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Nelson's Symmetry and the Infinite Volume Behavior of the Vacuum in $P(\phi)_2$*

F. GUERRA**, L. ROSEN, and B. SIMON***
Joseph Henry Laboratories, Princeton, New Jersey, USA

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Abstract. Let H_l be the Hamiltonian in a $P(\phi)_2$ theory with sharp space cutoff in the interval $(-l/2, l/2)$. Let $E_l = \inf\sigma(H_l)$, $\alpha(l) = -E_l/l$, and let Ω_l be the vacuum for H_l. We discuss properties of $\alpha(l)$ and Ω_l. In particular, as $l\to\infty$, there are finite constants $\beta_\infty < 0$ and α_∞ such that $\alpha(l)\uparrow\alpha_\infty$, $(\alpha(l)-\alpha_\infty)\, l\downarrow\beta_\infty$, and hence $\alpha(l)=\alpha_\infty+\beta_\infty/l+o(l^{-1})$. Moreover $\exp(-c_1 l)\leqq\|\Omega_l\|_1\leqq\exp(-c_2 l)$ for c_1, c_2 positive constants, where $\|\Omega_l\|_1$ is the $L^1(Q, d\mu_0)$ norm of Ω_l with respect to the Fock vacuum measure. We also present a new proof of recent estimates of Glimm and Jaffe on local perturbations of H_l in the infinite volume limit.

§ 1. Introduction

In this paper, we deal with the by now standard $P(\phi)_2$ field theory [4]. A polynomial $P(X)$ which has real coefficients and which is bounded from below will be called *semi-bounded.* If $P(0)=0$ and $P\not\equiv 0$, we will say P is *normalized.* Our spatially cutoff Hamiltonian will have a sharp space cutoff. We fix P semibounded and let $H_l = H_0 + V_l$ where

$$V_l = \int_{-l/2}^{l/2} :P(\phi(x)):dx$$

and where H_0 is the free Hamiltonian of mass $m_0 > 0$. By using techniques of "Markov field theory", Nelson recently proved [10]:

$$\langle\Omega_0, e^{-tH_l}\Omega_0\rangle = \langle\Omega_0, e^{-lH_t}\Omega_0\rangle \tag{1}$$

where Ω_0 is the Fock ($\equiv H_0$) vacuum. While this space-time symmetry looks innocent, it is extremely deep; in particular, it has the "exponential decoupling of distant regions" built into it via the exponential bound on the semigroup. The usefulness of (1) was noted by Nelson [10] who used it to prove the "linear lower bound" of Glimm-Jaffe [2]: $E_l\equiv\inf\sigma(H_0+V_l)\geqq -cl$ for some c. In [8] Guerra realized the possibility

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** On leave from Istituto di Fisica Teorica, Universitá di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli.

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of controlling the infinite volume limit by means of (1) and he obtained these results: Let Ω_l be the ground state of H_l (which we know to be unique [1, 19]) and define

$$\alpha(l) = -E_l/l. \tag{2}$$

Then [8],

$$\lim_{l\to\infty} \alpha(l) \quad \text{exists and equals} \quad \alpha_\infty = \sup_l \alpha(l) \tag{3a}$$

and, for any $p<2$ and m, there exists a constant $c_{m,p}$ so that for large l:

$$\|\Omega_l\|_p \leqq c_{m,p} l^{-m}. \tag{3b}$$

Here $\|\cdot\|_p$ is the $L^p(Q, d\mu_0)$ norm in "Q space" [14, 19, 5]. If P is normalized, $\alpha_\infty > 0$ so that Guerra concluded that E_l obeys a "linear upper bound" $E_l < -c'l(c'>0)$, and that the Van Hove phenomenon takes place: $\Omega_l \xrightarrow{w} 0$ in L^2.

The way in which $E(H) = \inf\sigma(H)$ depends on H, especially when we take H to be H_l perturbed by a local polynomial in ϕ, is of critical importance in removing the space cutoff and controlling the limiting physical theory (see [3] and the consequences of Theorem 2 below). Before Nelson's symmetry, this dependence seemed to be a difficult question since as $l\to\infty$, the vacuum renormalization constant, E_l, is a non-super-renormalizable infinity, i.e. it is infinite in every order in perturbation theory. Thus E_l must be defined implicitly and cannot be expressed in closed form. We will show that Nelson's symmetry provides a remarkable hold on E_l. Its most striking consequence is:

Theorem 1. *For any P, $\alpha(l)$ is a monotone increasing function of l.*

This result puts Eq. (3) in perspective. We note that it is predicted by perturbation theory (see § 6). A second consequence of Nelson's space-time symmetry (in the more general form (5c) below) is a simple proof of a result recently proved by Glimm and Jaffe [3]:

Theorem 2. *Let $W = \int g(x) : Q(\phi(x)) : dx$ where*
(i) *g is measurable, nonnegative and $\|g\|_\infty \leqq 1$,*
(ii) *$\operatorname{supp} g \subset [-a, b]$; $a, b < \infty$,*
(iii) *$P+Q$ is semibounded.*
Then if $[-a, b] \subset (-l/2, l/2)$,

$$-W \leqq (H_l - E_l) + c \tag{4}$$

where c is a constant independent of l and g but dependent on a, b and Q. As $a, b\to\infty$, c is bounded by $\tilde{c}|a+b|$.

