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NEGATIVE BINOMIAL MULTIPLICITY DISTRIBUTIONS IN HIGH ENERGY HADRON COLLISIONS*)

A. Giovannini
Istituto di Fisica Teorica, Università di Torino
INFN, Sezione di Torino

and

L. Van Hove
CERN - Geneva

ABSTRACT

This paper concerns the results recently obtained by the UA5 Collaboration on charged particle multiplicity distributions at the CERN $p\bar{p}$ collider ($\sqrt{s}$ = 540 GeV) and their comparison with lower energy data[1),2)]. Our aim is to interpret in general terms the experimental finding that the distributions are of negative binomial (NB) form both for total multiplicities and in finite pseudorapidity intervals. This can be understood in terms of partial stimulated emission of bosons, or of a simple form of cascade process, or (more artificially) with both mechanisms. The cascade interpretation appears to be the most attractive. It leads to a new concept of clustering, the empirical data imply unexpected properties for the clusters, and a novel type of high energy asymptotics seems to emerge. We briefly discuss the relation with QCD-based models for hadron production and the case of e^+e^- annihilation.

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1. - INTRODUCTION

With the advent of the proton-antiproton collider at CERN, there has been a revival of interest in the study of charged particle multiplicity distributions. In particular, the UA5 collaboration has recently given detailed information on charged particle multiplicity distributions of inelastic, non-single-diffractive $p\bar{p}$ collisions at centre-of-mass energy $\sqrt{s}$ = 540 GeV, for various intervals of pseudorapidity[1] and for total multiplicities[2]. All these multiplicities are found to be remarkably well fitted by negative binomial (NB) distributions, i.e., by distributions of the type

$$P(n,\bar{n},k) = \frac{k(k+1)\dots(k+n-1)}{n!} \; \frac{\bar{n}^n \, k^k}{(\bar{n}+k)^{n+k}} \tag{1}$$

where $\bar{n}$ is the average multiplicity and k is related to $D^2 = \overline{n^2} - \bar{n}^2$ by

$$\frac{D^2}{\bar{n}^2} = \frac{1}{\bar{n}} + \frac{1}{k} \tag{2}$$

The parameter k is found to increase from 1.6 for the interval $|\eta| < 0.2$ (η is the centre-of-mass pseudorapidity) to 3.7 for the total multiplicity.

The latter value is less than half of the k values of 8 to 11 found for total multiplicities of pp collisions at the ISR ($\sqrt{s}$ = 30 - 63 GeV), which again are well fitted by NB distributions[2]. As discussed in Ref. 2), this reveals a strong violation of KNO scaling[3] above the ISR energy range. Little is known at the ISR on multiplicity distributions for limited pseudorapidity intervals. Thomé et al.[4] give the distribution for $|\eta| < 1.5$ at $\sqrt{s}$ from 23.6 to 62.8 GeV. Their data include a certain amount of single diffraction dissociation. Except for an excess at low multiplicities, the distribution is of NB type, $\bar{n}$ increasing from 4.5 to 6.3 and k decreasing from 4.6 to 3.3. This happens to correspond to approximate KNO scaling over the ISR energy range. At $\sqrt{s}$ = 540 GeV ($p\bar{p}$ collider), the value of k for $|\eta| < 1.5$ is 1.95, again much lower than at the ISR.

While many authors pointed out that total multiplicities follow NB distributions[5], the occurrence of this type of distribution for limited η intervals had not been noticed before. Since the phenomenon appears to be very general, one naturally expects it to be related to some general characteristics of particle production. The interpretation of NB distributions in terms of a <u>stochastic cell</u>

model has been proposed in Ref. 6) and again recently in Ref. 7). It assumes stimulated emission of identical bosons by identical cells; each cell produces a Bose-Einstein distribution (which is a NB distribution with $k = 1$), and with K identical cells one then obtains a NB distribution with $k = K$[8]. In addition to the rather drastic nature of its basic assumptions, this type of model leads to integer k, in clear disagreement with the $p\bar{p}$ collider results[1] where k varies smoothly with the η interval considered[*].

On the other hand, there are mechanisms not involving stimulated emission which give rise to NB distributions. This holds for certain cascade processes studied in connection with cosmic ray phenomena[11] and for certain jet fragmentation processes in perturbative QCD[12], and the resulting NB distributions have a continuously varying parameter k. Since cascading and fragmentation can obviously play a role in particle production processes, any attempt to understand the occurrence of NB distributions in high-energy collisions must allow for these possibilities as well as for stimulated emission.

In the present paper, we point out and exploit a characteristic property of NB distributions which leads to a general understanding of their occurrence with continuous parameter k in both cases of stimulated emission and cascading. This property is described in the next section. Section 3 explains how stimulated emission, under the simplest assumptions, gives rise to this property and therefore to NB distributions. What we are led to is a general concept of partial stimulated emission allowing all values $k > 1$. Cascade effects are treated in Section 4. Starting again from the simplest assumptions, cascading gives rise to the same characteristic property and therefore to NB distributions. This time all $k > 0$ are allowed. Our cascading model leads naturally to a new concept of particle clusters. We derive from the ISR and $p\bar{p}$ collider data various properties of these clusters, which turn out to be quite interesting and make the cascading explanation of the NB character of multiplicity distributions in our view the most attractive one. Section 5 considers the possibility of simultaneous occurrence of stimulated emission and cascading; we find the resulting description to be complicated and we do not pursue it. A summary and some further remarks are presented in Section 6. We note that all QCD-based models of hadron production involve some form of cascading, and we suggest which class of models might be closest to the simple cascade picture of Section 4. A brief discussion of e^+e^-

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*) In photon counting experiments, it has been argued by Mandel[9] and Helstrom[10] that the stochastic cell model can lead to continuous k. However, their considerations apply only to large k, whereas here we are concerned with small non-integer values of k.

annihilation into hadrons completes Section 6. Some additional material is grouped in four short Appendices.

