESSARY AND SUFFICIENT CONDITIONS FOR EVENTUALLY VANISHING OSCILLATORY SOLUTIONS OF FUNCTIONAL EQUATIONS WITH SMALL DELAYS

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(Received March 28, 1978 and in Revised form June 30, 1978)

ABSTRACT. Necessary and sufficient conditions are found for all oscillatory solutions of the equation

$$(r_{n-1}(t)(r_{n-2}(t)(---(r_2(t)(r_1(t)y'(t))')')'---)' + a(t)h(y(g(t))) = b(t)$$

to approach zero. Sufficient conditions are also given to ensure that all solutions of this equation are unbounded.

KEY WORDS AND PHRASES. Oscillatory, Nonoscillatory, Delay, Functional.

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES.

1. INTRODUCTION.

Recently, T. Kusano and H. Onose [6] studied the equation

$$(r_{n-1}(t)(r_{n-2}(t)(---(r_2(t)(r_1(t)y'(t))')'---)')')' + a(t)h(y(g(t))) = b(t) \quad (1)$$

and found sufficient conditions which force all bounded nonoscillatory solutions of (1) to approach zero when a(t) is oscillatory. For positive a(t), the same conditions ensured that all nonoscillatory solutions of (1) approached zero. Kartsatos [4, Theo. 1] also found sufficient criteria for all bounded nonoscillatory solutions of (1) to, asymptotically, vanish generalizing results of this author and Dahiya [7, Theo. 1]. In fact, since the work of Hammett [3] such asymptotic results about the nonoscillatory solutions of ordinary and retarded differential equations have been obtained by many authors such as Kartsatos [4], Kusano and Onose [5, 6], this author and Dahiya [7], this author [9, 10, 11] and many others. A fairly exhaustive list of references on oscillation can be found in Graef [2]. Most of these results relate to nonoscillation properties of solutions. Very little has been said about the asymptotic nature of the corresponding oscillatory solutions of these equations. This author's work [8, 9, 12] is devoted to this type of study about the oscillatory solutions of such equations.

Our purpose in this paper is to further the study initiated by Kusano and Onose [6] and find necessary and sufficient conditions to ensure that all oscillatory solutions of equation (1) tend to zero as $t \to \infty$. In the last section, we give sufficient conditions which cause all solutions of (1) to be unbounded. Chen [1] studied a similar problem but our results are different and more extensive.

In what follows, we shall restrict our study to those solutions of (1) which can be continuously extended on some positive half line, say for $t \geq t_0 > o$. We shall, therefore, assume the point t_0 fixed for the rest of this paper. The term "solution" applies only to continuously extendable solutions on $R^+ = [t_0, \infty)$.

2. DEFINITIONS AND ASSUMPTIONS.

The following conditions hold for the rest of this paper:

(i) $a(t)$, $b(t)$, $g(t)$, $r_1(t)$, ---, $r_{n-1}(t)$ are real valued and continuous on $[t_0, \infty)$, $h \in C(-\infty, \infty)$;

(ii) $g(t) \leq t$, $g(t) \to \infty$ as $t \to \infty$;

(iii) $t\,h(t) > 0$, $t \neq 0$;

(iv) there exists a number m such that

$$\frac{h(t)}{t} \leq m \tag{2}$$

(v) $r_i(t) \geq \alpha > 0$, $i = 1, 2, ---, n-1$; for large t on R^+.

A solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. To further shorten notations we designate:

$$Z_1y(t) \equiv (r_1(t)y'(t))',\ Z_2y(t) \equiv (r_2(t)(r_1(t)y'(t))')',\ ----,$$
$$Z_iy(t) \equiv (r_i(t)(r_{i-1}(t)(---(r_2(t)(r_1(t)y'(t))')'---)')')',$$
$$i = 1, 2, ---, n-1. \tag{3}$$

3. MAIN RESULTS.

LEMMA (3.1). Suppose $\int^\infty |a(t)|dt < \infty$ and $\int^\infty |b(t)|dt < \infty$. Let $y(t)$ be a bounded oscillatory solution of (1). Then

$$\frac{r_{i+1}(Z_iy(t))}{t^{n-i-2}} \to 0 \tag{4}$$

as $t \to \infty$, $i = 1, 2, ---, n-2$.