Theorem 2 has a number of important applications. By a limiting argument, (4) transfers to a bound on the physical Hilbert space: $-W \leqq H_{\text{ren}} + c$. The bounds for $Q(X) = \pm X$ alone imply that:

(a) [3, 11] $\int f(x,t)\,\phi(x,t)\,dx\,dt$ is self-adjoint on the physical Hilbert space if $f \in \mathscr{S}(\mathbb{R}^2)$ is real;

(b) [3, 11] The physical vacuum expectation values are tempered distributions;

(c) [11] Green's functions exist;

(d) [3] The physical vacuum vector which was picked to be cyclic in the sense of bounded quasi-local operators is cyclic in the Wightman sense.

The proof of theorem 2 depends on ideas of Glimm-Jaffe and most importantly on an extension of (1), again due to Nelson; namely, if $P_1, P_2, \ldots, P_n$ are semibounded polynomials, let

$$V_l^{(i)} = \int_{-l/2}^{l/2} :P_i(\phi(x)):dx \tag{5a}$$

and let

$$Y = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} :P_i(\phi(x)):dx \tag{5b}$$

where $-\infty < a_0 < a_1 < \cdots < a_n < \infty$. Then

$$\langle \Omega_0, e^{-t(H_0+Y)}\Omega_0\rangle = \left\langle \Omega_0, \prod_{i=1}^{n} e^{-(a_i-a_{i-1})(H_0+V_t^{(i)})}\Omega_0 \right\rangle. \tag{5c}$$

Intuitively, Eq. (5) and its special case (1) come from a rotation by $\pi/2$ in (x, it) space. They are a reflection of the Lorentz invariance of the free theory and of the interaction density. It is interesting to see the role played by the relativistic properties of the free field in controlling the infinite volume limit. We have known for several years [7, 16] that the locality of the interaction and the causality of H_0 allow one to control the $l\to\infty$ limit of the time automorphisms. It now appears that properties of the free theory are also critical in controlling properties of the states $\omega_l(\cdot) = \langle \Omega_l, \cdot\, \Omega_l\rangle$ in the infinite volume limit.

In the remainder of this paper, we take (5) as given and otherwise do not use Markov field techniques. We do so because we find the probabilistic techniques less familiar than the L^p methods of Fock space and Q-space and we suspect the same is true for other quantum field theorists. More significantly, if there are fermions present one has a good candidate for Q-space [6] but as yet, no Markovian field theory. Thus, it is useful to isolate the methods in a form which may be applicable to the Yukawa theory.

The plan of this paper is as follows: in §§ 2, 3 we prove Theorems 1 and 2 respectively. In § 4 we develop some "hypercontractive machinery" to be used in later sections and, in particular, we provide a detailed version of Nelson's proof of the linear lower bound. In § 5 we show that $\|\Omega_l\|_1$

goes to 0 exponentially, thereby improving (3b). In § 6 we establish various properties of $\alpha(l)$ suggested by second order perturbation theory; e.g., to order l^{-1}, $\alpha(l)$ has an asymptotic expansion, $\alpha(l)=\alpha_\infty+\beta_\infty l^{-1}+o(l^{-1})$.

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§ 2. Monotonicity of the Vacuum Energy per Unit Volume

Theorem 1. *For any $P(\phi)_2$ theory with P a semibounded polynomial, $\alpha(l)$ is a monotone non-decreasing function of l.*

Proof. Let μ be a probability measure on $[0,\infty)$ and let $0<a<1$. Then, by Hölder's inequality, $\|x^a\|_1 \leqq \|x^a\|_{a^{-1}}\|1\|_{(1-a)^{-1}}$ or

$$\int_0^\infty x^a d\mu(x) \leqq \left(\int_0^\infty x\, d\mu(x)\right)^a. \tag{6}$$

By the spectral theorem, (6) implies that for any positive selfadjoint operator A and any unit vector ψ in its form domain, $\langle\psi, A^a\psi\rangle \leqq \langle\psi, A\psi\rangle^a$, so that, in particular

$$\langle\Omega_0, e^{-alH_t}\Omega_0\rangle \leqq \langle\Omega_0, e^{-lH_t}\Omega_0\rangle^a \tag{7a}$$

or by Nelson's symmetry,

$$\langle\Omega_0, e^{-tH_{al}}\Omega_0\rangle \leqq \langle\Omega_0, e^{-tH_l}\Omega_0\rangle^a. \tag{7b}$$

