Neighboring-Patch Integrals in Transient Electromagnetic Scattering

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Abstract—**The integrals over patches that are close to the self-patch are calculated by expanding the factors in the integrand in power series. The values are computed analytically up to first order in the linear size of the patch. This procedure applies to patches for which the distance between the centers is of the same order of magnitude as the size of the patch. The same formulas are useful in steady-state scattering problems.**

I. INTRODUCTION

IN THIS PAPER we continue to consider the problems associated with the numerical solution of the integral equations that arise in the theory of the electromagnetic scattering of transient fields. These integral equations involve vector and scalar functions defined on the surface of the scatterer, here assumed to be sufficiently smooth so that the necessary derivatives are well defined.

The integrands in these integral equations are singular, and the contributions of the self-patch, where the singularities occur, have to be treated separately from the rest of the contributions. We previously determined [1] these contributions to first order in the linear dimension of the patches for orthogonal curvilinear coordinates on the surface.

The contributions to the integrals from neighboring patches also deserve special attention [2]. The functions of the distance R between the field point and the source point can hardly be considered constant as the source point ranges over the patch; R varies by a factor greater than three over a nearest neighbor of the self-patch. The integrals we have to consider are no longer singular, as is the case for the self-patch, and here we have zeroth-order contributions that vanish for the self-patched integrals due to symmetry. The linear size of the patch is of the same order of magnitude as the distance between the centers of the patches. We should calculate these neighboring-patch integrals accurately if we determine that the contributions from these integrals to the overall surface integral are significant.

In Section II we present the expansions of the integrals that occur in these equations to terms that are of first order in the linear size of the patch. In Section III we present an example of these terms for the simple case of a sphere. Most of the calculations and formulas that permit the actual evaluation of the integrals are shown in the Appendix. The notation is similar to the one we used in [1].

The integrals and the results are essentially the same as those needed for monochromatic waves.

II. NEIGHBORING-PATCH INTEGRALS

The magnetic field integral equation (MFIE) and electric field integral equation (EFIE) for perfect conductors show the typical terms found in the integral equations of electromagnetic scattering. The MFIE and EFIE are

$$\vec{J}_s(\vec{x}, t) = (2/\mu_0)\hat{n} \times \vec{B}^{\text{in}}(\vec{x}, t) - \frac{1}{2\pi}\hat{n} \times \oint_S dS'\, \vec{R} \times \left[\frac{1}{R^2 c}\frac{\partial \vec{J}_s(\vec{x}', \tau)}{\partial t'} + \frac{1}{R^3}\vec{J}_s(\vec{x}', \tau)\right], \quad (1)$$

$$0 = \hat{n} \times \vec{E}^{\text{in}}(\vec{x}, t) + \frac{1}{4\pi}\hat{n} \times \oint_S dS' \left[\frac{\vec{R}}{\epsilon_0 R^2 c}\frac{\partial \rho_s(\vec{x}', \tau)}{\partial t'} + \frac{\vec{R}}{R^3}\rho_s(\vec{x}', \tau) - \frac{\mu_0}{R}\frac{\partial \vec{J}_s(\vec{x}', \tau)}{\partial t'}\right], \quad (2)$$

where $\vec{J}_s$ is the surface current density, ρ_s the surface charge density, τ the retarded time

$$\tau = t - R/c, \qquad R = |\vec{R}|, \quad \vec{R} = \vec{x} - \vec{x}', \quad (3)$$

ϵ_0 the permittivity of free space, μ_0 its permeability, c the speed of light, S the surface of the conductor, $\hat{n}$ the unit normal, $\vec{B}^{\text{in}}$ the magnetic induction of the incident pulse, and $\vec{E}^{\text{in}}$ its electric field. The field point $\vec{x}$ is in a patch S_1 and the source point $\vec{x}'$ is in a neighboring patch S_2, as shown in Fig. 1. The center of the patch is defined by the midvalue of the curvilinear coordinates.

We first consider the integral

$$\vec{I}_1 = \int_{S_2} dS' \frac{\vec{R}}{R^3} \times \vec{J}_s(\vec{x}', \tau). \quad (4)$$

We expand the functions in the integrand about the center $\vec{x}_0' = \vec{x}(u_0', v_0')$ of the patch S_2 at the corresponding retarded time $\tau_0 = t - R_0/c$. We use (30) for $\vec{R}$, (35) for R^{-3}, (36) for $\vec{J}_s$, and (37) for dS' to obtain

$$\vec{I}_1 \approx \vec{J}_s(\vec{x}_0', \tau_0) \times \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \left\{\frac{\beta_0'\tilde{u}\vec{x}_u' + \alpha_0'\tilde{v}\vec{x}_v'}{(\tilde{u}^2 + \tilde{v}^2)^{3/2}} \cdot \left[1 + \frac{K_3}{\alpha_0'\beta_0'} + \frac{3K_2}{2(\tilde{u}^2 + \tilde{v}^2)}\right] - \frac{\alpha_0'\beta_0'\vec{K}_1}{(\tilde{u}^2 + \tilde{v}^2)^{3/2}}\right\}$$
$$- \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\beta_0'\tilde{u}\vec{x}_u' + \alpha_0'\tilde{v}\vec{x}_v'}{(\tilde{u}^2 + \tilde{v}^2)^{3/2}} \times \left\{(\vec{x}_u'\tilde{u} + \vec{x}_v'\tilde{v}) \cdot \nabla'\vec{J}_s(\vec{x}_0', \tau_0) + \frac{1}{c}\left[\left(\frac{(\Delta u_0)^2}{\alpha_0'^2} + \frac{(\Delta v_0)^2}{\beta_0'^2}\right)^{1/2} - (\tilde{u}^2 + \tilde{v}^2)^{1/2}\right]\frac{\partial \vec{J}_s(\vec{x}_0', \tau_0)}{\partial t'}\right\}, \quad (5)$$

