Nerve Axon Equations: III Stability of the Nerve Impulse

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1. Introduction. This is the third in a series of papers treating nerve axon equations of the type

$$(1)\qquad \begin{bmatrix} \dfrac{\partial W^0}{\partial t}(x, t) \\ \vdots \\ \dfrac{\partial W^n}{\partial t}(x, t) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 W^0}{\partial x^2}(x, t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} f^0(W^0(x, t), \cdots, W^n(x, t)) \\ \vdots \\ f^n(W^0(x, t), \cdots, W^n(x, t)) \end{bmatrix}$$

where $f^0, \cdots, f^n$ are twice continuously differentiable functions of $n + 1$ real variables. The equations describe the behavior of voltage $W^0(x, t)$ in the axon at location x and time t and of auxiliary parameters $W^1(x, t), \cdots, W^n(x, t)$ which together with $W^0(x, t)$ determine the value of a source term f^0 in the diffusion equation for $W^0(x, t)$.

Of importance in physiology are a resting state where $f^0 = \cdots = f^n = 0$ which we take to be at $W^0 = \cdots = W^n = 0$ and a solution to (1) representing a nerve impulse which on an infinite axon has the form $\phi^0(x - vt), \cdots, \phi^n(x - vt)$ where the velocity v is a constant.

In the first paper in this series [3] it is shown that if $(\phi^0(x - vt), \cdots, \phi^n(x - vt))$ is a solution to (1) then a small perturbation of the initial values $(\phi^0(x), \cdots, \phi^n(x))$ will give rise to a solution which tends exponentially fast in time to some translate in x of $(\phi^0(x - vt), \cdots, \phi^n(x - vt))$ if and only if a related linear system has the property that solutions tend exponentially fast to a multiple of $(d\phi^0/dx, \cdots, d\phi^n/dx)$. In the second paper [4] a detailed study of a linearization of (1) about the resting state is made.

In this paper we return to the related linear system of [3] which is treated as an operator perturbation of the linear system studied in [4] and by use of the spectral theory of linear operators the stability of the nerve impulse under small perturbations of the initial conditions is shown to depend on the absence

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of bounded solutions to a family of ordinary differential equations derived from (1) and $\phi^0, \cdots, \phi^n$.

In Section 2 some preliminaries are given and in Section 3 the main results are stated. Proof of the main results is deferred until the final section, Section 11. Section 4 is devoted to background information and Sections 5 and 6 to a formulation of some relevant equations in operator form. In Sections 7, 8, 9 and 10 properties of these operators are developed which are used in the proof of the main results in Section 11.

2. Preliminaries. By setting

$$\tilde{x} = vx, \qquad \tilde{t} = v^2 t, \qquad \tilde{W}(\tilde{x}, \tilde{t}) = W\left(\frac{\tilde{x}}{v}, \frac{\tilde{t}}{v^2}\right)$$

where W is the transpose of $(W^0, \cdots, W^n)$ and $\tilde{f}^i(W) = v^{-2} f^i(W)$, $i = 0, \cdots, n$, we obtain a system equivalent to (1)

$$\frac{\partial \tilde{W}}{\partial \tilde{t}} = \begin{bmatrix} \dfrac{\partial^2 \tilde{W}^0}{\partial \tilde{x}^2} \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} \tilde{f}^0(\tilde{W}) \\ \vdots \\ \tilde{f}^n(\tilde{W}) \end{bmatrix}$$

with a solution

$$\tilde{W}(\tilde{x}, \tilde{t}) = \phi(\tilde{x} - \tilde{t}) = \phi\left(\frac{\tilde{x}}{v} - \frac{\tilde{t}}{v^2}\right).$$

We will therefore assume that $v = 1$ in the introduction. Under the coordinate change $y = x - t$ with $U(y, t) = W(y + t, t)$ we obtain the system

$$\frac{\partial U}{\partial t} = \begin{bmatrix} \dfrac{\partial^2 U^0}{\partial y^2} \\ 0 \\ \vdots \end{bmatrix} + \frac{\partial U}{\partial y} + \begin{bmatrix} f^0(U) \\ \vdots \\ f^n(U) \end{bmatrix} \tag{2}$$

with $U(y, t) = \phi(y)$ as a standing solution. The linearization of (2) about ϕ is given by

$$\frac{\partial U}{\partial t} = \begin{bmatrix} \dfrac{\partial^2 U^0}{\partial y^2} \\ 0 \\ \vdots \end{bmatrix} + \frac{\partial U}{\partial y} + (A + P)U \tag{3}$$

where A and P are $(n + 1) \times (n + 1)$ matrices with $A = ((\partial f^i/\partial U_j)(0))_{i,j=0}^n$ and $A + P(y) = ((\partial f^i/\partial U_j)(\phi(y)))_{i,j=0}^n$. We recall from Section 4 of [3] that the function $U(y, t) = (d\phi/dy)(y)$ for $t \geqq 0$ and all y is a solution of (3) and the definition of the exponentially stability of (3) at $d\phi/dy$. That is that (3) is said to be exponentially stable at $d\phi/dy$ if there are constants P, $\bar{\alpha} > 0$ such that given any solution U to (3) with $\|U(\cdot\,, 0)\|_\infty \leqq \beta$ there is an $\beta P \leqq h \leqq \beta P$ such that $\|U(\cdot\,, t) - h(d\phi/dy)\|_\infty \leqq \beta P e^{-\bar{\alpha} t}$ for $t \geqq 0$.

