Bulletin of the **S**ection of **L**ogic
Volume 37:3/4 (2008), pp. 143–160

Janusz Ciuciura*

NEGATIONS IN THE ADJUNCTIVE DISCURSIVE LOGIC

Abstract

In the logical literature, *Discursive* (or *Discussive*) Logic introduced by Stanisław Jaśkowski is seen as one of the earliest examples of the so-called paraconsistent logic. Nevertheless, there is some confusion over what discursive logic actually is. One of the possible sources of the confusion may be easily discerned; it comes from the fact that Jaśkowski published his two papers in Polish and their English translations appeared many years later.[1] Up till 1999, no one but a Polish reader was able to read Jaśkowski's paper on the discursive conjunction and, consequently some authors took discursive logic to be a foremost example of a *non-adjunctive* logic.[2]

The situation became even more complicated when da Costa, Dubikajtis and Kotas presented an axiomatization with discursive connectives as primitive symbols. It turned out that a connective of the discursive conjunction they considered did not correspond to any of Jaśkowski's connectives. Thus, their axiomatization contained some axiom schemata that were not generally valid in Jaśkowski's logic.[3]

The purpose of this paper is to clarify the confusion surrounding the discursive logic. We will present a direct semantics and axiomatization of Jaśkowski's adjunctive discursive logic and show how to define and axiomatize two additional connectives of negation.

Keywords: discursive (discussive) logic, D_2, paraconsistent logic.

*Research leading to the results presented in this paper and the stay of the author at the Centre for Logic and Philosophy of Science were supported by a stipend from the Foundation for Polish Science (FNP, Program Kolumb 2007).

[1] For details, see *References* ([17] and [18]).

[2] See, for example, [12], [23] and [26].

[3] See, [1], [9] and [13].

1 Introduction

In 1949, Jaśkowski published his second paper on the discursive logic. It was the first time when the discursive conjunction appeared. The language of the resulting calculus is as follows.

Definition 1. Let *var* denote a non-empty denumerable set of all propositional variables $\{p_1, p_2, ...\}$. For_{D2} is defined to be the smallest set for which the following holds

(i) if $\alpha \in var$ then $\alpha \in For_{D2}$
(ii) if $\alpha \in For_{D2}$ then $\sim \alpha \in For_{D2}$
(iii) if $\alpha \in For_{D2}$ and $\beta \in For_{D2}$ then $\alpha \bullet \beta \in For_{D2}$,
where $\bullet \in \{\vee, \wedge_d, \rightarrow_d\}$.

The symbols: $\sim$, $\vee$, $\wedge_d$, $\rightarrow_d$ denote negation, disjunction, discursive conjunction and discursive implication, respectively. The discursive equivalence, $\alpha \leftrightarrow_d \beta$, is defined by $(\alpha \rightarrow_d \beta) \wedge_d (\beta \rightarrow_d \alpha)$.

Now we determine a translation function of the language of the new calculus, D_2 for short, into the language of $S5$ of Lewis, $f : For_{D2} \longrightarrow For_{S5}$, i.e.

(i) $f(p_i) = p_i$ if $p_i \in var$ and $i \in N$
(ii) $f(\sim \alpha) =\sim f(\alpha)$
(iii) $f(\alpha \vee \beta) = f(\alpha) \vee f(\beta)$
(iv) $f(\alpha \wedge_d \beta) = f(\alpha) \wedge \Diamond f(\beta)$
(v) $f(\alpha \rightarrow_d \beta) = \Diamond f(\alpha) \rightarrow f(\beta)$

and additionally

$(*)$ $\forall \alpha \in For_{D2} : \alpha \in D_2$ iff $\Diamond f(\alpha) \in S5$.[4]

Notice that $(*)$ does not belong to a recurrent definition of the function f, but it is a part of the definition of the logic D_2 by means of f.

Interesting that Jaśkowski did not propose any philosophical reading for $\alpha \wedge_d \beta$. We conjecture that his intention was to mark a distinction in which the first *conjunct* was a voice in the discussion (we do not know exactly who said α, but we know that someone did it), while the second reflected an opinion of the particular discussant (we know who said β).

[4]See [17] p. 44, [18] p. 57. We use here the English translations of the Jaśkowski papers that appeared in Logic and Logical Philosophy (see *References* for details).

In [23], we read: "Let us start with non-adjunctive systems, so called because the inference from A and B to A & B fails. The first of these to be produced was also the first formal paraconsistent logic. This was Jaskowski's discussive (or discursive) logic."

This claim is correct as long as we ignore the discursive conjunction. To prove that, it is enough to note the rule

(AdR) $\alpha,\ \beta\ /\ \alpha \wedge_d \beta$

is admissible in D_2 (for any formulas α, β). D_2 is closed under *adjunction* since S_5 is closed under

(AR) $\Diamond\alpha,\ \Diamond\beta\ /\ \Diamond(\alpha \wedge \Diamond\beta)$.[5]

At the first sight the rule (AR) may look oddly asymmetrical and we can even wish to replace $f(\alpha \wedge_d \beta) = f(\alpha) \wedge \Diamond f(\beta)$ with one of the following definitions

(iv)* $f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta)$
(iv)** $f(\alpha \wedge_d \beta) = \Diamond f(\alpha) \wedge \Diamond f(\beta)$.

The conjunction seems to be more elegant now, but there is a price to be paid for it. If we adopted a different strategy and replaced $f(\alpha \wedge_d \beta) = f(\alpha) \wedge \Diamond f(\beta)$ with (iv)* we would obtain a non-adjunctive system (and then the author of [23] was right) plus $(p \wedge \sim p) \rightarrow_d q$ as a thesis.[6]

The definition (iv)** is not a far better choice despite the fact that the resulting system would still be closed under *adjunction*. It suffices to say that the formula

$(p \wedge_d q) \rightarrow_d (\sim (p \wedge_d q) \rightarrow_d r)$

is a thesis of the system and the presence of $p \wedge_d q$ and $\sim (p \wedge_d q)$ trivializes it. Jaśkowski would never approve of the strategy.[7]

Hiding every translated formula behind $\Diamond$ is a result of Jaśkowski's philosophy, which protects our system from collapsing into *deductive insufficiency* (surely not in a sense of *trivialization*, but in a sense that the set of theses of the system would be very limited).