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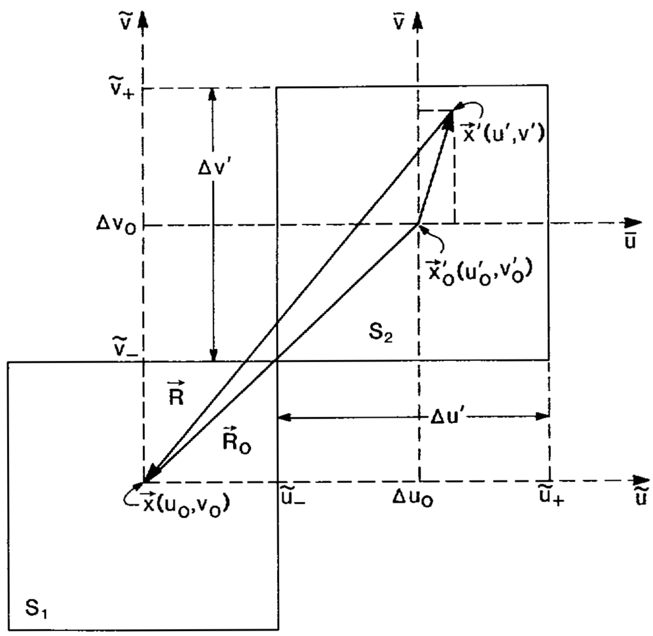

Fig. 1. Neighboring patches S_1 and S_2 showing the different variables used. To simplify the notation, we have suppressed the scale factors α and β.

where the limits of integration are

$$u_\pm = u_0' \pm \tfrac{1}{2}\Delta u',\quad v_\pm = v_0' \pm \tfrac{1}{2}\Delta v', \tag{6}$$

$\tilde{u}$ and $\tilde{v}$ are variables defined in (28), and $\vec{K}_1$, K_2, and K_3 are polynomials in $\tilde{u}$ and $\tilde{v}$ defined in (31), (33), and (38). The integrals can be carried out by changing the variables of integration to $\tilde{u}$ and $\tilde{v}$ and then using the appropriate $C(l, m, n)$ from the Appendix. The first term, as expected, is of zeroth order, and the others that we have kept are of first order. We write down the first few terms in the expansion to show some of the contributions; we have

$$\begin{aligned}\vec{I}_1 &\approx \vec{J}_s(\vec{x}_0', \tau_0) \times [C(1,0,3)\vec{x}_u'/\alpha_0' + C(0,1,3)\vec{x}_v'/\beta_0' \\ &\quad + (\kappa_1'\beta_0'/\alpha_0'^2 + \kappa_3'/\beta_0')C(2,0,3)\vec{x}_u'/\alpha_0'^2\beta_0' + \cdots] \\ &\quad - C(2,0,3)\vec{x}_u' \times [\vec{x}_u' \cdot \nabla'\vec{J}_s(\vec{x}_0', \tau_0)]/\alpha_0'^2 + \cdots. \end{aligned}\tag{7}$$

The next term is similar to but of higher order than (4), and it gives a first-order contribution only. We have

$$\begin{aligned}\vec{I}_2 &= \int_{S_2} dS' \frac{\vec{R}}{R^2} \times \frac{\partial \vec{J}_s(\vec{x}', \tau)}{\partial t'} \\ &\approx \frac{\partial \vec{J}_s(\vec{x}_0', \tau_0)}{\partial t'} \times \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \, \frac{\beta_0'\tilde{u}\vec{x}_u' + \alpha_0'\tilde{v}\vec{x}_v'}{\tilde{u}^2 + \tilde{v}^2}, \end{aligned}\tag{8}$$

which reduces to

$$\begin{aligned}\vec{I}_2 &\approx \partial\vec{J}_s(\vec{x}_0', \tau_0)/\partial t' \times [C(1,0,2)\vec{x}_u'/\alpha_0' \\ &\quad + C(0,1,2)\vec{x}_v'/\beta_0']. \end{aligned}\tag{9}$$

The terms that come from the EFIE are

$$\begin{aligned}\vec{I}_3 &= \int_{S_2} dS' \frac{\vec{R}}{R^3} \rho_s(\vec{x}', \tau) \\ &\approx -\rho_s(\vec{x}_0', \tau_0) \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \left\{ \frac{\beta_0'\tilde{u}\vec{x}_u' + \alpha_0'\tilde{v}\vec{x}_v'}{(\tilde{u}^2 + \tilde{v}^2)^{3/2}} \right. \\ &\quad \left. \cdot \left[1 + \frac{K_3}{\alpha_0'\beta_0'} + \frac{3K_2}{2(\tilde{u}^2 + \tilde{v}^2)}\right] - \frac{\alpha_0'\beta_0'\vec{K}_1}{(\tilde{u}^2 + \tilde{v}^2)^{3/2}} \right\} \\ &\quad - \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\beta_0'\tilde{u}\vec{x}_u' + \alpha_0'\tilde{v}\vec{x}_v'}{(\tilde{u}^2 + \tilde{v}^2)^{3/2}} \\ &\quad \cdot \left\{ (\vec{x}_u'\tilde{u} + \vec{x}_v'\tilde{v}) \cdot \nabla'\rho_s(\vec{x}_0', \tau_0) + \frac{1}{2}\left[\left(\frac{(\Delta u_0)^2}{\alpha_0'^2} \right.\right.\right. \\ &\quad \left.\left.\left. + \frac{(\Delta v_0)^2}{\beta_0'^2}\right)^{1/2} - (\tilde{u}^2 + \tilde{v}^2)^{1/2}\right] \frac{\partial\rho_s(\vec{x}_0', \tau_0)}{\partial t'} \right\}, \end{aligned}\tag{10}$$

