NEGATIVELY SUBSCRIPTED FIBONACCI AND LUCAS NUMBERS AND THEIR COMPLEX FACTORIZATIONS

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Abstract. In this paper, we find families of $(0, -1, 1)$ −tridiagonal matrices whose determinants and permanents equal to the negatively subscripted Fibonacci and Lucas numbers. Also we give complex factorizations of these numbers by the first and second kinds of Chebyshev polynomials.

1. Introduction

The well-known Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 2$

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

where $F_1 = F_2 = 1$. The Lucas Sequence, $\{L_n\}$, is defined by the recurrence relation, for $n \geq 2$

$$L_{n+1} = L_n + L_{n-1} \tag{1.2}$$

where $L_1 = 1$, $L_2 = 3$.

Rules (1.1) and (1.2) can be used to extend the sequence backward, respectively, thus

$$\begin{aligned} F_{-1} &= F_1 - F_0, \quad F_{-2} = F_0 - F_{-1} \\ L_{-1} &= L_1 - L_0, \quad L_{-2} = L_0 - L_{-1}, \ldots, \end{aligned}$$

and so on. Clearly

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n, \tag{1.3}$$

$$L_{-n} = L_{-n+2} - L_{-n+1} = (-1)^{n} L_n. \tag{1.4}$$

In [9] and [5], the authors give complex factorizations of the Fibonacci numbers by considering the roots of Fibonacci polynomials as follows

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i \cos \frac{\pi k}{n}\right), \quad n \geq 2. \tag{1.5}$$

2000 *Mathematics Subject Classification.* 11B37, 15A15, 15A36.

Key words and phrases. Fibonacci and Lucas numbers, tridiagonal matrices, determinant, permanent, factorization.

In [10] and [11], the authors establish the following forms:

$$F_n = i^{n-1}\frac{\sin\left(n\cos^{-1}\left(-\frac{i}{2}\right)\right)}{\sin\left(\cos^{-1}\left(-\frac{i}{2}\right)\right)}, \quad n \geq 1 \tag{1.6}$$
$$L_n = 2i^n\cos\left(n\cos^{-1}\left(-\frac{i}{2}\right)\right), \quad n \geq 1.$$

In [3], the authors prove (1.5) by considering how to the Fibonacci numbers can be connected to Chebyshev polynomials by determinants of a sequence of matrices, and then show that a connection between the Lucas numbers and Chebyshev polynomials by using a slightly different sequence of matrices as follows

$$L_n = \prod_{k=1}^{n}\left(1 - 2i\cos\frac{\pi\left(k-\frac{1}{2}\right)}{n}\right), \quad n \geq 1.$$

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [12] defines a $n \times n$ super diagonal matrix $F(n,k)$ for $n > k \geq 2$, and shows that the permanent of $F(n,k)$ equals the generalized order-k Fibonacci numbers. In [14], the author gives the same result of Minc by the same matrix $F(n,k)$ and using different a computing method of permanent, contraction. In particular, when $k = 2$, the matrix $F(n,k)$ is reduced to the tridiagonal toeplitz matrix

$$F(n,2) = \begin{bmatrix} 1 & 1 & & 0 \\ 1 & 1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 1 \end{bmatrix}$$

and $perF(n,2) = F_{n+1}$.

In [15], Lehmer proves a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries are somewhat arbitrary.

Also in [16] and [17], the authors define the $n \times n$ tridiagonal matrix M_n and show that the determinant of $M(n)$ is the Fibonacci number F_{2n+2}. In [2] and [3], the authors define the $n \times n$ tridiagonal matrix $H(n)$ and show that the determinant of $H(n)$ is the Fibonacci number F_n. In a similar family of matrices, the $(1,1)$ element of $H(n)$ is replaced with a 3, thus the determinants, [18], now generate the Lucas sequence L_n.

Recently, in [7], the authors find families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrix is a sum of consecutive Fibonacci or Lucas numbers." Also, in [8], the authors define two tridiagonal matrices and then give the relationships between the permanents and determinants of these matrices, and the terms of second order linear recurrences.