PROOF. Since $y(t)$ is oscillatory, $Z_iy(t)$ is oscillatory for $i = 1, 2, ---, n-1$. Let $\varepsilon > 0$ be arbitrary and let $t_1 > t_0$ be so large that $Z_{n-2}y(t_1) = 0$,

$$mM\int_{t_1}^\infty |a(t)|dt < \varepsilon/2 \tag{5}$$

and

$$\int_{t_1}^\infty |b(t)|dt \leq \varepsilon/2, \tag{6}$$

where $|y(g(t))| \leq M$ for $t \geq t_1$.

Integrating equation (1) we have

$$|r_{n-1}(t)Z_{n-2}y(t)| \le mM\int_{t_1}^{t}|a(x)|dx + \int_{t_1}^{t}|b(x)|dx \le \varepsilon \tag{7}$$

Thus $Z_{n-2}y(t) \to o$ as $t \to \infty$.

Let now $t_2 > t_1$ be a zero of $Z_{n-3}y(t)$ so that $|r_{n-1}(t)Z_{n-2}y(t)| \le \varepsilon$ for $t \ge t_2$. Now

$$r_{n-2}(t)Z_{n-3}y(t) = \int_{t_2}^{t}Z_{n-2}y(s)ds \tag{8}$$

which readily gives

$$\frac{|r_{n-2}(t)Z_{n-3}y(t)|}{t} \le \frac{\varepsilon\ (t-t_2)}{t} \le \varepsilon. \tag{9}$$

Proceeding this way, the proof is completed.

LEMMA (3.2). Suppose $\int^{\infty}|a(t)|dt < \infty$, $\int^{\infty}|b(t)|dt < \infty$ and

$$\frac{1}{r_1(t)} = O\left(\frac{1}{t^{n-\beta}}\right)$$

for some $\beta \in [0, 1)$. Then oscillatory solutions of (1) are bounded.

PROOF. Let $T > t_0$ be large enough so that for $t \ge T$, $g(t) > t_0$. Integrating (1) (for $t \ge T$) over $[t_0, t]$ we have

$$Z_{n-2}(t)y(t) = \frac{1}{r_{n-1}(t)}\ r_{n-1}(t_0)Z_{n-2}y(t_0) - \frac{1}{r_{n-1}(t)}\int_{t_0}^{t}a(x)h(y(g(x)))dx + \frac{1}{r_{n-1}(t)}\int_{t_0}^{t}b(x)dx. \tag{10}$$

On repeated integration (10) yields

$$\begin{aligned} r_1(t)y'(t) = {} & r_1(t_0)y'(t_0) + r_2(t_0)Z_1y(t_0)\int_{t_0}^{t}1/r_2(x)dx \\ & + r_3(t_0)Z_2(y(t_0)) \int_{t_0}^{t}1/r_2(x_2)\int_{t_0}^{x_2}1/r_3(x\ )dxdx_2 + --- \\ & + --- + r_{n-1}(t_0)Z_{n-2}y(t_0)\int_{t_0}^{t}1/r_2(x_2)\int_{t_0}^{x_2}1/r_3 --- \int_{t_0}^{x_{n-2}}1/r_{n-1}(x)dx --- dx_2 \\ & - \int_{t_0}^{t}1/r_2(x_2)\int_{t_0}^{x_2}1/r_3(x_3)\int_{t_0}^{x_3}--- \int_{t_0}^{x_{n-2}}\frac{1}{r_{n-1}(x_{n-1})}\int_{t_0}^{x_{n-1}}a(x)h(y(g(x)))dx --- dx_2 \end{aligned}$$

$$+ \int_{t_0}^{t} 1/r_2(x_2) \int_{t_0}^{x_2} 1/r_3(x_3) \int_{t_0}^{x_3} 1/r_4(x_4) \cdots \int_{t_0}^{x_{n-2}} \frac{1}{r_{n-1}(x_{n-1})} \int_{t_0}^{x_{n-1}} b(x)dx \cdots dx_2 \tag{11}$$

