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Nekhoroshev stability estimates for different models of the Trojan asteroids

Christos Efthymiopoulos

Research Center for Astronomy and Applied Mathematics, Academy of Athens
Soranou Efessiou 4, 115 27 Athens, Greece
e-mail: cefthim@cc.uoa.gr

Abstract. Estimates of the region of Nekhoroshev stability of Jupiter's Trojan asteroids are obtained by a direct (i.e. without use of the normal form) construction of formal integrals near the Lagrangian elliptic equilibrium points. Formal integrals are constructed in the Hamiltonian model of the planar circular restricted three body problem (PCRTBP), and in a mapping model (Sándor *et al.* 2002) of the same problem for small orbital eccentricities of the asteroids. The analytical estimates are based on the calculation of the size of the remainder of the formal series by a computer program. An analysis is made of the accumulation of small divisors in the series. The most important divisors introduce competing Fourier terms with sizes growing at similar rates as the order of truncation increases. This makes impossible to improve the estimates by considering nearly resonant forms of the formal integrals for particular near-resonances. Improved estimates were obtained in a mapping model of the PCRTBP. The main source of improvement is the use of better variables (Delaunay). Our best estimate represents a maximum libration amplitude $D_p = 10.6^0$. This is a quite realistic value which demonstrates the usefulness of Nekhoroshev theory.

Keywords. Trojan asteroids; Nekhoroshev stability

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1. Introduction

Stability estimates in nonlinear Hamiltonian dynamical systems are obtained by applications of either the KAM theorem (Kolmogorov 1954, Arnold 1963a,b, Moser 1962), or the Nekhoroshev theorem (Nekhoroshev 1977, Benettin et al. 1985, , Lochak 1992, Pöschel 1993). A connection between the two theorems is provided by the theorem of superexponential stability (Morbidelli and Giorgilli 1995a,b).

The question of obtaining analytical stability estimates for the orbits in specific subsystems of the solar system has raised considerable interest in recent years. In the case of KAM stability, the goal is to prove theoretically the existence of a KAM torus with fixed frequencies. Stability is guaranteed for all times for orbits with initial conditions on the torus. In the case of two degrees of freedom systems, KAM stability implies also the stability of all the orbits with initial conditions in an open domain inside the last KAM librational torus. This is no longer true for systems of three or more degrees of freedom, because of the a priori possibility of chaotic orbits to exhibit Arnold diffusion. Estimates of KAM stability refer to upper bounds with respect to either a) the distance from a local equilibrium solution, or b) the size of the effective perturbation (e.g. the mass parameter or the eccentricity) for which the existence of a torus is guaranteed. Examples of this approach, in Celestial Mechanics, were given by Robutel (1995), Celletti and Chierchia (1997, 1998), Locatelli (1998), and Locatelli and Giorgilli (2000).

Nekhoroshev stability, on the other hand, is applicable to all the orbits in open domains of initial conditions, independently of whether a particular orbit is regular, i.e., lies on a KAM torus, or chaotic. The Nekhoroshev formula predicts a finite time of stability for both types of orbits, regular or chaotic. This time is a lower limit, i.e., an underestimate of the real time of stability.

Nekhoroshev estimates applied to solar system dynamics refer to a) analytical upper bounds of the perturbation parameter so that the conditions for the application of Nekhoroshev theorem are fulfilled (e.g. Benettin et al. 1998), b) numerical verifications that there are open sets of the Arnold web being in a 'Nekhoroshev regime', i.e., where resonances do not overlap (Guzzo and Morbidelli 1997, Guzzo et al. 2000), and, finally c) estimates of the size of the stability region for times equal to the age of the solar system. A particular example of the latter approach are the estimates of the stability region of the Trojan asteroids in the neighborhood of Jupiter's triangular points given by Simó (1989), Celletti and Giorgilli (1991), Giorgilli and Skokos (1997) and Skokos and Dokoumentzidis (2001).