In this paper, we consider negatively subscripted Fibonacci and Lucas numbers and find associated families of tridiagonal matrices whose determinants or permanents equal to these numbers. Then we give the complex factorizations of these numbers by Chebyshev polynomial.

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

Also one can find more applications of permanents in [13].

A matrix is said to be a $(-1, 0, 1)$-*matrix* if each of its entries are $-1, 0$ or 1.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is *contractible on column* (resp. *row*.) k *if column* (resp. *row*.) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called *the contraction of* A *on column* k *relative to rows* i *and* j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A_{ij:k}^T\right]^T$ is called *the contraction of* A *on row* k *relative to columns* i *and* j. Every contraction used in this paper will be on the first column using the first and second rows. We say that A can be contracted to a matrix B if either $B = A$ or exist matrices $A_0, A_1, \ldots A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, 2, \ldots, t$.

Let we consider the following result (*see* [1]): Let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then

$$perA = perB. \tag{1.7}$$

2. Negatively Subscripted Fibonacci and Lucas numbers

In this section, we define families of tridiagonal matrices and then show that the determinants and permanents of these matrices equal to the negatively subscripted Fibonacci and Lucas numbers.

We start with negatively subscripted Fibonacci numbers. Now we define a $n \times n$ tridiagonal toeplitz $(0, -1, 1)$ $-$matrix $A_n = [a_{ij}]$ with $a_{ii} = -1$ for

$1 \le i \le n$, $a_{i,i+1} = a_{i+1,i} = 1$ for $1 \le i \le n-1$ and 0 otherwise. That is,

$$A_n = \begin{bmatrix} -1 & 1 & & 0 \\ 1 & -1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -1 \end{bmatrix}. \tag{2.1}$$

Then we give following Theorem.

Theorem 1. *Let the matrix A_n have the form (2.1). Then, for $n \ge 1$*

$$perA_n = F_{-(n+1)}$$

where F_{-n} is the nth negatively subscripted Fibonacci number.

Proof. If $n = 1$, then $perA_1 = per\,[-1] = F_{-2} = -1$.

If $n = 2$, then

$$A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and hence $perA_2 = F_{-3} = 2$.

Let A_n^p be pth contraction of A_n, $1 \le p \le n-2$. From the definition of A_n, the matrix A_n can be contracted on column 1 so that

$$A_n^1 = \begin{bmatrix} 2 & -1 & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}.$$

Since the matrix A_n^1 can be contracted on column 1 and $F_{-4} = -3$, $F_{-3} = 2$,

$$A_n^2 = \begin{bmatrix} -3 & 2 & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix} = \begin{bmatrix} F_{-4} & F_{-3} & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}.$$

Continuing this process, we obtain

$$A_n^r = \begin{bmatrix} F_{-(r+2)} & F_{-(r+1)} & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}$$

for $3 \le r \le n-4$. Hence,

$$A_n^{n-3} = \begin{bmatrix} F_{-(n-1)} & F_{-(n-2)} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

which, by contraction of A_n^{n-4} on column 1, gives

$$A_n^{n-2} = \begin{bmatrix} F_{-(n-2)} - F_{-(n-1)} & F_{-(n-1)} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} F_{-n} & F_{-(n-1)} \\ 1 & -1 \end{bmatrix}.$$

By the Eq. (1.7) and the definition of the negatively subscripted Fibonacci numbers, we obtain

$$per A_n = per A_n^{n-2} = F_{-(n-1)} - F_{-n} = F_{-(n+1)}.$$

So the proof is complete. □

Second, we define a $n \times n$ tridiagonal $(0, -1, 1)$ −matrix $B_n = [b_{ij}]$ with $b_{ii} = -1$ for $2 \le i \le n$, $b_{i,i+1} = b_{i+1,i} = 1$ for $1 \le i \le n-1$, $b_{11} = -\frac{1}{2}$ and 0 otherwise. That is,

$$B_n = \begin{bmatrix} -\frac{1}{2} & 1 & & 0 \\ 1 & -1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -1 \end{bmatrix}. \tag{2.2}$$

Now we give following Theorem.

Theorem 2. *Let the matrix B_n has the form (2.2). Then*

$$per B_n = \frac{L_{-n}}{2}$$

where L_{-n} is the nth negatively subscripted Lucas number.