Now, for any fixed l, we know $\langle\Omega_l, \Omega_0\rangle>0$, so that

$$e^{-tE_l} \geqq \langle\Omega_0, e^{-tH_l}\Omega_0\rangle \geqq e^{-tE_l}|\langle\Omega_l,\Omega_0\rangle|^2$$

implies

$$-E_l=\lim_{t\to\infty} t^{-1}\log\langle\Omega_0, e^{-tH_l}\Omega_0\rangle \tag{8}$$

a formula used extensively by Glimm and Jaffe in [3]. (7b) and (8) imply that for any $0<a<1$ and $l>0$, $-E_{al}\leqq -aE_l$ or $\alpha(al)\leqq\alpha(l)$. □

Corollary. *If P is normalized, E_l is monotone decreasing in l.*

Proof. If P is normalized, $\langle\Omega_0, H_l\Omega_0\rangle=0$ and Ω_0 is not an eigenvector, so that $E_l<0$. Thus $-E_{al}\leqq -aE_l<-E_l$. □

§ 3. Volume Independent Bounds on Local Perturbations

It is our goal in this section to prove theorem 2. We first note several simplifications:

(i) We may as well suppose that P and Q are normalized since adding a constant to P doesn't affect H_l-E_l and constant terms in Q can be absorbed in the constant c in (4).

(ii) It is enough to prove the estimate (4) when g is of the form $\sum_{i=1}^{n} \gamma_i \chi_{(a_{i-1}, a_i)}$ with $-a=a_0<a_1<\cdots<a_n=b$ and $0 \leqq \gamma_i \leqq 1$. For any g satisfying the conditions of theorem 2 can be approximated in L^2 by such functions and if $g_n \to g$ in L^2, it is known that $\int g_n(x):Q(\phi(x)):dx$ converges weakly to $\int g(x):Q(\phi(x)):dx$ on a form core of H_l.

(iii) Let W be of the form:

$$W=\sum_{i=1}^{n} \gamma_i \int_{a_{i-1}}^{a_i} :Q(\phi(x)):dx \tag{9}$$

and write E_l^W for the ground state energy of H_l+W. Following [3], we note that to prove

$$-W \leqq (H_l-E_l)+c \tag{10}$$

it is enough to prove

$$-E_l^W \leqq -E_l+c\,. \tag{11}$$

For if (11) holds, then adding H_l to both sides we see that $(H_l+W-E_l^W)-W \leqq H_l-E_l+c$. Since $H_l+W-E_l^W \geqq 0$, (10) follows. This observation of Glimm and Jaffe which reduces a hard operator inequality (10) to a hard numerical inequality (11) is crucial for the proof of theorem 2.

Remark. Glimm and Jaffe also establish the inequality $-E_l \leqq -E_l^W+c$. For general Q we do not prove this reverse inequality but we note (following [3]) that it follows at once from (11) in case both $P+Q$ and $P-Q$ are semibounded. For then $+W \leqq (H_l-E_l)+c$ so that $\langle \Omega_l, W\Omega_l\rangle \leqq c$ and $E_l^W \leqq \langle \Omega_l, (H_l+W)\Omega_l\rangle = E_l+\langle \Omega_l, W\Omega_l\rangle \leqq E_l+c$.

Proof of Theorem 2. We need only prove (11) for W of the form (9). Let $Y_t=\int_{-t/2}^{t/2} :Q(\phi(x)):dx$. By (5),

$$\langle \Omega_0, e^{-t(H_l+W)}\Omega_0\rangle = \left\langle \Omega_0, e^{-(\frac{1}{2}l-a)H_t} \prod_{i=1}^{n} e^{-(a_i-a_{i-1})(H_t+\gamma_i Y_t)} e^{-(\frac{1}{2}l-b)H_t}\Omega_0\right\rangle$$

$$\leqq \prod_{i=1}^{n} \|e^{-(a_i-a_{i-1})(H_t+\gamma_i Y_t)}\| \langle \Omega_0, e^{-(l-2a)H_t}\Omega_0\rangle^{\frac{1}{2}} \langle \Omega_0, e^{-(l-2b)H_t}\Omega_0\rangle^{\frac{1}{2}}\,.$$