Hiding, however, is not enough. "Can a discussive system be based on ordinary two-valued logic? - Jaśkowski asked - It can easily be seen that it

[5] Or more accurately (in this context) – under $\Diamond f(\alpha),\ \Diamond f(\beta)\ /\ \Diamond(f(\alpha) \wedge \Diamond f(\beta))$.

[6] See [17] p. 47.

[7] Cf. ibidem p. 44. Observe that $(p \vee \sim p) \rightarrow_d (\sim (p \vee \sim p) \rightarrow_d q)$ is still valid in D_2.

is not so. Even such an elementary form of reasoning as the rule of modus ponens fails. If implication is interpreted so as it is done in two-valued logic, then out of the two theses one of which is $P \to Q$, and thus states: "it is possible that if P, then Q'', and the other is P, and thus states: "it is possible that P, it does not follow that "it is possible that Q, so that the thesis Q, does not follow intuitively, as the rule of modus ponens requires."[8]

So we really need *diamonds* and the two-step translation method.

It is worth adding, for those who might be wondering about why Jaśkowski did not accept

(v)* $f(\alpha \to_d \beta) = \Diamond f(\alpha) \to \Diamond f(\beta)$

nor

(v)** $f(\alpha \to_d \beta) = f(\alpha) \to \Diamond f(\beta)$

as a definition of the discursive implication, that a different notion of the discursive implication can cause some problems in interpretation (at least from the paraconsistent point of view). More precisely, if we replace $f(\alpha \to_d \beta) = \Diamond f(\alpha) \to f(\beta)$ with (v)* we obtain a system in which $(p \to_d q) \to_d (\sim (p \to_d q) \to_d r)$ is valid and, similarly to the case of (iv)**, the presence of $p \to_d q$ and $\sim (p \to_d q)$ will trivialize it. This is the main reason why Jaśkowski rejected the definition.[9]

The situation becomes even more interesting when we deal with (v)**. Firstly, the rule of detachment is *unsafe* since $S5$ of Lewis is not closed under $\Diamond\alpha$, $\Diamond(\alpha \to \Diamond\beta)$ / $\Diamond\beta$. Secondly, Duns Scotus' thesis is valid in such a system because the formula $\Diamond(p \to \Diamond(\sim p \to \Diamond q))$ is valid in S_5. An intriguing question arises as a consequence: What is the nature of paraconsistency? Some authors calls a system paraconsistent if it is not closed under Duns Scotus' thesis.[10] In this sense, the resulting system is not paraconsistent at all. However, we are not able to deduce an arbitrary sentence from $\{\alpha, \sim \alpha\}$ because the system is not closed under detachment (neither is closed under the rule of explosion, i.e. $\alpha, \sim \alpha$ / β). The standard way to trivialize the system fails.[11]

---

[8] [17] p. 43.

[9] Cf. ibidem p. 44.

[10] See, for example, [21] p. 233.

[11] Unfortunately, the objective of this paper is not a detailed philosophical study of paraconsistency. For more detailed information on the topic see, for example, G. Priest, R. Routley, and J. Norman (eds) *Paraconsistent Logic: Essays on the Inconsistent*, Philosophia Verlag, Mnchen, 1989 and D. Batens, J. P. Van Bendegem, G. Priest (eds)

2 Discursive Logic's New Clothes

The guiding idea behind the semantics we present is to eliminate the translation rules.

A frame (D_2-frame) is a pair $< W, R >$, where W is a non-empty set of points (or possible worlds) and R is the equivalence relation on W. A model (D_2-model) is a triple $< W, R, v >$, where v is a mapping from propositional variables to sets of worlds, $v : var \Rightarrow 2^W$. The satisfaction relation $\models_m$ is defined as follows

(var) $x \models_m p_i$ iff $x \in v(p_i)$ and $i \in N$
$(\sim)$ $x \models_m \sim \alpha$ iff $x \not\models_m \alpha$
$(\vee)$ $x \models_m \alpha \vee \beta$ iff $x \models_m \alpha$ or $x \models_m \beta$
$(\wedge_d)$ $x \models_m \alpha \wedge_d \beta$ iff $x \models_m \alpha$ and $\exists_{y \in W}(xRy$ and $y \models_m \beta)$
$(\rightarrow_d)$ $x \models_m \alpha \rightarrow_d \beta$ iff $\forall_{y \in W}($ if xRy then $y \not\models_m \alpha)$ or $x \models_m \beta$.

A formula α is valid in D_2, $\models \alpha$ for short, iff for any model $< W, R, v >$, for every $x \in W$, there exists $y \in W$ such that: xRy and $y \models_m \alpha$.

Notice that the semantics we presented is straightforwardly adopted from the definition of the translation function. The conditions (i) – (v) of the translation are, respectively, replaced with (var) – $(\rightarrow_d)$ and $(*)$ finds its expression in the definition of $\models$.

Since the accessibility relation defined on D_2-frames is reflexive, symmetric and transitive (just as in case of $S5$ of Lewis), we can restrict our attention to frames in which the *accessibility* relation includes every pair of worlds. This fact enables us to simplify the notion of the D_2-model.

A model (D_2-model) is a pair $< W, v >$, where W is a non-empty set (of points) and a function, $v : For_{D2} \times W \longrightarrow \{1, 0\}$, is inductively defined

$(\sim)$ $v(\sim \alpha,\ x) = 1$ iff $v(\alpha,\ x) = 0$
$(\vee)$ $v(\alpha \vee \beta,\ x) = 1$ iff $v(\alpha,\ x) = 1$ or $v(\beta,\ x) = 1$
$(\wedge_d)$ $v(\alpha \wedge_d \beta,\ x) = 1$ iff $v(\alpha,\ x) = 1$ and $\exists_{y \in W}(v(\beta,\ y) = 1)$
$(\rightarrow_d)$ $v(\alpha \rightarrow_d \beta,\ x) = 1$ iff $\forall_{y \in W}(v(\alpha,\ y) = 0)$ or $v(\beta,\ x) = 1$.

