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NEIGHBORHOODS, BASES AND CONTINUITY IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT. It is shown that fuzzy continuous functions can be characterized by the closure of fuzzy sets, a subbasis for a fuzzy topology and fuzzy neighborhoods. Additional results are obtained concerning the collection of all fuzzy topologies on a fixed set, the interior of a fuzzy set, the closure of a fuzzy set, a fuzzy limit point, the derived fuzzy set and the relative fuzzy topology.

1. **Introduction.** Fuzzy topological spaces were first introduced in the literature by Chang [1] who studied a number of the basic concepts, including fuzzy continuous maps and compactness. Nazaroff [4] has used the fuzzy topological machinery of Chang as the starting point for developing a generalized theory of optimal control and has contributed the basic ideas of exterior and closure of a fuzzy set. Within a broader framework, Goguen [2] presented the fundamental ideas of basis, subbasis and product in an investigation of compactness. Fuzzy spaces are surveyed by Wong [6].

This paper is a thorough study of the basic notions in fuzzy topological spaces. We establish six characterizations of fuzzy continuous maps by introducing the notion of a neighborhood of a point and by building on the earlier concepts of subbasis and closure. We introduce and develop the new concepts of derived fuzzy set and relative fuzzy topology. We show that the collection of all fuzzy topologies on a fixed set is a complete lattice, give four characterizations of an open fuzzy set, characterize a neighborhood of a fuzzy set and establish two characterizations of a basis for a fuzzy topology. All of the contributions in this paper which are not referenced are original.

Fuzzy topological spaces are a very natural generalization of topological spaces in the following sense. As a result of the (1-1)-correspondence between the family of all subsets of a set X and the set of all characteristic functions which have domain X, a topology on X can be regarded as a family of characteristic functions with the usual set operations of $\subset$, $\cup$, $\cap$ and complementation replaced by the function operations of $\leqq$, $\vee$, $\wedge$ and $1 - \mu_E$, respectively. A fuzzy

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topological space allows more general functions to be members of the topology.

At the present time there is a great deal of activity in the area of fuzzy topological spaces. In [6] there is a fairly complete bibliography through 1974. Since that time, [7]-[19] have appeared.

2. Fuzzy Topological Spaces.

The results in this section are the foundations for a systematic treatment of fuzzy topological spaces.

DEFINITION 2.1. [4, p. 478]. Let X be a set. A *fuzzy set* in X is a function from X into $[0, 1]$, the closed unit interval. So g is a fuzzy set in X iff $g : X \to [0, 1]$.

Retaining the notation that a fuzzy set is a function, as we have done, facilitates the discussion of a fuzzy set in relation to an element of X, which will be important in the sequel. This idea was begun by Goguen in [2].

Since a function is a set which can be related to other sets by set operations, we will not introduce new set operations for fuzzy sets as has been done in [1] and [4]. Rather, since fuzzy sets are real valued functions, we will use the existing function operations of $=$, $\leqq$, $\vee$, $\wedge$, $+$ and $-$ to relate fuzzy sets to other fuzzy sets. The notation $f \leqq g$ means $f(x) \leqq g(x)$ for all $x \in X$ and the notation $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$, with similar definitions for the other operations. We assume that a supremum (infimum) of fuzzy sets taken over an empty index set is $\mu_{\varnothing}$ (μ_X).

If $E \subset X$, then the fuzzy set μ_E is the characteristic function whose values are 1 at the points of E, and 0 at the points of $X - E$. Conversely, if μ_E is a fuzzy set in X, then μ_E is the characteristic function on X defined by its subset E.

DEFINITION 2.2. [1, p. 183]. Let X be a set and let T be a family of fuzzy sets in X. Then T is called a *fuzzy topology* on X iff it satisfies the conditions:

(a) $\mu_{\varnothing}, \mu_X \in T$;
(b) if $g_i \in T, i \in I$, then $\vee_I g_i \in T$;
(c) if $g, h \in T$, then $g \wedge h \in T$.