which is similar to the result in (7),

$$\begin{aligned}\vec{I}_4 &= \int_{S_2} dS' \frac{\vec{R}}{R^2} \frac{\partial\rho_s(\vec{x}', \tau)}{\partial t'} \\ &\approx -\frac{\partial\rho_s(\vec{x}_0', \tau_0)}{\partial t'} \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\beta_0'\tilde{u}\vec{x}_u' + \alpha_0'\tilde{v}\vec{x}_v'}{\tilde{u}^2 + \tilde{v}^2} \end{aligned}\tag{11}$$

similar to (9), and

$$\begin{aligned}\vec{I}_5 &= \int_{S_2} dS' \frac{1}{R} \frac{\partial\vec{J}_s(\vec{x}', \tau)}{\partial t'} \\ &\approx \frac{\partial\vec{J}_s(\vec{x}_0', \tau_0)}{\partial t'} \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha_0'\beta_0'}{(\tilde{u}^2 + \tilde{v}^2)^{1/2}} \\ &= C(0,0,1)\partial\vec{J}_s(\vec{x}_0', \tau_0)/\partial t'. \end{aligned}\tag{12}$$

Other integral equations related to electromagnetic scattering have the same type of terms.

For monochromatic fields, the expansions are somewhat different because the fields are independent of time and differentiation with respect to time is replaced by multiplication by $-i\omega$. The surface integrals remain unchanged and the contributions from neighboring patches involve the same $C(l, m, n)$.

III. NEIGHBORING-PATCH INTEGRALS ON A SPHERE

To obtain an estimate of the size of the error made by assuming that the integrand is constant over the patch, we compute some of the expressions that occur in these integrals. We consider two neighboring patches on a unit sphere, and we choose

$$\theta_0 = \pi/3, \phi_0 = 0, \theta_0' = 23\pi/60, \phi_0' = \pi/15, \Delta\theta_0 = \Delta\theta' = \pi/20,$$
$$\Delta\phi_0 = \Delta\phi' = \pi/15, \tag{13}$$

which gives

$$\alpha_0' = 1, \beta_0' = 0.934, \tilde{\theta}_+ = 0.236, \tilde{\theta}_- = 0.079, \tilde{\phi}_+ = 0.293,$$
$$\tilde{\phi}_- = 0.098, \tilde{\rho}_1 = 0.125, \tilde{\rho}_2 = 0.255, \tilde{\rho}_3 = 0.376,$$
$$\tilde{\rho}_4 = 0.304. \tag{14}$$

If we assume that the integrand of $\vec{I}_1$ in (4) is constant, we obtain

$$\vec{I}_1^c = \frac{\vec{R}_0}{R_0^3} \times \vec{J}_s(\vec{x}_0', \tau_0)\Delta S_2, \tag{15}$$

where $\Delta S_2 \approx \sin\theta_0'\Delta\theta'\Delta\phi'$ is the area of the patch S_2. The factor multiplying $\vec{J}_s$ is

$$\Delta S_2\vec{R}_0/R_0^3 \approx -0.099\hat{i} - 0.406\hat{j} + 0.296\hat{k}. \tag{16}$$

We compare this result with the zeroth-order terms in (7),

$$-C(1,0,3)\vec{x}_\theta'/\alpha_0' - C(0,1,3)\vec{x}_\phi'/\beta_0' = -0.036\hat{i} - 0.424\hat{j} + 0.321\hat{k}, \tag{17}$$

a vector that is close to the one in (16). A complete comparison would involve also the first-order terms, which include terms proportional to $\nabla_s'\vec{J}_s$ and $\partial\vec{J}_s/\partial t'$.

For $\vec{I}_2$ we compare

$$\Delta S_2\vec{R}_0/R_0^2 \approx -0.024\hat{i} - 0.099\hat{j} + 0.073\hat{k}, \tag{18}$$

with

$$-C(1,0,2)\vec{x}_\theta'/\alpha_0' - C(0,1,2)\vec{x}_\phi'/\beta_0' \approx -0.008\hat{i} - 0.099\hat{j} + 0.074\hat{k}, \tag{19}$$

which are very close. The forms of $\vec{I}_3$ and $\vec{I}_4$ are essentially those of $\vec{I}_1$ and $\vec{I}_2$. For $\vec{I}_5$ we get excellent agreement between

$$\Delta S_2/R_0 \approx 0.125 \tag{20}$$

and

$$C(0,0,1) \approx 0.126. \tag{21}$$

The importance of the discrepancies in these individual terms depends on the total number of patches that contribute to the integral and the relative size of the field and its derivatives in the different patches. In a stepping-in-time procedure, the number of patches that contribute to the integrals increases gradually starting from the time the wave first hits the scatterer, and we can have significant errors at early times.

IV. CONCLUSION

The method developed here allows us to compute more accurately the contributions to singular surface integrals from patches close to the self-patch in an arbitrary system of orthogonal curvilinear coordinates on the surface.

The error made by approximating the integrand by its value at the center of the patch is not negligible, and the need to take these corrections into account depends on the overall accuracy that is required, the method of integration, and the nature of the fields.

APPENDIX

In this Appendix, we give the details of the expansions of the surface fields and the computations of the neighboring-patch integrals.

