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NEGATIVE THEOREMS ON MONOTONE APPROXIMATION

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ABSTRACT. In this paper we show that for f continuous on $[-1,+1]$ and satisfying $(f(x_2)-f(x_1))/(x_2-x_1) \geqq \delta > 0$, it is possible to have infinitely many of the polynomials of best uniform approximation to f not increasing on $[-1,+1]$.

1. **Introduction.** Monotone approximation in its simplest form is the study of the uniform approximation of continuous functions on a closed interval by algebraic polynomials which are increasing there. Of particular interest is the case when f is continuous and increasing on the interval since in this case f may be uniformly approximated as close as desired by polynomials which are also increasing on the interval.

We now introduce some notation. We will work on the closed interval $[-1,+1]$ since no loss of generality is imposed by this. If $f \in C[-1,+1]$ we define the uniform norm by

$$\|f\| = \max\{|f(x)|; -1 \leqq x \leqq 1\}.$$

For each nonnegative integer n define

$$H_n = \{p; p \text{ is an algebraic polynomial of degree less than or equal to } n\},$$

$$M_n = \{p \in H_n; p'(x) \geqq 0 \text{ for all } x \text{ in } [-1,+1]\}.$$

Now define the *degree of approximation*

$$E_n(f) = \inf\{\|f-p\|; p \in H_n\}$$

and the *degree of monotone approximation*

$$D_n(f) = \inf\{\|f-p\|; p \in M_n\}.$$

It is well known that for each f in $C[-1,+1]$ and for each nonnegative integer n that there is a unique $p \in H_n$ such that $\|f-p\| = E_n(f)$. It is also known that there is a unique $q \in M_n$ such that $\|f-q\| = D_n(f)$. See G. G. Lorentz and K. L. Zeller [4]. The polynomials p and q above are called the *polynomial of best approximation* and the *monotone polynomial of best approximation* to f respectively.

One might ask if the study of these concepts is trivial if f is increasing. That is, if f is increasing then is $p = q$? Or equivalently is $E_n(f) = D_n(f)$? The answer to these questions is in general no. Numerous examples are given to

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verify this. See J. A. Roulier [6] and [7] and G. G. Lorentz [3] and G. G. Lorentz and K. L. Zeller [5]. These references show that even if n is large we need not have $D_n(f) = E_n(f)$. All of these examples have one thing in common:

There is no $\delta > 0$ for which

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geqq \delta \quad \text{for all } x_2 \neq x_1 \text{ in } [-1, +1] \tag{1}$$

$$(\text{or } f'(x) \geqq \delta \quad \text{if } f \in C^1[-1, +1]).$$

On the other hand, J. A. Roulier in [7] proves:

THEOREM 1.1. *If $f \in C^2[-1, +1]$, and if there is $\delta > 0$ so that $f'(x) \geqq \delta > 0$ for $-1 \leqq x \leqq 1$, and if $f'' \in \text{Lip } \alpha$ for some $0 < \alpha \leqq 1$, then, for n sufficiently large, $E_n(f) = D_n(f)$.*

The purpose of this paper is to show that if f satisfies (1), *it still may happen that $D_n(f) \neq E_n(f)$ even for large n.*

2. **A negative theorem.** The main theorem of this section involves the modulus of continuity of the 2π-periodic function $\phi(t) = f(\cos t)$ induced by f $(\omega(\phi, h) = \sup\{|\phi(t) - \phi(t')|; |t - t'| \leqq h\})$.

THEOREM 2.1. *Let f be monotone increasing and continuous on* $[-1, +1]$ *and let $\phi(t) = f(\cos t)$. Assume that*

$$\limsup_{k \to \infty} \frac{k\,\omega(\phi, 1/k)}{\log k} = +\infty. \tag{2}$$

Then there are infinitely many positive integers n for which the polynomial p in H_n of best approximation to f is not increasing.

In order to prove this theorem we need two lemmas.