The pair (X, T) is called a *fuzzy topological space* (abbreviated as fts). The elements of T are called *open* fuzzy sets. A fuzzy set f is *closed* iff $1 - f$ is open. By a fuzzy set in a fts (X, T), we mean a fuzzy set in X.

THEOREM 2.1. *For every fts (X, T), the system $(T, \geqq)$ is a complete distributive lattice.*

PROOF. Goguen [2, p. 736] points out that the system $(T, \geqq)$ is a cl_∞-monoid (in our case, the operation $* = \wedge$) from which the result is immediate.

THEOREM 2.2. *If T_i, $i \in I$, are fuzzy topologies on X, then $\cap_I T_i$ is a fuzzy topology on X.*

PROOF. It is easily verified that $\cap T_i$ satisfies requirements (a), (b), and (c) of Definition 2.2.

THEOREM 2.3. *For a fixed X, let θ be the collection of all fuzzy topologies on X. Then $(\theta, \supset)$ is a complete lattice.*

PROOF. Since θ has a greatest element, it follows from Theorem 2.2 that every nonempty subset of θ has a least upper bound.

DEFINITION 2.3 [5]. A fuzzy set n in a fts (X, T) is a *neighborhood of a point* $x \in X$ iff there is $g \in T$ such that $g \leqq n$ and $n(x) = g(x) > 0$. A neighborhood of a point x is frequently denoted by n_x. A neighborhood n_x is called an *open neighborhood* of x iff $n_x \in T$.

THEOREM 2.4. *If n_x and p_x are neighborhoods of x, then $n_x \wedge p_x$ is also a neighborhood of x.*

PROOF. Let g, h be elements of T satisfying $g \leqq n_x$, $h \leqq p_x$, $n_x(x) = g(x) > 0$ and $p_x(x) = h(x) > 0$. Then $g \wedge h \leqq n_x \wedge p_x$ and $(n_x \wedge p_x)(x) = (g \wedge h)(x) > 0$.

DEFINITION 2.4. [1, p. 183]. Let (X, T) be a fts. A fuzzy set h in X is a *neighborhood of a fuzzy set* a in X iff there is $g \in T$ such that $a \leqq g \leqq h$.

It should be noted that Definition 2.4 is a generalization of the definition of a neighborhood of a set in [3, p. 112].

THEOREM 2.5. *Let (X, T) be a fts and let h, a be two fuzzy sets in X. Then h is a neighborhood of a iff, given $x \in X$ satisfying $a(x) > 0$, then there exists $n_x \in T$ such that $a(x) \leqq n_x(x)$ and $n_x \leqq h$.*

PROOF. $(\Rightarrow)$ Let $g \in T$ satisfying $a \leqq g \leqq h$. Given $x \in X$ for which $a(x) > 0$, then choose $n_x = g$.

$(\Leftarrow)$ If $g = \bigvee \{\text{open } n_x : 0 < a(x) \leqq n_x(x) \text{ and } n_x \leqq h\}$, then $a \leqq g \leqq h$ and $g \in T$.

THEOREM 2.6. [1, p. 183]. *Let (X, T) be a fts. Then $a \in T$ iff, whenever b is a fuzzy set in X satisfying $b \leqq a$, then a is neighborhood of b.*

THEOREM 2.7. *A fuzzy set a in X is open in the fts (X, T) iff, for every $x \in X$ satisfying $a(x) > 0$, there is $n_x \leqq a$ such that $n_x(x) = a(x)$.*

PROOF. ($\Rightarrow$) Let $n_x = a$.
($\Leftarrow$) Let $g = \bigvee\{\text{open } n_x \leqq a : a(x) > 0\}$. Then $g \in T$ and $g = a$.

By Theorem 2.7 a fuzzy set is open iff it is a neighborhood of each point at which it assumes a positive value.