By the linear lower bound for $H_0+V_t+\gamma_i Y_t$, which exists since $P+\gamma_i Q$ is semibounded, $\|e^{-(a_i-a_{i-1})(H_t+\gamma_i Y_t)}\| \leqq e^{c(\gamma_i)(a_i-a_{i-1})t}$. By the concavity of the energy in γ, $\sup_{0\leqq\gamma\leqq 1} c(\gamma)=\max(c(0), c(1))=c<\infty$. Thus, using $\sum_{i=1}^{n} a_i-a_{i-1}=b+a$,

$$\langle \Omega_0, e^{-t(H_l+W)}\Omega_0\rangle \leqq e^{+tc(b+a)}\langle \Omega_0, e^{-(l-2a)H_t}\Omega_0\rangle^{\frac{1}{2}} \langle \Omega_0, e^{-(l-2b)H_t}\Omega_0\rangle^{\frac{1}{2}}\,. \tag{12}$$

From Nelson's symmetry (1) and Eq. (8), we find that

$$-E_l^W \leqq -\tfrac{1}{2}E_{l-2a} - \tfrac{1}{2}E_{l-2b} + c|b+a|\,. \tag{13}$$

Since P is normalized, and $a, b \geqq 0$, it follows from the corollary to theorem 1 that $-E_{l-2a} < -E_l$ and $-E_{l-2b} < -E_l$. Thus (13) implies (11). □

§ 4. Hypercontractive Bounds

In this section we will prove various estimates on $\|\cdot\|_{p,q}$ norms of e^{-tH_l}, the bounds of e^{-tH_l} as maps from $L^p(Q, d\mu_0)$ to $L^q(Q, d\mu_0)$. Our first result involves the abstract theory of hypercontractive semigroups. We remind the reader that a hypercontractive semigroup [16, 19] is a self-adjoint semigroup of operators, $\{e^{-tH}\}_{t \geqq 0}$ on $L^2(X, d\mu)$ for some probability measure space (X, μ) such that (i) e^{-tH} is a contraction on each L^p and (ii) for some T and B, $\|e^{-tH}\|_{2,4} \leqq B$ for all $t > T$. We recall that on $L^2(Q, d\mu_0)$, e^{-tH_0} generates a hypercontractive semigroup with $B = 1$ which in addition is positivity preserving, i.e. if $f \geqq 0$ pointwise on Q, then $e^{-tH_0} f \geqq 0$ pointwise on Q. The general theory of perturbations of hypercontractive semigroups [16, 19] involves perturbing H_0 with a real-valued multiplication operator V satisfying

$$V \in L^p, \quad \text{some} \quad p > 2, \quad \text{and} \quad e^{-V} \in \bigcap_{p<\infty} L^p. \tag{14}$$

Then $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$ (or even on $C^\infty(H_0) \cap D(V)$ [18]) and $H_0 + \beta V$ defined for $\beta \in \mathbb{C} \backslash (-\infty, 0]$ is an analytic family of type (b) [19] (in the sense of Kato [9]) which has $D(H_0) \cap D(V)$ as a core for any β [13]. We first note that:

Lemma. *Let e^{-tH_0} be a positivity preserving hypercontractive semigroup and let V, W satisfy* (14). *Then for $t \geqq 0$, $x \geqq 0$, y real, and any p, q,*

$$\|e^{-t(H_0 + xV + iyW)}\|_{p,q} \leqq \|e^{-t(H_0 + xV)}\|_{p,q}\,. \tag{15}$$

Proof. Since e^{-txV} and e^{-tH_0} are positivity preserving, so is $e^{-t(H_0 + xV)}$ by the Trotter product formula. Thus $|e^{-t(H_0 + xV)} f| \leqq e^{-t(H_0 + xV)} |f|$. Since $|e^{-iyW} f| = |f|$ the Trotter product formula implies that

$$|e^{-t(H_0 + xV + iyW)} f| \leqq e^{-t(H_0 + xV)} |f| \tag{16}$$

and (15) follows. □

Theorem 3. *Let e^{-tH_0} be a positivity preserving semigroup with $\|e^{-TH_0}\|_{2,4} \leqq B$. Let V obey* (14) *and let Ω_0 be the function identically* 1. *Then, for any $t \geqq \frac{1}{2}T$*

$$\|e^{-t(H_0 + V)}\|_{2,2} \leqq B \langle \Omega_0, e^{-t(H_0 + 4V)} \Omega_0 \rangle^{\frac{1}{4}}\,. \tag{17}$$

Remarks. 1. Similar estimates hold for any $\|\cdot\|_{p,q}$ norm, and we can replace 4 and $\frac{1}{4}$ in (17) by α and α^{-1} for arbitrary $\alpha > q$ and $(1-p^{-1})^{-1}$.

2. In the case where H_0 is the free Boson Hamiltonian, (17) can be proven by path space integration using Hölder's inequality and can be found implicitly in Nelson [10].