Dividing (11) by $r_1(t)$ and integrating between t_0 and $g(t)$ we have

$$y(g(t)) = y(g(t_0)) + r_1(t_0)y'(t_0) \int_{t_0}^{g(t)} 1/r_1(x)dx + r_2(t_0)Z_1y(t_0) \int_{t_0}^{g(t)} 1/r_1(x_1) \int_{t_0}^{x_1} 1/r_2(x)dxdx_1$$

$$+ r_3(t_0)Z_2y(t_0) \int_{t_0}^{g(t)} 1/r_1(x_1) \int_{t_0}^{x_1} 1/r_2(x_2) \int_{t_0}^{x_2} 1/r_3(x)dxdx_2dx_1$$

$$+ \cdots + r_{n-1}(t_0)Z_{n-2}y(t_0) \int_{t_0}^{g(t)} 1/r_1(x_1) \int_{t_0}^{x_1} 1/r_2(x_2) \cdots \int_{t_0}^{x_{n-2}} \frac{1}{r_{n-1}(x)} dxdx_{n-2} \cdots dx_1$$

$$- \int_{t_0}^{g(t)} 1/r_1(x_1) \int_{t_0}^{x_1} 1/r_2(x_2) \int_{t_0}^{x_2} 1/r_3(x_3) \cdots + \int_{t_0}^{x_{n-2}} 1/r_{n-1}(x_0) \int_{t_0}^{x_0} a(x)h(y(g(x)))dxdx_0 \cdots dx_1$$

$$+ \int_{t_0}^{g(t)} 1/r_1(x_1) \int_{t_0}^{x_1} 1/r_2(x_2) \int_{t_0}^{x_2} 1/r_3(x_3) \cdots \int_{t_0}^{x_{n-2}} 1/r_{n-1}(x_{n-1}) \int_{t_0}^{x_{n-1}} b(x)dxdx_{n-1}dx_{n-2}dx_{n-3} \cdots dx_1 \tag{12}$$

Since each $1/r_i(t) \le \frac{1}{\alpha}$, $i = 2, 3, \cdots n-1$ and $g(t) \le t$ we have from above

$$|y(g(t))| \le |y(g(t_0))| + |r_1(t_0)y'(t_0)| \int_{t_0}^{t} 1/r_1(x)dx$$

$$+ \frac{1}{\alpha} |r_2(t_0)Z_1y(t_0)| \int_{t_0}^{t} (x-t_0)/r_1(x)dx$$

$$+ \frac{1}{2!\alpha^2} |r_3(t_0)Z_2y(t_0)| \int_{t_0}^{t} (x-t_0)^2/r_1(x)dx$$

$$+ ----$$

$$+ \frac{1}{(n-2)!\alpha^{n-2}} |r_{n-1}(t_0)Z_{n-2}y(t_0)| \int_{t_0}^{t} (x-t_0)^{n-2}/r_1(x)dx$$

$$+ \frac{1}{(n-2)!\alpha^{n-2}} \int_{t_0}^{t} \frac{1}{r_1(x)} \int_{t_0}^{x} (x-s)^{n-2}|a(s)||h(y(g(s)))|dsdx$$

$$+ \frac{1}{(n-2)!} \frac{1}{\alpha^{n-2}} \int_{t_0}^{t} \frac{1}{r_1(x)} \int_{t_0}^{x} (x-s)^{n-2}|b(s)|dsdx. \qquad (13)$$

Due to conditions on $r_1(t)$ we find that each term on the right hand side of (13) except possibly last two are bounded, and since $h(t)/t \leq m$, there exist constants K_1, K_2 and K_3 such that

$$|y(g(t))| \leq K_1 + K_2 \int_{t_0}^{t} \int_{t_0}^{x} \frac{(x-s)^{n-2}}{x^{n-\beta}} |a(s)||y(g(s))|dsdx$$

$$+ K_3 \int_{t_0}^{t} \int_{t_0}^{x} \frac{(x-s)^{n-2}}{x^{n-\beta}} |b(s)|dsdx.$$

Rearranging constants still further we get

$$|y(g(t))| \leq K_1 + K_4 \int_{t_0}^{t} \int_{t_0}^{x} \frac{|a(s)||y(g(s))|dsdx}{x^{2-\beta}}$$

$$+ K_5 \int_{t_0}^{t} \int_{t_0}^{x} \frac{|b(s)|dsdx}{x^{2-\beta}}$$

$$= K_1 + K_4 \int_{t_0}^{t} \left(\int_{s}^{t} \frac{1}{x^{2-\beta}} dx \right) |a(s)||y(g(s))|ds$$

$$+ K_5 \int_{t_0}^{t} \left(\int_{s}^{t} \frac{1}{x^{2-\beta}} dx \right) |b(s)|ds \qquad (14)$$

by change of order of integration. Now

$$\int_s^t \frac{1}{x^{2-\beta}}\, dx \le C \tag{15}$$

for some constant $C > o$ since $o \le \beta < 1$. Thus the last term in (14) is bounded, since $\int^\infty |b(s)|ds < \infty$.