THEOREM 2.8. *Let (X, T) be a fts.*
(i) *Then $\mu_\emptyset$, μ_X are closed fuzzy sets.*
(ii) *If c_i, $i \in I$, are closed fuzzy sets, then $\bigwedge_I c_i$ is a closed fuzzy set.*
(iii) *If d, e are closed fuzzy sets, then $d \vee e$ is a closed fuzzy set.*

PROOF. We will show that (ii) holds. If S is a nonempty set of real numbers, then it is well known that $\inf\{x : x \in S\} = -\sup\{-x : x \in S\}$ and $\inf\{1 + x : x \in S\} = 1 + \inf\{x : x \in S\}$. Hence, if $b_i = 1 - c_i$, then $\bigwedge\{c_i : i \in I\} = \bigwedge\{1 - b_i : i \in I\} = 1 - \bigvee\{b_i : i \in I\}$, where each b_i is open.

THEOREM 2.9. *A fuzzy set a in X is closed in the fts (X, T) iff, given $x \in X$ for which each n_x satisfies $n_x(x) \neq 1 - a(x)$ or $n_x(y) > 1 - a(y)$ for some $y \in X$, then $a(x) = 1$.*

PROOF. The proof is an application of Theorem 2.7 to the fuzzy set $1 - a$.

DEFINITION 2.5. [1, p. 184]. Let a and b be fuzzy sets in a fts (X, T) and let $a \geqq b$. Then b is called an *interior fuzzy set* of a iff a is a neighborhood of b. The least upper bound of all interior fuzzy sets of a is called the *interior* of a and is denoted by a°.

THEOREM 2.10. [1, p. 184]. *Let a be a fuzzy set in a fts (X, T). Then a° is open and is the largest open fuzzy set less than or equal to a. The fuzzy set a is open iff $a = a^\circ$.*

COROLLARY 2.1. *Given a fts (X, T) and an infinite set I, then in the lattice $(T, \geqq)$ the infimum of $\{g_i : i \in I\} \subset T$ is $(\bigwedge\{g_i : i \in I\})^\circ$.*

PROOF. This result follows from Theorem 2.10.

DEFINITION 2.6. [4, p. 483]. Let a be a fuzzy set in a fts (X, T). Then $\bigwedge\{\text{closed fuzzy sets } b \text{ in } X : b \geqq a\}$ is called the *closure* of a and is denoted by $\bar{a}$.

Theorem 2.11. *Let a be a fuzzy set in a fts (X, T). Then $\bar{a}$ is closed and is the least closed fuzzy set which is greater than or equal to a. Also, a is closed iff $a = \bar{a}$.*

Proof. If $a = \bar{a}$, then there is a sequence of closed fuzzy sets b_i such that $0 \leqq b_i(x) - a(x) < 1/i$ for all x in X. Hence, $a = \bigwedge \{b_i\}$ which is closed.

The other results are easily verified.

Theorem 2.12. *Let a and b be fuzzy sets in a fts (X, T). If $a \leqq b$, then $\bar{a} \leqq \bar{b}$ and $a^\circ \leqq b^\circ$. Also $\bar{\bar{a}} = \bar{a}$ and $(a^\circ)^\circ = a^\circ$.*

Proof. These results follow from the appropriate definitions.

Theorem 2.13. *Let a and b be fuzzy sets in a fts (X, T). Then*

(i) $\bar{a} \vee \bar{b} = \overline{a \vee b}$ *and* $\bar{a} \wedge \bar{b} \geqq \overline{a \wedge b}$.

(ii) $a^\circ \wedge b^\circ = (a \wedge b)^\circ$ *and* $a^\circ \vee b^\circ \leqq (a \vee b)^\circ$.

(iii) $(1 - a)^\circ = 1 - \bar{a}$.

(iv) $\overline{1 - a} = 1 - a^\circ$.

Proof. (i) Since $\bar{a} \vee \bar{b}$ is closed and $\bar{a} \vee \bar{b} \geqq a \vee b$, it follows from Definition 2.6 that $\bar{a} \vee \bar{b} \geqq \overline{a \vee b}$. On the other hand, $a \vee b \geqq a$ implies $\overline{a \vee b} \geqq \bar{a}$. Similarly, $\overline{a \vee b} \geqq \bar{b}$. Hence, $\overline{a \vee b} \geqq \bar{a} \vee \bar{b}$. By Theorem 2.12, $\bar{a} \wedge \bar{b} \geqq \overline{a \wedge b}$.