Proof. If $n = 1$, then

$$per B_1 = per \left[-\frac{1}{2} \right] = L_{-1}/2 = -1/2.$$

If $n = 2$, then

$$B_2 = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix}$$

and hence $per B_2 = L_{-2}/2 = 3/2$.

Let B_n^p be pth contraction of B_n, $1 \leq p \leq n-2$. From the definition of B_n, the matrix B_n can be contracted on column 1 so that

$$B_n^1 = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & & & 0 \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -1 \end{bmatrix}.$$

Since the matrix B_n^1 can be contracted on column 1 and $L_{-3} = -4,\ L_{-2} = 3$,

$$B_n^2 = \begin{bmatrix} -\frac{4}{2} & \frac{3}{2} & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{L_{-3}}{2} & \frac{L_{-2}}{2} & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}.$$

Continuing this process, we obtain

$$B_n^r = \begin{bmatrix} \frac{L_{-(r+1)}}{2} & \frac{L_{-r}}{2} & & & \\ 1 & -1 & 1 & & \\ & 1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -1 \end{bmatrix}$$

for $3 \leq r \leq n-4$. Hence,

$$B_n^{n-3} = \begin{bmatrix} L_{-(n-2)}/2 & L_{-(n-3)}/2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

which, by contraction of B_n^{n-4} on column 1, gives

$$B_n^{n-2} = \begin{bmatrix} \left(L_{-(n-3)} - L_{-(n-2)}\right)/2 & L_{-(n-2)}/2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} L_{-(n-1)}/2 & L_{-(n-2)}/2 \\ 1 & -1 \end{bmatrix}.$$

By the Eq. (1.7) and the definition of the negatively subscripted Lucas numbers, we obtain

$$per B_n = per B_n^{n-2} = \left(L_{-(n-2)} - L_{-(n-1)}\right)/2 = L_{-n}/2.$$

So the proof is complete. □

A matrix A is called *convertible* if there is an $n \times n$ $(1,-1)$ −matrix H such that $per A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let S be a $(1,-1)$ −matrix of order n, defined by

$$S=\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

Then we have that $F_{-(n+1)}=\det(A_n\circ S)$ and $L_{-n}/2=\det(B_n\circ S)$ where F_{-n} and L_{-n} are the nth negatively subscripted Fibonacci and Lucas number, respectively.

Let we denote the matrices $A_n\circ S$ and $B_n\circ S$ by C_n and D_n, respectively. Thus

$$C_n=\begin{bmatrix} -1 & 1 & & 0 \\ -1 & -1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & -1 \end{bmatrix}$$

and

$$D_n=\begin{bmatrix} -\frac{1}{2} & 1 & & 0 \\ -1 & -1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & -1 \end{bmatrix}$$

Also it is clear that the value of following determinant is independent of x : (see p.105, [19])

$$\begin{vmatrix} a & x & & 0 \\ \frac{1}{x} & a & \ddots & \\ & \ddots & \ddots & x \\ 0 & & \frac{1}{x} & a \end{vmatrix}.$$

Using the above result and considering the following matrices

$$\hat{C}_n=\begin{bmatrix} -1 & -\sqrt{1} & & 0 \\ \sqrt{1} & -1 & \ddots & \\ & \ddots & \ddots & -\sqrt{1} \\ 0 & & \sqrt{1} & -1 \end{bmatrix}$$

and

$$\hat{D}_n = \begin{bmatrix} -\frac{1}{2} & -\sqrt{1} & & 0 \\ \sqrt{1} & -1 & \ddots & \\ & \ddots & \ddots & -\sqrt{1} \\ 0 & & \sqrt{1} & -1 \end{bmatrix},$$

we can write that

$$\begin{aligned} \det \hat{C}_n &= \det C_n = per A_n = F_{-(n+1)}, \\ \det \hat{D}_n &= \det D_n = per B_n = L_{-n}/2. \end{aligned}$$