From (14) and (15), there exists a positive constant K_6 such that

$$|y(g(t))| \le K_6 + K_5\, C \int_{t_0}^t |a(s)|\,|y(g(s))|ds.$$

By Gronwall's inequality $y(g(t))$ is bounded and the proof is complete.

THEOREM (3.1). Subject to the conditions of Lemma 3.2 all oscillatory solutions of equation (1) approach zero.

PROOF. Suppose to the contrary that some oscillatory solution $y(t)$ of (1) is such that

$$\limsup_{t \to \infty} |y(t)| > 3d > o \tag{16}$$

for some number d. Let $T > t_o$ be large enough so that for $t \ge T$ we have (from lemma 3.1)

$$\frac{r_{i+1}(t)(Z_i y(t))}{i!\alpha^i t^{n-i-2}} < \frac{d}{n}\ , \ i = 1, 2, ---, n-2 \tag{17}$$

It follows from (4) that

$$\frac{r_1(t)y'(t)}{t^{n-2}} \to o \ \text{ as } \ t \to \infty. \tag{18}$$

Let now $T_0 > T$ be a zero of $y(t)$ so that for $t \ge T_0$, (17) and (18) imply

$$\frac{r_{i+1}(t)Z_i(y(t))}{i!\alpha^i t^{n-i-2}} < \frac{d}{n}\ , \ i = 1, 2, ---, n-2,$$

and

$$\frac{r_1(t)y'(t)}{t^{n-2}} < \frac{d}{n}\ . \tag{19}$$

Integrating (1) between $[T_0, t]$ we have

$$y(t) = r_1(T_0)y'(T_0)\int_{T_0}^{t} 1/r_1(x)dx + r_2(T_0)Z_1(T_0)\int_{T_0}^{x_1} 1/r_2(x_1)\int_{T_0}^{x_1} 1/r_2(x)dxdx_1$$

$$+ r_3(T_0)Z_2y(T_0)\int_{T_0}^{t} 1/r_1(x_1)\int_{T_0}^{x_1} 1/r_2(x_2)\int_{T_0}^{x_2} 1/r_3(x)dxdx_2dx_1$$

$$+ \text{------} + r_{n-1}(T_0)Z_{n-2}y(T_0)\int_{T_0}^{t} 1/r_1(x_1)\int_{T_0}^{x_1} 1/r_2(x_2) --- \int_{T_0}^{x_{n-2}} 1/r_{n-1}(x)dxdx_{n-2} --- dx_1$$

$$-\int_{T_0}^{t} 1/r_1(x_1)\int_{T_0}^{x_1} 1/r_2(x_2) ---- \int_{T_0}^{x_{n-2}} 1/r_{n-1}(x_{n-1})\int_{T_0}^{x_{n-1}} a(x)h(y(g(x)))dxdx_{n-1} --- dx_1$$

$$+\int_{T_0}^{t} 1/r_1(x_1)\int_{T_0}^{x_1} 1/r_2(x_2)\int_{T_0}^{x_2} 1/r_3(x_3) --- \int_{T_0}^{x_{n-1}} b(x)dxdx_{n-1} --- dx_1. \quad (20)$$