$\models \alpha$ iff for any model $< W, R, v >$, there exists $y \in W$ such that $v(\alpha,\ y)$=1.

Frontiers of Paraconsistent Logic, Research Studies Press, 2000.

Proposition 1. $\forall\alpha \in For_{D2}$: $\models \alpha$ iff $\alpha \in D_2$ (iff $\Diamond f(\alpha) \in S5$).

Proof. By induction.

It follows from Proposition 1 that our semantics is equivalent to the familiar translation procedure. The translation is rendered redundant.

Now let us focus on the syntactic analysis of D_2.

(A_1) $\alpha \to_d (\beta \to_d \alpha)$
(A_2) $(\alpha \to_d (\beta \to_d \gamma)) \to_d ((\alpha \to_d \beta) \to_d (\alpha \to_d \gamma))$
(A_3) $(\alpha \wedge_d \beta) \to_d \alpha$
(A_4) $(\alpha \wedge_d \beta) \to_d \beta$
(A_5) $(\alpha \to_d \beta) \to_d ((\alpha \to_d \gamma) \to_d (\alpha \to_d (\beta \wedge_d \gamma)))$
(A_6) $\alpha \to_d (\alpha \vee \beta)$
(A_7) $\beta \to_d (\alpha \vee \beta)$
(A_8) $(\alpha \to_d \gamma) \to_d ((\beta \to_d \gamma) \to_d ((\alpha \vee \beta) \to_d \gamma))$
(A_9) $\alpha \vee (\alpha \to_d \beta)$
(A_{10}) $\sim (\sim \alpha \wedge_d \sim\sim \alpha \wedge_d \sim (\alpha \vee \sim \alpha))$
(A_{11}) $\sim(\sim \alpha \wedge_d \sim \beta \wedge_d \sim(\alpha \vee \beta)) \to_d \sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim(\alpha \vee \beta \vee \gamma))$
(A_{12}) $\sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim (\alpha \vee \beta \vee \gamma)) \to_d$
$\sim (\sim \alpha \wedge_d \sim \gamma \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \gamma \vee \beta))$
(A_{13}) $\sim(\sim \alpha \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim(\alpha \vee \beta \vee \gamma)) \to_d ((\alpha \vee \beta \vee \sim \gamma) \to_d (\alpha \vee \beta))$
(A_{14}) $\sim (\sim \alpha \wedge_d \sim \beta) \to_d (\alpha \vee \beta)$
(A_{15}) $(\alpha \vee (\beta \vee \sim \beta)) \to_d \sim (\sim \alpha \wedge_d \sim (\beta \vee \sim \beta))$.

The sole rule of inference is *Detachment Rule*

$(MP)^*$ $\alpha,\ \alpha \to_d \beta\ /\ \beta$.

The set of axiom schemata and $(MP)^*$ define $\vdash_{D2}$ (the consequence relation).

The axiomatization we presented is in fact the first axiomatization of D_2 (with discursive connectives as primitive symbols and positive and negation fragments to be separated).[12]

Let D_2^+, for the sake of brevity, denote $\{(A_1), \ldots (A_9)\}$.[13]

[12] Most of the authors who dealt with this subject were interested in an alternative strategy. They treated Jaśkowski's calculus as a starting point for pure modal analysis. See [3], [4], [5], [6], [14], [15], [16], [19], [20] and [22].

[13] As before, the discursive equivalence is a definable connective.

THEOREM 1. $\vdash_{D2} \alpha$ *iff* $\models \alpha$.

PROOF. See Section 6.

3 Negation as a $Possible - not$ Connective

In this section we introduce a new connective of negation. This move allows some of the weaker form of Duns Scotus' thesis to be theorems of the modified calculus. The definition is the following

DEFINITION 2. $\sim_d \alpha = (p_1 \vee \sim p_1) \wedge_d \sim \alpha$

Observe that we can apply the translation function to transform the connective into its modal counterpart

(ii)' $f(\sim_d \alpha) = \Diamond \sim f(\alpha)$

and extend our semantics by the additional condition

($\sim_d$) $x \models_m \sim_d \alpha$ iff $\exists_{y \in W}(xRy$ and $y \not\models_m \alpha)$.

We will henceforth regard $\sim_d$ as a primitive symbol that has replaced the connective of $\sim$. This exchange results in obtaining a quite new calculus, called ND_2^+.

A model (ND_2^+-model) is a pair $< W, v >$, where W is a non-empty set (of points) and a function, $v : For_{ND2^+} \times W \longrightarrow \{1, 0\}$, is inductively defined

($\sim_d$) $v(\sim_d \alpha,\ x) = 1$ iff $\exists_{y \in W}(v(\alpha,\ y) = 0)$
($\vee$) $v(\alpha \vee \beta,\ x) = 1$ iff $v(\alpha,\ x) = 1$ or $v(\beta,\ x) = 1$
($\wedge_d$) $v(\alpha \wedge_d \beta,\ x) = 1$ iff $v(\alpha,\ x) = 1$ and $\exists_{y \in W}(v(\beta,\ y) = 1)$
($\rightarrow_d$) $v(\alpha \rightarrow_d \beta,\ x) = 1$ iff $\forall_{y \in W}(v(\alpha,\ y) = 0)$ or $v(\beta,\ x) = 1$.

$\models \alpha$ iff for any model $< W, R, v >$, there exists $y \in W$ such that $v(\alpha,\ y)$=1.

The idea of treating negation as "possibly-not" is not altogether new, having been examined by many authors,[14] but it has hardly been studied in relation to D_2 and even in those cases neither an axiomatization nor a direct semantics for the resulting system has been given.

[14] See, for instance, [2], [14] and [26].