2. - A CHARACTERIZATION OF NEGATIVE BINOMIAL DISTRIBUTIONS

We consider the frequencies F(n) of occurrence of multiplicities n in a large ensemble of collisions. The multiplicity n refers to some type of particles (e.g., charged particles) and to some phase space region (e.g., an interval of pseudorapidity). In the case of total multiplicities, a complication arises because of conservation laws (e.g., charged particles are always produced in pairs). We shall bypass it in our presentation by imagining that the total multiplicity is approximated by the multiplicity in a large phase space region, an approximation which should be good for most inelastic collisions at high energy. Our discussion may be restricted to multiplicities in a finite interval $n_0 < n < n_1$, excluding, for example, low n values where problems related to the separation of diffraction dissociation may become severe, and high n values where different physics and limitations of available energy may affect the distribution. For the present, the normalization of F(n) may be left arbitrary.

Our approach is to characterize the multiplicity distribution by a recurrence relation between F(n) and F(n+1). To define this relation we note that, although many of the observed particles may be identical, they can be distinguished in practice by their momenta. When relating collisions of multiplicities n and n+1, one must therefore take into account that a given collision of multiplicity n+1 can be related to n+1 collisions of multiplicity n, namely those obtained by removing any one of n+1 particles of the given collisions. One therefore expects that the simplest form of the recurrence relation will be between F(n) and (n+1)F(n+1), which leads us to express it in terms of the ratio

$$g(n) = (n+1)\, F(n+1) / F(n) \tag{3}$$

This expectation is immediately confirmed in the case of independent emission of particles, for which the distribution is Poissonian:

$$F(n) \propto a^n / n! \tag{4}$$

a is the average multiplicity $\bar{n}$ if (4) holds for all n. The distribution (4) is characterized by the property that g(n) = a. This constancy of g(n) simply means that the production of an additional particle does not depend on the particles already present, which is of course the very concept of independent emission.

As seen immediately from Eq. (1), NB distributions are characterized by a linear dependence of g(n) on n,

$$g(n) = a + b\,n \tag{5}$$

More precisely, Eqs. (1) and (3) imply (5) with

$$k = a/b \ , \quad \bar{n} = a/(1-b) \tag{6}$$

$$a = \bar{n}k/(\bar{n}+k) \ , \quad b = \bar{n}/(\bar{n}+k) \tag{7}$$

Conversely, iterating (3) with (5) for $n > 0$ reproduces (1) if F(0) is chosen to normalize the resulting distribution. Experimental values of a and b are given in the Table and in Fig. 1 (on the expected behaviour of k, a and b for very small η intervals, see Appendix I, where we also estimate k for small η intervals at the ISR). We note in Fig. 1 that at the $p\bar{p}$ collider b becomes constant for intervals $|\eta| < \eta_0$ with $\eta_0 \gtrsim 2.5$. Reference 1) also gives NB fits for the multiplicity distributions in off-centre pseudorapidity intervals ($|\eta-\delta\eta| < \eta_0$, $\delta\eta < 3$, $\eta_0 < 1$). The fits give only small changes of the parameters. We shall not discuss them.

In the following sections, we show that under simple assumptions a linear ratio g(n) can be expected for stimulated emission as well as for cascade effects. We first discuss the two cases separately and then consider the possibility that they could operate together.

3. - PARTIAL STIMULATED EMISSION

In its simplest form, stimulated emission occurs for identical bosons emitted in the same quantum state and it results from Bose-Einstein statistics. The emission of a boson is enhanced by a factor n+1 in probability when n bosons are already present. In our notation, this corresponds to a = b in (5), i.e.,

$$g(n) = a(n+1) \tag{8}$$

The resulting distribution is $F(n) \propto a^n$ or the NB with $k = 1$, which is of course the familiar Bose-Einstein distribution for a single state. If the bosons are not all identical and/or are not all emitted in the same state, it is natural to expect that stimulated emission will only be partial.

We introduce a simple model of partial stimulated emission as follows. Depending on its momentum, the additional particle can be emitted independently of the n particles already present, and we represent this effect by the constant term a in g(n). The emission can also be enhanced by Bose-Einstein interference with one or more of the particles already present; the average effect of this enhancement is represented by an additional term in g(n) which we can reasonably assume to be proportional to n, i.e., the term bn with constant b in (5). Clearly, in this model of partial stimulated emission, the parameter $k = a/b$ can be any real number ≥ 1, with the value $k = 1$ corresponding to the case (8) of full stimulated emission.

For the stochastic cell model with k identical cells (k integer), the NB distribution is usually derived by the convolution of k Bose-Einstein distributions, each with mean multiplicity $\bar{n}/k$. In the approach we are using, it is obtained by noting that if n particles are already present, the average number of these particles per cell is n/k, so that the emission of the (n+1)-st particle is enhanced on average from the independent emission value $g(n) = a$ to

$$g(n) = a(1 + n/k) = a + bn$$

The main point here is that n/k is the average number of particles, among the n already present, to be active in stimulating the emission of the (n+1)-st particle. In other words, k^{-1} is the average fraction of particles already present stimulating the emission of an additional particle. Evidently, this interpretation can be adopted also for non-integer $k > 1$.

If we assume that partial stimulation emission is entirely responsible for the NB character of the multiplicity distributions in given pseudorapidity intervals, the above analysis leads to the following qualitative consequences. Increase first the pseudorapidity interval at constant collision energy $\sqrt{s}$. Since the pseudorapidity range of Bose-Einstein interference is presumably finite, we expect that the fraction k^{-1} of particles already present stimulating the emission of an additional particle will decrease, i.e., we expect k to increase. Consider next a fixed pseudorapidity interval for increasing $\sqrt{s}$ and take the interval to be in the central region where the average multiplicity $\bar{n}$ increases. Since more particles are crowded in the same interval, their phase space density increases (we neglect a small variation of mean transverse momentum), so that more Bose-Einstein interference and hence a smaller k is expected. The latter argument should also hold for total multiplicities. Partial stimulated emission therefore accounts readily for both variation trends of k found experimentally. What we do not know, however, is how to interpret in this model the fact that the parameter b becomes constant for large pseudorapidity intervals (Fig. 2).

4. - CASCADE PROCESSES

4.1 General Considerations

In this section we discuss cascade processes under the assumption that there is no stimulated emission. Cascading refers to the possibility for particles already produced to emit additional particles, where the former particles can change their momentum and quantum numbers, as happens for example in fragmentation and decay processes. In the case of cascading, one can usually define a grouping of particles in clusters. A cluster is formed of all particles originating directly or indirectly from one particle regarded as being originally produced in the collision. The latter particle will be called the ancestor of the cluster. An originally-produced particle which has not emitted any additional particle forms a one-particle cluster and is its own ancestor. In the simplest model, ancestors and therefore clusters are assumed to be produced independently. As we shall see, this is compatible with the data.