Given a continuous 2π-periodic function g we define $E_n^*(g)$ to be the degree of approximation to g by trigonometric polynomials of degree n or less. The following lemma is due to S. B. Stečkin [8]. See also the book of G. G. Lorentz [2, p. 59].

LEMMA 2.2. *There is a constant M such that for each continuous 2π-periodic function g*

$$\omega(g, h) \leqq Mh \sum_{0 \leqq n \leqq h^{-1}} E_n^*(g). \tag{3}$$

The next lemma makes use of (3).

LEMMA 2.3. *Let g be a continuous 2π-periodic function for which*

$$\limsup_{k \to \infty} \frac{k\,\omega(g, 1/k)}{\log k} = +\infty. \tag{4}$$

Then there does not exist a positive constant C so that $E_n^*(g) \leqq C/n$ for $n = 1, 2, 3, \ldots$.

PROOF. Assume that there is C so that

$$E_n^*(g) \leqq C/n, \qquad n = 1, 2, 3, \ldots.$$

Then by Lemma 2.2 with $h = 1/N$ we have a constant M for which

$$N\omega\left(g, \frac{1}{N}\right) \leqq M \sum_{n=0}^{N} E_n^*(g). \tag{5}$$

But by the above assumption we have

$$\sum_{n=0}^{N} E_n^*(g) \leqq E_0^*(g) + C \sum_{n=1}^{N} \frac{1}{n} \leqq D + 2C \log M \quad \text{for } N \geqq 3.$$

D depends on g but not on N.

Thus for N sufficiently large

$$\sum_{n=0}^{N} E_n^*(g) \leqq 3C \log N.$$

This together with (5) gives for N sufficiently large

$$N\omega(g, 1/N)/\log N \leqq 3CM.$$

But this contradicts (4).

Hence, the assumption is false and the lemma is proven. □

PROOF OF THEOREM 2.1. Assume that f satisfies the hypotheses of the theorem and that there is $N > 0$ so that for all $n \geqq N$ the $p \in H_n$ of best approximation to f is increasing. Let $n \geqq N$ be given. It follows from the Chebyshev alternation theorem (Lorentz [2, p. 30]) that there are $n + 2$ points $x_0 < x_1 < \cdots < x_{n+1}$ in $[-1, +1]$ where $|f(x_i) - p(x_i)| = E_n(f)$ and $f(x_i) - p(x_i)$ alternate in sign. If $p(x_{i+1}) = f(x_{i+1}) - E_n(f)$ then $p(x_i) = f(x_i) + E_n(f)$ and

$$p(x_{i+1}) - p(x_i) = f(x_{i+1}) - f(x_i) - 2E_n(f).$$

By our assumption that p is increasing, we must have

$$E_n(f) \leqq (f(x_{i+1}) - f(x_i))/2$$

for all such i. Since there are at least $[(n+1)/2]$ such pairs of points we have

$$\frac{n}{2} E_n(f) \leqq \left[\frac{n+1}{2}\right] E_n(f) \leqq \frac{1}{2} \sum (f(x_{i+1}) - f(x_i))$$
$$\leqq \frac{1}{2}(f(1) - f(-1)).$$

Thus for $n \geqq N$

$$E_n(f) \leqq (f(1) - f(-1))/n. \tag{6}$$

Now $E_n^*(\phi) = E_n(f)$ where $\phi(t) = f(\cos t)$. Thus (6) gives

$$E_n^*(\phi) \leqq (f(1) - f(-1))/n \quad \text{for } n \geqq N. \tag{7}$$

But this contradicts Lemma 2.3 since (7) implies the existence of a constant C for which $E_n^*(\phi) \leqq C/n$ for $n = 1, 2, 3, \ldots$. Thus the assumption is false and the theorem is proven. □

It is clear from the above proof that the following stronger but less constructive theorem is true.