Observe that some of the ND_2^+-valid formulas do not correspond to their D_2-counterparts (i.e. the result of replacing $\sim_d$ with $\sim$), for example,

(1) $\sim_d p \rightarrow_d (\sim_d\sim_d p \rightarrow_d q)$
(2) $\sim_d p \rightarrow_d (\sim_d\sim_d p \rightarrow_d (\sim_d\sim_d\sim_d p \rightarrow_d q))$
(3) $(\sim_d p \wedge_d \sim_d\sim_d p) \rightarrow_d q$
(4) $(\sim_d p \rightarrow_d \sim_d q) \rightarrow_d ((\sim_d p \rightarrow_d \sim_d\sim_d q) \rightarrow_d p)$
(5) $(p\vee \sim_d q) \rightarrow_d ((p\vee \sim_d\sim_d q) \rightarrow_d p)$.[15]

On the other hand, there are many D_2-valid formulas that are not valid in ND_2^+ (the result of replacing $\sim$ with $\sim_d$), for example,

(6) $(p \rightarrow_d q) \rightarrow_d \sim\sim (p \rightarrow_d q)$
(7) $p \rightarrow_d \sim\sim p$
(8) $\sim (\sim p \wedge_d p)$
(9) $p \rightarrow_d \sim (\sim p \wedge_d \sim q)$
(10) $(p \vee q) \rightarrow_d (p\vee \sim\sim q)$.

Proposition 2. *ND_2^+ (with $\sim_d$ as primitive) is not a conservative extension of D_2.*

ND_2^+ is axiomatizable by the rule of $(MP)^*$ plus the set of axiom schemata

(A_1) α, if $\alpha \in D_2^+$
(A_2) $\sim_d (\alpha\wedge_d \sim_d \beta) \rightarrow_d\sim_d\sim_d (\sim_d \alpha \vee \beta)$
(A_3) $\sim_d (\alpha\wedge_d \sim_d \alpha)$
(A_4) $(\alpha\vee \sim_d \beta) \rightarrow_d ((\alpha\vee \sim_d\sim_d \beta) \rightarrow_d \alpha)$
(A_5) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d (\alpha\vee \sim_d\sim_d \beta)$
(A_6) $\sim_d\sim_d \alpha \rightarrow_d \alpha$
(A_7) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d \sim_d\sim_d (\alpha \vee \beta \vee \gamma)$
(A_8) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d \sim_d\sim_d (\beta \vee \alpha)$.

The consequence relation $\vdash_{ND2+}$ is defined by the set and $(MP)^*$.

Theorem 2. $\vdash_{ND2+} \alpha$ *iff* $\models \alpha$.

Proof. See [10] for details.

[15] Cf. [17] pp. 46-50, [18] p. 58.

4 Collapse into the Classical Logic

Like in the previous section, we start by characterizing a new connective of negation:

DEFINITION 3. $\neg_d \alpha = \alpha \rightarrow_d \sim (p_1 \vee \sim p_1)$.

The formula $\sim (p_1 \vee \sim p_1)$ has the same meaning as falsum and Definition 3 looks like the one for intuitionistic logic. It is, however, a bit deceptive since the formula $\alpha \vee \neg_d \alpha$ is generally valid in D_2; in view of that, the translation rules may be of much help, especially the rule

(ii)$''$ $f(\neg_d \alpha) = \sim \Diamond f(\alpha)$.

is of particular interest.

In what follows, we will use $\neg_d$ as a primitive symbol that has replaced $\sim$ and SD_2^+ to denote the resulting calculus.

Here is a direct semantics for SD_2^+.

A model (SD_2^+-model), as before, is a pair $< W, v >$, where W is a non-empty set (of points) and a valuation function, $v : For_{SD2^+} \times W \longrightarrow \{1, 0\}$, is defined

($\neg_d$) $v(\neg_d \alpha,\ x) = 1$ iff $\forall_{y \in W}(v(\alpha,\ y) = 0)$
($\vee$) $v(\alpha \vee \beta,\ x) = 1$ iff $v(\alpha,\ x) = 1$ or $v(\beta,\ x) = 1$
($\wedge_d$) $v(\alpha \wedge_d \beta,\ x) = 1$ iff $v(\alpha,\ x) = 1$ and $\exists_{y \in W}(v(\beta,\ y) = 1)$
($\rightarrow_d$) $v(\alpha \rightarrow_d \beta,\ x) = 1$ iff $\forall_{y \in W}(v(\alpha,\ y) = 0)$ or $v(\beta,\ x) = 1$.

$\models \alpha$ iff for any model $< W, R, v >$, there exists $y \in W$ such that $v(\alpha,\ y)$=1.

It is remarkable that among the theses of SD_2^+ there are all the laws of the classical propositional calculus (including Duns Scotus' thesis) and the semantics we introduced can be viewed as a new semantics for the classical propositional logic.

A deductive structure of SD_2^+ is given by the set of axiom schemata

(A_1) α, if $\alpha \in D_2^+$
(A_2) $\neg_d \neg_d \alpha \rightarrow_d \alpha$
(A_3) $\alpha \rightarrow_d \neg_d \neg_d \alpha$
(A_4) $(\alpha \rightarrow_d \beta) \rightarrow_d (\neg_d \beta \rightarrow_d \neg_d \alpha)$

and the rule

$(MP)^*$ $\alpha,\ \alpha \rightarrow_d \beta\ /\ \beta$.

The consequence relation $\vdash_{SD2+}$ is determined by the set and $(MP)^*$.

The axiomatization coincides with the well-known axiom system originated by Hilbert and Bernays, but one might just as well *adopt* a different set of the axiom schemata of the classical propositional calculus (CPC for short) and use, for example, $(\neg_d\alpha \rightarrow_d \neg_d\beta) \rightarrow_d (\beta \rightarrow_d \alpha)$ instead of (A_2)–(A_4).

PROPOSITION 3. *Each thesis α of SD_2^+ becomes a thesis α' of the classical propositional logic after replacing in α the connectives $\neg_d$, $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$ by $\neg$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$, respectively.*

PROPOSITION 4. *Each thesis α' of the classical propositional logic becomes a thesis α of SD_2^+ after replacing in α' the connectives $\neg$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$ by $\neg_d$, $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$, respectively.*

PROOF. Apply the method described in [17], pp. 45–46.

THEOREM 3. $\vdash_{SD2+} \alpha$ *iff* $\models \alpha$.

PROOF. See Section 6.

5 Da Costa, Dubikajtis and Kotas' system of the Discursive Logic

In the late seventies, da Costa, Dubikajtis and Kotas published a few papers concerned with an axiomatization of the discursive logic. Their axiomatization contains, among others, the formulas

$\sim ((p \wedge_d q) \vee r) \rightarrow_d (p \rightarrow_d \sim (q \vee r))$
$\sim (\sim (p \wedge_d q) \vee r) \rightarrow_d (p \wedge_d \sim (\sim q \vee r))$

as axioms. Notice, however, that they are not valid in Jaśkowski's calculus.