In the y, t coordinate system the linearization of (1) about the resting state takes the form

$$\frac{\partial W}{\partial t} = \begin{bmatrix} \dfrac{\partial^2 W^0}{\partial y^2} \\ 0 \\ \vdots \end{bmatrix} + \frac{\partial U}{\partial y} + AU.$$

We make the critical assumption that this system is exponentally stable at zero so that for some M, $\alpha > 0$ we have that $\|U(\cdot\,, t)\|_\infty \leqq \|U(\cdot\,, 0)\|_\infty M e^{-\alpha t}$ for $t \geqq 0$ for solutions U.

The main results are a characterization of the exponential stability of (3) in terms of the nature of solutions to the ordinary differential equation

$$\begin{bmatrix} \dfrac{d^2 \psi^0}{dy^2} \\ 0 \\ \vdots \end{bmatrix} + \frac{d\psi}{dy} + (A + P)\psi = \lambda\psi \tag{4}$$

for complex λ and the formal adjoint differential equation

$$\begin{bmatrix} \dfrac{d^2 \psi^0}{dy^2} \\ 0 \\ \vdots \end{bmatrix} - \frac{d\psi}{dy} + ({}^tA + {}^tP)\psi = \lambda\psi \tag{5}$$

where ${}^tA$, ${}^tP$ are the transposes of A, P.

We note here that we show in Theorem 4 that given $0 < \sigma < \alpha$ there are only a finite number of λ with $\operatorname{Re} \lambda > -\sigma$ for which (4) has a bounded solution. We assume that $\phi(y)$ and therefore $\|P(y)\|$ tend to zero as $|y|$ tends to infinity.

3. Main results. The main results consist of theorems giving the equivalence of exponential stability to the character of certain solutions of (4) and (5).

Theorem 1. *The system* (3) *is exponentially stable at* $d\phi/dy$ *if and only if the maximum of the real parts of the* $\lambda \neq 0$ *such that* (4) *has a bounded solution is negative and there is no bounded solution* ψ *to*

$$\begin{bmatrix} \frac{d^2\psi^0}{dy^2} \\ 0 \\ \vdots \end{bmatrix} + \frac{d\psi}{dy} + (A + P)\psi = \frac{d\phi}{dy}. \tag{6}$$

Theorem 2. There is a bounded solution ψ *to* (6) *if and only if* $\int_{-\infty}^{\infty} \phi^0(y) \cdot (d^2\gamma^0/dy^2)(y)\, dy = 0$ *where* $\gamma = {}^t(\gamma^0, \cdots, \gamma^n)$ *is the unique* (*up to multiplication by a constant*) *bounded solution to* (5) *with* $\lambda = 0$.

In the proof of the main results equations (4) and (5) are put in operator form. The statements in Theorems 1 and 2 then correspond to properties or the spectrum and spectral projections of these operators. By the results of [3] Theorems 1 and 2 can be applied to give necessary and sufficient conditions for the exponential stability under small perturbations of the initial conditions ϕ in (1).

4. Mathematical background. We use the theory of closed operators and of compact operators on a Banach space. If X is a Banach space we denote by X^* the dual space of bounded linear functionals on X. For C a bounded linear map from X into X we denote by $||C||$ the usual operator norm of C. If T is a closed linear operator into X with a dense domain D in X we denote by T^* the adjoint of T and by $\mathcal{P}(T)$ the resolvent set of T. This is the set of complex λ for which $T - \lambda I$ has a bounded two sided inverse which we denote by $(T - \lambda I)^{-1}$. To denote the complementary set of complex numbers, the spectrum of T, we use $\Sigma(T)$ as usual [7].

We now give the standard properties of compact operators which are used in the proofs that follow. If K is a compact operator on a Banach space X, then the range of $I + K$ is closed with codimension equal to the dimension of the null space of $I + K$ which is finite and is equal to the dimension of the null space of $(I + K)^*$. Also ψ is in the range of $I + K$ if and only if $\eta^*(\psi) = 0$ for all η^* in the null space of $(I + K)^*$. If $K(\lambda)$ is a holomorphic family of compact operators defined for λ in a connected open set $\mathcal{O}$, then $(I + K(\lambda))^{-1}$ either fails to exist for all $\lambda \in \mathcal{O}$ or there are at most a finite number of distinct λ in any compact subset of $\mathcal{O}$ for which $(I + K(\lambda))^{-1}$ fails to exist. This material is given in [5] and [7].