It is easiest here to discuss first the case of total multiplicities. Assume that n particles are already present. The (n+1)-st one can be originally produced, forming an extra one-particle cluster; this effect is represented by the constant a in g(n), see Eqs. (3) and (5). The (n+1)-st particle can also be emitted by one of the particles already produced, which puts it as an extra particle in an already existing cluster. The average effect of this cascade emission can be reasonably assumed to be proportional to the number n of particles already present and is therefore represented by the term bn of g(n).

We now consider the case of the multiplicity distribution in a limited domain of phase space, e.g., a limited η interval. We call this domain D. A cluster can have all, some, or none of its particles in D; in the first two cases, the particles of the cluster contained in D are said to form a D-cluster. Assume that n particles are already present in D. The (n+1)-st particle in D can form an extra one-particle D-cluster (either because it is originally produced, or because it is emitted by a particle belonging to a cluster totally outside D); this effect is represented by the constant a in g(n). The (n+1)-st particle can also be emitted as an extra particle in a pre-existing D-cluster (the particle emitting it may have been in the D-cluster, or elsewhere in the cluster containing the D-cluster); the average effect of this cascade emission can again be assumed to be proportional to n, i.e., to be represented by the term bn in g(n).

As in the case of partial stimulated emission (see the previous section), it is easy to understand in the cascading model why k increases when one increases the domain D at constant $\sqrt{s}$. If this is done for given n, it becomes more likely for the (n+1)-st particle to be produced as an extra one-particle D-cluster than to be emitted as an extra particle in a pre-existing D-cluster. Hence we expect an increase of a/(bn) = k/n, and therefore of k, since the comparison is made at given n. In the case of cascading, we have no simple prediction for the variation of k with $\sqrt{s}$. On the other hand, as we now show, the model implies simple and interesting properties for the cluster, which turn out to be controlled by the parameter b.

4.2 Properties of Clusters

We consider first the multiplicity distribution $F_c(n_c)$ of particles in a single cluster or a single D-cluster. The two cases are entirely similar; we present the discussion for clusters. To apply it to D-clusters, replace "cluster" by "D-cluster" everywhere and take n,.. b to refer to the phase space domain D. The distribution F_c is characterized by the following properties:

$$F_c(0) = 0 \tag{9}$$

$$g_c(n_c) \equiv (n_c+1) F_c(n_c+1)/F_c(n_c) = b n_c \quad , \quad n_c \geqslant 1 \tag{10}$$

Equation (9) corresponds to the fact that by definition a cluster contains at least one particle. Equation (10) where p is a constant expresses the assumption that, on average, the emission effect is proportional to the number n_c of particles already present in the cluster. Iteration of (10) gives

$$F_c(n_c) = F_c(1)\, p^{n_c - 1} / n_c \quad , \quad n_c \geq 1 \tag{11}$$

As for the number N of clusters, we adopt the simplest assumption of independent emission, which gives the Poissonian distribution

$$\tilde{F}(N) \propto \bar{N}^N / N! \tag{12}$$

If (11) and (12) hold for all n_c and N up to some maximum M, and if the $F_c(n_c)$ are normalized probabilities, one easily verifies (see Appendix II) that the resulting distribution of the overall multiplicity n has for n < M the NB form

$$F(n) \propto a(a+b) \ldots [a + b(n-1)] / n! \tag{13}$$

with the parameters

$$a = \bar{N} F_c(1) \quad , \quad b = p \tag{14}$$

(13) gives g(n) = a + bn for n < M and the identifications (14) are in agreement with the physical meaning of the various parameters. This shows the consistency of the properties we have assumed for the clusters with the overall NB distribution of n.

Consider the distribution (9)-(11), where following (14) we replace p by b, and assume it to hold now for all n_c. Then

$$\bar{n}_c = F_c(1)/(1-b) \tag{15}$$

$$F_c(1) = -b/\ln(1-b) \tag{16}$$

where (16) is calculated from the normalization condition $\Sigma F_c(n_c) = 1$. The first equation (14) combined with (6), (15) and (16) then gives

$$\bar{N} = \bar{n}/\bar{n}_c \tag{17}$$

as it should be, since $\bar{N}$ in (12) is the average number of clusters. One also finds

$$\overline{n_c^2} = \bar{n}_c \;/(1-b) \tag{18}$$

Note that in this model k = a/b can take any positive value, whereas it was restricted to k > 1 for (partial) stimulated emission (Section 3). In both cases b is restricted to 0 < b < 1 as seen from Eq. (7).

We mention two limiting cases of these equations. If $\bar{n}$ << k (as is the case for D-clusters pertaining to very small domains of phase space), one has b ≃ $\bar{n}$/k << 1, and hence by (15)-(18)

$$\bar{n}_c \simeq \overline{n_c^2} \simeq 1 \;, \quad \bar{N} \simeq \bar{n} \tag{19}$$

This means that the D-clusters are mostly composed of a single particle. The other extreme is $\bar{n}$ >> k, as occurs at the p$\bar{p}$ collider for large phase space domains. Then b ≃ 1-(k/$\bar{n}$) ≃ 1, so that

$$\left.\begin{aligned} \bar{n}_c &\simeq \bar{n}/[k \ln(\bar{n}/k)] \\ \overline{n_c^2}/\bar{n}_c^{\,2} &\simeq \bar{N}/k \simeq \ln(\bar{n}/k) \end{aligned}\right\} \tag{20}$$

and both quantities are large. The clusters are therefore big and have large dispersion.