We easily fix the problem by shifting the diamond from the right to the left side of the conjunction, i.e.

(iv)' $f(\alpha \wedge_d \beta) = \Diamond f(\alpha) \wedge f(\beta)$[16]

and then replacing the item $(\wedge_d)$ with

$(\wedge_d)$' $v(\alpha \wedge_d \beta,\ x) = 1$ iff $\exists_{y \in W}(v(\alpha,\ y) = 1)$ and $v(\beta,\ x) = 1$.

[16] We have decided not to introduce a new symbol for the *left* discursive conjunction since our remarks on the da Costa, Dubikajtis and Kotas' system are limited to this section only.

At cursory glance, the change seems just cosmetic but dig a little deeper into formulas to realize that it is not the point. For example, the formulas

$$\sim ((q \wedge_d p) \vee r) \rightarrow_d (p \rightarrow_d \sim (q \vee r))$$
$$\sim (\sim (q \wedge_d p) \vee r) \rightarrow_d (p \wedge_d \sim (\sim q \vee r))$$

are valid in Jaśkowski's calculus, whereas it is not so in the case of da Costa, Dubikajtis and Kotas' system of the discursive logic (hereinafter referred to as D_2^* for short).

There is an advantage of using da Costa, Dubikajtis and Kotas' definition of the discursive conjunction. And this is not because of the axioms mentioned above but due to the connection between the discursive implication and conjunction. It is sufficient to say that the formulas

$$(p \rightarrow_d q) \rightarrow_d \sim (p \wedge_d \sim q)$$
$$\sim (p \wedge_d \sim q) \rightarrow_d (p \rightarrow_d q)$$

are not valid in Jaśkowski's calculus. They are valid in D_2^*.

On the other hand, the formulas

$$(p \rightarrow_d q) \rightarrow_d \sim (\sim q \wedge_d p)$$
$$\sim (p \wedge_d \sim q) \rightarrow_d (q \rightarrow_d p)$$

are valid in D_2 (but not in D_2^*).

In fact, Jaśkowski lost more than he expected: the elegant *classical-like* definition of the discursive implication.

Since there are, nevertheless, striking similarities between the two approaches let us just list a few of them, without trying to be complete.

PROPOSITION 5. *(i) Each of the axiom schemata of D_2^+ is valid in D_2^* and $(MP)^*$ preserves validity.*

(ii) Assume that α includes, besides variables, at most the connectives $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$ and $\vee$. If α is valid in D_2^* (or D_2), then α_{cpc} is valid in CPC, where α_{cpc} is obtained from α by replacing $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$ with $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$, respectively.

(iii) Suppose that α contains, besides variables, at most the connectives $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\vee$. If α is valid in CPC, then α_d is valid in D_2^* (and D_2), where α_d is obtained from α by replacing $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$ with $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$, respectively.

(iv) Let α contain, besides variables, at most the connectives $\vee$ and $\sim$. If α is valid in CPC, then both α and $\alpha \rightarrow_d q$ is valid in D_2^* (and D_2).

As long as we deal with negation-free formulas, there is no difference between D_2 and D_2^*; it does not matter which definition of the discursive conjunction we use.

In practice, as stated in Proposition 5, we may read off the validity of some formulas of For_{D2^*} (and For_{D2}) directly from a classical true-value analysis.

Now we focus on a new axiomatization of D_2^* which is reformulation of the axiomatization presented in [9].

(A_1) α, if $\alpha \in D_2^{*+}$
(A_{10}) $\sim(\sim(\alpha\vee\sim\alpha)\wedge_d\sim\sim\alpha\wedge_d\sim\alpha)$
(A_{11}) $\sim(\sim(\alpha\vee\beta)\wedge_d\sim\beta\wedge_d\sim\alpha)\rightarrow_d\sim(\sim(\alpha\vee\beta\vee\gamma)\wedge_d\sim\gamma\wedge_d\sim\beta\wedge_d\sim\alpha)$
(A_{12}) $\sim(\sim(\alpha\vee\beta\vee\gamma)\wedge_d\sim\gamma\wedge_d\sim\beta\wedge_d\sim\alpha)\rightarrow_d$
$\sim(\sim(\alpha\vee\gamma\vee\beta)\wedge_d\sim\beta\wedge_d\sim\gamma\wedge_d\sim\alpha)$
(A_{13}) $\sim(\sim(\alpha\vee\beta\vee\gamma)\wedge_d\sim\gamma\wedge_d\sim\beta\wedge_d\sim\alpha)\rightarrow_d((\alpha\vee\beta\vee\sim\gamma)\rightarrow_d(\alpha\vee\beta))$
(A_{14}) $\sim(\sim\alpha\wedge_d\sim\beta)\rightarrow_d(\alpha\vee\beta)$
(A_{15}) $(\alpha\vee(\beta\vee\sim\beta))\rightarrow_d\sim(\sim(\beta\vee\sim\beta)\wedge_d\sim\alpha)$
plus $(MP)^*$ $\alpha,\ \alpha\rightarrow_d\beta\ /\ \beta$ as the sole rule of inference.[17]

The axiom schemata and $(MP)^*$ define $\vdash_{D2^*}$ (the consequence relation).

The differences with respect to the axiomatization of D_2 appear in $(A_{10})-(A_{13})$ and (A_{15}) where nothing but the variation of the components of the discursive conjunction does change. Metaphorically speaking, the discursive conjunction changes its flow.

THEOREM 4. $\vdash_{D2^*}\alpha$ *iff* $\models\alpha$.

PROOF. See [9].

The discursive conjunction has also changed its flow direction in the following:

DEFINITION 2*. $\sim_d\alpha=\sim\alpha\wedge_d(p_1\vee\sim p_1)$
and, consequently in $(\wedge_d)$, (A_2) and (A_3) of ND_2^+.

There is no difference which definition of the discursive conjunctive is preferable after having introduced $\neg_d$ (Definition 3); the collapse into the classical logic is inevitable.