The present paper reports Nekhoroshev stability estimates for the Trojan asteroids obtained by calculating formal integrals in the neighborhood of the triangular points with a direct method due to Whittaker (1916), Cherry (1924) and Contopoulos (1960). The Hamiltonian model is the planar circular restricted three body problem. Then the discrete analog of the direct method is used to construct formal integrals in a mapping model (Sándor et al. 2002) which describes the dynamics of the PCRTBP for low orbital eccentricities of the asteroids. Nekhoroshev estimates are obtained for this mapping model which are expressed in terms of the proper elements of the orbits. Our best result corresponds to a libration amplitude $D_p = 10.6^o$ which is quite realistic (most asteroids have $D_p < 35^o$).

2. Nekhoroshev estimates in the Hamiltonian model

Following Giorgilli and Skokos (1997), the Hamiltonian of the planar, circular restricted three-body problem is written in heliocentric polar canonical coordinates $(\rho, \theta, p_\rho, p_\theta)$ as:

$$H = \frac{1}{2}\left(p_\rho^2 + \frac{p_\theta^2}{\rho^2}\right) - p_\theta - \mu\rho\cos\theta - \frac{1-\mu}{\rho} - \frac{\mu}{\sqrt{\rho^2+1+2\rho\cos\theta}} \tag{2.1}$$

The coordinates of L4 are $\rho = 1$, $\theta = 2\pi/3$, $p_\rho = 0$, $p_\theta = 1$, and the mass parameter $\mu = 0.00095387536$.

Following the canonical change of variables:

$$\rho = 1 + 1.00599x_1 + 0.329451x_2 \tag{2.2}$$

$$\theta = \frac{2\pi}{3} + 0.00125186x_1 + 0.0473548x_2 + 2.01417y_1 - 6.15034y_2$$

$$p_\rho = 1.00273y_1 - 0.0265079y_2$$

$$p_\theta = 1 + 1.00599x_1 + 0.329451x_2 \tag{2.3}$$

the Hamiltonian (2.1) can be expanded as

$$H(x_1, x_2, y_1, y_2) = H_2 + H_3 + \ldots \tag{2.4}$$

where the quadratic part H_2 is given by

$$H_2 = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) \tag{2.5}$$

i.e., it is diagonal in the canonically conjugate pairs (x_i, y_i), $i = 1, 2$. The frequencies are $\omega_1 = 0.99675752552$ and $\omega_2 = -0.080463875837$. The time unit is equal to $T_J/2\pi$, with

$T_J = 11.84yr$. The period $T_1 = 2\pi/\omega_1$ corresponds to the period of oscillation of the quantity $\omega - M'$ in the linearized approximation around L4, where ω is the argument of the perihelion of the asteroid and M' is the mean anomaly of Jupiter. Since the precession of the perihelion is very slow, the frequency ω_1 is very close to 1, i.e., to the mean motion of Jupiter. On the other hand, the frequency ω_2 corresponds to the frequency of oscillation of the major semi-axis of the asteroid in the linearized approximation around L4, which is about twelve times smaller than the frequency ω_1.

A second formal integral $\Phi = \Phi_2 + \Phi_3 + \ldots$ of the hamiltonian (2.4) can be constructed by solving recursively the equations:

$$D_\omega \Phi_s = -\{\Phi_{s-1}, H_3\}, \qquad D_\omega \cdot = \{\cdot, H_2\} \tag{2.6}$$

where $\{\cdot,\cdot\}$ denotes Poisson brackets and the linear differential operator D_ω is defined as

$$D_\omega = \{\cdot, H_2\} \tag{2.7}$$

The construction of a formal integral starts with a particular choice for the second order term, i.e., $\Phi_{i,2} = \frac{1}{2}(y_i^2 + x_i^2)$, where $i = 1$ or $i = 2$. Then, Eq. (2.6) can be solved order by order and it ensures that the quantity $\Phi_i = \Phi_{i,2} + \Phi_{i,3} + \ldots$ has zero Poisson bracket with the Hamiltonian, i.e. it is a formal integral of motion. This 'direct' method of calculating the integrals step by step via Eq.(2.6) (Whittaker 1916, Cherry 1924, Contopoulos 1960) is as old as the Birkhoff - Gustavson method (Birkhoff 1927, Gustavson 1966) of determining the formal integrals via normal forms. Its extension to deal with resonance cases was given by Contopoulos (1963).