Proof. We need only use the Stein interpolation theorem twice. Consider first the operator-valued analytic function $A(z) = \exp[-t(H_0 + zV)]$ in the strip $0 \leqq \operatorname{Re} z \leqq 4$. By (15), when $\operatorname{Re} z = 0$, $\|A(z)\|_{2,2} \leqq \|A(0)\|_{2,2} = 1$ and when $\operatorname{Re} z = 4$, $\|A(z)\|_{2,1} \leqq \|A(4)\|_{2,1}$. Letting $\beta(s) = \langle \Omega_0, e^{-s(H_0+4V)}\Omega_0\rangle$ we see that

$$\begin{aligned}\|A(4)f\|_1 &\leqq \|A(4)|f|\|_1 = \langle \Omega_0, A(4)|f|\rangle \\ &= \langle A(4)\Omega_0, |f|\rangle \leqq \|f\|_2 \|A(4)\Omega_0\|_2 = \beta(2t)^{\frac{1}{2}} \|f\|_2 .\end{aligned}$$

By the Stein interpolation theorem, if $\operatorname{Re} z = 2$, $\|A(z)\|_{2,4/3} \leqq \beta(2t)^{\frac{1}{4}}$. By duality, $\|A(z)\|_{4,2} \leqq \beta(2t)^{\frac{1}{4}}$ when $\operatorname{Re} z = 2$.

Now let $C(z) = A(z)A(2-z)$ in the strip $0 \leqq \operatorname{Re} z \leqq 2$. Let $t > T$. Then when $\operatorname{Re} z = 2$,

$$\|C(z)\|_{2,2} \leqq \|A(z)\|_{4,2} \, \|A(2-z)\|_{2,4} \leqq \|A(2)\|_{4,2} \|A(0)\|_{2,4} \leqq B\beta(2t)^{\frac{1}{4}} .$$

Similarly when $\operatorname{Re} z = 0$, $\|C(z)\|_{2,2} \leqq B\beta(2t)^{\frac{1}{4}}$. By the Stein interpolation theorem (actually by an operator valued Phragmen-Lindelöf theorem), $\|C(1)\|_{2,2} \leqq B\beta(2t)^{\frac{1}{4}}$. Since $C(1) = \exp(-2t(H_0 + V))$, Eq. (17) holds. □

As a corollary of Theorem 3, we can give a hypercontractive version of Nelson's proof [10] of the linear lower bound:

Corollary. *$-E_l \leqq cl$ for some constant c.*

Proof. Since for the free Boson Hamiltonian the constant $B = 1$ [4],

$$\begin{aligned}e^{-TE_l} &= \|e^{-TH_l}\|_{2,2} \leqq \langle \Omega_0, e^{-T(H_0+4V_l)}\Omega_0\rangle^{\frac{1}{4}} \\ &= \langle \Omega_0, e^{-l(H_0+4V_T)}\Omega_0\rangle^{\frac{1}{4}} \leqq e^{-\frac{1}{4}lE(H_0+4V_T)} .\end{aligned}$$

We conclude that $-E_l \leqq \left[-\frac{1}{4T} E(H_0 + 4V_T)\right] l$. □

The second theorem on $\|\cdot\|_{p,q}$ norms that we shall need is a consequence of the linear lower bound:

Theorem 4. *There are constants c, d, and T such that for all $l \geqq 0$,*

$$\|e^{-tH_l}\|_{4,4} \leqq e^{ctl}, \qquad t \geqq 0, \tag{18a}$$

$$\|e^{-tH_l}\|_{2,4} \leqq e^{dtl}, \qquad t \geqq T. \tag{18b}$$

Proof. Consider the analytic operator valued function

$$\exp[-t(H_0 + zV_l)] .$$

By (15), when $\operatorname{Re} z = 0$,

$$\|\exp[-t(H_0 + zV_l)]\|_{\infty,\infty} \leqq \|\exp(-tH_0)\|_{\infty,\infty} = 1 .$$

By the linear lower bound for $H_0 + 2V_l$,

$$\|\exp[-t(H_0 + 2V_l)]\|_{2,2} \leqq e^{2clt} . \tag{19}$$

Then (18a) follows from the Stein interpolation theorem. Now write

$$2TH_l = TH_0 + T(H_0 + 2V_l) .$$

By (18a) (for $H_0 + 2V_l$) $\|\exp[-t(H_0 + 2V_l)]\|_{p,p} \leqq e^{dlt}$ for all $l, t \geqq 0$ and $2 \leqq p \leqq 4$. By the Trotter product formula and the smoothing property of H_0, (18b) holds. □

Remarks. 1. Similar results hold for any $\|\cdot\|_{p,q}$ norm.

2. It is worth emphasizing that the self-adjointness of $(H_0 + 2V_l)$ on $L^2(Q)$ has been used to deduce that (19) holds for all $t \geqq 0$ if it holds for large t.