[17] D_2^{*+} denotes the positive part of D_2^*.

6 Metalogic of the Discursive Systems

In this section we concentrate on the metalogical properties of the discursive systems.

Note that the (schemata of) formulas

$\alpha \rightarrow_d (\beta \rightarrow_d \alpha)$
$(\alpha \rightarrow_d (\beta \rightarrow_d \gamma)) \rightarrow_d ((\alpha \rightarrow_d \beta) \rightarrow_d (\alpha \rightarrow_d \gamma))$

constitute the implicational fragment of all the systems mentioned in this paper and each of them is closed under $(MP)^*$ which is the sole rule of inference. Therefore a proof of the deduction theorem is standard.

THEOREM 5. *$\Phi \vdash_{D_2}$ ($\vdash_{SD_2^+}$, $\vdash_{ND_2^+}$) $\alpha \rightarrow_d \beta$ iff $\Phi \cup \{\alpha\} \vdash_{D_2}$ (resp. $\vdash_{SD_2^+}$, $\vdash_{ND_2^+}$) β, where α, $\beta \in For_{D_2}$ ($For_{SD_2^+}$, $For_{ND_2^+}$), $\Phi \subseteq For_{D_2}$ ($For_{SD_2^+}$, $For_{ND_2^+}$).*

PROPOSITION 6. *The (schemata of) formulas listed below are provable in all the discursive adjunctive systems*

(T_1) $(\alpha \vee \alpha) \leftrightarrow_d \alpha$
(T_2) $(\alpha \vee \beta) \leftrightarrow_d (\beta \vee \alpha)$
(T_3) $((\alpha \vee \beta) \vee \gamma) \leftrightarrow_d (\alpha \vee (\beta \vee \gamma))$
(T_4) $(\alpha \vee (\beta \wedge_d \gamma)) \leftrightarrow_d ((\alpha \vee \beta) \wedge_d (\alpha \vee \gamma))$
(T_5) $(\alpha \rightarrow_d \beta) \rightarrow_d ((\alpha \vee \gamma) \rightarrow_d (\beta \vee \gamma))$
(T_6) $(\beta \vee \alpha \vee \beta) \leftrightarrow_d (\alpha \vee \beta)$
(T_7) $(\alpha \wedge_d (\alpha \rightarrow_d \beta)) \rightarrow_d \beta$

and the set $\{\alpha : \vdash_{D_2^+} \alpha\}$ is closed under the rules:

(R_1) $\alpha,\ \beta\ /\ \alpha \wedge_d \beta$
(R_2) $\alpha \wedge_d \beta\ /\ \alpha\ (\beta)$
(R_3) $\alpha\ (\beta)\ /\ \alpha \vee \beta$,

where $\vdash_{D_2^+}$ is the consequence relation defined by D_2^+ and $(MP)^*$.

PROOF. Straightforward.

PROPOSITION 7. *The formulas*

(T_8) $\alpha \vee \sim \alpha$
(T_9) $\sim(\sim \alpha \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \beta)) \rightarrow_d (\sim(\sim \alpha \wedge_d \sim\sim \beta \wedge_d \sim(\alpha \vee \sim \beta)) \rightarrow_d \alpha)$

$(T_{10}) \sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \beta)) \rightarrow_d ((\alpha \vee \sim \beta) \rightarrow_d \alpha)$
$(T_{11})\ (\alpha \vee \sim \alpha) \rightarrow_d \sim (\sim \alpha \wedge_d \sim\sim \alpha \wedge_d \sim (\alpha \vee \sim \alpha))$.
are (schemata of the) theses of D_2.

PROPOSITION 8. *The formulas*

$(T_{12})\ \alpha \vee \neg_d \alpha$
$(T_{13})\ \neg_d(\alpha \wedge_d \neg_d \alpha)$
$(T_{14})\ (\alpha \vee \beta) \rightarrow_d (\neg_d \beta \rightarrow_d \alpha)$

are (schemata of the) theses of SD_2^+.

THEOREM 6. *(i) $\vdash_{D_2} \alpha$ iff α is valid in D_2.*
(ii) $\vdash_{SD_2^+} \alpha$ iff α is valid in SD_2^+.

SOUNDNESS. By induction.

The initial idea of the proof we present below traces back to [24]. The crucial point is to construct a canonical valuation that falsifies a non-thesis. However, contrary to Henkin's method, we do not verify, but falsify the sets of formulas we build.

COMPLETENESS. Assume that $\nvdash_{D_2} \alpha$ (resp. $\nvdash_{SD_2^+} \alpha$) and α is valid in D_2 (SD_2^+). Define a sequence of all the formulas of D_2 (SD_2^+) as follows:

$\Gamma = \gamma_1, \gamma_2, \gamma_3, ...$

The only restriction is that the first element of Γ is α (i.e. $\alpha = \gamma_1$).
Define a family of (finite) subsequences of Γ:

$\Delta_1 = \delta_1$, where $\delta_1 = \gamma_1 = \alpha$.

Now assume that $\Delta_k = \delta_1, \dots, \delta_k$ has been defined, we put

$$\Delta_{k+1} = \begin{cases} \Delta_{k.\gamma_{k+1}}, & \text{if } \nvdash_{D_2} (\nvdash_{SD_2^+})\ \delta_1 \vee ... \vee \delta_k \vee \gamma_{k+1} \\ \Delta_{k.\delta_k}, & \text{otherwise.} \end{cases}$$