THEOREM 2.4. *Let f be continuous and monotone increasing on $[-1, +1]$ and assume that $E_n(f) \neq O(1/n)$. Then there are infinitely many positive integers n for which the polynomial p in H_n of best approximation to f is not increasing.*

3. **Further negative results.** In the previous section we showed that all continuous increasing functions whose moduli of continuity satisfied a certain condition would fail to have increasing polynomials of best approximation even for n large. One might ask if f satisfies (1) but not (2) whether for n large enough the polynomials of best approximation must be increasing.

The purpose of this section is to provide a constructive proof that this need not happen no matter how nice the modulus of continuity of f is.

THEOREM 3.1. *Let ω be any modulus of continuity. Then there is an increasing function f in $C[-1, +1]$ which satisfies* (1) *and for which*

$$\omega(h) \leqq \omega(f, h) \leqq K\omega(h), \tag{8}$$

and yet there are infinitely many positive integers n for which the polynomial of best approximation from H_n to f is not increasing.

In order to prove this theorem we need several lemmas.

The following two lemmas are found in Lorentz [2, p. 45 and p. 94 respectively].

LEMMA 3.2. *Let ω be any modulus of continuity. Then there is a concave modulus of continuity $\overline{\omega}$ with the same domain of definition as ω for which*

$$\tfrac{1}{2}\overline{\omega}(h) \leqq \omega(h) \leqq \overline{\omega}(h). \tag{9}$$

LEMMA 3.3. *If $g(x) = |x|$ on $[-1, +1]$ then there is a constant $M > 0$ independent of n for which*

$$E_n(g) \geqq M/n, \qquad n = 1, 2, \ldots. \tag{10}$$

The next lemma is due to M. I. Kadec [1]. We first make a few comments.

Let ϕ be a 2π-periodic continuous even function, and for each positive integer n let T_n be the nth degree trigonometric polynomial of best approximation to ϕ. Then T_n is an even trigonometric polynomial.

By the Chebyshev alternation theorem then there are $2n + 2$ points in $[-\pi, \pi]$ at which $\phi - T_n$ assumes its maximum absolute value with alternating signs. This together with the fact that ϕ and T_n are even shows the existence of at least $n + 2$ points $t_{0,n} < t_{1,n} < \cdots < t_{n+1,n}$ in $[0, \pi]$ of maximum deviation at which the signs alternate. Let

$$\Delta_n = \max_{0 \leqq k \leqq n+1} \left| t_{k,n} - \frac{k\pi}{n+1} \right|. \tag{11}$$

LEMMA 3.4. *Let ϕ be a continuous even 2π-periodic function and let the notation be as above, and let $\varepsilon > 0$ be given. Then*

$$\liminf_{n\to\infty} \Delta_n n^{\frac{1}{2}-\varepsilon} = 0. \tag{12}$$

The following corollary to Lemma 3.4 will be used in the sequel.

COROLLARY. *Let $f \in C[-1,+1]$ and for each $n = 0, 1, 2, \ldots$ let $x_{0,n} < \cdots < x_{n+1,n}$ be a Chebyshev alternation for f. Let*

$$\delta_n = \max_{0\leqq k\leqq n+1} \left| x_{k,n} - \cos\left(\frac{k\pi}{n+1}\right)\right|.$$

Then there is a sequence $\{n_j\}_{j=0}^{\infty}$ of positive integers for which

$$\lim_{j\to\infty} \delta_{n_j} = 0. \tag{13}$$

PROOF. Let $\phi(t) = f(\cos t)$ and use Lemma 3.4. □

PROOF OF THEOREM 3.1. Let M be as in (10) and let $\varepsilon = \min(1, M/10)$. Let ω be any modulus of continuity. We define

$$f(x) = \begin{cases} x - \omega(1) + \omega(x+2) & \text{on } [-2,-1], \\ 2x + |x| & \text{on } [-1,+1], \\ 3 + \varepsilon(x-1) & \text{on } [1,2]. \end{cases}$$

Clearly $f \in C[-2,2]$ and is increasing and satisfies (1) on $[-2,2]$ with $\delta = \min(1,\varepsilon)$. We will work on $[-2,+2]$ for simplicity of notation although everything could be "shrunk" to $[-1,+1]$ with no difficulty.