(iii)
$$\begin{aligned} 1 - \bar{a} &= 1 - \bigwedge \{d : 1 - d \in T \text{ and } d \geqq a\} \\ &= \bigvee \{1 - d : 1 - d \in T \text{ and } d \geqq a\} \\ &= \bigvee \{c : c \in T \text{ and } c \leqq 1 - a\} \\ &= (1 - a)^\circ \text{ where } c = 1 - d. \end{aligned}$$

The proofs of (ii) and (iv) are similar.

Definition 2.7. Let a be a fuzzy set in a fts (X, T). A point $x \in X$ is called a *fuzzy limit point* of a iff whenever $a(x) = 1$, then for each n_x $\exists\, y \in X - \{x\}$ such that $n_x(y) \wedge a(y) \neq 0$; or whenever $a(x) \neq 1$, then $\bar{a}(x) > 0$ and for each open n_x satisfying $1 - n_x(x) = a(x)$ $\exists\, y \in X - \{x\}$ such that $n_x(y) \wedge a(y) \neq 0$. We define the *derived fuzzy set* of a (denoted by a') as:

$$a'(x) = \begin{cases} \bar{a}(x) & \text{if } x \text{ is a fuzzy limit point of } a \\ 0 & \text{otherwise.} \end{cases}$$

If all fuzzy sets are restricted to be characteristic functions (i.e., the usual concept of general topology), then Definition 2.7 agrees with the general topology concept of limit point and derived set.

Theorem 2.14. *Let a be a fuzzy set in a fts (X, T) and let $x \in X$. Then x is a fuzzy limit point of a iff $a'(x) > 0$.*

Proof. This result is a consequence of Definition 2.7.

Theorem 2.15. *Let a be a fuzzy set in a fts* (X, T). *Then a is closed iff* $a' \leqq a$. *Furthermore,* $a'' \leqq \bar{a} = a \vee a'$.

Proof. If a is closed, then $\bar{a} = a$. Since $a' \leqq \bar{a}$, it follows that $a' \leqq a$.

Assuming that $a' \leqq a$, we will show that $a = \bar{a}$. If $a'(x) = \bar{a}(x)$, then since $a' \leqq a \leqq \bar{a}$, it follows that $a(x) = \bar{a}(x)$. If $a'(x) \neq \bar{a}(x)$, then x is not a fuzzy limit point of a and $\bar{a}(x) > 0$. When $a(x) = 1$, then $\bar{a}(x) = a(x)$. Thus we may also assume that x is not a fuzzy limit point of a, $\bar{a}(x) > 0$ and $a(x) \neq 1$. By Definition 2.7 there is an open n_x such that $1 - n_x(x) = a(x)$ and if $y \in X - \{x\}$, then $n_x(y) \wedge a(y) = 0$. Hence, if $y \neq x$, then $1 - n_x(y) \geqq a(y)$. Since $1 - n_x$ is closed, $\bar{a} \leqq 1 - n_x$. Therefore, $\bar{a}(x) \leqq 1 - n_x(x) = a(x)$.

Next we verify that $a'' \leqq \bar{a}$. By Definition 2.7 $a'' \leqq \overline{a'}$. From $a' \leqq \bar{a}$, it follows that $\overline{a'} \leqq \bar{a}$.

To see that $\bar{a} = a \vee a'$, we note that if $a'(x) = \bar{a}(x)$, then $\bar{a}(x) = (a \vee a')(x)$. On the other hand, if $a'(x) \neq \bar{a}(x)$, then $a'(x) = 0$ and by the argument in the previous paragraph $\bar{a}(x) = a(x)$.