Furthermore, from [13], we have that let A be a tridiagonal matrix, and let $\check{A} = (\check{a}_{ij})$ be defined by $\check{a}_{st} = ia_{st}$ if $s \neq t$ and $\check{a}_{ss} = a_{ss}$, for all s and t $\left(i = \sqrt{-1}\right)$. Then we have

$$per\,(A) = \det\left(\check{A}\right).$$

Also let we define the following matrices;

$$\check{C}_n = \begin{bmatrix} -1 & i & & 0 \\ i & -1 & \ddots & \\ & \ddots & \ddots & i \\ 0 & & i & -1 \end{bmatrix} \tag{2.3}$$

and

$$\check{D}_n = \begin{bmatrix} -\frac{1}{2} & i & & 0 \\ i & -1 & \ddots & \\ & \ddots & \ddots & i \\ 0 & & i & -1 \end{bmatrix}. \tag{2.4}$$

Thus we have following Corollaries without proof.

Corollary 1. *Let the $n \times n$ tridiagonal toeplitz matrix $\check{C}_n$ as in (2.3). Then, for $n \geq 1$*

$$\det \check{C}_n = F_{-(n+1)}.$$

Corollary 2. *Let the $n \times n$ tridiagonal matrix $\check{D}_n$ be as in (2.4). Then, for $n \geq 1$*

$$\det \check{D}_n = L_{-n}/2.$$

3. Complex Factorization of the negatively subscripted Fibonacci numbers

In [3], the authors consider the relationships between the certain tridiagonal determinants, and, the usual Fibonacci and Lucas numbers. Then using the eigenvalues of these tridiagonal matrices, the authors give the complex factorizations of the usual Fibonacci and Lucas numbers. Following the method of [3], we find the eigenvalues of the two tridiagonal matrices whose determinants associated with the negatively subscripted Fibonacci and Lucas numbers. Therefore, we give the complex factorizations of the negatively subscripted Fibonacci and Lucas numbers.

There are variety of ways of computing matrix determinants (see [4] and [6] for more details). In addition to the method of cofactor expansion, the determinant of a matrix can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $\check{C}_n$ to find an alternative representation of $\det \check{C}_n$.

Now we define another $n \times n$ tridiagonal toeplitz matrix $V_n = [v_{ij}]$ with $v_{ii} = 0$ for $1 \leq i \leq n$ and $v_{i,i-1} = v_{i-1,i} = 1$ for $2 \leq i \leq n$ and 0 otherwise. Clearly

$$V_n = \begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}. \tag{3.1}$$

So it is clear that $\check{C}_n = -I + iV_n$. Then we give following Theorem.

Theorem 3. *Let F_{-n} be the nth negatively subscripted Fibonacci number. Then, for $n \geq 1$*

$$F_{-(n+1)} = \prod_{j=1}^{n} \left(-1 - 2i \cos \left(\frac{\pi j}{n+1} \right) \right).$$

Proof. Let $\lambda_j,\ j = 1, 2, \ldots, n$, be the eigenvalues of V_n with respect to eigenvectors x_i. Then, for all j

$$\check{C}_n x_j = (-I + iV_n)\, x_j = -Ix_j + iV_n x_j = -x_j + i\lambda_j x_j = (-1 + i\lambda_j)\, x_j. \tag{3.2}$$

Therefore, $\mu_j = -1 + i\lambda_j,\ j = 1, 2, \ldots, n$, are the eigenvalues of $\check{C}_n$. Hence, for $n \geq 1$

$$\det \check{C}_n = \prod_{j=1}^{n} (-1 + i\lambda_j)\,. \tag{3.3}$$

To compute the λ_j 's, we recall that each λ_j is a zero of the characteristic polynomial $p_n(\lambda) = |V_n - \lambda I|$. Since $V_n - \lambda I$ is a tridiagonal toeplitz matrix, i.e.,

$$V_n - \lambda I = \begin{pmatrix} -\lambda & 1 & & \\ 1 & -\lambda & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -\lambda \end{pmatrix}, \tag{3.4}$$

we can establish a recursive formula for the characteristic polynomials V_n :

$$\begin{aligned} p_1(\lambda) &= -\lambda, \\ p_2(\lambda) &= \lambda^2 - 1, \\ p_n(\lambda) &= -\lambda p_{n-1}(\lambda) - p_{n-2}(\lambda). \end{aligned} \tag{3.5}$$