3. A direct proof of (18a) using the N^{loc} operators of [17] is also possible. Since it helps explain why (18) holds, let us sketch it. Write $V_l = \sum_{n=-l}^{l-1} V^{(n)}$ where $V^{(n)} = \int_{n/2}^{(n+1)/2} :P(\phi(x)):dx$. Let N_n^{loc} be the operator $d\Gamma(P_n)$ where P_n is the projection in the one particle space onto those x-space functions with support in $(n/2, (n+1)/2)$. We can find a with $a \sum_{n=-\infty}^{\infty} N_n^{\text{loc}} \leqq m_0 N \leqq H_0$. Thus $\exp[-t(H_0 - a\Sigma N_n^{\text{loc}})]$ is a contraction on L^4 so by the Trotter product formula, one need only prove that $\|\exp[-t(N_n^{\text{loc}} + V^{(n)})]\|_{4,4} \leqq e^{ct}$ for all t and n. P_n induces a decomposition of Fock space $\mathscr{F}$ into $\mathscr{F} = \mathscr{F}_n \otimes {}^{\perp}\mathscr{F}_n$ and so of Q-space into $Q = Q_n \times {}^{\perp}Q_n$. As a map of $L^4(Q_n)$ to $L^4(Q_n)$, $\exp[-t(N_n^{\text{loc}} + V^{(n)})]$ is bounded by e^{ct} because $e^{-tN_n^{\text{loc}}}$ generates a hypercontractive semigroup on $L^2(Q_n)$. Since $L^4(Q) = L^4(Q_n) \otimes L^4({}^{\perp}Q_n)$, the results follows.

§ 5. Properties of Ω_l in the Limit $l \to \infty$

In this section, we use a result which we do not prove until § 6: namely, that $\alpha(l)$ is not constant if P is not constant. Since Ω_l is invariant when a constant is added to P, we shall suppose that P is normalized. The basic result on the falloff of $\|\Omega_l\|_1$ was conjectured in [19]:

Theorem 5. *There exist constants* $c, d > 0$ *so that for* l *large*, $\exp(-cl) \leqq \|\Omega_l\|_1 \leqq \exp(-dl)$.

Remark. From these bounds, Hölder's inequality, and the normalization condition $\|\Omega_l\|_2 = 1$, it follows that for any p there exist c_p, d_p with $\exp(-c_p l) \leqq \|\Omega_l\|_p \leqq \exp(-d_p l)$ where $c_p, d_p > 0$, if $p < 2$, and $c_p, d_p < 0$, if $p > 2$.

Proof. Let us first derive the fundamental equation of [8]:

$$\alpha(l) \geqq \alpha(t) + 2(lt)^{-1} \log \|\Omega_t\|_1 . \tag{20}$$

(20) follows upon taking logarithms in the inequality

$$\begin{aligned} e^{-tE_l} &\geqq \langle \Omega_0, e^{-tH_l} \Omega_0 \rangle = \langle \Omega_0, e^{-lH_t} \Omega_0 \rangle \\ &= \|e^{-\frac{1}{2}lH_t} \Omega_0\|^2 \geqq \langle \Omega_t, e^{-\frac{1}{2}lH_t} \Omega_0 \rangle^2 = e^{-lE_t} \|\Omega_t\|_1^2 . \end{aligned}$$

Since $\alpha(l)$ is monotone and non-constant, pick $l_0 \neq 0$ and $l_1 > l_0$ with $\alpha(l_0) < \alpha(l_1) < \alpha_\infty$. Then for $l > l_1$,

$$\alpha(l_0) \geqq \alpha(l) + 2(l_0 l)^{-1} \log \|\Omega_l\|_1$$

or

$$\log \|\Omega_l\|_1 \leqq \tfrac{1}{2} l_0 (\alpha(l_0) - \alpha(l))\, l \leqq [\tfrac{1}{2} l_0 (\alpha(l_0) - \alpha(l_1))]\, l$$

Since $d = -\frac{1}{2} l_0 (\alpha(l_0) - \alpha(l_1)) > 0$, the bound $\|\Omega_l\|_1 \leqq \exp(-dl)$ follows.

On the other hand, by Theorem 4, $\|e^{-TH_l}\|_{2,4} \leqq e^{cl/2}$. It follows that $\|\Omega_l\|_4 = e^{+TE_l} \|e^{-TH_l} \Omega_l\|_4 \leqq e^{cl/2}$ (since $E_l < 0$). Thus, by Hölder's inequality, $1 = \|\Omega_l\|_2 \leqq \|\Omega_l\|_4^{2/3} \|\Omega_l\|_1^{1/3}$, and $\|\Omega_l\|_1 \geqq e^{-cl}$. □

Corollary. [8] *$\Omega_l \to 0$ weakly in L^2.*

Proof. If $\psi \in L^\infty$, $\langle \psi, \Omega_l \rangle \to 0$. Since L^∞ is dense in L^2 and $\|\Omega_l\|_2 = 1$, $\Omega_l \xrightarrow{w} 0$ in L^2. □