Clearly for $0 \leqq h \leqq 1$

$$\omega(h) \leqq \omega(f,h) \leqq 4h + \omega(h). \tag{14}$$

Let $\bar{\omega}$ be a concave modulus of continuity for which (9) holds. It is easy to see that $\bar{\omega}(h)/h$ is a decreasing function of h (Lorentz [2, p. 44]). Thus for $0 < h \leqq 1$, $\bar{\omega}(h)/h \geqq \bar{\omega}(1)$. Hence, for $0 \leqq h \leqq 1$,

$$h \leqq C_1 \bar{\omega}(h). \tag{15}$$

Thus using (9), (14) and (15) we have for $0 \leqq h \leqq 1$

$$\omega(h) \leqq \omega(f,h) \leqq K\omega(h), \tag{16}$$

where $K = 8C_1 + 1$.

Let $g(x) = 2x + |x|$ on $[-1,+1]$. Then for $n \geqq 1$, $E_n(g) = E_n(|x|)$. Thus (10) gives

$$E_n(g) \geqq M/n \qquad \text{for } n = 1, 2, \ldots. \tag{17}$$

Let $E_n(f)$ be the best approximation to f by polynomials from H_n on $[-2,+2]$. It is easy to see that for $n = 0, 1, 2, \ldots$, $E_n(g) \leqq E_n(f)$. This combines with (17) to give

$$(18) \qquad E_n(f) \geqq M/n \quad \text{for } n = 1, 2, 3, \ldots.$$

Now use the corollary to Lemma 3.4 to find a sequence $\{n_j\}_{j=0}^{\infty}$ of positive integers so that (13) holds. We note that we are using this corollary on f on the interval $[-2, 2]$. This is accomplished by first applying the corollary to $f(2z)$ for $-1 \leqq z \leqq 1$ and then replacing z by $x/2$ throughout to return to $[-2, +2]$. In this case we are working with

$$\delta_n = \max_{0 \leqq k \leqq n+1} \left| x_{k,n} - 2 \cos\left(\frac{k\pi}{n+1}\right)\right|$$

where the $x_{k,n}$ are the Chebyshev alternation points for f on $[-2, +2]$. It is easy to see from this corollary that for j sufficiently large at least $(n_j + 2)/5 + 2$ of the $n_j + 2$ alternation points are in the interval $[1, 2]$.

For each $n = 0, 1, 2, \ldots$ let p_n be the polynomial from H_n of best approximation to f on $[-2, 2]$. Assume that for n sufficiently large p_n is increasing on $[-2, 2]$.

We may now use the same argument on $[1, 2]$ as is used to arrive at inequality (6) to show that for j sufficiently large

$$E_{n_j}(f) \leqq 5(f(2) - f(1))/(n_j + 2).$$

Thus since $f(2) - f(1) = \varepsilon \leqq M/10$ we have

$$E_{n_j}(f) \leqq 5\varepsilon/(n_j + 2) \leqq M/2(n_j + 2).$$

This together with (18) gives for j sufficiently large

$$M/n_j \leqq E_{n_j}(f) \leqq M/2(n_j + 2).$$

But this implies that $(n_j + 2)/n_j \leqq \frac{1}{2}$ for j sufficiently large. This is a contradiction since $\lim_{j\to\infty}(n_j + 2)/n_j = 1$. This proves Theorem 3.1. □

4. **Conclusions.** There is a gap between the main theorems of this paper and the main theorem in [7]. That is, if we assume that $f \in C^1[-1, +1]$ and $f'(x) \geqq \rho > 0$ on $[-1, +1]$ then does it follow that for n sufficiently large the polynomial from H_n of best approximation to f is increasing? This author conjectures that the answer is no, but this remains an open question.

The condition on f'' in Theorem 1.1 can be relaxed, but this theorem will appear elsewhere.

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