We consider two patches, S_1 and S_2, close together on a surface S. The field point $\vec{x}$ is in S_1 and has coordinates u_0 and v_0. The source point $\vec{x}'$ is in S_2 and has coordinates u' and v'. The center $\vec{x}_0'$ of the patch S_2 has coordinates u_0' and v_0', and we define

$$\bar{u} = u' - u_0', \bar{v} = v' - v_0'. \tag{22}$$

We expand the functions about the point $\vec{x}_0'$; for instance,

$$\vec{x}' = \vec{x}_0' + \vec{x}_u'\bar{u} + \vec{x}_v'\bar{v} + \tfrac{1}{2}(\vec{x}_{uu}'\bar{u}^2 + 2\vec{x}_{uv}'\bar{u}\bar{v} + \vec{x}_{vv}'\bar{v}^2) + \cdots, \tag{23}$$

where the derivatives such as $\vec{x}_u'$ are evaluated at $\vec{x}_0'$. We then have

$$\vec{R} = \vec{x} - \vec{x}' = \vec{R}_0 - \vec{x}_u'\bar{u} - \vec{x}_v'\bar{v} - \tfrac{1}{2}(\vec{x}_{uu}'\bar{u}^2 + 2\vec{x}_{uv}'\bar{u}\bar{v} + \vec{x}_{vv}'\bar{v}^2) + \cdots. \tag{24}$$

We also expand $\vec{R}_0$ about $\vec{x}_0'$ and obtain

$$\vec{R}_0 = \vec{x} - \vec{x}_0' = -\vec{x}_u'\Delta u_0 - \vec{x}_v'\Delta v_0 + \tfrac{1}{2}[\vec{x}_{uu}'(\Delta u_0)^2 + 2\vec{x}_{uv}'\Delta u_0\Delta v_0 + \vec{x}_{vv}'(\Delta v_0)^2] + \cdots, \tag{25}$$

where

$$\Delta u_0 = u_0' - u_0, \Delta v_0 = v_0' - v_0. \tag{26}$$

The quantities Δu_0, Δv_0, $\bar{u}$, and $\bar{v}$ are all small, and we assume that they are of the same order of magnitude for neighboring patches. We combine (24) and (25) to obtain

$$\vec{R} = -(\Delta u_0 + \bar{u})\vec{x}_u' - (\Delta v_0 + \bar{v})\vec{x}_v' + \tfrac{1}{2}\{[(\Delta u_0)^2 - \bar{u}^2]\vec{x}_{uu}' + 2(\Delta u_0\Delta v_0 - \bar{u}\bar{v})\vec{x}_{uv}' + [(\Delta v_0)^2 - \bar{v}^2]\vec{x}_{vv}'\} + \cdots. \tag{27}$$

We use the expansion (25) for $\vec{R}_0$ because otherwise this constant vector remains in the integrals that have to be evaluated. These integrals become more complicated and they involve small constants that may cause problems in numerical computations. An expansion based on (24) instead of (27) may be useful in an intermediate region where the expansion (25) is inaccurate but $\vec{R}$ still varies significantly as $\vec{x}'$ ranges over S_2.

We define the new variables

$$\tilde{u} = \alpha_0'(\Delta u_0 + \bar{u}),\ \tilde{v} = \beta_0'(\Delta v_0 + \bar{v}), \tag{28}$$

where

$$\alpha_0' = |\vec{x}_u'|, \beta_0' = |\vec{x}_v'|, \tag{29}$$

and rewrite (27) as

$$\vec{R} \approx -\tilde{u}\vec{x}_u'/\alpha_0' - \tilde{v}\vec{x}_v'/\beta_0' + \vec{K}_1, \tag{30}$$

where

$$\vec{K}_1 = \tfrac{1}{2}[(2\tilde{u}\Delta u_0/\alpha_0' - \tilde{u}^2/\alpha_0'^2)\vec{x}_{uu}' + 2(\tilde{u}\Delta v_0/\alpha_0' + \tilde{v}\Delta u_0/\beta_0' - \tilde{u}\tilde{v}/\alpha_0'\beta_0')\vec{x}_{uv}' + (2\tilde{v}\Delta v_0/\beta_0' - \tilde{v}^2/\beta_0'^2)\vec{x}_{vv}']. \tag{31}$$

From this equation we obtain

$$R^2 \approx \tilde{u}^2 + \tilde{v}^2 - K_2, \tag{32}$$

where

$$K_2 = \kappa_1'(2\tilde{u}^2\Delta u_0/\alpha_0' - \tilde{u}^3/\alpha_0'^2) + \kappa_2'[2\tilde{u}^2\Delta v_0/\alpha_0' + (\tilde{u}\tilde{v}\Delta u_0 - \tilde{u}^2\tilde{v}/\alpha_0')(2/\beta_0' - 1/\alpha_0')] + \kappa_3'[2\tilde{v}^2\Delta u_0/\beta_0' + (\tilde{u}\tilde{v}\Delta v_0 - \tilde{u}\tilde{v}^2/\beta_0')(2/\alpha_0' - 1/\beta_0')] + \kappa_4'(2\tilde{v}^2\Delta v_0/\beta_0' - \tilde{v}^3/\beta_0'^2), \tag{33}$$

and

$$\kappa_1' = \vec{x}_u'\cdot\vec{x}_{uu}', \kappa_2' = \vec{x}_u'\cdot\vec{x}_{uv}', \kappa_3' = \vec{x}_v'\cdot\vec{x}_{uv}', \kappa_4' = \vec{x}_v'\cdot\vec{x}_{vv}'. \tag{34}$$

From (32) we obtain

$$R^{\nu} \approx (\tilde{u}^2 + \tilde{v}^2)^{\nu/2}[1 - (\nu/2)K_2/(\tilde{u}^2 + \tilde{v}^2)]. \tag{35}$$

A surface field, which is a function of the retarded time $\tau = t - R/c$, can also be expanded about the point $\vec{x}_0'$ at the time $\tau_0 = t - R_0/c$. For instance, the surface current density becomes

$$\begin{aligned} \vec{J}_s(\vec{x}', \tau) &= \vec{J}_s(\vec{x}_0', \tau_0) + (\vec{x}' - \vec{x}_0') \cdot \nabla' \vec{J}_s(\vec{x}_0', \tau_0) \\ &\quad + (\tau - \tau_0)\partial \vec{J}_s(\vec{x}_0', \tau_0)/\partial t' + \cdots \\ &\approx \vec{J}_s(\vec{x}_0', \tau_0) + (\vec{x}_u'\tilde{u} + \vec{x}_v'\tilde{v}) \cdot \nabla' \vec{J}_s(\vec{x}_0', \tau_0) \\ &\quad + \{[(\Delta u_0)^2/\alpha_0'^2 + (\Delta v_0)^2/\beta_0'^2]^{1/2} \\ &\quad - (\tilde{u}^2 + \tilde{v}^2)^{1/2}\}c^{-1}\partial \vec{J}_s(\vec{x}_0', \tau_0)/\partial t'. \end{aligned} \tag{36}$$