This family of characteristic polynomials can be transformed into another family $\{U_n(x), \ n \geq 1\}$ by taking $\lambda \equiv -2x$:

$$\begin{aligned} U_1(x) &= 2x, \\ U_2(x) &= 4x^2 - 1, \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x). \end{aligned} \tag{3.6}$$

The family $\{U_n(x), \ n \geq 1\}$ is the set of Chebyshev polynomials of second kind. It is a well-known fact (see [10]) that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as:

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{3.7}$$

From (3.7), we can see that the roots of $U_n(x) = 0$ are given by $\theta_k = \frac{\pi k}{n+1}, \ k = 1, 2, \ldots, n$, or $x_k = \cos\theta_k = \cos\frac{\pi k}{n+1}, \ k = 1, 2, \ldots, n$. Applying the transformation $\lambda \equiv -2x$, we have the eigenvalues of V_n :

$$\lambda_k = -2\cos\left(\frac{\pi k}{n+1}\right), \ k = 1, 2, \ldots, n. \tag{3.8}$$

Considering Corollary 1, the Eqs. (3.3) and (3.8), we obtain

$$F_{-(n+1)} = \det \check{C}_n = \prod_{j=1}^{n}\left(-1 - 2i\cos\left(\frac{\pi j}{n+1}\right)\right)$$

which is desired. □

Theorem 4. *Let F_{-n} be nth negatively subscripted Fibonacci number. Then, for $n \geq 1$*

$$F_{-(n+1)} = i^n \frac{\sin\left((n+1)\cos^{-1}\left(\frac{i}{2}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{i}{2}\right)\right)}.$$

Proof. From (3.4), we can think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

$$K_n(x) = \begin{bmatrix} 2x & 1 & & \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{bmatrix}, \tag{3.9}$$

where $K_n(x)$ is $n \times n$. If we denote that $\check{C}_n = iK_n\left(\frac{i}{2}\right)$, then we obtain:

$$\det \check{C}_n = i^n \det K_n\left(\frac{i}{2}\right) = i^n U_n\left(\frac{i}{2}\right). \tag{3.10}$$

Combining the result of Corollary 1, the Eqs. (3.7) and (3.10) yields, for $n \geq 1$

$$F_{-(n+1)} = i^n \frac{\sin\left((n+1)\cos^{-1}\left(\frac{i}{2}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{i}{2}\right)\right)}.$$

So the proof is complete. □

Theorem 5. *Let L_{-n} be nth negatively subscripted Lucas number. Then, for $n \geq 1$*

$$L_{-n} = \prod_{k=1}^{n}\left(-1 - 2i\cos\frac{\pi\left(k-\frac{1}{2}\right)}{n}\right).$$

Proof. From Corollary 2, we have that $2\det \check{D}_n = L_{-n}$. We will not compute the spectrum of $\check{D}_n$ directly. Instead, we will note that the following: $(\det\left(I + e_1 e_1^T\right) = 2)$

$$\det \check{D}_n = \frac{1}{2}\det\left(\left(I + e_1 e_1^T\right)\check{D}_n\right), \tag{3.11}$$

where e_j is the jth column of the identity matrix. Thus we can write that the right-side of (3.11) as follows

$$\frac{1}{2}\det\left(\left(I + e_1 e_1^T\right)\check{D}_n\right) = \frac{1}{2}\det\left(-I + i\left(V_n + e_1 e_2^T\right)\right) \tag{3.12}$$

where the matrix V_n is given by (3.1). Let γ_j, $j = 1, 2, \ldots, n$, be the eigenvalues of $V_n + e_1 e_2^T$ with respect to eigenvectors y_j. Then, for all j

$$\left(-I + i\left(V_n + e_1 e_2^T\right)\right) y_j = -Iy_j + i\left(V_n + e_1 e_2^T\right) y_j = -y_j + i\gamma_j y_j = \left(-1 + i\gamma_j\right) y_j.$$