Since $y(t)$ is bounded, let $|y(g(t))| \le C_1$ for some positive constant C_1. From (20) we have

$$|y(t)| \le |r_1(T_0)y'(T_0)|\int_{T_0}^{t} 1/r_1(x)dx + \frac{1}{\alpha}|r_2(T_0)Z_1y(T_0)|\int_{T_0}^{t}\frac{(x-T_0)}{r_1(x)}dx$$

$$+ \frac{1}{2!\alpha^2}|r_3(T_0)Z_2y(T_0)|\int_{T_0}^{t}(x-T_0)^2/r_1(x)dx$$

$$+ ----$$

$$+ \frac{1}{(n-2)!\alpha^{n-2}}\int_{T_0}^{t}(x-T_0)^{n-2}/r_1(x)dx$$

$$+ \frac{C_1m}{(n-2)!\alpha^{n-2}}\int_{T_0}^{t} 1/r_1(x)\int_{T_0}^{x}(x-s)^{n-2}|a(s)|dsdx$$

$$+ \frac{1}{(n-2)!\alpha^{n-2}}\int_{T_0}^{t} 1/r_1(x)\int_{T_0}^{x}(x-s)^{n-2}|b(s)|dsdx. \quad (21)$$

Now there exists a constant $D_i > o$ such that for each i

$$\left|\frac{r_{i+1}(T_0)Z_iy(T_0)}{\alpha^i i!}\int_{T_0}^{t}\frac{(x-T_0)^{i-2}}{r_1(x)}dx\right|$$

$$\leq \left|\frac{r_{i+1}(T_0)Z_i y(T_0)}{\alpha^i i!}\right| D_i \int^t \frac{(x-T_0)^{i-2}}{x^{n-\beta}} dx$$

$$\leq \left|\frac{r_{i+1}(T_0)Z_i y(T_0)}{\alpha^i i!}\right| D_i \int_{T_0}^t \frac{1}{x^{n-i+2-\beta}} dx$$

$$\leq \left|\frac{r_{i+1}(T_0)Z_i y(T_0)}{\alpha^i i!}\right| D_i \cdot \frac{1}{n-i+1-\beta} \left[\frac{1}{t^{n-i+1-\beta}} + \frac{1}{T_0^{n-i+1-\beta}}\right]$$

$$< \frac{d}{n} \tag{22}$$

in view of (17), conveniently large enough choice of T_0) and the fact that $\beta < 1$.

Similarly, as it was shown in the later part of inequality (14) (by changing the order of integration) it is easily shown that a large choice of T_0 results in

$$\left|\frac{C_1 m}{(n-2)!\alpha^{n-2}} \int_{T_0}^t 1/r_1(x) \int_{T_0}^x (x-s)^{n-2}|a(s)|dsdx\right| \leq \frac{d}{n} \tag{23}$$

and

$$\left|\frac{1}{(n-2)!\alpha^{n-2}} \int_{T_0}^t 1/r_1(x) \int_{T_0}^x (x-s)^{n-2}|b(s)|dsdx\right| \leq \frac{d}{n}\,. \tag{24}$$

From (21), (22), (23) and (24) we get

$$|y(t)| \leq \left(\frac{d}{n} + \frac{d}{n} + \text{---} \frac{d}{n}\right) = d \tag{25}$$

From (25) we see that if we choose a large enough T_0, then for all $t \geq T_0$, $|y(t)| \leq d$. But this contradicts (16) for any positive d. The proof is now complete.

EXAMPLE (3.1). The equation

$$(t(e^t y'(t))')' + e^{-t-2\pi}y(t-\pi) = 2e^{-t}t \sin t + e^{-t}\sin t$$
$$+ 4te^{-t} \cos t - 3e^{-t} \cos t - e^{-3t} \sin t, \tag{26}$$

$t > \pi$ has $y = e^{-2t} \sin t$ as an oscillatory solution approaching zero. All conditions of Theorem 3.1 are satisfied. Hence all oscillatory solutions of (26) vanish at ∞.

Our next theorem leads to a necessary and sufficient criteria for all oscillatory solutions of equation (1) to vanish at ∞.

THEOREM (3.2). Suppose $a(t) > o$, $\frac{1}{r_1(t)} = O\left(\frac{1}{t^{n-\beta}}\right)$, for $o \leq \beta < 1$ and $\int^{\infty} a(t)dt < \infty$. Further suppose that $b(t)/a(t)$ approaches a finite limit as $t \to \infty$. Then a necessary and sufficient condition for all oscillatory solution of (1) to approach zero is

$$\lim_{t\to\infty} \frac{|b(t)|}{a(t)} = o \ . \tag{27}$$

PROOF. (SUFFICIENCY). Suppose that $\frac{b(t)}{a(t)} \to o$ as $t \to \infty$. Since $\int^{\infty} a(t)dt < \infty$, we have $\int^{\infty} |b(t)|dt < \infty$. By Theorem 3.1 all oscillatory solutions approach zero.