The series Φ is, in general, divergent (Siegel 1941). However, a proper truncation of it, say at order N, gives a polynomial function $\Phi^{(N)}$ which represents an approximate integral of the Hamiltonian H. The time derivative of the truncated series is given by

$$\frac{d\Phi^{(N)}}{dt} = R^{(N)} = \sum_{j=N+1}^{\infty} U_j \tag{2.8}$$

where

$$U_r = \sum_{k=2}^{N} \{\Phi_k, H_{r-k}\}, \qquad r > N \tag{2.9}$$

It can be shown that the series $R^{(N)}$, called the remainder of the integral, is convergent (Giorgilli 1988). The size of the remainder can be bounded from above by a quantity (Giorgilli 1988)

$$||R^{(N)}|| \leqslant \frac{AN!}{\prod_{s=2}^{N} a_s} \tag{2.10}$$

where the norm $||\cdot||$ is defined as the sum of the moduli of the polynomial coefficients, A is a positive constant, depending essentially on the size of the Hamiltonian perturbation $H_3 + \ldots$, and the sequence a_s, $s = 2, 3, \ldots$ refers to the smallest divisors that may appear in the formal series at order s. These are given by

$$a_s = \min\{|k\cdot\omega|,\ k \equiv (k_1,k_2),\ \omega \equiv (\omega_1,\omega_2),\ |k_1|+|k_2| \leqslant s, (|k_1|+|k_2|) mod 2 = s mod 2\} \tag{2.11}$$

where the modulo 2 restriction comes from the fact that the order $|k_1| + |k_2|$ of any Fourier mode $\exp(ik\cdot\phi)$ in the formal series at order s has the same parity as s.

We say that the frequencies ω_i satisfy a diophantine condition, if there are positive constants γ, τ such that $|k\cdot\omega| \geqslant \gamma/|k|^\tau$ for all $k \equiv (k_1,k_2)$ with $k_1, k_2 \in \mathbb{Z}$, $|k| \equiv |k_1| + |k_2| \neq 0$ and $\tau \geqslant n-1$ where n is the number of degrees of freedom. In the diophantine case, the sequence a_s satisfies the estimate $a_s \sim 1/s^\tau$. This will be called a 'diophantine sequence'. The product $\prod_{s=2}^N a_s$ in Eq.(2.10) is estimated as $\sim 1/N!^\tau$. Thus, at distance ρ from the equilibrium point, Eq.(2.10) yields the estimate

$$||R_N||_\rho \leqslant BN!^{\tau+1}\rho^N \tag{2.12}$$

where $||R_N||_\rho = ||R_N||\rho^N$ measures the size of the remainder terms at the distance ρ, and B is a positive constant. The optimal estimate is found by finding the optimal order of truncation N for which the r.h.s. of (2.12) has a minimum with respect to N. This minimum value is $O(\exp(-1/\rho^{\frac{1}{\tau+1}}))$. Thus, in view of Eq.(2.8), the time of stability is exponentially long in the inverse of the distance ρ, i.e.,

$$T \sim O(\exp(1/\rho^{\frac{1}{\tau+1}})) \tag{2.13}$$

If T is fixed, say the age of the solar system, Eq.(2.13) can be used to estimate the size of the Nekhoroshev stability region around the equilibrium points.