4.3 Interpretation of Experimental Data

We can now come back to the interpretation of the experimental trends. Whereas k was the parameter of physical significance in the partial stimulated emission model, in the cascading model it is clearly b because it controls the multiplicity distribution of a cluster. The experimental values of $\bar{n}_c$ and $\bar{n}/\bar{n}_c$ are given in the Table in Section 2 and in Fig. 2. The Table shows that the main difference between the ISR and the $p\bar{p}$ collider is the strong growth of $\bar{n}_c$ and $\overline{n_c^2}/\bar{n}_c^2$, i.e., of cluster size. A big surprise is that the number of clusters $\bar{n}/\bar{n}_c$ is about the same at $\sqrt{s}$ = 62 and 540 GeV, being $\sim$ 8 for total multiplicities; we return to this point later on. The fact that b approaches a constant limit in Fig. 1 can now be interpreted; in our cascade model, it is related to the distinction between clusters and D-clusters. The latter are those parts of clusters which fall in the phase space domain D considered. If D is small, most D-clusters are small parts of clusters. As D becomes large enough, more and more clusters are expected to fall almost entirely inside D, i.e., more and more D-clusters should become full clusters. This corresponds to a growth of b towards a limiting value and allows us to estimate the average size of the clusters in pseudorapidity.

As seen in the Table, at the ISR the D-clusters for $|\eta| < 1.5$ are about the same as the full clusters obtained from the total multiplicities, whereas $\bar{n}/\bar{n}_c$ differs by more than a factor 2. This means that the pseudorapidity extent of the clusters should be $\lesssim 1.5$. These small clusters may be essentially the same as the clusters found to describe the two-particle pseudorapidity correlations[13].

The situation is totally different at the $p\bar{p}$ collider ($\sqrt{s}$ = 540 GeV). While b for $|\eta| < \eta_0$ approaches its constant limit around $\eta_0 \sim 2.5$ (Fig. 1), for $\bar{n}_c$ the approach takes place only at $\eta_0 \sim 3.5$ (Fig. 2). This is due to the occurrence of the small quantity 1-b in the denominator of Eq. (15). The clusters we find at the $p\bar{p}$ collider have therefore a pseudorapidity extent of order 3.5. Since the mean transverse momentum of the particles is not much larger than at the ISR, these large clusters must be strongly aligned longitudinally. They are not seen at the level of the two-particle pseudorapidity correlations, which the UA5 collaboration[14] finds to be almost the same as at the ISR (the two-particle correlations may mostly reflect the average effect of common resonances). There is no contradiction, since our cluster analysis is entirely based on the NB shape of multiplicity distributions, a property which involves the higher moments of these distributions and is not reflected in two-particle correlations.

The clusters derived from our cascade model at $\sqrt{s}$ = 540 GeV are not only large, their multiplicity distribution is also wide ($\overline{n_c^2}/\bar{n}_c^2 \sim 2.4$). The dispersion of their pseudorapidity width may also be large. All this could help to account for the pronounced forward-backward multiplicity correlations observed by the UA5 Collaboration[14),15]; they could be due to the fact that large clusters tend to extend rather far on both sides of the point $\eta = 0$.

The analysis carried out by the UA5 Collaboration[2)] suggests the following energy variation of the NB parameter k

$$k^{-1} = \lambda \ln(s/s_0)\ , \quad \lambda = 0.028\ , \quad \sqrt{s_0} = 5.7\ \text{GeV} \tag{21}$$

On the other hand, the average number of clusters $\bar{N} = \bar{n}/\bar{n}_c$ is found to increase to the value ~ 8 at the highest ISR energy, and is again ~ 8 at the $p\bar{p}$ collider. One might therefore speculate that $\bar{N}$ could be ~ 8 for all energies $\sqrt{s} \gtrsim 63$ GeV*) (although we do not know of any reason for such behaviour). A simple calculation using the equations of this section then gives, for the mean multiplicity, the asymptotic growth

$$\bar{n} \sim 8\,\bar{n}_c \sim \left[(s/s_0)^{8\lambda} - 1 \right] \Big/ \left[\lambda \ln(s/s_0) \right] \tag{22}$$

with $8\lambda \sim 0.23$. On the other hand, the asymptotic behaviour (21), together with a slower growth of $\bar{n}$, would imply a decrease of the mean number $\bar{N}$ of clusters at higher energies.

At the beginning of this section, we used very simple arguments to justify the linear behaviour of g(n) and consequently the NB distribution for our cascading model. The derivation of NB distributions given in the literature for cascade models of stochastic type[11)] is quite different. The relation between the two approaches is given in Appendix III.

5. - STIMULATED EMISSION AND CASCADING

We must still consider the possibility that the particle production processes involve both emission mechanisms considered separately in the previous sections. Do we still then expect the ratio g(n) to be linear in n? We attempt to answer this question by starting from cascade emission (as described in Section 4 in the absence of stimulated emission), and by discussing how stimulated emission would affect it. As before, we consider the ratio g(n) of Eq. (3),

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*) This possibility has also been noted by G. Ekspong, private communication.

corresponding to the emission of a (n+1)-st particle when n particles are already present. In Section 4, the emission of the (n+1)-st particle as an extra one-particle cluster was represented by the constant a in g(n). We now allow this process to be enhanced by partial stimulated emission; as in Section 3, we expect its contribution to g(n) to be increased to $a+b_1 n$ with b_1 constant. The (n+1)-st particle can also be emitted as an extra particle in an already existing cluster. In the absence of stimulated emission, we described this by the term bn of g(n). Partial stimulated emission would enhance this further, and the simplest expectation is now that the enhancement is of the form $b_2 n + b_3 n^2$, with a term quadratic in n, giving in total

$$g(n) = a + (b_1 + b_2) n + b_3 n^2 \tag{23}$$

For $b_3 \neq 0$, the resulting multiplicity distribution is no longer of NB type (it is not even normalizable if holding for unlimited n).

The data show that b_3 is negligible. A reason may be that a particle emitted in a cascade process tends to be more separated from the particles already present (except, of course, the emitting one), making stimulated emission ineffective. If we accept this, we are left with

$$g(n) = a + bn \quad , \quad b = b_1 + b_2 \tag{24}$$

where b_1 is that part of b caused by partial stimulated emission. Existing data do not allow us to decide how b would be shared between b_1 and b_2.

If (24) holds with b_1 and $b_2 \neq 0$, one may be tempted still to define clusters as we did in Section 4, but using $p = b_2$ instead of $p = b$ in Eq. (11). If this is done the distribution for the number of clusters is no longer Poissonian. Its generating function is readily calculated when multiplicities are assumed to be unrestricted (see Appendix IV), but we have not found for it a satisfactory physical interpretation. More generally, the problems related to the possible interplay between stimulated emission and cascading seem to be difficult also experimentally. The study of second-order interference between pions can at best only provide partial information. This form of interference can take place even if the original production does not involve stimulated emission and, conversely, stimulated emission can occur without causing second-order interference between final state pions (e.g., it can occur for gluons which then undergo fragmentation and hadronization).