The expansion of the surface element dS' at the point $\vec{x}'$ is

$$\begin{aligned} dS' &= \alpha'\beta'\, du'\, dv' \\ &= \left[\alpha_0' + \left(\frac{\partial \alpha'}{\partial u'}\right)_0 \tilde{u} + \left(\frac{\partial \alpha'}{\partial v'}\right)_0 \tilde{v} + \cdots\right] \\ &\quad \cdot \left[\beta_0' + \left(\frac{\partial \beta'}{\partial u'}\right)_0 \tilde{u} + \left(\frac{\partial \beta'}{\partial v'}\right)_0 \tilde{v} + \cdots\right] du'\, dv' \\ &\approx (\alpha_0'\beta_0' + K_3)du'\, dv', \end{aligned} \tag{37}$$

where

$$\begin{aligned} K_3 &= (\kappa_1'\beta_0'/\alpha_0'^2 + \kappa_3'/\beta_0')\tilde{u} + (\kappa_2'/\alpha_0' + \kappa_4'\alpha_0'/\beta_0'^2)\tilde{v} \\ &\quad - (\kappa_1'\Delta u_0 + \kappa_2'\Delta v_0)\beta_0'/\alpha_0' - (\kappa_3'\Delta u_0 \\ &\quad + \kappa_4'\Delta v_0)\alpha_0'/\beta_0'. \end{aligned} \tag{38}$$

The quantities α' and β' are the magnitudes of the tangent vectors $\vec{x}_u$ and $\vec{x}_v$ at $\vec{x}'$, and their derivatives follow from (29) and (34); we evaluate them at the point $\vec{x}_0'$ and obtain

$$\begin{aligned} &(\partial\alpha'/\partial u')_0 = \kappa_1'/\alpha_0', (\partial\alpha'/\partial v')_0 = \kappa_2'/\alpha_0', (\partial\beta'/\partial u')_0 = \kappa_3'/\beta_0', \\ &(\partial\beta'/\partial v')_0 = \kappa_4'/\beta_0'. \end{aligned} \tag{39}$$

We need to be able to do integrals of the form

$$C(l, m, n) = \int_{\tilde{u}_-}^{\tilde{u}_+} d\tilde{u} \int_{\tilde{v}_-}^{\tilde{v}_+} d\tilde{v}\, \frac{\tilde{u}^l \tilde{v}^m}{(\tilde{u}^2 + \tilde{v}^2)^{n/2}}, \tag{40}$$

where

$$\tilde{u}_{\pm} = \alpha_0'(\Delta u_0 \pm \tfrac{1}{2}\Delta u'), \tilde{v}_{\pm} = \beta_0'(\Delta v_0 \pm \tfrac{1}{2}\Delta v'), \tag{41}$$

$\Delta u'$ and $\Delta v'$ being the coordinate increments of the patch S_2.

We do these integrals by transforming to polar coordinates in the $\tilde{u}\tilde{v}$-plane. For instance,

$$C(0, 0, 1) = \sum_{i=1}^{4} \int_{\tilde{\phi}_{i1}}^{\tilde{\phi}_{i2}} d\phi \int_0^{\rho_i} d\rho, \tag{42}$$

where the limits of integration are (see Fig. 2)

$$\begin{aligned} &\tilde{\phi}_{11} = \tilde{\phi}_{42} = \tilde{\phi}_1 = \arctan(\tilde{v}_-/\tilde{u}_-), \\ &\tilde{\phi}_{12} = \tilde{\phi}_{21} = \tilde{\phi}_2 = \arctan(\tilde{v}_-/\tilde{u}_+), \\ &\tilde{\phi}_{22} = \tilde{\phi}_{31} = \tilde{\phi}_3 = \arctan(\tilde{v}_+/\tilde{u}_+), \\ &\tilde{\phi}_{32} = \tilde{\phi}_{41} = \tilde{\phi}_4 = \arctan(\tilde{v}_+/\tilde{u}_-), \end{aligned} \tag{43}$$

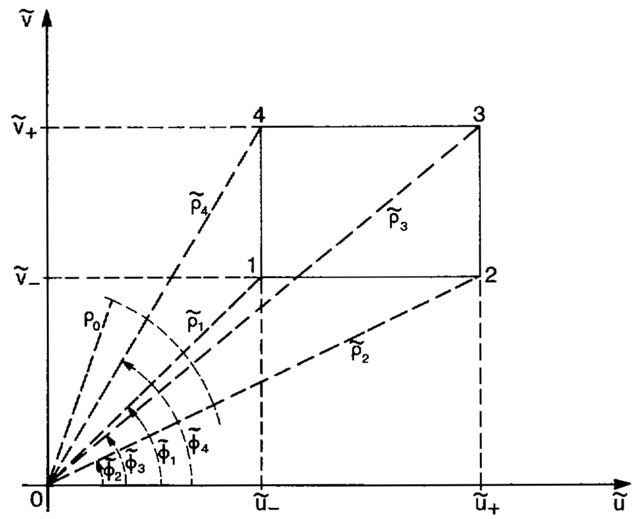

Fig. 2. Region of integration in the $\tilde{u}\tilde{v}$-plane.

$$\rho_1 = \tilde{v}_- \csc\phi, \rho_2 = \tilde{u}_+ \sec\phi, \rho_3 = \tilde{v}_+ \csc\phi, \rho_4 = \tilde{u}_- \sec\phi. \tag{44}$$