We shall show that the following statement is false: if $a \leqq b$, then $a' \leqq b'$. Let X be a nonempty set and designate x_0 in X. Define fuzzy sets a and b as $a(x_0) = 1/4$, $b(x_0) = 1/2$ and $a(x) = 0 = b(x)$ if $x \in X - \{x_0\}$. Let the fuzzy topology on X be $\{\mu_\emptyset, \mu_X, b\}$. Then $a'(x_0) = 1/2$ and $b'(x_0) = 0$. To analyze why the statement is false, it is noted that x_0 is a fuzzy limit point of a, but is not a fuzzy limit point of b.

In the above example, $a'(x_0) \vee b'(x_0) > (a \vee b)'(x_0)$ and $a'(x_0) \wedge b'(x_0) < (a \wedge b)'(x_0)$. However, the following result is valid.

Theorem 2.16. *Let a and b be fuzzy sets in a fts* (X, T) *and let* $x \in X$.

(i) *If x is a fuzzy limit point of* $a \vee b$, *then* $a'(x) \vee b'(x) \leqq (a \vee b)'(x)$.

(ii) *If x is a fuzzy limit point of a and b, then* $a'(x) \wedge b'(x) \geqq (a \wedge b)'(x)$.

Proof. (i) Since $a'(x) \leqq \bar{a}(x) \leqq \bar{a}(x) \vee \bar{b}(x) = \overline{a \vee b}(x) = (a \vee b)'(x)$ and since $b'(x) \leqq (a \vee b)'(x)$, the result follows.

(ii) Since $(a \wedge b)'(x) \leqq \overline{a \wedge b}(x) \leqq \bar{a}(x) \wedge \bar{b}(x) = a'(x) \wedge b'(x)$, the result follows.

Similarly, there is an example in which $a'(x_0) \vee b'(x_0) < (a \vee b)'(x_0)$. However, if x is a fuzzy limit point of a and b, then $(a \vee b)'(x)$

$\leqq a'(x) \vee b'(x)$ and x is a fuzzy limit point of $a \vee b$. So, by Theorem 2.16, $(a \vee b)'(x) = a'(x) \vee b'(x)$.

THEOREM 2.17. *Let (X, T) be a fts and let $A \subset X$. Then the family $T_A = \{g|_A : g \in T\}$ is a fuzzy topology on A, where $g|_A$ is the restriction of g to A.*

PROOF. It is easily verified that T_A satisfies properties (a) through (c) of Definition 2.2.

DEFINITION 2.8. The fuzzy topology T_A is called the *relative fuzzy topology* on A or the fuzzy topology on A *induced* by the fuzzy topology T on X. Also, (A, T_A) is called a *subspace* of (X, T).

THEOREM 2.18. *Let (A, T_A) be a subspace of the fts (X, T) and let a be a fuzzy set in A. Further, let b be the fuzzy set in X defined by $b(x) = a(x)$ if $x \in A$* and *$b(x) = 0$ if $x \in X - A$. Then $\bar{a} = \bar{b}|_A$ and $b^\circ|_A = a^\circ \wedge (\mu_A{}^\circ)|_A$ where $\bar{a}$ and a° are with respect to T_A and $\bar{b}$, b° and $\mu_A{}^\circ$ are with respect to T.*

PROOF.
$$\begin{aligned}\bar{a} &= \wedge \{c|_A : c|_A \geqq a \text{ and } 1 - c|_A \in T_A\}\\ &= (\wedge \{d : d \geqq b \text{ and } 1 - d \in T\})|_A\\ &= \bar{b}|_A.\end{aligned}$$
The verification of the other result is similar.

3. **Basis and Subbasis for a Given Fuzzy Topology.** In this section we prove results analogous to those in general topology for a basis and subbasis. The task of specifying a fuzzy topology is simplified by giving only enough open fuzzy sets to generate all the open fuzzy sets.

DEFINITION 3.1. [2, p. 737]. Let T be a fuzzy topology on X and let $B, S \subset T$. Then B is called a *basis* for T iff each element of T is the supremum of members of B. Also, S is called a *subbasis* for T iff the family of all finite infimums of elements of S is a basis for T.