6. - SUMMARY AND FURTHER REMARKS

The main purpose of this paper was to point out that the occurrence of negative binomial multiplicity distributions can be understood in terms of very simple assumptions of considerable generality. We noted in Section 2 that NB distributions are characterized by the linearity in n of the ratio $g(n) = (n+1)P(n+1)/P(n)$ relating the probabilities for production of n and n+1 particles. This is the simplest generalization of the case of independent particle production, the latter corresponding to constant g(n) and Poissonian P(n). We then explained that, under the simplest assumptions, a linear $g(n) = a + bn$ is expected to occur when the correlations among particles result either from partial stimulated emission (Section 3) or from the cascading nature of the production process (fragmentation, decay; Section 4). If the two effects are present simultaneously, they both contribute to the term bn and one also expects g to contain an n^2-term, which is not seen experimentally (Section 5).

From data on multiplicities one cannot find out whether partial stimulated emission, cascading, or both are responsible for the very wide occurrence of NB distributions observed experimentally. A very attractive feature of the cascading model without stimulated emission is that it suggests an interesting form of clustering of final state particles, with remarkably simple distribution laws for the number of clusters (Poissonian) and for the number n_c of particles per cluster ($\propto b^{n_c}/n_c$). At the ISR, these clusters are small and not very different from those deduced from the familiar two-particle correlation analysis. At the $p\bar{p}$ collider ($\sqrt{s}$ = 540 GeV), while the average number of clusters is the same (~ 8) as at the highest ISR energy ($\sqrt{s}$ = 63 GeV), the clusters are much larger in multiplicity and in pseudorapidity extent. This we regard to be the most remarkable result of our analysis. It suggests a new and puzzling asymptotic behaviour for high-energy hadroproduction.

Our approach has been to try to understand the wide occurrence of NB distributions in terms of general properties, not in terms of specific dynamical models. To our knowledge, none of the current QCD-based models for low p_T processes have predicted the NB shape of multiplicity distributions for finite pseudorapidity intervals, but they are quite flexible and may well be adjusted to reproduce the new experimental facts.

Basically these models are of fragmentation type, i.e., they involve cascading but no stimulated emission. Despite the complication due to the eventual hadronization of partons, these models might be close to the very simple cascading picture described in Section 4. The parton cascade models[16] would seem to go most naturally in this direction if they could be justified for low-p_T processes.

Our final remark concerns the applicability of our considerations to e^+e^- annihilation into hadrons. We have examined the TASSO Collaboration data for total charge multiplicities[17]. They follow NB distributions quite well at $\sqrt{s}$ = 14 GeV ($\bar{n} \simeq 9$, $k \simeq 58$) and $\sqrt{s}$ = 22 GeV ($\bar{n} \simeq 11$, $k \simeq 38$) and rather well at $\sqrt{s}$ = 34 GeV ($\bar{n} \simeq 13.5$, $k \simeq 28$). If one applies the cascading model of Section 4, the mean number of particles per cluster is found to be very close to the minimum $\bar{n}_c = 1$ ($\bar{n}_c = 1.08$, 1.14 and 1.22 respectively), which means that there is little cascading, corresponding to the fact that $\bar{n} \ll k$. Remarkably, the observed variation of k is compatible with a linear increase of k^{-1} with ℓns; Eq. (21) holds with $\lambda \simeq 0.01$, $\sqrt{s_0} \simeq 6$ GeV. Extrapolating this relation and the three fits given in Ref. 17) for the s dependence of $\bar{n}$, we find that $\bar{n}$ and k become comparable ($\bar{n} \simeq k \simeq 20$) at $\sqrt{s} \sim 67 - 77$ GeV, depending on the function adopted for $\bar{n}(s)$. Experiments at LEP energies ($\sqrt{s}$ = 100 - 200 GeV) will therefore offer a chance to study a region where $\bar{n}$ should be substantially larger than k, as is the case at the $p\bar{p}$ collider. It will be interesting to see whether the behaviour revealed for hadron collisions by the UA5 Collaboration then also appears for e^+e^- annihilation.

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APPENDIX I

As is well known, the dispersion of the multiplicity distribution in a pseudorapidity interval is expressible as

$$D^2 \equiv \overline{n^2} - \bar{n}^2 = \bar{n} + \int d\eta_1 \int d\eta_2 \, C_2(\eta_1, \eta_2) \qquad \text{(I.1)}$$

where both integrals extend over the η interval considered and where

$$C_2(\eta_1, \eta_2) = \rho_2(\eta_1, \eta_2) - \rho_1(\eta_1)\, \rho_1(\eta_2)$$

is the two-particle correlation function, $\rho_1 = \sigma^{-1} d\sigma/d\eta$ and $\rho_2 = \sigma^{-1} d^2\sigma/d\eta_1 d\eta_2$ being the one- and two-particle densities (σ is the appropriate inelastic cross-section, non-single-diffractive in our case). If one knows C_2, one can use (I.1) to find, for various pseudorapidity intervals, the parameter k as defined by Eq. (2), the result being

$$k^{-1} = \bar{n}^{-2} \int d\eta_1 \int d\eta_2 \, C_2(\eta_1, \eta_2) \qquad \text{(I.2)}$$

Whether the multiplicity distribution in the η interval considered is of NB type is, of course, another question which concerns, not C_2, but the higher correlation functions.