Specifically we let $X(\bar{X})$ denote the Banach space of bounded continuous complex $(n + 1)$-vector ($(n + 2)$-vector) valued functions of a real variable with the sup norm $|| \ ||_\infty$ used in the definition of exponential stability and let $D(\bar{D})$ be the subspace of those $\psi = {}^t(\psi^0, \cdots, \psi^n) \in X(\bar{\psi} \in \bar{X})$ with $\partial^2\psi^0/\partial y^2$ and $\partial\psi/\partial y(\partial\bar{\psi}/\partial y)$ bounded and continuous. We note that $D(\bar{D})$ is dense in $X(\bar{X})$.

We make use of the following compactness property in X. If $\{\psi_m\}_{m=1}^{\infty} \subset X$ is a sequence with the property that there is a (finite) uniform bound on $||d\psi_m/dy||_\infty$ for $m = 1, 2, \cdots$ and if there is a function g with $g(y)$ tending to zero as $|y|$ tends to infinity such that $|\psi_m(y)| \leqq g(y)$ for $m = 1, 2, \cdots$ and all y, then $\{\psi_m\}$ contains a subsequence convergent in $\| \ \|_\infty$. This follows at once from the Ascoli Theorem [5] which gives a subsequence which converges uniformly on compact subsets of the real line. The dominance of g assures that this same subsequence converges in $\| \ \|_\infty$.

5. The differential equations in operator form. The differential equation (4) is a special case of the first order equation

$$\frac{d}{dy}\bar{\psi} + (\bar{A}_\lambda + \bar{P})\bar{\psi} = 0 \tag{7}$$

where

$$\bar{A}_\lambda = \begin{bmatrix} 1 & a - \lambda & r \\ -1 & 0 & 0 \\ 0 & c & B - \lambda I \end{bmatrix} \quad \text{for} \quad A = \begin{bmatrix} a & r \\ c & B \end{bmatrix}$$

with B an $n \times n$ matrix and

$$\bar{P} = \begin{bmatrix} 0 & p_{11} & p_{12} \\ 0 & \cdots & \\ \vdots & p_{21} & p_{22} \end{bmatrix} \quad \text{for} \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

with p_{22} an $n \times n$ matrix valued function. The formal adjoint differential equation of (7) is then

$$-\frac{d\bar{\psi}}{dy} + {}^t(\bar{A}_\lambda + \bar{P})\bar{\psi} = 0. \tag{8}$$

We define the operator $T_\lambda(T_\lambda{}^a, \bar{T}_\lambda)$ with domain $D(D, \bar{D})$ by

$$T_\lambda\psi = \begin{bmatrix} \frac{d^2\psi^0}{dy^2} \\ 0 \\ \vdots \end{bmatrix} + \frac{d\psi}{dy} + (A - \lambda I)\psi$$

$$\cdot \left[T_\lambda^a\psi = \begin{bmatrix} \frac{d^2\psi^0}{dy^2} \\ 0 \\ \vdots \end{bmatrix} - \frac{d\psi}{dy} + {}^t(A - \lambda I)\psi, \ \bar{T}_\lambda\bar{\psi} = \frac{d\bar{\psi}}{dy} + \bar{A}_\lambda\bar{\psi} \right]$$

and we set $T = T_\lambda|_{\lambda=0}$. We use $P({}^tP, \bar{P})$ to denote both the matrix valued function and the operator on $X(X, \bar{X})$ obtained by multiplying by this matrix valued function and note that $||P|| = ||P(\cdot)||_\infty$ where $||P||$ is the norm on bounded operators on X.

Equations (4), (5), (6) and (7) are given by $(T_\lambda + P)\psi = 0$, $(T_\lambda^a + {}^tP)\psi = 0$, $(T + P)\psi = d\phi/dy$ and $(\bar{T}_\lambda + \bar{P})\bar{\psi} = 0$ respectively in operator form. Equations (7) and (8) are given here because it is in this form that the inverse operators are most conveniently studied. It is easily verified that T_λ, T_λ^a, $\bar{T}_\lambda$, $T_\lambda + P$, $T_\lambda^a + {}^tP$, and $\bar{T}_\lambda + \bar{P}$ are closed operators for all complex λ.