The integrations over ϕ can be done using the integrals

$$\int d\phi \sec\phi = \frac{1}{2}\log\frac{1 + \sin\phi}{1 - \sin\phi}, \tag{45}$$

$$\int d\phi \csc\phi = -\frac{1}{2}\log\frac{1 + \cos\phi}{1 - \cos\phi}, \tag{46}$$

and we obtain

$$\begin{aligned} C(0, 0, 1) &= \frac{\tilde{v}_-}{2}\log\frac{(\tilde{\rho}_1 + \tilde{u}_-)(\tilde{\rho}_2 - \tilde{u}_+)}{(\tilde{\rho}_1 - \tilde{u}_-)(\tilde{\rho}_2 + \tilde{u}_+)} \\ &\quad + \frac{\tilde{u}_+}{2}\log\frac{(\tilde{\rho}_3 + \tilde{v}_+)(\tilde{\rho}_2 - \tilde{v}_-)}{(\tilde{\rho}_3 - \tilde{v}_+)(\tilde{\rho}_2 + \tilde{v}_-)} \\ &\quad + \frac{\tilde{v}_+}{2}\log\frac{(\tilde{\rho}_3 + \tilde{u}_+)(\tilde{\rho}_4 - \tilde{u}_-)}{(\tilde{\rho}_3 - \tilde{u}_+)(\tilde{\rho}_4 + \tilde{u}_-)} \\ &\quad + \frac{\tilde{u}_-}{2}\log\frac{(\tilde{\rho}_1 + \tilde{v}_-)(\tilde{\rho}_4 - \tilde{v}_+)}{(\tilde{\rho}_1 - \tilde{v}_-)(\tilde{\rho}_4 + \tilde{v}_+)}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} &\tilde{\rho}_1 = (\tilde{u}_-^2 + \tilde{v}_-^2)^{1/2}, \tilde{\rho}_2 = (\tilde{u}_+^2 + \tilde{v}_-^2)^{1/2}, \\ &\tilde{\rho}_3 = (\tilde{u}_+^2 + \tilde{v}_+^2)^{1/2}, \rho_4 = (\tilde{u}_-^2 + \tilde{v}_+^2)^{1/2}. \end{aligned} \tag{48}$$

To avoid the divergence of $1/\rho$ at $\rho = 0$ (not in the original region of integration) when calculating $C(0, 0, 3)$, we start the integration at an arbitrary radius $\rho_0 < \tilde{\rho}_i$, $i = 1, 2, 3, 4$ so that

$$\begin{aligned} C(0, 0, 3) &= \sum_{i=1}^{4} \int_{\tilde{\phi}_{i1}}^{\tilde{\phi}_{i2}} d\phi \int_{\rho_0}^{\rho_i} \frac{d\rho}{\rho^2} \\ &= -\frac{\tilde{\rho}_1}{\tilde{u}_-\tilde{v}_-} + \frac{\tilde{\rho}_2}{\tilde{u}_+\tilde{v}_-} - \frac{\tilde{\rho}_3}{\tilde{u}_+\tilde{v}_+} + \frac{\tilde{\rho}_4}{\tilde{u}_-\tilde{v}_+}. \end{aligned} \tag{49}$$

We do the integrals $C(1, 0, 3)$ and $C(0, 1, 3)$ in rectangular coordinates, and obtain

$$C(1,0,3)=\log\frac{(\tilde\rho_4+\tilde v_+)(\tilde\rho_2+\tilde v_-)}{(\tilde\rho_3+\tilde v_+)(\tilde\rho_1+\tilde v_-)}, \tag{50}$$

$$C(0,1,3)=\log\frac{(\tilde\rho_2+\tilde u_+)(\tilde\rho_4+\tilde u_-)}{(\tilde\rho_3+\tilde u_+)(\tilde\rho_1+\tilde u_-)}. \tag{51}$$

We do $C(2, 0, 3)$, $C(1, 1, 3)$, and $C(0, 2, 3)$ in polar coordinates, and obtain

$$C(2,0,3)=-\frac{\tilde v_-}{2}\log\frac{(\tilde\rho_2+\tilde u_+)(\tilde\rho_1-\tilde u_-)}{(\tilde\rho_2-\tilde u_+)(\tilde\rho_1+\tilde u_-)}+\frac{\tilde v_+}{2}\log\frac{(\tilde\rho_3+\tilde u_+)(\tilde\rho_4-\tilde u_-)}{(\tilde\rho_3-\tilde u_+)(\tilde\rho_4+\tilde u_-)}, \tag{52}$$

$$C(1,1,3)=-\tilde\rho_1+\tilde\rho_2-\tilde\rho_3+\tilde\rho_4, \tag{53}$$

$$C(0,2,3)=-\frac{\tilde u_-}{2}\log\frac{(\tilde\rho_4+\tilde v_+)(\tilde\rho_1-\tilde v_-)}{(\tilde\rho_4-\tilde v_+)(\tilde\rho_1+\tilde v_-)}+\frac{\tilde u_+}{2}\log\frac{(\tilde\rho_3+\tilde v_+)(\tilde\rho_2-\tilde v_-)}{(\tilde\rho_3-\tilde v_+)(\tilde\rho_2+\tilde v_-)}. \tag{54}$$