Take first a very small pseudorapidity interval, of length $\Delta\eta \ll 1$. We then expect

$$\bar{n} = L\,\Delta\eta \;, \quad C_2(\eta_1, \eta_2) = L' - L''(\eta_1 - \eta_2)^2 \qquad \text{(I.3)}$$

L, L' and L" being constants, L always positive and L', L" usually positive (attractive correlation). Since we are in the central region, we have neglected the $\Delta\eta^2$ term in $\bar{n}$. Inserting into (I.2), we find the following small $\Delta\eta$ behaviour of k

$$k = (L^2/L')(1 + L''\Delta\eta^2/12L') \qquad \text{(I.4)}$$

For small $\Delta\eta$ and to lowest order, the parameters a, b of Eq. (7) vary therefore as

$$a = \bar{n} = L\Delta\eta\,,\quad b = \bar{n}/k = (L'/L)\Delta\eta \qquad \text{(I.5)}$$

By means of an empirical parametrization of C_2, we have also used (I.2) to estimate k at the top ISR energy ($\sqrt{s}$ = 63 GeV) for pseudorapidity intervals $|\eta| < \eta_0$ with η_0 ranging from 0 to 1.5 (as mentioned before, except for $\eta_0 = 1.5$, we do not know whether the distributions are of NB type). We have adopted the parametrization of Giacomelli and Jacob[18] which fits the published data reasonably well:

$$C_2(\eta_1, \eta_2) = \rho_1^2 \beta^{-1} \left\{ \gamma \exp\left[-\frac{(\eta_1 - \eta_2)^2}{4\delta^2}\right] + 1 - \beta \right\} \qquad \text{(I.6)}$$

$\rho_1(\eta)$ is taken to be constant and drops out in (I.2). We chose $\gamma = 0.23$ and $\delta = 0.75$ as in Ref. 18), and also took a second value for the range, $\delta = 0.61$, to test the sensitivity of the result. The remaining constant β is determined by fitting the observed value k = 3.32 for the interval $|\eta| < 1.5$ [see Ref. 4)]. One finds $\beta = 0.87$ and 0.88 respectively.

Figure 3 shows the resulting variation of k with the interval $|\eta| < \eta_0$, the upper curve corresponding to the longer range ($\delta = 0.75$). The k values for η_0 = 0.2 (the smallest η_0 considered by the UA5 Collaboration) are 2.41 and 2.55 for the two choices of δ. This is clearly larger than the value k = 1.57±0.09 determined for the same η_0 at $\sqrt{s}$ = 540 GeV by UA5[1].

APPENDIX II

Combining the distributions (11) and (12), one obtains for the overall multiplicity distribution

$$F(n) \propto \sum_{N=1}^{n} \left[(\bar{N}^N/N!) \Sigma^* F_c(n_1) \dots F_c(n_N) \right]$$

$$= p^n \sum_{N=1}^{n} \left[(L^N/N!) \Sigma^* (n_1 \dots n_N)^{-1} \right]$$

Σ^* denotes the sum over all partitions $n = n_1 + \dots n_N$ with all n_i integers ≥ 1, and L is given by

$$L = \bar{N} F_c(1)/p$$

One then uses the identity

$$\sum_{N=1}^{n} \left[(L^N/N!) \Sigma^* (n_1 \dots n_N)^{-1} \right] = L(L+1)\dots(L+n-1)/n!$$

which is derived by equating the coefficients of z^n in the Taylor expansion of the obvious relation

$$(1-z)^{-L} = \exp[-L \ln(1-z)] \qquad \text{(II.1)}$$

The results (13) and (14) follow immediately. Use of (II.1) is analogous to the use of generating functions, but contrary to the generating function argument, our reasoning also applies when the validity of (11) and (12) is limited to n_c and N below a maximum M, in which case one obtains (13) for $n < M$.

APPENDIX III

This Appendix discusses the difference between the cascade emission model presented in Section 4 and the cascade processes which were shown in earlier work to lead to NB multiplicity distributions. The latter processes are of stochastic type, i.e., they involve a parameter describing the degree of advancement or of depth of the cascade. This parameter may be a time[11)], or the effective mass of a fragmenting parton[12)], or an energy variable[19)]. On closer consideration, these stochastic models turn out to be special cases of our model of Section 4. We show this for the time-dependent cascade of Ref. 11). In this model, one distinguishes between the emitted particles of the first generation which we call g_1, and the further particles which we call g_2. The g_1 particles are emitted by a source with probability one per unit time. The g_2 particles are emitted by g_1 particles and previously emitted g_2 particles, each such emission occurring with probability β per unit time. At any time t, the resulting overall multiplicity distribution is shown in Ref. 11) to be of NB type with

$$k = 1/\beta\ , \quad \bar{n} = [\exp(\beta t) - 1]/\beta \tag{III.1}$$

Consider a g_1 particle and call τ the time of its emission. By time $t > \tau$ it will have generated a "cluster", the multiplicity distribution of which is easily found to have generating function (g.f.)

$$q(z, t-\tau) = z/[z + (1-z)\exp\{\beta(t-\tau)\}] \tag{III.2}$$

For given t and for an emission process that starts at time zero, the average cluster has the g.f. obtained by averaging (III.2) over τ

$$\left.\begin{aligned} \bar{q}(z,t) &= t^{-1}\int_0^t q(z, t-\tau)\,d\tau = \ln(1-p_t z)/\ln(1-p_t) \\ p_t &= 1 - \exp(-\beta t) \end{aligned}\right\} \tag{III.3}$$

This g.f. corresponds to the multiplicity distribution (11) characteristic of a cluster in our model. As to the distribution of the number of clusters, it is identical to the multiplicity distribution of g_1 particles, i.e., it is Poissonian with average value t. The overall multiplicity distribution is therefore of NB type and one easily verifies that its parameters are given by (III.1).

APPENDIX IV

We take up the consideration on clusters at the end of Section 5, after Eq. (24), and we assume unrestricted multiplicities. The multiplicity distribution of a single cluster is taken to be (11) with $p = b_2$. Its generating function (g.f.) is

$$Q_c(z) = \ln(1-b_2 z)/\ln(1-b_2) \qquad \text{(IV.1)}$$

The overall multiplicity distribution is the NB of (13), which has the g.f.

$$Q(z) = (1-b)^k/(1-bz)^k, \quad k = a/b \qquad \text{(IV.2)}$$

We must determine the g.f. $\tilde{Q}(z)$ for the distribution of the number of clusters. It satisfies

$$Q(z) = \tilde{Q}[Q_c(z)]$$

and is therefore given by

$$\tilde{Q}(z) = Q(\zeta) \qquad \text{(IV.3)}$$

where ζ is the solution of

$$z = Q_c(\zeta)$$

(IV.1) gives

$$\zeta = [1-(1-b_2)^z]/b_2 \qquad \text{(IV.4)}$$

With (IV.2) and (IV.3) we find the result

$$\tilde{Q}(z) = \left[1 - \beta + \beta \exp\{\alpha(1-z)\}\right]^{-k}$$

$$\alpha = -\ln(1-b_2), \quad \beta = (b/b_2)\left[(1-b_2)/(1-b)\right] \tag{IV.5}$$

Since $0 < b_2 < b < 1$, one sees that

$$\alpha > 0 \quad , \quad \beta > 1$$

When $b_2 = b$, one has $\beta = 1$ and $\tilde{Q}(z)$ reduces to the g.f. of the Poisson distribution (12).