Define in addition:

$$\nabla_1 = \underbrace{\delta_1,}_{\Delta_1}\ \underbrace{\delta_1, \delta_2,}_{\Delta_2}\ \underbrace{\delta_1, \delta_2, \delta_3,}_{\Delta_3}\ \dots\ ,\ \underbrace{\delta_1, \delta_2, \delta_3,\ \dots\ , \delta_n,}_{\Delta_n}\ \dots$$

$$\nabla_2 = \underbrace{\delta_1, \delta_2}_{\Delta_2},\ \underbrace{\delta_1, \delta_2, \delta_3}_{\Delta_3},\ \ldots\ ,\ \underbrace{\delta_1, \delta_2, \delta_3,\ \ldots\ \delta_n}_{\Delta_n},\ \ldots$$
$$\nabla_3 = \underbrace{\delta_1, \delta_2, \delta_3}_{\Delta_3},\ \underbrace{\delta_1, \delta_2, \delta_3, \delta_4}_{\Delta_4},\ \ldots\ ,\ \underbrace{\delta_1, \delta_2, \delta_3,\ \ldots\ , \delta_n}_{\Delta_n},\ \ldots$$
$$\vdots$$
$$\nabla_n = \underbrace{\delta_1, \delta_2,\ \ldots\ , \delta_n}_{\Delta_n},\ \ldots\ ,\ \underbrace{\delta_1, \delta_2,\ \ldots\ , \delta_n,\ \ldots\ , \delta_{n+k}}_{\Delta_{n+k}},\ \ldots$$
$$\vdots$$

Let ∇_i, where $i \in N$, denote both the i-sequence and the set of formulas which contains all the elements of the i-sequence and $\nabla = \{\nabla_1, \nabla_2, ..., \nabla_i, ...\}$.

LEMMA 1. *(i)* $\nvdash_{D_2}$ $(\nvdash_{SD_2^+})$ $\delta_1 \vee \ldots \vee \delta_n$, *for any* $n \in N$
(ii) if $\beta \notin \nabla_i$, *then* $\vdash_{D_2}$ $(\vdash_{SD_2^+})$ $\delta_1 \vee \ldots \vee \delta_k \vee \beta$, *for some* $k \in N$
(iii) $\beta \in \nabla_i$ *iff* $\exists_{k \in N}$ $(\Delta_k \subset \nabla_i$ *and* $\beta \in \Delta_k)$, *for any* $i \in N$.

PROOF. Straightforward.

LEMMA 2. *For every* $\beta, \gamma \in For_{D2}$ (For_{SD2+}), *for any* $i, k \in N$:
(i) $\beta \vee \gamma \in \nabla_i$ *iff* $\beta \in \nabla_i$ *and* $\gamma \in \nabla_i$
(ii) $\beta \wedge_d \gamma \in \nabla_i$ *iff* $\beta \in \nabla_i$ *or* $\forall_{k \in N}$ $(\gamma \in \nabla_k)$
(iii) $\beta \rightarrow_d \gamma \in \nabla_i$ *iff* $\exists_{k \in N}$ $(\beta \notin \nabla_k)$ *and* $\gamma \in \nabla_i$.

PROOF. See [10].

LEMMA 3. *For every* $\beta \in For_{SD2+}$, *for any* $i \in N$:
(i) $\neg_d \beta \in \nabla_i$ *iff* $\exists_{k \in N}(\beta \notin \nabla_k)$.

PROOF. (i) Assume that (1) $\neg_d \beta \in \nabla_i$ and (2) $\forall_{k \in N}$ $(\beta \in \nabla_k)$. In particular, (3) $\beta \in \nabla_i$. Apply Lemma 1(i) to receive (4) $\nvdash_{SD2+}$ $\beta \vee \neg_d \beta$. A contradiction due to (T_{12}).
Now suppose that (1) $\exists_{k \in N}$ $(\beta \notin \nabla_k)$ and (2) $\neg_d \beta \notin \nabla_i$. Obviously, $i \geq k$ or $k > i$.

Let $i \geq k$. Since $\nabla_k = \Delta_k,\ \ldots\ ,\ \underbrace{\Delta_i, \Delta_{i+1},\ \ldots}_{\nabla_i}$ then $\nabla_i \subseteq \nabla_k$ and (3) $\beta \notin \nabla_i$. Now use Lemma 1(ii), to obtain (4) $\vdash_{SD2+}$ $\delta_1 \vee \ldots \vee \delta_m \vee \beta$, for some $m \in N$, and (5) $\vdash_{SD2+}$ $\delta_1 \vee \ldots \vee \delta_n \vee \neg_d \beta$, for some $n \in N$. Observe that $m \geq n$ or $n > m$. Suppose then that $m \geq n$ (the case $n > m$ is similar to $m \geq n$). Apply (R_3), (T_2), (T_3), $(MP)^*$ to (5), to deduce (6) $\vdash_{SD2+}$ $\delta_1 \vee$

$\ldots \vee \delta_m \vee \neg_d \beta$. Use (R_1), to get (7) $\vdash_{SD2+} (\delta_1 \vee \ldots \vee \delta_m \vee \beta) \wedge_d (\delta_1 \vee \ldots \vee \delta_m \vee \neg_d \beta)$ and (T_4) to receive (8) $\vdash_{SD2+} (\delta_1 \vee \ldots \vee \delta_m) \vee (\beta \wedge_d \neg_d \beta)$. Since (T_{13}), (T_{14}) are theses of SD_2^+ and the system is closed under $(MP)^*$, we finally obtain (11) $\vdash_{SD2+} \delta_1 \vee \ldots \vee \delta_m$. Notice, however, that $\delta_1, \ldots, \delta_m \in \nabla_i$. A contradiction due to Lemma 1(i).

Let $k > i$. Then $\nabla_k \subseteq \nabla_i$ and (3) $\neg_d \beta \notin \nabla_k$ since $\nabla_i = \Delta_i, \ \ldots \ , \underbrace{\Delta_k, \Delta_{k+1}, \ \ldots}_{\nabla_k}$ Next proceed analogously to $i \geq k$.

Lemma 4. *For every $\beta \in For_{D2}$, for any $i \in N$:*
(i) $\sim \beta \in \nabla_i$ iff $\beta \notin \nabla_i$.

Proof. See [11].

Let $< \nabla, v_c >$ be a canonical model for D_2 (SD_2^+). The canonical valuation $v_c : For_{D2(SD2^+)} \times \nabla \longrightarrow \{1, 0\}$ is defined:

$$v_c(\beta, \nabla_i) = \begin{cases} 1, & \text{if } \beta \notin \nabla_i \\ 0, & \text{if } \beta \in \nabla_i \end{cases}$$

Apply Lemma 2 and Lemma 4 (Lemma 3) to show that the conditions $(\sim)$, $(\vee)$, $(\wedge_d)$ and $(\rightarrow_d)$ $((\neg_d), (\vee), (\wedge_d)$ and $(\rightarrow_d))$ hold for v_c.