THEOREM 3.1. *Let T be a fuzzy topology on X and let $B \subset T$.* Then *the following two properties of B are equivalent*:

(i) *B is a basis for T.*

(ii) *For each $g \in T$, for each $x \in X$ such that $g(x) > 0$ and for each real number $\epsilon > 0$, there is $b \in B$ such that $b \leqq g$ and $g(x) - b(x) < \epsilon$.*

PROOF. (i) $\Rightarrow$ (ii). Since $g \in T$, $g(x) > 0$ and B is a basis for T, it follows that $g(x) = \vee \{b(x) : b \leqq g \text{ and } b \in B\}$. Thus there is $b \in B$ satisfying $b \leqq g$ and $g(x) - b(x) < \epsilon$.

(ii) $\Rightarrow$ (i). Let $g \in T$ and let $g(x) > 0$. Then there is $b_{x,n} \in B$ such that $b_{x,n} \leqq g$ and $g(x) - b_{x,n}(x) < 1/n$. Hence $g = \bigvee_{x,n}\{b_{x,n} : g(x) > 0$ and $n = 1, 2, \cdots\}$.

Corollary 3.1. *Let $B \subset T$ be a basis for the fuzzy topology T on X and let a be a fuzzy set in X. Then a is open iff for each $x \in X$ such that $a(x) > 0$ and for each $\epsilon > 0$, there is $b \in B$ such that $b \leqq a$ and $a(x) - b(x) < \epsilon$.*

By specifying a basis for T, all the open fuzzy sets are generated as supremums. However, Corollary 3.1 is a more convenient way to describe the open fuzzy sets.

Next we consider two general methods for introducing fuzzy topologies on sets. Goguen [2, p. 737] has pointed out the first method.

Theorem 3.2. *Given any family $F = \{a_\alpha : \alpha \in I\}$ of fuzzy sets in X, there is always a unique, smallest fuzzy topology S on X such that $S \supset F$. The collection S can be described as follows: It consists of $\mu_\varnothing$, μ_X, all finite infimums of the a_α, and all arbitrary supremums of these finite infimums. Furthermore, F is a subbasis for S.*

Proof. Using Theorem 2.2, these results are easily verified.

Theorem 3.3. *Let $B = \{b_\alpha : \alpha \in J\}$ be any family of fuzzy sets in X such that for each $(\gamma, \lambda) \in J \times J$, for each $x \in X$ for which $(b_\gamma \wedge b_\lambda)(x) > 0$ and for each $\epsilon > 0$, there is some b_α satisfying $b_\alpha \leqq b_\gamma \wedge b_\lambda$ and $(b_\gamma \wedge b_\lambda)(x) - b_\alpha(x) < \epsilon$. Then the collection S consisting of $\mu_\varnothing$, μ_X and all supremums of members of B is a fuzzy topology on X. Also, S is unique and is the smallest fuzzy topology on X containing B. In addition, B is a basis for S.*

Proof. The verification is straightforward.

We note that if B is a basis for a fuzzy topology on X, then B satisfies the hypothesis of Theorem 3.3. Therefore, this gives another characterization of a basis for a fuzzy topology.

Though each basis is associated with a unique fuzzy topology, it is obvious that a fuzzy topology may have several distinct bases. We now determine when this will occur.

Theorem 3.4. *Let B, B' be bases for fuzzy topologies S, S' on X, respectively. Then $S \subset S'$ iff, for each $b \in B$, each $x \in X$ for which $b(x) > 0$ and each $\epsilon > 0$, there is $b' \in B'$ such that $b' \leqq b$ and $b(x) - b'(x) < \epsilon$.*

Proof. If $S \subset S'$, then given $b \in B$ and $x \in X$ for which $b(x) > 0$, it follows that $b \in S'$. Since B' is a basis for S', the existence of a suitable b' follows from Theorem 3.1.

For the converse, it follows from Corollary 3.1 that $B \subset S'$. Hence $S \subset S'$.