We will use two standard results from the theory of the asymptotic behavior of nonlinear systems of ordinary differential equations. First (since $P(y)$ tends to zero as $|y|$ tends to infinity), if λ is such that A_λ has no imaginary eigenvalues then the dimension of the subspace of solutions of $d\bar{\psi}/dy + (\bar{A}_\lambda + \bar{P})\bar{\psi} = 0$ with $\bar{\psi}(y)$ bounded as y tends to infinity is equal to the dimension of the subspace of solutions to the unperturbed equation $d\bar{\psi}/dy + \bar{A}_\lambda\bar{\psi} = 0$ with the same property and any bounded solution ψ of $d\bar{\psi}/dy + (\bar{A}_\lambda + \bar{P})\bar{\psi}$ must have $||\bar{\psi}(y)||$ tending to zero exponentially as $|y|$ tends to infinity. Second $||e^{tB}|| \leqq M_1e^{-\alpha t}$ for $t \geqq 0$ and some $M_1 > 0$ by the assumption of exponential stability at rest. Given $0 < \sigma < \alpha$ it follows that if $\epsilon(t)$ is a continuous $n \times n$ matrix valued function with $||\epsilon(t)|| \leqq (\alpha - \sigma)/4M_1$ for $t \geqq 0$, then any nonsingular matrix valued solution $\Psi(t)$ of $(d\Psi/dt)(t) = (B + \epsilon(t))\Psi(t)$ satisfies $||\Psi(t)\Psi^{-1}(s)|| \leqq 2M_1e^{\sigma(s-t)}$ for $t \geqq s$. This material is found in chapter 13 of [1].

6. The solutions in operator form. The existence and uniqueness of solutions to (3) is given in the first paper in this series [3] and in [2]. Solutions are obtained by an iterative procedure. We now outline this construction giving those details which figure in later proofs. For a fixed $T > 0$ let Y_T be the space of complex $(n + 1)$-vector valued functions $u(x, t) = {}^t(u^0(x, t), \cdots, u^n(x, t))$ defined for $0 \leqq t \leqq T$ and all x. We define a map G from Y_T into Y_T by

$$\begin{aligned}(Gu)^0(x, t) &= \int_{-\infty}^{\infty} F(x + t, y, t)u^0(y, t)\,dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} F(x + t, y, t - s)\left(\sum_{j=0}^{n} \frac{\partial f^0}{\partial \phi^j}(\phi(y - s))u^j(y - s, s)\right) dy\,ds,\end{aligned} \tag{9a}$$

$$\begin{aligned}(Gu)^i(x, t) &= u^i(x + t, 0) \\ &\quad + \int_0^t \left(\sum_{j=0}^{n} \frac{\partial f^i}{\partial \phi^j}(\phi(x + t - s))u^j(x + t - s, s)\right) ds, \qquad i = 1, \cdots, n,\end{aligned} \tag{9b}$$

where $F(x, y, t)$ is the fundamental solution of the heat equation used in (6a) of [3]. Given $\psi \in X$ if we set $U_0(x, t) = \psi(x)$ for $0 \leqq t \leqq T$ and all x, then $\lim_{m\to\infty} (G^mU_0)(x, t) = U(x, t)$ exists with uniform convergence for $0 \leqq t \leqq T$ and all x. Moreover U is the unique solution to $GU = U$ with $U(\cdot, 0) = \psi$. If in addition $\psi \in D$, then $U(\cdot, t) \in D$ for $0 \leqq t \leqq T$ and U satisfies (3).

In a similar manner we set Hu for $u \in Y_T$ equal to the right hand side of (9a, b) with $(\partial f^i/\partial\phi^j)(\phi)$ replaced by $(\partial f^i/\partial\phi^j)(0)$ for $0 \leqq i, j \leqq n$. Again if

$U_0(\cdot\,, t) = \psi \in X$ for $0 \leqq t \leqq T$, then $(H^m U_0)$ converges uniformly to U the unique solution to $HU = U$ with $U(\cdot\,, 0) = \psi$ and if $\psi \in D$, then $U(\cdot\,, t) \in D$ for $0 \leqq t \leqq T$ and U satisfies the linearization of (1) about the resting state treated in [4] viewed in the (y, t) coordinate system of Section 2 of this paper.

For $t \geqq 0$ and $\psi \in X$ we denote by $\Lambda_t \psi (\Gamma_t \psi)$ the function $U(\cdot\,, t)$ where U is the unique solution to $GU = U(HU = U)$ with $U(\cdot\,, 0) = \psi$. We have from section 4 of [3] that $\|\Lambda_t \psi\|_\infty \leqq e^{Lt} \|\psi\|_\infty$ for $t \geqq 0$ where $L/(n + 1)$ is a bound on $|(\partial f^i / \partial \phi^j)(\phi(x))|$ for $0 \leqq i, j \leqq n$ and all x. The bound $\|\Gamma_t \psi\|_\infty \leqq M \|\psi\|_\infty e^{-\alpha t}$ is given by the assumption of exponential stability at rest.

We note that $\Lambda_t(\Gamma_t)$ is a semigroup of operators on X with infinitesimal generator $T + P(T)$ (see [6]). In the sequel we use from semigroup theory the fact that a continuous square-matrix valued function $M(t)$ defined for $t \geqq 0$ with $M(0) = I$ and $M(t)M(s) = M(t + s)$ for all $t, s \geqq 0$ satisfies $M(t) = e^{tR}$ for all $t \geqq 0$ and some matrix R.