Thus

$$\frac{1}{2}\det\left(-I + i\left(V_n + e_1 e_2^T\right)\right) = \frac{1}{2}\prod_{k=1}^{n}\left(-1 + i\gamma_j\right). \tag{3.13}$$

To compute the γ_j's, we recall that all γ is a zero of the characteristic polynomial $t_n(\gamma) = \det\left(V_n + e_1 e_2^T - \gamma I\right)$. Since $\det\left(I - \frac{1}{2} e_1 e_1^T\right) = \frac{1}{2}$, we can alternately write the characteristic polynomial as

$$t_n(\gamma) = 2 \det\left[\left(I - \tfrac{1}{2} e_1 e_1^T\right)\left(V_n + e_1 e_2^T - \gamma I\right)\right]. \tag{3.14}$$

Since $t_n(\gamma)$ is twice the determinant of a tridiagonal matrix, that is,

$$t_n(\gamma) = 2 \det \begin{bmatrix} -\frac{\gamma}{2} & 1 & & \\ 1 & -\gamma & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -\gamma \end{bmatrix}, \tag{3.15}$$

one can derive a recursive formula for $\frac{t_n(\gamma)}{2}$:

$$\begin{aligned} \tfrac{t_1(\gamma)}{2} &= -\tfrac{\gamma}{2} \\ \tfrac{t_2(\gamma)}{2} &= \tfrac{\gamma^2}{2} - 1 \\ \tfrac{t_n(\gamma)}{2} &= -\gamma t_{n-1}(\gamma) - t_{n-2}(\gamma). \end{aligned}$$

This family of polynomials can be transformed into another family $\{T_n(x),\ n \geq 1\}$ by taking $\gamma \equiv -2x$:

$$\begin{aligned} T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_n(x) &= 2x T_{n-1}(x) - T_{n-2}(x). \end{aligned}$$

The family $\{T_n(x),\ n \geq 1\}$ is the set of Chebyshev polynomials of first kind. In [10], Rivlin presents that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the first kind to be written as

$$T_n(x) = \cos n\theta. \tag{3.16}$$

From the Eq. (3.16), one can see that the roots of $T_n(x) = 0$ are given by

$$\theta_k = \frac{\pi\left(k - \frac{1}{2}\right)}{n} \quad \text{or} \quad x_k = \cos\theta_k = \cos\frac{\pi\left(k - \frac{1}{2}\right)}{n} \text{ for } k = 1, 2, \ldots, n.$$

Applying the transformation $\gamma \equiv -2x$ and considering the roots of the (3.14) are also roots of $\det\left(V_n + e_1 e_2^T - \gamma I\right) = 0$, we have the eigenvalues of $V_n + e_1 e_2^T$:

$$\gamma_k = -2\cos\frac{\pi\left(k - \frac{1}{2}\right)}{n} \text{ for } k = 1, 2, \ldots, n. \tag{3.17}$$

From Corollary 2, the Eqs. (3.17) and (3.13), we obtain

$$L_{-n} = \prod_{k=1}^{n}\left(-1 - 2i\cos\frac{\pi\left(k - \frac{1}{2}\right)}{n}\right).$$

So the proof is complete. □

Theorem 6. *Let L_{-n} be nth negatively subscripted Lucas number. Then, for $n \geq 1$*

$$L_{-n} = 2i^n \cos\left(n \cos^{-1}\left(\frac{i}{2}\right)\right).$$

Proof. From (3.15), we think of Chebyshev polynomials of the first kind as being generated by determinants of successive matrices of the form

$$G_n(x) = \begin{bmatrix} x & 1 & & \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{bmatrix}_{n\times n}.$$

We note that $\det \check{D}_n = iG_n\left(\frac{i}{2}\right)$, thus

$$\det \check{D}_n = i^n \det G_n\left(\frac{i}{2}\right) = i^n T_n\left(\frac{i}{2}\right). \tag{3.18}$$

From Corollary 2, the Eqs. (3.16) and (3.18), we obtain

$$L_{-n} = 2i^n \cos\left(n \cos^{-1}\left(\frac{i}{2}\right)\right).$$

So the proof is complete. □

Acknowledgements

The authors would like to thank to the anonymous referee for helpful suggestions.

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