§ 6. Additional Properties of $\alpha(l)$

To determine what properties we expect for $\alpha(l)$, we expand $E_l(\beta) = E(H_0 + \beta V_l)$ in a formal power series in β. For $P(X) = X^n$, the lowest order non-trivial term in the series for $\alpha(l)$ is (up to a numerical constant)

$$\alpha^{(2)}(l) = \int \frac{dk_1 \ldots dk_n}{\mu(k_1) \ldots \mu(k_n)} \frac{1}{\sum_{i=1}^{n} \mu(k_i)} \frac{\sin^2\left(\frac{1}{2} l \sum_{i=1}^{n} k_i\right)}{l(\Sigma k_i)^2} .$$

Letting

$$g(k) = \int_{k_n = k - \sum_{i=1}^{n-1} k_i} \frac{dk_1 \ldots dk_{n-1}}{\mu(k_1) \ldots \mu(k_n)} \frac{1}{\sum_{i=1}^{n} \mu(k_i)}$$

we note several properties of g:

(1) $g \in L^1(\mathbb{R})$;

(2) g has only polynomial falloff;

(3) g is of positive type;

(4) g is analytic in a strip about $\mathbb{R}$.

With the exception of (3), these are all elementary and (3) follows from suitable contour pushing. These properties of g transfer to simple properties of

$$\alpha^{(2)}(l) = \int_{-\infty}^{\infty} dk\, g(k) \sin^2(lk/2)/lk^2$$

$$= c \int_{-l}^{l} (1 - |x|/l)\, \hat{g}(x)\, dx$$

where $c > 0$ is some constant:

(1′) Since $|\sin^2(lk/2)/lk^2| \leqq l/4$, we see that when $l \to 0$, $\alpha^{(2)}(l) \to 0$; in fact, $\alpha^{(2)}(l) \sim O(l)$. It is easy to see that $\alpha^{(m)}(l) \sim O(l^{m-1})$ as $l \to 0$. Thus perturbation theory "predicts" that $\alpha(l) \sim O(l)$ as $l \to 0$.

(2′) $\alpha^{(2)}(l)$ is C^∞ but not analytic in l, so perturbation theory "predicts" that $\alpha(l)$ is C^∞.

(3′) $\alpha^{(2)}(l)$ is monotone increasing in l (cf. Theorem 1) and $(\alpha^{(2)}(l) - \alpha^{(2)}_\infty)\, l$ monotone decreasing.

$$(4') \qquad \alpha^{(2)}(l) = c \int_{-\infty}^{\infty} \hat{g}(x)\, dx - \frac{c}{l} \int_{-\infty}^{\infty} |x|\, \hat{g}(x)\, dx + O(e^{-dl}).$$

It is not hard to see that in general $\alpha^{(m)}(l) = c_1^{(m)} + l^{-1} c_2^{(m)} + \cdots + c_n^{(m)} l^{-n+1} + O(e^{-dl})$. This suggests that $\alpha(l)$ has an asymptotic series, $\alpha_\infty + \beta_\infty l^{-1} + \gamma_\infty l^{-2} + \cdots$, as $l \to \infty$. In particular, by (3′), β_∞ should not be 0. Thus perturbation theory suggests that, unlike Ω_l, the approach of $\alpha(l)$ to α_∞ is not exponential.

In this section, we go part way toward verifying the predictions of perturbation theory. We shall prove:

(1″) For some $c, d > 0$, $cl < \alpha(l) < dl$ when l is small;

(2″) $\alpha(l)$ is Lipschitz continuous;

(3″) $\beta(l) = (\alpha(l) - \alpha_\infty)\, l$ is monotonically decreasing;

(4″) $\alpha(l) = \alpha_\infty + \beta_\infty l^{-1} + o(l^{-1})$ where $\alpha_\infty > 0$, $\beta_\infty < 0$.

On the basis of perturbation theory and the results (3″) and (4″), it is an attractive conjecture that the coefficients in the asymptotic expansion for $\alpha(l)$ alternate in sign. Thus we expect that $\gamma(l) = (\alpha(l) - \alpha_\infty - \beta_\infty l^{-1})\, l^2$ is monotonically increasing and bounded. These facts would establish the validity of the expansion of $\alpha(l)$ to $o(l^{-2})$ and one could hope to continue in this way to all orders, alternating between increasing and decreasing monotonicity.