It should be stressed that the above rigorous estimates are rather pessimistic. A rigorous improvement of these estimates was given by Fassò et al. (1998), Guzzo et al. (1998) and Niederman (1998). On the other hand, a careful analysis of the accumulation of small divisors in the formal series (Efthymiopoulos et al. 2004) has shown that this cannot be in the form of products $\prod_{s=2}^N a_s$. Let d_s $s = 2,\dots N$ denote the sequence of divisors appearing in the fastest growing terms of the remainder. Then, at most every second divisor can satisfy the equality $d_s = a_s$, i.e., be equal to the minimum divisor at the same order s. This leads to an improved estimate $R_{opt} \sim O(\exp(-1/\rho^{\frac{2}{\tau+1}}))$ which is in close agreement with estimates found by computer experiments (Contopoulos et al. 2003, Efthymiopoulos et al. 2004). But even this formula is a simplification, because the sequence a_s gives divisors equal to the diophantine limit γ/s^τ only at particular orders s determined by the continued fraction approximation of the frequency ratio ω_2/ω_1. A further complication is introduced by the 'inversion' and 'delay' effects (Efthymiopoulos et al. 2004).

On the other hand, it is possible to provide Nekhoroshev estimates by calculating precisely the size of the remainder with a computer program performing the algebraic manipulations. This allows one to determine the size of the region of stability in terms of a radius ρ given by $\rho = \min(\rho_1, \rho_2)$, where the radii ρ_1 and ρ_2 correspond to disks around the origin in the subspaces (x_1, y_1) and (x_2, y_2) respectively defined so that the variations of the corresponding formal integrals do not exceed an upper limit (Giorgilli and Skokos 1997). The final stability region is defined as the interior of a torus corresponding to the product of two circles of radius ρ. In the present calculations, the precise values of the radii ρ_1 and ρ_2 are specified by the same procedure as in Giorgilli and Skokos (1997). The radius ρ is given essentially by the formula

$$\rho \sim \left(\frac{1}{TCs||R_{s+1}^{(s)}||}\right)^{1/s} \tag{2.14}$$

where $T = 10^9$ (in periods of Jupiter) is the Nekhoroshev time, $||R_{s+1}^{(s)}||$ denotes the size of the first order of the remainder $R^{(s)}$, and C is a constant equal to the ratio of the total size of the remainder $||R^{(s)}||$ over $||R_{s+1}^{(s)}||$. The radius ρ is, thus, a function of the order of truncation s. The optimal order of truncation N_{opt} corresponds to the order $s = N_{opt}$ at which ρ, given by (2.14), is maximum.

By calculating the formal integrals as described above we found $N_{opt} = 34$ and $\rho = 0.0304$. This radius is marginally better than the radius found by computer

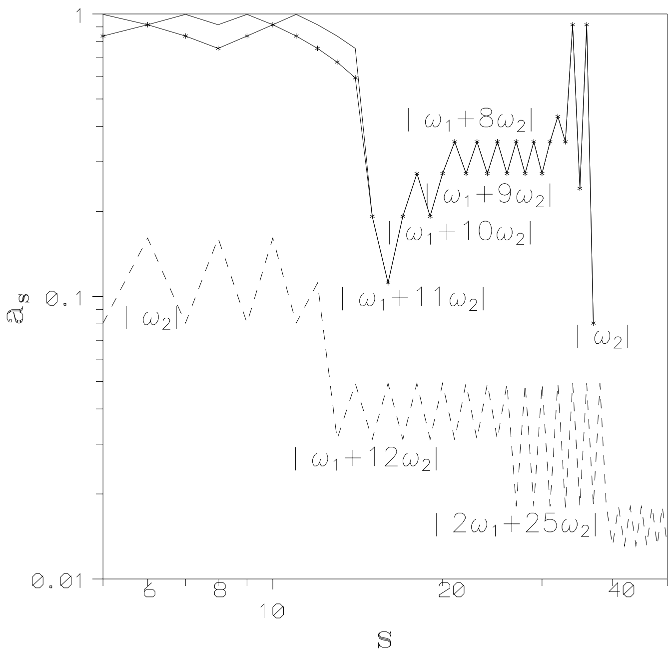

Figure 1. The sequence of small divisors a_s corresponding to the continued fraction expansion of the frequency ratio $a = \omega_2/\omega_1$, as a function of the order of truncation s (dashed line). The solid lines correspond to the sequence of divisors $|k\cdot\omega|$ for which the corresponding Fourier term $\exp(ik\cdot\phi)$ is the leading term of the remainder at order s, for the formal integrals Φ_1 (solid line with stars) and Φ_2 (solid line without stars).

implementation of the normal form method (Giorgilli and Skokos 1997). In physical units, it represents about 1/10 of the distance from L4 to Jupiter.