In the following sections we establish a correspondence between the spectrum of Λ_t outside the circle about the origin of radius $e^{-\alpha t}$ and the spectrum of $T + P$ with real part greater than $-\alpha$. To accomplish this we study $T + P$ as a perturbation of T and Λ_t as a perturbation of Γ_t. The results depend in an essential manner on the fact that $P(y)$ tends to zero as $|y|$ tends to infinity.

7. The operators T and T^a. In this section we state and prove a proposition and three corollaries.

Proposition 1. *$\Sigma(T)$ consists of those complex numbers λ for which $\bar{A}_\lambda$ has a purely imaginary eigenvalue.*

Corollary 1. *$\Sigma(T)$ consists of the roots of $(\alpha_0 - (\lambda + i\theta)) \cdots (\alpha_n - (\lambda + i\theta)) - \theta^2(\beta_1 - (\lambda + i\theta)) \cdots (\beta_n - (\lambda + i\theta))$ for θ real where $\alpha_0, \cdots, \alpha_n$ are the eigenvalues of $A = \left(\begin{smallmatrix} a & r \\ c & B \end{smallmatrix}\right)$ and $\beta_1, \cdots, \beta_n$ are the eigenvalues of B.*

Corollary 2. $\sup_{\lambda \in \Sigma(T)} (\operatorname{Re} \lambda) < -\alpha$.

Corollary 3. $\Sigma(T^a) = \Sigma(T)$.

Proof of Proposition 1. If $\bar{A}_\lambda$ has no purely imaginary eigenvalues let R_λ^+ be the projection associated with the eigenvalues with positive real part and $R_\lambda^- = I - R_\lambda^+$. Necessarily there are constants $M_2, \delta > 0$ which depend on λ such that

$$\|R_\lambda^- e^{y\bar{A}_\lambda}\| \leqq M_2 e^{-\delta y} \quad \text{for} \quad y \geqq 0 \quad \text{and} \quad \|R_\lambda^+ e^{y\bar{A}_\lambda}\| \leqq M_2 e^{\delta y} \quad \text{for} \quad y \leqq 0.$$

It can be verified directly that $\bar{T}_\lambda^{-1}$ exists and is given by

$$(\bar{T}_\lambda^{-1}\bar{\gamma})(y) = \int_{-\infty}^{y} R_\lambda^+ e^{(x-y)\bar{A}_\lambda} \bar{\gamma}(x)\, dx - \int_{y}^{\infty} R_\lambda^- e^{(x-y)\bar{A}_\lambda} \bar{\gamma}(x)\, dx. \tag{13}$$

We note that the right hand side of (13) is the unique solution in $\bar{D}$ to $\bar{T}_\lambda \bar{\psi} = \bar{\gamma}$ since any other solution to $d\bar{\psi}/dy + \bar{A}_\lambda \bar{\psi} = \bar{\gamma}$ differs from this one by a non zero linear combination of the columns of $e^{-y\bar{A}_\lambda}$ which cannot be bounded.

Now T_λ^{-1} is given by

$$(T_\lambda^{-1}\gamma) = \begin{bmatrix} 0 & \\ \vdots & I \\ \cdot & \end{bmatrix} \bar{T}_\lambda^{-1}\bar{\gamma}$$

where $\bar{\gamma} = {}^t(\gamma^0, 0, \gamma^1, \cdots, \gamma^n)$ for $\gamma \in X$ and where I is $(n+1) \times (n+1)$.

If $\bar{A}_\lambda$ has a purely imaginary eigenvalue, then a bounded solution $\bar{\psi}(\psi)$ to $\bar{T}_\lambda\bar{\psi} = 0$ $(T_\lambda\psi = 0)$ is easily constructed.

Thus $\Sigma(T)$ is exactly the set given in the proposition.

Proof of Corollary 1. λ is in $\Sigma(T)$ by Proposition 1 if and only if for some real θ

$$0 = \det \begin{bmatrix} 1 - i\theta & a - \lambda & r \\ -1 & -i\theta & 0 \\ 0 & c & B - (\lambda + i\theta)I \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & a - (\lambda + i\theta) - \theta^2 & r \\ -1 & -i\theta & 0 \\ 0 & c & B - (\lambda + i\theta)I \end{bmatrix}$$

$$= \det \begin{bmatrix} a - (\lambda + i\theta) & r \\ c & B - (\lambda + i\theta)I \end{bmatrix} - \theta^2 \det (B - (\lambda + i\theta)I)$$

$$= (\alpha_0 - (\lambda + i\theta)) \cdots (\alpha_n - (\lambda + i\theta))$$

$$- \theta^2(\beta_1 - (\lambda + i\theta)) \cdots (\beta_n - (\lambda + i\theta))$$

as given.