(NECESSITY). Let $y(t)$ be an oscillatory solution of (1).

Dividing (1) by $a(t)$ we have

$$\frac{1}{a(t)} (r_{n-1}(r_{n-2}(---(r_1 y'(t))')')'--)' + h(y(g(t))) = \frac{b(t)}{a(t)} \tag{28}$$

Now $y(t) \to o$ as $t \to \infty$. Suppose to the contrary that

$$\lim_{t\to\infty} \frac{|b(t)|}{a(t)} \geq \lambda > o. \tag{29}$$

Since $h(y(g(t))) \to o$, (28) from (29) reveals that there exists a large T such that for $t \geq T$, $Z_{n-1}y(t) > o$. But then $y(t)$ is nonoscillatory, a contradiction. The proof is now complete.

EXAMPLE (3.2). Consider the equation

$$(t^2 y'(t))' + \frac{1}{t^2} y(t) = \frac{5}{t^3} (\sin(\ell_n t) - \cos(\ell_n t)) + \frac{\sin(\ell_n t)}{t^5} \tag{30}$$

Here all conditions of Theorem 3.2 are satisfied. Hence all oscillatory solutions approach zero. In fact $y(t) = \sin(\ell_n t)/t^3$ is an oscillatory solution of (30).

The necessity part of Theorem 3.2 leads us to the following theorem.

THEOREM (3.3). Suppose $\frac{1}{r_1(t)} = O(1/t^{n-\beta})$ for some β such that $o \leq \beta < 1$. Further suppose that $\int^{\infty} a(t)dt < \infty$, $a > o$ and $b(t)/a(t)$ is bounded. Then all oscillatory solution of equation (1) approach zero as $t \to \infty$.

PROOF. Since $\int^{\infty} a(t) < \infty$, boundedness of $b(t)/a(t)$ implies $\int^{\infty} |b(t)|dt < \infty$. Conditions of Theorem 3.1 hold. The proof is complete.

Our next Theorem gives conditions when oscillatory solutions do not approach limits.

THEOREM (3.4). Suppose $a(t) > o$ and $\liminf_{t \to \infty} |b(t)|/a(t) > o$.

Let $y(t)$ be an oscillatory solution of equation (1). Then $\limsup_{t \to \infty} |y(t)| > o$.

PROOF. Suppose to the contrary that $y(t) \to o$ as $t \to \infty$. Then $h(y(g(t))) \to o$. From equation (1)

$$\frac{1}{a(t)} |(r_{n-1}(r_{n-2}(--(r_2(t)(r_1(t)y'(t))')'--)'| + |h(y(g(t)))| \geq |b(t)|/a(t).$$

This shows that $Z_{n-1}y(t)$ is eventually positive contradicting the fact that $y(t)$ is oscillatory.

REMARK. It is to be noted that the conditions

$$\frac{1}{r_1(t)} = O\left(\frac{1}{t^{n-\beta}}\right), \int^{\infty} a(t)dt < \infty \text{ and } \int^{\infty} |b(t)|dt < \infty$$

are not needed here.

EXAMPLE (3.3). All oscillatory solutions of the equation

$$y''(t) + y(t-2\pi) = 2 \tag{31}$$

satisfy $\limsup_{t \to \infty} |y(t)| > o$ since this equation satisfies all conditions of Theorem 3.4. $y(t) = 2 + 2\cos(t)$ is one such solution.

Next theorem gives nonoscillation criterion.

THEOREM (3.5). Suppose $a(t) > o$, $\int^{\infty} a(t)dt < \infty$ and $\frac{1}{r_1(t)} = O\left(\frac{1}{t^{n-\beta}}\right)$, $o \leq \beta < 1$. Further suppose that $\liminf_{t \to \infty} |b(t)|/a(t) > o$ and $b(t)/a(t)$ is bounded. Then all solutions of equation (1) are nonoscillatory.

PROOF. Suppose to the contrary that $y(t)$ is an oscillatory solution of (1). Since all conditions of Theorem 3.3 are satisfied, $y(t) \to o$ as $t \to \infty$. Thus $h(y(g(t))) \to o$. From equation (1)

$$\frac{1}{a(t)} \left|(r_{n-1}(r_{n-2}(t)(---(r_1(t)y'(t))')')'---)'\right| \geq |b(t)|/a(t) - |h(y(g(t)))|. \quad (32)$$

(32) suggests that $z_{n-1}y(t) > o$ eventually, contradicting the fact that $y(t)$ is oscillatory. The proof is now complete.