We examine now how do small divisors accumulate in the formal series so as to produce the observed asymptotic behavior of the series.

In the case of the Trojan problem, the frequency ratio $a = |\omega_2/\omega_1| = 0.0807256266212$ is expressed as a continued fraction expansion $a = [12, 2, 1, 1, 2, \ldots]$, with rational truncations $0/1, 1/12, 2/25, 3/37, 5/62, \ldots$. As shown in Figure 1, as the order of the expansion s increases, newborn small divisors $|k_1\omega_1 + k_2\omega_2|$ appear at the orders $s = k_1 + k_2$, where k_1/k_2 belongs to the sequence of rational truncations of a. Thus, newborn small divisors appear at the orders $1 = 1 + 0$, $13 = 1 + 12$, $27 = 2 + 25$, $40 = 3 + 37$, etc.

Nevertheless, a detailed analysis of the leading Fourier modes in the series shows that the divisors belonging to the diophantine sequence a_s are not the most important, in the sense that they are not the ones producing the terms of maximum size in the series. The solid lines in Figure 1 represent the sequences of divisors $|k\cdot\omega|$ corresponding to the dominant Fourier modes $\exp ik\cdot\phi$ in the remainder (where $\phi \equiv (\phi_1, \phi_2)$ are canonical angles), at successive orders of truncation s of the formal integrals $\Phi_1 = (x_1^2+y_1^2)/2+\ldots$, and $\Phi_2 = (x_2^2 + y_2^2)/2 + \ldots$. After a few transient steps, the two lines (for Φ_1 and Φ_2) coincide, beyond the order $s = 14$. This implies that, as s increases, the same Fourier modes become dominant at successive steps in the two series.

It can be observed that from order $s = 14$ to $s = 33$, there is a sequence of divisors with values of the same order of magnitude $\sim 10^{-1}$, namely $|\omega_1+11\omega_2| = 0.1116$, $|\omega_1+10\omega_2| = 0.1921$, $|\omega_1 + 9\omega_2| = 0.2726$, $|\omega_1 + 8\omega_2| = 0.3505$, which produce dominant Fourier terms in the series, namely the terms $\exp(i(\phi_1 + 11\phi_2))$, $\exp(i(\phi_1 + 10\phi_2))$, $\exp(i(\phi_1 + 9\phi_2))$, and $\exp(i(\phi_1 + 8\phi_2))$ respectively. At all orders s in the interval $14 \leqslant s \leqslant 33$, these

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terms have comparable size in the series. On the other hand, the size of the Fourier term corresponding to the diophantine divisor $a_{27} = |2\omega_1 + 25\omega_2| = 0.01808$ is much smaller than the size of the previous Fourier terms, at least up to order 40 (which is beyond the optimal order of truncation of the series). This is despite the fact that the divisor a_{27} is one order of magnitude smaller ($\sim 10^{-2}$) than the above listed divisors, which are of order 10^{-1}. This phenomenon is caused by the 'delay' mechanism (Efthymiopoulos et al. 2004). Namely, although the Fourier term $\exp(i(2\phi_1 + 25\phi_2))$ is the fastest growing term beyond order $s = 27$, when this term appears for the first time, at order $s = 27$, it has size much smaller than the size of other Fourier terms which are temporarily dominant at $s = 27$, and many iterations of the recurrent relation (2.6) are required before the size of the term $\exp(2\phi_1 + 25\phi_2)$ becomes dominant. This phenomenon appears even for good diophantine frequency ratios, because the intervals of orders s separating the appearance of successive new small denominators increases exponentially with s.

Another phenomenon observed near the order $s = 27$ is 'inversion'. Namely, a Fourier term with relatively large divisor becomes temporarily dominant while other terms, with smaller divisors, are temporarily less important. This is shown as two consecutive peaks of the solid lines at orders $s = 32$ and $s = 34$ (Figure 1).