Proof of Corollary 2. This is immediate from Corollary 1 and from Theorem 2 of the previous paper [4].

Proof of Corollary 3. Identical methods show that the results of Corollary 1 hold for $\Sigma(T^a)$.

8. The operators $T + P$ **and** $T^a + {}^tP$. In this section we give important properties of $\Sigma(T + P)$. We note first that $\Sigma(T) \subset \Sigma(T + P)$ since if $0 \neq \psi \in D$ satisfies $T_\lambda\psi = 0$, then ψ is periodic and $\lim_{m\to\infty} (T_\lambda + P)f_m\psi = 0$ where $f_m(y) = 1 - e^{-(y/m)^2}$. This can be verified directly. Similarly $\Sigma(T^a) = \Sigma(T) \subset \Sigma(T^a + {}^tP)$; also $0 \in \Sigma(T + P)$ since $(T + P)(d\phi/dy) = 0$ as shown previously. Let $\mathcal{P}^+(T)$ denote the component of $\mathcal{P}(T)$ containing 0 and for complex λ let $N_\lambda \subset D$ $(N_\lambda^a \subset D)$ be the null space of $T_\lambda + P(T_\lambda^a + {}^tP)$. We now give the main results of this section in the following theorem.

Theorem 3.

(i) $\Sigma(T + P) = \Sigma(T^a + {}^tP)$.

(ii) *For* $\lambda \in \mathcal{P}(T)$, $\lambda \in \Sigma(T + P)$ *if and only if* N_λ *is not the zero subspace.*

(iii) *For* $\lambda \in \mathcal{P}(T)$ *the dimension of* N_λ *is not greater than the minimum of the number of eigenvalues of* $\bar{A}_\lambda$ *with positive real part and the number with negative real part (counting multiplicities in both instances).*

(iv) *For* $\lambda \in \mathcal{P}(T)$ *the dimension of* N_λ *is equal to the dimension of* N_λ^a .

(v) *For* $\lambda \in \mathcal{P}(T)^+$ *the dimension of* N_λ *is at most one.*

(vi) *Any compact subset of* $\mathcal{P}(T^+)$ *contains only a finite number of points of* $\Sigma(T + P)$.

Proof. To prove the theorem we state and prove two lemmas.

Lemma 1. *For* $\lambda \in \mathcal{P}(T)$ *the operator* $T_\lambda^{-1}\,{}^tP(T_\lambda^{a^{-1}}\,{}^tP)$ *is compact.*

Lemma 2. $(I - T_\lambda^{-1}P)^{-1}$ *exists for* $\mathrm{Re}\,\lambda > L$ *and is equal to* $-\int_0^\infty e^{-\lambda t}\Lambda_t\,dt$.

Proof of Lemma 1. By the discussion in Section 4 we need only show that there is a bound on $||(T_\lambda^{-1}P\psi)(y)||$ tending to zero as $|y|$ tends to infinity and a bound on $||dT_\lambda^{-1}P\psi/dy||_\infty$ holding uniformly for $\psi \in X$ with $||\psi||_\infty \leq 1$. From the properties of T_λ^{-1} given in Proposition 1 we see that

$$||(T_\lambda^{-1}P\psi)(y)|| \leqq \int_{-\infty}^{y} M_2 e^{\delta(x-y)}\,||\bar{P}(x)||\,||\psi||_\infty\,dx + \int_{y}^{\infty} M_2 e^{\delta(y-x)}\,||\bar{P}(x)||\,||\psi||_\infty\,dx \tag{14}$$

where M_2 and δ are as given in the proof of Proposition 1. This easily gives the desired bound on $||T_\lambda^{-1}P\psi(y)||$ since $||\bar{P}(x)||$ tends to zero as $|x|$ tends to infinity. Also from the proof of Proposition 1 we have that

$$\frac{d}{dy}(T_\lambda^{-1}\psi) = \begin{bmatrix} 0 & \\ \vdots & I \\ \vdots & \end{bmatrix} \frac{d}{dy}(\bar{T}_\lambda^{-1}\psi) = \begin{bmatrix} 0 & \\ \vdots & I \\ \vdots & \end{bmatrix} (\psi - \bar{A}_\lambda \bar{T}_\lambda^{-1}\bar{\psi})$$

for

$$\bar{\psi} = t(\psi^0, 0, \psi^1, \cdots, \psi^n)$$

so

$$\left|\left|\frac{d}{dy}(T_\lambda^{-1}P\psi)(y)\right|\right| \leqq ||\bar{A}_\lambda||\,||\bar{T}_\lambda^{-1}||\,||\bar{P}||\,||\psi||_\infty + ||\bar{P}||\,||\psi||_\infty$$

for all y. A similar proof holds for $T_\lambda^{a^{-1}}\,{}^tP$.