COROLLARY 3.2. *Let B, B' be bases for fuzzy topologies S, S' on X, respectively. Then $S = S'$ iff both of the following conditions hold:*

(a) *For each $b \in B$, each $x \in X$ for which $b(x) > 0$ and each $\epsilon > 0$, there is $b' \in B'$ such that $b' \leqq b$ and $b(x) - b'(x) < \epsilon$.*

(b) *For each $b' \in B'$, each $x \in X$ for which $b'(x) > 0$ and each $\epsilon > 0$, there is $b \in B$ such that $b \leqq b'$ and $b'(x) - b(x) < \epsilon$.*

4. **Fuzzy Continuous Functions.** In this section we continue the study of a generalized notion of continuity begun by Chang in [1] and Goguen in [2].

DEFINITION 4.1. [1, p. 185] and [2, p. 737]. Let f be a function from X to Y. Let b be a fuzzy set in Y and let a be a fuzzy set in X. Then the *inverse image* of b under f is the fuzzy set $f^{-1}(b)$ in X defined by $f^{-1}(b)(x) = b(f(x))$ for $x \in X$. The *image* of a under f is the fuzzy set $f(a)$ in Y defined by $f(a)(y) = \bigvee \{a(x) : f(x) = y\}$ for $y \in Y$.

In addition to the mapping properties given in [1, Theorem 4.1], we contribute the following properties.

THEOREM 4.1. *Let f be a function from X to Y. If a and a_i, $i \in I$, are fuzzy sets in X and if b and b_j, $j \in J$, are fuzzy sets in Y, then the following relations are valid:*

(i) $f(f^{-1}(b)) = b$ *when f is onto Y.*

(ii) $f(\bigwedge a_i) \leqq \bigwedge f(a_i)$.

(iii) $f^{-1}(\bigwedge b_j) = \bigwedge f^{-1}(b_j)$.

(iv) $f(\bigvee a_i) = \bigvee f(a_i)$.

(v) $f^{-1}(\bigvee b_j) = \bigvee f^{-1}(b_j)$.

(vi) $f(f^{-1}(b) \wedge a) = b \wedge f(a)$.

PROOF. These relations are all consequences of the definition of $f(a)$ and $f^{-1}(b)$. We will verify (ii), (iv) and (vi).

(ii) $f(\bigwedge a_i)(y) = \bigvee_x \bigwedge_i \{a_i(x) : f(x) = y\} \leqq \bigwedge_i \bigvee_x \{a_i(x) : f(x) = y\} = \bigwedge f(a_i)(y)$.

(iv) $f(\bigvee a_i)(y) = \bigvee_x \bigvee_i \{a_i(x) : f(x) = y\} = \bigvee_i \bigvee_x \{a_i(x) : f(x) = y\} = \bigvee f(a_i)(y)$.

(vi)
$$\begin{aligned} f(f^{-1}(b) \wedge a)(y) &= \bigvee \{(f^{-1}(b) \wedge a)(x) : f(x) = y\} \\ &= \bigvee \{b(f(x)) \wedge a(x) : f(x) = y\} \\ &= b(y) \wedge (\bigvee \{a(x) : f(x) = y\}) \\ &= (b \wedge f(a))(y). \end{aligned}$$

DEFINITION 4.2. [1, p. 187] and [2, p. 737]. Let (X, T), (Y, S) be fts and let $f: X \to Y$. Then f is *F-continuous* iff, whenever $b \in S$, then $f^{-1}(b) \in T$.

THEOREM 4.2. *Let (X, T), (Y, S) be fts and let $f: X \to Y$. Then the following conditions are equivalent:*

(a) *The function f is F-continuous.*

(b) [1, p. 187]. *The inverse image of every closed fuzzy set is closed.*

(c) [5]. *The inverse image of every element of a subbasis for S is in T.*

(d) [5]. *For every $x \in X$ and every neighborhood n of $f(x)$, $f^{-1}(n)$ is a neighborhood of x.*

(e) [5]. *For every $x \in X$ and every neighborhood n of $f(x)$, there is a neighborhood m of x such that $f(m) \leqq n$ and $m(x) = f^{-1}(n)(x)$.*

(f) [5]. *For every fuzzy set a in X, $f(\bar{a}) \leqq \overline{f(a)}$.*

(g) [5]. *For every fuzzy set b in Y, $\overline{f^{-1}(b)} \leqq f^{-1}(\bar{b})$.*

(h) *Set $G = \{(x, f(x)) : x \in X\}$ and let G have the fuzzy topology inherited as a subspace of $(X \times Y, T \times S)$. If $g: X \to G$ by $g(x) = (x, f(x))$, then g is F-continuous.*

PROOF. (a) $\Longrightarrow$ (c). Clearly this is true.