EXAMPLE (3.5). The equation

$$(\tfrac{1}{2}\, t^2 y'(t))' + \frac{1}{t^2} y(t) = \frac{1}{t^2} + \frac{1}{t^4} \quad (33)$$

satisfies all conditions of this theorem. $y(t) = 1/t^2$ is a nonoscillatory solution of (33).

THEOREM (3.6). Suppose $\int^{\infty} |a(t)|dt < \infty$ and $\left|\int^{\infty} b(t)dt\right| = \infty$. Then all oscillatory solutions of (1) are unbounded.

PROOF. Suppose to the contrary that some oscillatory solution $y(t)$ satisfies $|y(t)| \leq C_0$ for some $C_0 > o$. From equation (1) on integration for $t \geq T$.

$$|r_{n-1}(t)z_{n-2}y(t)| + |r_{n-1}(T)z_{n-2}y(T)| + C_0 m \int_T^t |a(s)|ds \geq \left|\int_T^t b(s)ds\right|. \quad (34)$$

(34) yields that $z_{n-2}y(t)$ assumes a constant sign eventually, contradicting that $y(t)$ is oscillatory. The proof is now complete. The following example shows that under the conditions of Theorem 3.6, it is possible to have bounded nonoscillatory solutions.

EXAMPLE (3.6). The equation

$$(t^{5/2}(ty'(t)')' + \frac{1}{t^2}\, y(t) = \frac{1}{t^3} + \frac{1}{2\sqrt{t}}$$

satisfies all conditions of Theorem 3.6. It has $y(t) = 1/t$ as a bounded nonoscillatory solution.

THEOREM (3.7). Suppose $\int^{\infty}|a(t)|dt < \infty$ and $\int^{\infty} b(t)dt = \pm\infty$. Further suppose that $r_i(t)$ is bounded $i = 1, 2, ---, n-1$. Then all solutions of equation (1) are unbounded.

PROOF. Due to Theorem 3.6 we only need to prove it for a nonoscillatory solution. Let $y(t)$ be nonoscillatory and bounded. From inequality (34) in the proof of Theorem 3.6 it follows that $|z_{n-2}y(t)| \to \infty$ as $t \to \infty$. Since $z_{n-2}y = (r_{n-2}z_{n-3}y(t))'$ and r_{n-2} is bounded, we have $z_{n-3}y(t) \to \pm\infty$. Proceeding this way we find that $y'(t) \to \pm\infty$ forcing $y(t) \to \pm\infty$. The proof is now complete by contradiction.

FINAL REMARK. Theorem 3.1 improves our main result in [9] (c.f [11]) where it was shown that oscillatory solutions of

$$(r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = f(t) \tag{35}$$

approach zero subject to:

$$\int^{\infty}|a(t)|t^{n-2}dt < \infty, \quad \int^{\infty}|f(t)|t^{n-2}dt < \infty$$

and

$$\frac{1}{r(t)} = O\left(\frac{1}{t^{n-\beta}}\right), \quad o \le \beta < 1.$$

The restriction on $r(t)$ cannot be weakened i.e. β cannot be greater than or equal to 1 as the following example shows.

EXAMPLE (3.7). The equation

$$(t^2y'(t))'' + \frac{1}{t(\ell_n t)^2}\, y(t) = \frac{\cos(\ell_n(\ell_n t))}{t(\ell_n t)^3} + \frac{3\sin(\ell_n(\ell_n t))}{t(\ell_n t)^3} - \frac{\cos(\ell_n(\ell_n t))}{t(\ell_n t)^2}$$

$$t > o \tag{36}$$

has $y = \sin(\ell_n(\ell_n t))$ as an oscillatory solution which does not have a limit at ∞. Only the condition on $r(t)$ is violated. We see that for $n = 3$

$$\frac{1}{r(t)} = \frac{1}{t^{n-1}} \quad \text{so that} \quad \beta = 1$$

even though $\int^{\infty} \frac{1}{r(t)}\, dt < \infty$.

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