The fact that many Fourier terms, with small divisors of similar size, become successively dominant in a short interval of values of s, implies that it is not possible to obtain better Nekhoroshev stability estimates by considering near-resonant forms of the formal integrals innstead of non-resonant formal integrals. In fact, a near-resonant construction deals with only one near-resonance at the time, eliminating the effect of the small divisors associated with this resonance. This can improve the estimates up to a particular order of truncation s, on the condition that there are no other near-resonances producing terms of considerable size up to order s. But figure 1 shows that this condition is not fulfilled in our case.

3. Nekhoroshev estimates with Delaunay variables in a mapping model

The most relevant variables for the description of the stability region around L4 and L5 are the Delaunay action-angle variables

$$x = \sqrt{\frac{\alpha}{\alpha'}} - 1, \quad \tau = \lambda - \lambda'$$
$$x_2 = \sqrt{\frac{\alpha}{\alpha'}}\left(\sqrt{1-e^2} - 1\right), \varpi \tag{3.1}$$

where α, e are the semi-major axis and eccentricity of the asteroid, α' is the semi-major axis of Jupiter, τ is the critical argument, i.e., the difference between the mean longitude of the asteroid and of Jupiter, and ϖ is the longitude of the pericenter of the asteroid. These variables have the advantage that the pair of action-angle variables x, τ are immediately translated in the motion of the asteroid in configuration space. Namely, x gives the amplitude of librations perpendicularly to the circle of the 1:1 coorbital motion, while $\tau - \tau_0$ measures the synodic libration around the equilibrium values $\tau_0 = \pi/3$ (for L4) or $5\pi/3$ (for L5). If both librations are considered nearly harmonic, then, following Érdi (1988), the amplitude of librations is measured by the parameter D_p given by

$$\Delta\alpha = \alpha - \alpha' \simeq 2\alpha' x \simeq \sqrt{3\mu}\alpha' D_p \sin\phi, \qquad \Delta\tau = \tau - \tau_0 \simeq D_p \cos\phi \tag{3.2}$$