Now assume that $\nvdash_{D2} \alpha$ $(\nvdash_{SD2^+} \alpha)$. Notice that α is the very first element of each i-sequence we defined (i.e. for every $i \in N$, $\alpha \in \nabla_i$). Then the formula α is not valid in D_2 (SD_2^+) since there exists a model $< \nabla, v_c >$ such that $v_c(\alpha, \ \nabla_i) = 0$, for every $i \in N$.

Acknowledgements. I wish to express my gratitude to Anonymous Referee for his (or her) valuable assistance in connection with the publication.

References

[1] G. Achtelik, L. Dubikajtis, E. Dudek, J. Kanior, *On Independence of Axioms of Jaśkowski Discussive Propositional Calculus*, **Reports on Mathematical Logic** 11 (1981), pp. 3–11.

[2] J.-Y. Béziau, *Paraconsistent logic from a modal viewpoint*, **Journal of Applied Logic** 3 (2005), pp. 7–14.

[3] J. J. Błaszczuk, W. Dziobiak, *Remarks on Perzanowski's Modal System*, **Bulletin of the Section of Logic** 4(2) (1975), pp. 57–64.

[4] J. J. Błaszczuk, W, Dziobiak *Modal Systems Related to S4n of Sobociński*, **Bulletin of the Section of Logic** 4(3) (1975), pp. 103–108.

[5] J. J. Błaszczuk, W, Dziobiak *An Axiomatization of M-counterpart for Some Modal Calculi*, **Reports on Mathematical Logic** 6 (1976), pp. 3–6.

[6] J. J. Błaszczuk, W, Dziobiak *Modal Logics Connected with Systems S4n of Sobociński*, **Studia Logica** 36(3) 1977, pp. 151–164.

[7] J. Ciuciura, *History and Development of the Discursive Logic*, **Logica Trianguli** 3 (1999), pp. 3–31.

[8] J. Ciuciura, *Discursive Logic* (in Polish), **Principia** 35–36 (2003), pp. 279–291.

[9] J. Ciuciura, *On the da Costa, Dubikajtis and Kotas' system of the discursive logic, D_2^**, **Logic and Logical Philosophy** 14(2) (2005), pp. 235–253.

[10] J. Ciuciura, *A Quasi-Discursive System ND_2^+*, **Notre Dame Journal of Formal Logic** 47(3) (2006), pp. 371–384.

[11] J. Ciuciura, *Frontiers of the discursive logic*, **Bulletin of the Section of Logic** 37(2) (2008), pp. 81–92.

[12] H. A. Costa, *Non-Adjunctive Inference and Classical Modalities*, **Journal of Philosophical Logic** 34 (2005), pp. 581–605.

[13] N. C. A. da Costa, L. Dubikajtis, *New Axiomatization for the Discussive Propositional Calculus*, [in:] A.I. Arruda, N.C.A. da Costa, R. Chuaqui (ed): **Non Classical Logics, Model Theory and Computability**, North-Holland Publishing, Amsterdam 1977, pp. 45–55.

[14] N. C. A. da Costa, *Remarks on Jaśkowski's Discussive Logic*, **Reports on Mathematical Logic** 4 (1975), pp. 7–16.

[15] W. Dziobiak *Semantics of Kripke's Style for Some Modal Systems*, **Bulletin of the Section of Logic** 5/2 (1976), pp. 33–37.

[16] T. Furmanowski *Remarks on Discussive Propositional Calculus*, **Studia Logica** 34(1) (1975), pp. 39–43.

[17] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, **Studia Societatis Scientiarum Torunensis**, Sect. A, I, No 5 (1948), pp. 57–77. English translations: *Propositional Calculus for Contradictory Deductive Systems*, **Studia Logica** 24 (1969), pp. 143–157 and *A Propositional Calculus for Inconsistent Deductive Systems*, **Logic and Logical Philosophy** 7(1) (1999), pp. 35–56.

[18] S. Jaśkowski,*O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych*, **Studia Societatis Scientiarum Torunensis**, Sect. A, I, No 8 (1949), pp. 171–172. English translation: *On the Discussive Conjuntion in the Propositional Calculus for Inconsistent Deductive Systems*, **Logic and Logical Philosophy** 7(1) (1999), pp. 57–59.

[19] J. Kotas, *Discussive Sentential Calculus of Jaśkowski*, **Studia Logica** 34(2) (1975), pp. 149–168.

[20] J. Kotas, *The Axiomatization of S. Jaśkowski's Discussive System*, **Studia Logica** 33(2) (1974), pp. 195–200.

[21] K. Mruczek-Nasieniewska, M. Nasieniewski, *Syntactical and Semantical Characterization of a Class of Paraconsistent Logics*, **Bulletin of the Section of Logic** 34(4) (2005), pp. 228–248.

[22] J. Perzanowski, *On M-fragments adn L-fragments of Normal Modal Propositional Logics*, **Reports on Mathematical Logic** 5 (1975), pp. 63–72.

[23] G. Priest, *Paraconsistent Logic*, **Stanford Encyclopedia of Philosophy**, available at http://plato.stanford.edu/entries/logic-paraconsistent/

[24] J. Reichbach, *On the first-order functional calculus and the truncation of models*, **Studia Logica** 7 (1957), pp. 181–220.

[25] M. Urchs, *Discursive logic: towards a logic of rational discourse*, **Studia Logica** 54 (1995), pp. 231–249.

[26] D. Vakarelov, *Consistency, Completeness and Negation*, in: G. Priest, R. Routley, J. Norman (eds): **Paraconsistent Logic. Essays on the Inconsistent**, Philosophia Velag, 1989, pp. 328–363.

[27] V. L. Vasyukov, *A New Axiomatization of Jaśkowski's Discussive Logic*, **Logic and Logical Philosophy** 9(3) (2002), pp. 35–46.

Department of Logic
University of Łódź
Kopcińskiego 16/18
90–232 Łódź, Poland
e-mail: janciu@uni.lodz.pl

Centre for Logic and Philosophy of Science
Universiteit Gent
Blandijnberg 2
B-9000 Gent, Belgium