Theorem 6. *Let P be a normalized polynomial. Then, for some $c, d > 0$ and l_0, $cl < \alpha(l) < dl$ for all $0 < l < l_0$. In particular, $\alpha(l)$ is not constant.*

Proof. By Eq. (17) and the Nelson symmetry,

$$\langle \Omega_0, e^{-lH_T}\Omega_0\rangle \leqq e^{-TE_l} \leqq \langle \Omega_0, e^{-l(H_0+4V_T)}\Omega_0\rangle^{\frac{1}{4}}$$

or

$$\frac{1}{Tl}\log\langle \Omega_0, e^{-lH_T}\Omega_0\rangle \leqq \alpha(l) \leqq \frac{1}{4Tl}\log\langle \Omega_0, e^{-l(H_0+4V_T)}\Omega_0\rangle .$$

By the analyticity and monotonicity of $\log z$ at $z=1$, we need only prove $l^{-2}(\langle \Omega_0, e^{-lH_T}\Omega_0\rangle - 1)$ and $l^{-2}(\langle \Omega_0, e^{-l(H_0+4V_T)}\Omega_0\rangle - 1)$ have positive limits as $l \to 0$. But since $\Omega_0 \in D(H_T)$ and $\langle \Omega_0, H_T\Omega_0\rangle = 0$, $l^{-2}(\langle \Omega_0, e^{-lH_T}\Omega_0\rangle - 1) \to \frac{1}{2}\|H_T\Omega_0\|^2 > 0$ and similarly

$$l^{-2}(\langle \Omega_0, e^{-l(H_0+4V_T)}\Omega_0\rangle - 1) \to 8\,\|V_T\Omega_0\|^2 > 0 . \quad \square$$

Remark. The theorem relies on $\|e^{-TH_0}\|_{2,4} \leqq 1$ and not merely on $\|e^{-TH_0}\|_{2,4} < \infty$.

Theorem 7. *Let P a normalized polynomial. Then, for any $l, l' \geqq 0$, $|E_l - E_{l'}| \leqq |l - l'|\,\alpha_\infty$, so that in particular E_l and $\alpha(l)$ are Lipschitz continuous. Moreover E_l is concave in l.*

Proof. For any $a, b, c, l \geqq 0$, we have

$$\begin{aligned} e^{-lE_{a+b+c}}\|\Omega_{a+b+c}\|_1^2 &\leqq \langle \Omega_0, e^{-lH_{a+b+c}}\Omega_0\rangle = \langle \Omega_0, e^{-(a+b+c)H_l}\Omega_0\rangle \\ &\leqq \|e^{-aH_l}\Omega_0\|_2\,\|e^{-bH_l}\|_{2,2}\,\|e^{-cH_l}\Omega_0\|_2 \leqq e^{-\frac{1}{2}lE_{2a}}e^{-bE_l}e^{-\frac{1}{2}lE_{2c}} \\ &\leqq e^{-l(\frac{1}{2}E_{2a} - b\alpha_\infty + \frac{1}{2}E_{2c})} . \end{aligned}$$

Therefore

$$E_{a+b+c} \geqq \tfrac{1}{2}E_{2a} - b\alpha_\infty + \tfrac{1}{2}E_{2c} .$$

To prove the first part of the theorem, let us suppose $l' \geqq l$. Since $E_{l'} \leqq E_l$ by Theorem 1, it is sufficient to show that

$$E_l - E_{l'} \leqq (l' - l)\,\alpha_\infty . \tag{22}$$

(22) follows from (21) upon setting $a = c = l/2$ and $b = l' - l$. The second part of the theorem is proved by putting $b = 0$ in (21). $\square$

Remark. In case P is not normalized but only bounded below, then the theorem holds with a different constant replacing α_∞.

Corollary. *The function $\alpha(l)$ is strictly increasing.*

Proof. Since we know that $\alpha(l)$ is monotone non-decreasing by Theorem 1, we need only show that it cannot be constant in any interval.

If $[l_0, l_1]$ is the first interval on which $\alpha(l)$ is constant, then, by Theorem 6, $l_0 > 0$. But this is incompatible with the concavity of E_l. □

Let $\beta(l) = l(\alpha(l) - \alpha_\infty) = -E_l - l\alpha_\infty$. Clearly $\beta(l) < 0$.

Theorem 8. *$\beta(l)$ is convex, monotone decreasing, and bounded below, and hence converges as $l \to \infty$ to $\beta_\infty = \inf \beta(l) < 0$.*

Proof. Convexity is an obvious consequence of the concavity of E_l, and monotonicity follows from the inequality (22). As for boundedness, we have from (20) that

$$\begin{aligned} \alpha(l) &\geqq \lim \alpha(t) + 2l^{-1} \overline{\lim}\, t^{-1} \log \|\Omega_t\|_1 \\ &= \alpha_\infty - 2l^{-1} c \end{aligned}$$

by Theorem 5. □

The convergence of $\beta(l)$ as $l \to \infty$ provides us with the beginning of an asymptotic series for $\alpha(l)$:

$$\alpha(l) = \alpha_\infty + \beta_\infty l^{-1} + o(l^{-1}).$$

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F. Guerra, L. Rosen, and B. Simon
Princeton University
Jadwin Hall
Princeton, N. J. 08540, USA