Other integrals are computed in a similar way. Those needed for the expansions in Section II are

$$C(1,0,2)=\tilde u_+\arctan\frac{\tilde u_+(\tilde v_+-\tilde v_-)}{\tilde u_+^2+\tilde v_+\tilde v_-}+\tilde v_+\log\frac{\tilde\rho_3}{\tilde\rho_4}-\tilde u_-\arctan\frac{\tilde u_-(\tilde v_+-\tilde v_-)}{\tilde u_-^2+\tilde v_+\tilde v_-}-\tilde v_-\log\frac{\tilde\rho_2}{\tilde\rho_1}, \tag{55}$$

$$C(0,1,2)=\tilde v_+\arctan\frac{\tilde v_+(\tilde u_+-\tilde u_-)}{\tilde v_+^2+\tilde u_+\tilde u_-}+\tilde u_+\log\frac{\tilde\rho_3}{\tilde\rho_2}-\tilde v_-\arctan\frac{\tilde v_-(\tilde u_+-\tilde u_-)}{\tilde v_-^2+\tilde u_+\tilde u_-}-\tilde u_-\log\frac{\tilde\rho_4}{\tilde\rho_1}, \tag{56}$$

$$C(3,0,5)=\frac{1}{3}\left(\frac{\tilde v_-}{\tilde\rho_2}-\frac{\tilde v_+}{\tilde\rho_3}-\frac{\tilde v_-}{\tilde\rho_1}+\frac{\tilde v_+}{\tilde\rho_4}\right)+\frac{2}{3}\log\frac{(\tilde\rho_4+\tilde v_+)(\tilde\rho_2+\tilde v_-)}{(\tilde\rho_3+\tilde v_+)(\tilde\rho_1+\tilde v_-)}, \tag{57}$$

$$C(2,1,5)=\frac{1}{3}\left(\frac{\tilde u_+}{\tilde\rho_3}-\frac{\tilde u_+}{\tilde\rho_2}-\frac{\tilde u_-}{\tilde\rho_4}+\frac{\tilde u_-}{\tilde\rho_1}\right)+\frac{1}{3}\log\frac{(\tilde\rho_2+\tilde u_+)(\tilde\rho_4+\tilde u_-)}{(\tilde\rho_3+\tilde u_+)(\tilde\rho_1+\tilde u_-)}, \tag{58}$$

$$C(1,2,5)=\frac{1}{3}\left(\frac{\tilde v_+}{\tilde\rho_3}-\frac{\tilde v_+}{\tilde\rho_4}-\frac{\tilde v_-}{\tilde\rho_2}+\frac{\tilde v_-}{\tilde\rho_1}\right)+\frac{1}{3}\log\frac{(\tilde\rho_4+\tilde v_+)(\tilde\rho_2+\tilde v_-)}{(\tilde\rho_3+\tilde v_+)(\tilde\rho_1+\tilde v_-)}, \tag{59}$$

$$C(0,3,5)=\frac{1}{3}\left(\frac{\tilde u_-}{\tilde\rho_4}-\frac{\tilde u_+}{\tilde\rho_3}-\frac{\tilde u_-}{\tilde\rho_1}+\frac{\tilde u_+}{\tilde\rho_2}\right)+\frac{2}{3}\log\frac{(\tilde\rho_2+\tilde u_+)(\tilde\rho_4+\tilde u_-)}{(\tilde\rho_3+\tilde u_+)(\tilde\rho_1+\tilde u_-)}, \tag{60}$$

$$C(4,0,5)=\tilde v_-\left[\frac{\tilde u_+^3}{3\tilde\rho_2^3}-\frac{\tilde u_-^3}{3\tilde\rho_1^3}-\frac{1}{2}\log\frac{(\tilde\rho_2+\tilde u_+)(\tilde\rho_1-\tilde u_-)}{(\tilde\rho_2-\tilde u_+)(\tilde\rho_1+\tilde u_-)}\right]+\tilde v_+\left[\frac{\tilde u_-^3}{3\tilde\rho_4^3}-\frac{\tilde u_+^3}{3\tilde\rho_3^3}-\frac{1}{2}\log\frac{(\tilde\rho_4+\tilde u_-)(\tilde\rho_3-\tilde u_+)}{(\tilde\rho_4-\tilde u_-)(\tilde\rho_3+\tilde u_+)}\right]-\tilde u_+\left(\frac{\tilde v_+^3}{3\tilde\rho_3^3}-\frac{\tilde v_-^3}{3\tilde\rho_2^3}\right)-\tilde u_-\left(\frac{\tilde v_-^3}{3\tilde\rho_1^3}-\frac{\tilde v_+^3}{3\tilde\rho_4^3}\right), \tag{61}$$

$$C(3,1,5)=-\tilde v_-\left(\frac{\tilde v_-^3}{3\tilde\rho_2^3}-\frac{\tilde v_-^3}{3\tilde\rho_1^3}-\frac{\tilde v_-}{\tilde\rho_2}+\frac{\tilde v_-}{\tilde\rho_1}\right)-\tilde u_+\left(\frac{\tilde u_+^3}{3\tilde\rho_3^3}-\frac{\tilde u_+^3}{3\tilde\rho_2^3}\right)-\tilde v_+\left(\frac{\tilde v_+^3}{3\tilde\rho_4^3}-\frac{\tilde v_+^3}{3\tilde\rho_3^3}-\frac{\tilde v_+}{\tilde\rho_4}+\frac{\tilde v_+}{\tilde\rho_3}\right)-\tilde u_-\left(\frac{\tilde u_-^3}{3\tilde\rho_1^3}-\frac{\tilde u_-^3}{3\tilde\rho_4^3}\right), \tag{62}$$

$$C(2,2,5)=-\tilde v_-\left(\frac{\tilde u_+^3}{3\tilde\rho_2^3}-\frac{\tilde u_-^3}{3\tilde\rho_1^3}\right)+\tilde u_+\left(\frac{\tilde v_+^3}{3\tilde\rho_3^3}-\frac{\tilde v_-^3}{3\tilde\rho_2^3}\right)-\tilde v_+\left(\frac{\tilde u_-^3}{3\tilde\rho_4^3}-\frac{\tilde u_+^3}{3\tilde\rho_3^3}\right)+\tilde u_-\left(\frac{\tilde v_-^3}{3\tilde\rho_1^3}-\frac{\tilde v_+^3}{3\tilde\rho_4^3}\right), \tag{63}$$