	Total Multiplicity			$\lvert\eta\rvert < 1.5$		
$\sqrt{s}$ (GeV)	30.4	62.2	540	23.6	62.8	540
$\bar{n}$	10.7	13.6	28.3	4.5	6.3	9.47
k	11.0	8.2	3.69	4.6	3.32	1.95
a	5.24	5.10	3.25	2.3	2.20	1.61
b	0.49	0.62	0.88	0.49	0.64	0.83
$\bar{n}_c$	1.43	1.69	3.46	1.4	1.74	2.75
$\overline{n_c^2}/\bar{n}_c^2$	1.37	1.56	2.41	1.4	1.60	2.14
$\bar{n}/\bar{n}_c$	7.48	8.05	8.18	3.1	3.56	3.44

Table

Parameters defined in Sections 1, 2 and 4 for the total multiplicity distribution and the multiplicity distribution in the pseudorapidity interval $\lvert\eta\rvert < 1.5$. The $\sqrt{s}$ = 540 GeV data are from the $p\bar{p}$ collider[1),2)]. The lower energy data are for pp collisions at the ISR, as quoted in Ref. 2) for total multiplicities and taken from Ref. 4) for $\lvert\eta\rvert < 1.5$.

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FIGURE CAPTIONS

Fig. 1 : The parameters a and b of Eq. (7) for pseudorapidity intervals $|\eta| < \eta_0$, $\eta_0 \leq 5$ at the $p\bar{p}$ collider, $\sqrt{s}$ = 540 GeV, from Ref. 1).

Fig. 2 : The mean multiplicity per cluster $\bar{n}_c$ and the mean number of clusters $\bar{n}/\bar{n}_c$ for pseudorapidity intervals $|\eta| < \eta_0$, $\eta_0 \leq 5$ at the $p\bar{p}$ collider, $\sqrt{s}$ = 540 GeV, deduced from the parameters a and b of Fig. 1.

Fig. 3 : Estimated variation of the parameter k derived from Eq. (I.2) for pseudorapidity intervals $|\eta| < \eta_0$, $\eta_0 \leq 1.5$ at the ISR, $\sqrt{s}$ = 63 GeV. The endpoint at η_0 = 1.5 is taken from experiment[4]. The two curves correspond to two values for the range δ of the two-particle correlation C_2, δ = 0.61 (lower curve) and δ = 0.75 (upper curve); see the text.