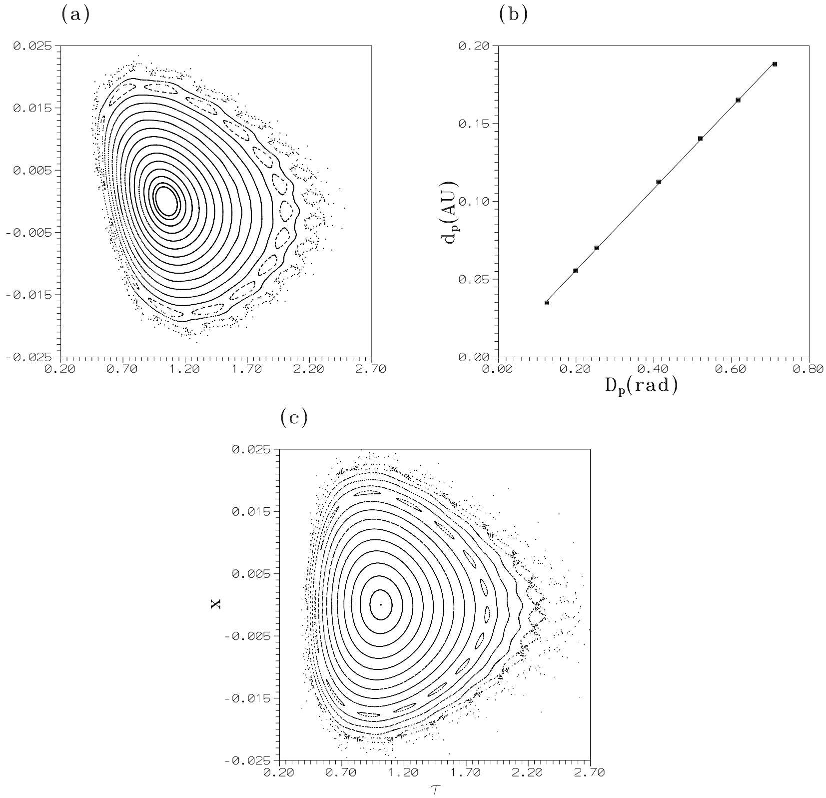

Figure 2. (a) The phase portrait of the mapping (3.3). (b) The relation d_p versus D_p (see text for definitions) as found by the invariant curves of the mapping (3.3). (c) The Poincaré surface of section for the equations of motion under the full Hamiltonian of the planar circular restricted three-body problem. The section is given by the variables (τ, x) when the surface $\varpi - \lambda' = 0$ is crossed by an orbit at the positive sense. The Jacobi constant is taken equal to $E = -1.49948$, which is very close to the Jacobi constant at $L4$, $E_{L4} = -1.49952$. This corresponds to eccentricities $e < 0.04$.

where ϕ is the phase of a libration, which is defined by the position of the guiding center of the motion of the asteroid along a tadpole-shaped orbit. According to Eq.(3.2) the parameter D_p is the amplitude of libration of the critical argument. For most asteroids the amplitude of libration D_p ranges from a few degrees up to about $D_p \simeq 35^o$ (Milani 1993, Érdi 1997, Levison et al. 1997), while in general D_p decreases as the eccentricity increases.

The usual analytic expansion of the Hamiltonian (2.1) cannot be transformed to an expansion in the variables (3.1) with analytically calculated coefficients. The most relevant expansion is the semi-analytic expansion of Beaugé and Roig (2001), which gives the averaged Hamiltonian over fast angles. A way to circumvent this problem is by using the 2D mapping model of Sándor et al. (2002) which accurately reproduces the dynamics at low orbital eccentricities of the asteroids. At the limit of zero eccentricity, the mapping reads

$$x_{n+1} = x_n + 2\pi\mu \sin\tau_n \left(1 - \frac{1}{(2 - 2\cos\tau_n)^{3/2}}\right)$$
$$\tau_{n+1} = \tau_n + 2\pi \left(\frac{1}{(1 + x_{n+1})^3} - 1\right) \quad (3.3)$$

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Figure 2a shows the phase portrait of the mapping (3.3). The invariant KAM curves of this mapping correspond to librations around L_4. Each invariant curve defines approximate proper elements corresponding to the amplitudes of libration $D_p = \tau_{max} - \tau_{min}$, and $d_p = \alpha_{max} - \alpha_{min}$, where α is defined in terms of x by Eq.(3.1). Figure 2b shows D_p as a function of d_p, calculated from Figure 2a, by considering ten invariant curves of Figure 2a with initial conditions $x = 0$ and $\tau = \pi/3 + n\Delta\tau$ with $n = 1, 2, ..., 10$ and $\Delta\tau = \pi/60$. Notice that the points of Figure 2b are almost on a straight line with slope $\simeq (1/0.273)$(rad/AU). The theoretical value given by Érdi (1988) is (1/0.2783)(rad/AU).

Figure 2c shows the Poincaré surface of section for the exact Hamiltonian model of the CRTBP, at the Jacobi constant $E = -1.49948$, which is very close to the value at L4 $E_{L4} = -1.49952$. The maximum eccentricity, at the outer invariant curve is small ($e = 0.04$), thus the surface of section can be compared with the mapping phase portrait, which corresponds to $e = 0$. The mapping portrait is angularly deformed with respect to the hamiltonian portrait. This phenomenon is an artifact of the method used to produce the mapping. However, the extend of the stability region is the same in the two portraits, and the resonant chains of same multiplicity are at approximately the same distances from the center. The rotation number of the central point of figure 2c can be used to compare the sequences of divisors found for the mapping model (3.3) with the corresponding sequences for the hamiltonian model. The two rotation numbers are found equal up to three significant figures.