$$C(1,3,5)=-\tilde u_-\left(\frac{\tilde u_-^3}{3\tilde\rho_4^3}-\frac{\tilde u_-^3}{3\tilde\rho_1^3}-\frac{\tilde u_-}{\tilde\rho_4}+\frac{\tilde u_-}{\tilde\rho_1}\right)-\tilde v_+\left(\frac{\tilde v_+^3}{3\tilde\rho_3^3}-\frac{\tilde v_+^3}{3\tilde\rho_4^3}\right)-\tilde u_+\left(\frac{\tilde u_+^3}{3\tilde\rho_2^3}-\frac{\tilde u_+^3}{3\tilde\rho_3^3}-\frac{\tilde u_+}{\tilde\rho_2}+\frac{\tilde u_+}{\tilde\rho_3}\right)-\tilde v_-\left(\frac{\tilde v_-^3}{3\tilde\rho_1^3}-\frac{\tilde v_-^3}{3\tilde\rho_2^3}\right), \tag{64}$$

$$C(0,4,5)=\tilde u_-\left[\frac{\tilde v_+^3}{3\tilde\rho_4^3}-\frac{\tilde v_-^3}{3\tilde\rho_1^3}-\frac{1}{2}\log\frac{(\tilde\rho_4+\tilde v_+)(\tilde\rho_1-\tilde v_-)}{(\tilde\rho_4-\tilde v_+)(\tilde\rho_1+\tilde v_-)}\right]+\tilde u_+\left[\frac{\tilde v_-^3}{3\tilde\rho_2^3}-\frac{\tilde v_+^3}{3\tilde\rho_3^3}-\frac{1}{2}\log\frac{(\tilde\rho_2+\tilde v_-)(\tilde\rho_3-\tilde v_+)}{(\tilde\rho_2-\tilde v_-)(\tilde\rho_3+\tilde v_+)}\right]-\tilde v_+\left(\frac{\tilde u_+^3}{3\tilde\rho_3^3}-\frac{\tilde u_-^3}{3\tilde\rho_4^3}\right)-\tilde v_-\left(\frac{\tilde u_-^3}{3\tilde\rho_1^3}-\frac{\tilde u_+^3}{3\tilde\rho_2^3}\right). \tag{65}$$

A relationship that is useful for deriving or verifying these integrals is

$$C(l+2,m,n+2)+C(l,m+2,n+2)=C(l,m,n). \quad (66)$$

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Egon Marx, for a photograph and biography please see page 172 of the February 1984 issue of this TRANSACTIONS.

TM Scattering by a Dielectric Cylinder in the Presence of a Half-Plane

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Abstract—**The problem considered is the transverse magnetic (TM) scattering by a dielectric cylinder in the presence of a perfectly conducting half-plane. An integral equation, involving the half-plane Green's function in its kernel, is obtained for the equivalent volume currents representing the dielectric cylinder. This integral equation is solved by the method of moments. Numerical results are compared with measurements for the echo width of a dielectric slab on a half-plane. The dielectric slab surface impedance and the fields inside the dielectric are also shown.**

I. INTRODUCTION

THIS PAPER WILL present a method of moments (MM) solution [1] to the two-dimensional problem of transverse magnetic (TM) scattering by a dielectric cylinder of arbitrary cross section near a perfectly conducting half-plane. In a conventional MM solution to this problem, one replaces all matter (half-plane and dielectric) by free space and equivalent currents. The advantage of this approach is that the resulting integral equation for the currents will involve the relatively simple free-space Green's function. A second approach is to replace only a portion of the matter by free space and equivalent currents. The advantage of this approach is that the number of unknowns in the MM solution is reduced. The disadvantage is that the resulting integral equation will involve the Green's function for that part of the matter which was not replaced by free space and equivalent currents. This second approach can be termed an MM/Green's function solution. When the geometrical theory of diffraction (GTD) is used as a high frequency approximation to the Green's function, it is termed an MM/GTD solution [2]. Here we employ the second approach. In particular, only the dielectric cyclinder will be replaced by free space and equivalent currents. As a result, the integral equation for the electric volume polarization currents representing the dielectric cylinder will involve the half-plane Green's function in its kernel. We chose to employ this MM/Green's function approach since the half-plane Green's function is well-known and available in a computationally efficient form.

Many authors have studied the problem of scattering by a dielectric cylinder. For example, Richmond presented an MM solution for scattering by a dielectric cylinder of arbitrary cross-section shape, for TM [3] and transverse electric (TE) [4] polarizations, and using the volume current approach. Scattering by dielectric cylinders using the surface current approach has been studied by Morita [5] and by Wu and Tsai [6]. Other approaches include Waterman's extended boundary condition method [7] and the unimoment method by Chang and Mei [8]. The work described here is new in that if includes the effects of a perfectly conducting half-plane in the vicinity of the dielectric cylinder. The method applies for the cases where the dielectric cylinder contacts the surface of the half-plane, or even completely surrounds the half-plane edge.

In Section II the integral equation for the equivalent volume polarization currents is obtained. An MM solution to this equation is presented. The half-plane Green's function is given, and efficient methods for its evaluation are discussed. Section III defines the details of the pulse basis and point matching MM solution. A comparison between measured and computed echo width of a rectangular dielectric cylinder on a half-plane is given. Computed values for the electric fields inside the dielectric, as well as the surface impedance of a dielectric slab on the half-plane edge are also shown.

II. THEORY

A. Derivation of the Integral Equation

This section will develop an integral equation for the two-dimensional scattering by a dielectric cylinder in the vicinity of a perfectly conducting half-plane. Fig. 1(a) shows a perfectly conducting half-plane located at $x \geqslant 0$ and $y = 0$. Confined to the region R is a dielectric cylinder with permeability and permittivity (μ_0, ϵ). The impressed electric and magnetic currents are denoted $(\mathbf{J}_i, \mathbf{M}_i)$. The ambient medium is homogeneous with parameters (μ_0, ϵ_0). All fields and currents are two-dimensional

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