Proof of Lemma 2. By the bound on $\mathrm{Re}\,\lambda$ for $\lambda \in \Sigma(T)$ we have that if $\mathrm{Re}\,\lambda \geq L$, then $\lambda \in \mathcal{P}(T)$ so that $T_\lambda^{-1}P$ is compact and $I + T_\lambda^{-1}P$ fails to have an inverse only if there is a $0 \neq \psi \varepsilon X$ such that $\psi = -T_\lambda^{-1}P\psi$. In this case $\psi \varepsilon D$ and $(T_\lambda + P)\psi = 0$ or $(T + P)\psi = \lambda\psi$. But then $e^{\lambda t}\psi$ is a solution to (3) not bounded in norm by $e^{Lt} ||\psi||_\infty$ for $t > 0$, a contradiction. The fact that $(T_\lambda + P)^{-1} = - \int_0^\infty e^{-\lambda t}\Lambda t\, dt$ when Re $\lambda > L$ is standard in semigroup theory and is easily derived from the fact that

$$\left((T + P - \lambda I) \int_0^t e^{-\lambda s}(\Lambda_s\psi)\, ds\right)(y) = \int_0^t e^{-\lambda s}(\partial/\partial s)(\Lambda_s\psi)(y)\, ds$$

$$- \lambda \int_0^t e^{-\lambda s}(\Lambda_s\psi)(y)\, ds = e^{-\Lambda t}(\Lambda_t\psi)(y) - \psi(y) \quad \text{for} \quad \psi \varepsilon D.$$

We now turn to the proof of the theorem; we give the proof in the order (ii), (iii), (iv), (i), (v), (vi).

Proof of (ii). For $\lambda \varepsilon \mathcal{P}(T)$ we have that $T_\lambda + P = T_\lambda(I + T_\lambda^{-1}P)$ and $T_\lambda + P$ has an inverse if and only if $I + T_\lambda^{-1}P$ has an inverse. Since $T_\lambda^{-1}P$ is compact $I + T_\lambda^{-1}P$ fails to have an inverse if and only if there is a $0 \neq \psi \varepsilon X$ such that $(I + T_\lambda^{-1}P)\psi = 0$. Now as in the proof of Lemma 2 we have that $(I + T_\lambda^{-1}P)\psi = 0$ if and only if $\psi \varepsilon D$ and $(T_\lambda + P)\psi = 0$. This completes the proof of (ii).

Proof of (iii). For $\lambda \varepsilon \mathcal{P}(T)$ the dimension of N_λ cannot exceed the dimension of the subspace of solutions to $d\bar{\psi}/dy + (\bar{A}_\lambda + \bar{P})\bar{\psi} = 0$ which are bounded as y tends to infinity and/or the dimension of the subspace of solutions bounded as y tends to minus infinity. The statement (iii) then follows from the remarks in Section 5 and the obvious relation of the dimension of these subspaces to the eigenvalues of $\bar{A}_\lambda$.

Proof of (iv). Suppose that $\lambda \varepsilon \mathcal{P}(T) = \mathcal{P}(T^a)$. If $\eta \varepsilon N_\lambda^a$, then by its asymptotic properties η is integrable and if we set $\eta^*(\psi) = \int_{-\infty}^\infty {}^t\eta(y)\psi(y)dy$ we have on integration by parts that $\eta^*(T_\lambda + P)\psi = 0$ or $\eta^*(T_\lambda\psi) = -\eta^*(P\psi)$ for all $\psi \varepsilon D$. This gives that η^* is in the domain of T_λ^* and that $T_\lambda^*\eta^*((I + T_\lambda^{-1}P)\psi) = 0$ for all $\psi \varepsilon D$. The map sending η to $T_\lambda^*\eta^*$ is thus a map of N_λ^a into the null space of $(I + T_\lambda^{-1}P)^*$. This map is one-one since T_λ is onto X so the dimension of N_λ which equals the dimension of the null space of $I + T_\lambda^{-1}P$ which in turn equals the dimension of the null space of $(I + T_\lambda^{-1}P)^*$ is greater than or equal to the dimension of N_λ^a . In an identical fashion it is seen that the dimension of N_λ^a is greater than or equal to the dimension of N_λ and the proof is complete.

Proof of (i). From the facts that $\Sigma(T) \subset \Sigma(T + P)$ and that $\Sigma(T) = \Sigma(T^a) \subset \Sigma(T^a + {}^tP)$, from statement (iv), statement (ii) and the obvious dual statement to (ii) that $\lambda \varepsilon \Sigma(T^a + {}^tP) \cap \mathcal{P}(T^a)$ if and only if N_λ^a is not the zero subspace we see that (i) holds.