(c) $\Longrightarrow$ (a). Let $s_1, \cdots, s_n$ be elements of a subbasis S_1 for S and let S_2 be the basis of finite infimums of elements of S_1. Then $f^{-1}(\wedge s_k) = \wedge f^{-1}(s_k)$ is in T. Hence, every member of S_2 has the property that its inverse image is in T. Now let $s \in S$. Then $s = \vee \{b_i : i \in I_s, b_i \in S_2\}$. It follows that $f^{-1}(s) = \vee \{f(b_i)\}$ is in T.

(a) $\Longrightarrow$ (d). Since n is a neighborhood of $f(x)$, there is $g \in S$ such that $g \leqq n$ and $n(f(x)) = g(f(x)) > 0$. So $f^{-1}(g) \leqq f^{-1}(n)$, $f^{-1}(n)(x) = f^{-1}(g)(x) > 0$ and $f^{-1}(g) \in T$. Therefore $f^{-1}(n)$ is a neighborhood of x.

(d) $\Longrightarrow$ (e). Given a neighborhood n of $f(x)$, then let $m = f^{-1}(n)$. Hence, $f(m) = f(f^{-1}(n)) \leqq n$.

(e) $\Longrightarrow$ (a). Let $g \in S$ and $x \in X$ such that $f^{-1}(g)(x) > 0$. Then $g(f(x)) > 0$ and g is a neighborhood of $f(x)$. Hence, there is a neighborhood m of x such that $f(m) \leqq g$ and $m(x) = f^{-1}(g)(x)$. Therefore, $m \leqq f^{-1}(f(m)) \leqq f^{-1}(g)$. By Theorem 2.7 $f^{-1}(g) \in T$.

(b) $\Longrightarrow$ (f). Now $\overline{f(a)} = \vee \{c : c \geqq f(a) \text{ and } 1 - c \in S\}$. Hence, $f^{-1}(\overline{f(a)}) = \vee \{f^{-1}(c) : c \geqq f(a) \text{ and } 1 - c \in S\}$. Since $f^{-1}(c)$ is a closed fuzzy set and $f^{-1}(c) \geqq a$, it follows that $f^{-1}(c) \geqq \bar{a}$. Therefore, $\vee \{f^{-1}(c)\} \geqq \bar{a}$. Thus, $f^{-1}(\overline{f(a)}) \geqq \bar{a}$. So $f(\bar{a}) \leqq \overline{f(a)}$.

(f) $\Rightarrow$ (g). Since $f^{-1}(b)$ is a fuzzy set in X, it follows that $\overline{f(f^{-1}(b))} \leqq \overline{f(f^{-1}(b))} \leqq \bar{b}$. Hence, $\overline{f^{-1}(b)} \leqq f^{-1}(\bar{b})$.

(g) $\Rightarrow$ (b). Let c be a closed fuzzy set in (Y, S). Then $\overline{f^{-1}(c)} \leqq f^{-1}(\bar{c}) = f^{-1}(c)$. We conclude that $f^{-1}(c)$ is a closed fuzzy set in (X, T).

(a) $\Rightarrow$ (h). Since $g =$ (identity map, f), g is F-continuous.

(h) $\Rightarrow$ (a). Let $p : X \times Y \rightarrow Y$ be the projection map which is F-continuous by Theorem 4.1 in [6]. Since $f = p \circ g$, f is F-continuous.

REMARK 4.1. In Theorem 4.2(h), G is the graph of f and it is easily verified that if f is F-continuous, then g is a fuzzy homeomorphism.

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