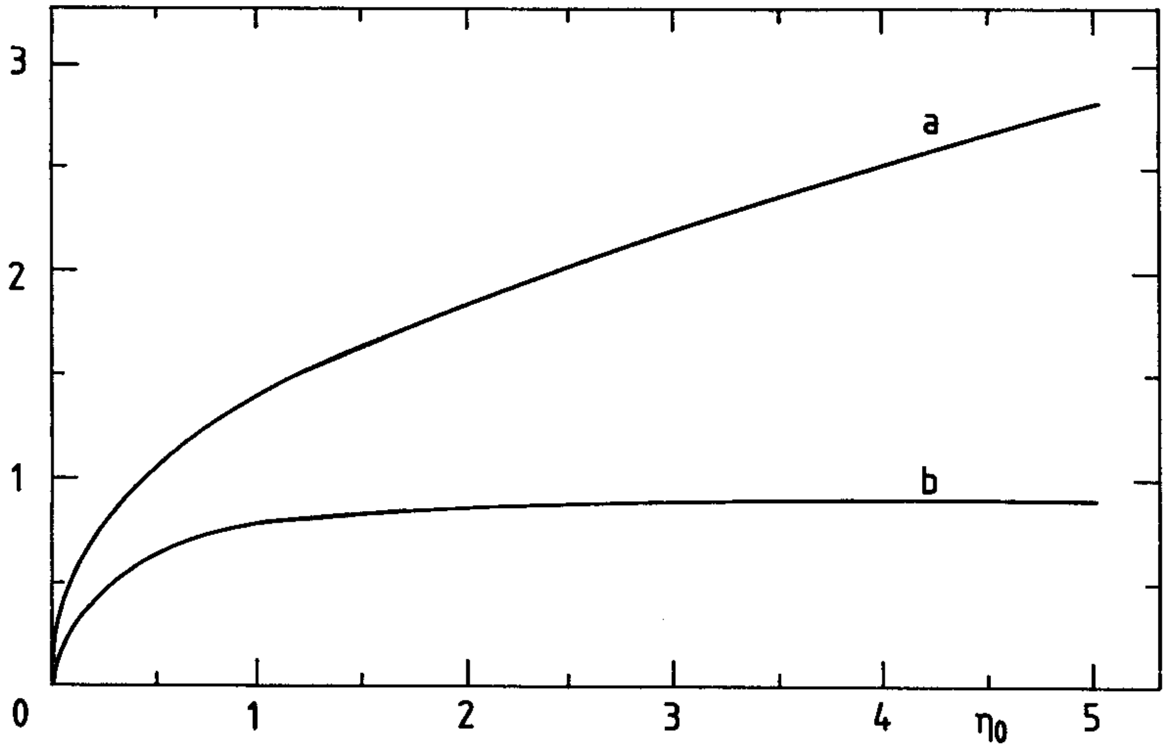

FIG. 1

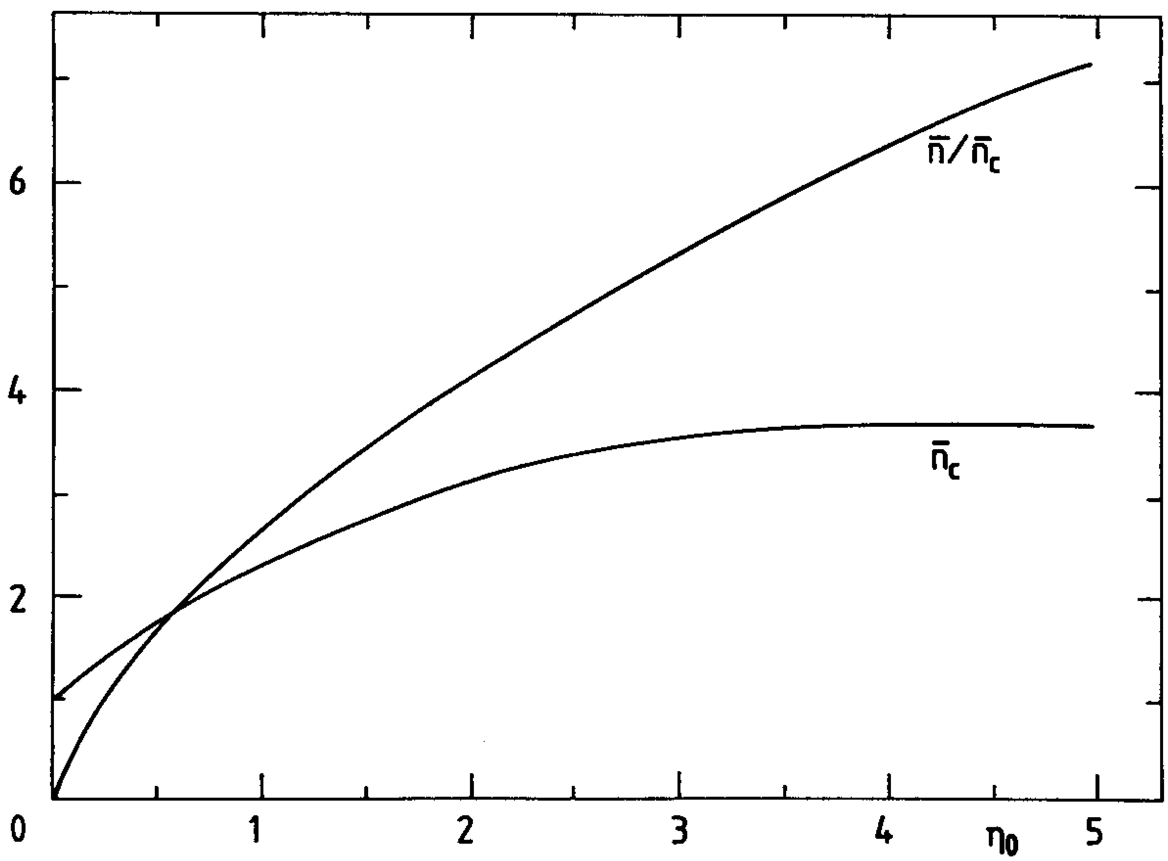

FIG. 2

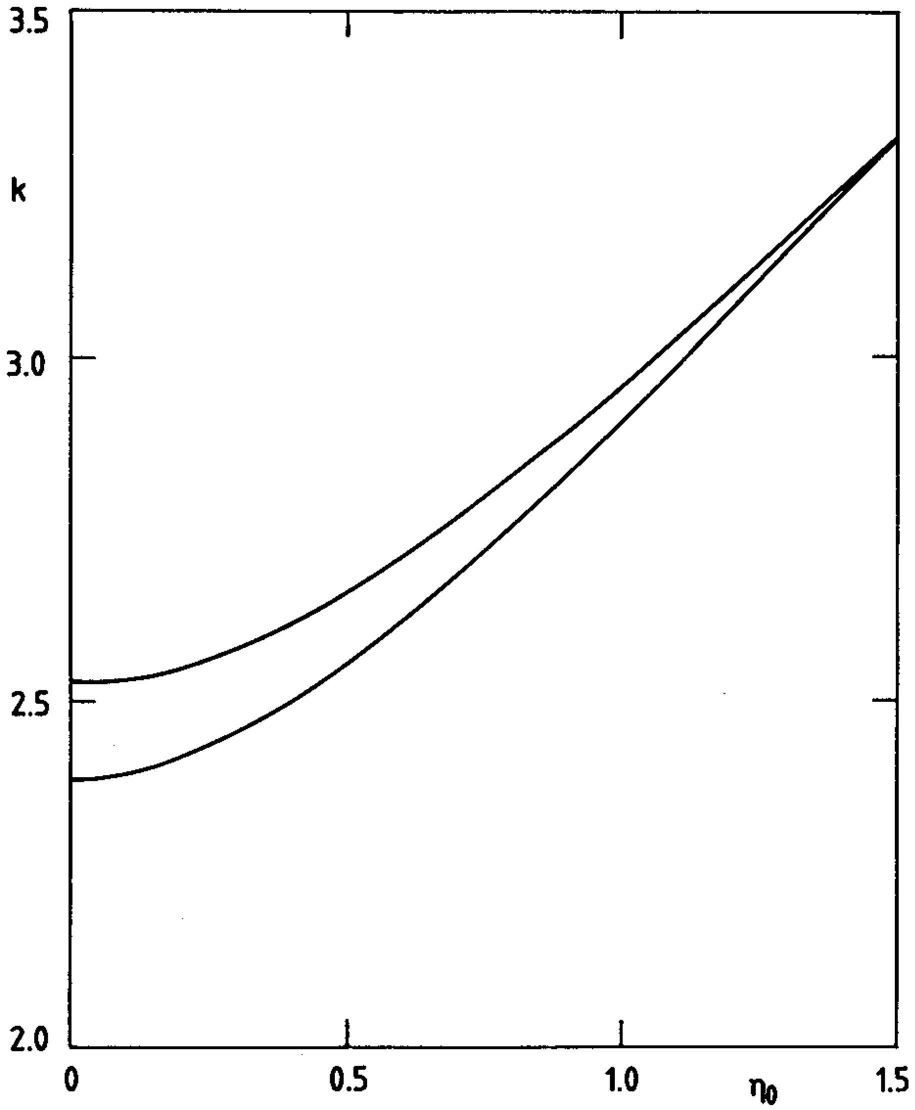

FIG. 3