A formal integral Φ for the mapping (3.3) is calculated by a computer program, by implementing a direct method for mappings which is the discrete analog of the direct method for Hamiltonian systems. This method consists of solving recurently the homological equation (Bazzani and Marmi 1991)

$$\Phi(z', z'_*) = \Phi(z, z_*) \tag{3.4}$$

where

$$z' = e^{i\omega}[z + F_2(z, z_*) + F_3(z, z_*) + ...] \tag{3.5}$$

is the Taylor-expanded mapping (3.3) in complex-conjugate coordinates, after a trivial normalization which transforms the ellipses of the linearized mapping around the elliptic equilibrium into circles. Our calculations were extended to the order of expansion $N = 60$. Details of the algorithm, as well as an analytical treatment of the Nekhoroshev stability estimates for mappings are given in Efthymiopoulos (2004). It turns out that the size of the region of stability is determined by a formula very similar to Eq.(2.14). Precisely, we have

$$I_s = \left(\frac{s-1}{s+1}\right)\left(\frac{2(1-A)^{\frac{s+1}{2}}}{(s+1)B_{\rho_*}||U_{s+1}^{(s)}||T}\right)^{\frac{2}{s-1}} \tag{3.6}$$

where $||U_{s+1}^{(s+1)}||$ is the size of the first order of the remainder at the order of truncation s, B_{ρ_*} is a constant with similar meaning as the constant C in Eq.(2.14), and I_s is the value of the outermost level curve of the integral Φ for which stability is guaranteed for all times $t \leqslant T$. Thus, the stability estimates are given in terms of a curve which is a deformed circle, while the stability domain is the interior of this curve. The constant A measures the degree of deformation of the level curve $\Phi(z, z_*) = I_s$ from a perfect circle (see Efthymiopoulos 2004 for details).

The main advantage of this approach is that it allows to express the results directly in terms of the proper elements D_p or d_p. As in the case of formula (2.14), the estimate (3.6) is optimized with respect to s. We found the optimal order $s = N_{opt} = 38$, yielding the stability estimates

$$D_p \leqslant 10.6^o, \qquad d_p \leqslant 0.0512 \text{ AU} \tag{3.7}$$

This result represents an improvement over previously obtained Nekhoroshev estimates of the region of effective stability (Giorgilli and Skokos 1997, Skokos and Dokoumentzidis 2001). The region of stability where real asteroids are observed extends to $D_p \simeq 35^o$, meaning that the region given in Eq.(3.7), by analytical methods, has a size equal to about one third the real size of the observed region of stability. Previous estimates were giving a size smaller by a factor 10 for most asteroids, and up to a factor 30 in the worst case (Giorgilli and Skokos 1997). It should be stressed, however, that the mapping model used here is also a simplification of the Hamiltonian problem, which reproduces approximately the dynamics only at low proper eccentricities. Thus, in a strict sense, the two models are not comparable.

Finally, let us note that the planar circular restricted three body model represents a great simplification of the real problem of stability of the Trojans. In particular, the border of the stability region is shaped by the overlapping of secular resonances caused either by the elliptic motion of Jupiter or by the direct or indirect effects of other major planets (e.g. Tsiganis et al. 2002, Robutel 2004).

Obtaining analytical Nekhoroshev stability estimates by adding more degrees of freedom to the problem represents a challenge from many points of view. First, the number of terms of a formal series up to order s depends on the number of degrees of freedom n as $O(s^{n+1})$. In the 4 DOF case, relevant results should involve about 10^8 terms, which is at the limit of the present computational capacities. Second, some secular variations of the orbit of Jupiter can be introduced in the Hamiltonian only as time-dependent terms. It is unclear how these terms should be treated from the point of view of Nekhoroshev theory. Finally, Nekhoroshev theory itself is far from giving optimal estimates of the time (or size of the area) of stability, even in simple Hamiltonian models.

In conclusion, the optimization of the Nekhoroshev stability estimates in the case of the Trojan asteroids represents a mathematically, computationally and physically interesting question. However, even the presently obtained Nekhoroshev estimates are realistic, and demonstrate the usefulness of Nekhoroshev theory.

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