Proof of (v). Statement (v) follows from (iii) as soon as we show that $\bar{A}_\lambda$ has only one eigenvalue with positive real part for $\lambda \varepsilon \mathcal{P}(T)^+$. But the number of eigenvalues of $\bar{A}_\lambda$ with positive real part must remain constant for $\lambda \varepsilon \mathcal{P}(T)^+$ and in the study of the linearization about the resting state in the previous paper [4] it is shown in Theorem 3 that

$$\begin{bmatrix} -v & -a & -r \\ 1 & 0 & 0 \\ 0 & -\frac{1}{v}c & -\frac{1}{v}B \end{bmatrix}$$

has just one eigenvalue with negative real part for all $v > 0$ and this matrix is $-\bar{A}_\lambda|_{\lambda=0}$ when $v = 1$.

Proof of (vi). This follows from the facts that $\lambda \in \mathcal{P}(T) \cap \Sigma(T + P)$ if and only if $I + T_\lambda^{-1}P$ has no inverse and that $T_\lambda^{-1}P$ is holomorphic in λ. By the discussion on the identity plus a holomorphic family of compact operators in Section 4 we see that (vi) holds if $I + T^{-1}P$ has an inverse for some $\lambda \in P(T)^+$ but this is assured for Re $\lambda > L$ by Lemma 2.

9. The operator Λ_t. In this section we relate the spectrum of Λ_t to the spectrum of $T + P$. It is apparent that if $\lambda \in \Sigma(T + P) \cap \mathcal{P}^+(T)$, then there is a $\psi \neq 0$ in N_λ and $\Lambda_t\psi = e^{\lambda t}\psi$ so that $e^{\lambda t} \in \Sigma(\Lambda_t)$ for $t \geqq 0$. An important partial converse of this fact is included in the theorem below. We set $N_{\lambda,t}(N_{\lambda,t}^*)$ equal to the null space of $\Lambda_t - \lambda I(\Lambda_t^* - \lambda I^*)$ for $t \geqq 0$ and for $\lambda \in \mathcal{P}(T)$ set N_λ^* equal to the subspace in X^* of elements ν^* with $\nu^*(\psi) = \int_{-\infty}^{\infty} {}^t\nu(y)\psi(y)dy$ for some $\nu \in N_\lambda^a$.

Theorem 4. *Suppose that* $|\lambda| > e^{-\alpha t}$ *for fixed* $t > 0$. *Then*

(i) $\lambda \in \Sigma(\Lambda_t)$ *if and only if there is a* $\eta \in \Sigma(T + P)$ *with* $\lambda = e^{\eta t}$.
(ii) $\lambda \in \Sigma(\Lambda_t)$ *if and only if* $N_{\lambda,t}$ *is not the zero subspace.*
(iii) $N_{\lambda,t}(N_{\lambda,t}^*)$ *is the subspace generated by elements of* $N_\eta(N_\eta^*)$ *for all* η *with* $\lambda = e^{t\eta}$.
(iv) *The dimension of* $N_{\lambda,t}$ *is equal to the dimension of* $N_{\lambda,t}^*$.
(v) *For any* $0 < \sigma < \alpha$ *there are at most a finite number of* $\lambda \in \Sigma(\Lambda_t)$ *with* $|\lambda| > e^{-\sigma t}$ *and a finite number of* $\lambda \in \Sigma(T + P)$ *such that* Re $\lambda > -\sigma$.

Proof. To prove the theorem we state and prove several lemmas. Let E^0 be the map which sends $\psi = {}^t(\psi^0, \cdots, \psi^n) \in X$ to ${}^t(\psi^0, 0, \cdots, 0) \in X$.

Lemma 3. $E^0(\Lambda_t - \Gamma_t)$ *is a compact operator for fixed* $t > 0$.

Lemma 4. *Given* $0 < \sigma < \alpha$ *there is a constant* M^σ *such that* $\Lambda_t = Q_t + K_t$ *where* K_t *is compact and* $||Q_t|| \leqq M^\sigma e^{-\sigma t}$ *for all* $t > 0$.

Lemma 5. *If* $\lambda \in \Sigma(\Lambda_t)$ *and* $|\lambda| > M^\sigma e^{-\sigma t}$, *then there are distinct complex numbers* $\eta_1, \cdots, \eta_k \in \Sigma(T + P) \cap \mathcal{P}^+(T)$ *where* $k \geqq 1$ *is the dimension of* $N_{\lambda,t}$ *such that* $e^{t\eta_1} = \cdots = e^{t\eta_k} = \lambda$. *Moreover if* $0 \neq \psi_i \in N_{\eta_i}$ *and* $0 \neq \nu_i \in N_{\eta_i}^a$ *for* $1 \leqq i \leqq k$, *then* $\psi_1, \cdots, \psi_k$ *is a basis for* $N_{\lambda,t}$ *and* $\nu_1^*, \cdots, \nu_k^*$ *is a basis for* $N_{\lambda,t}^*$ *where* $\nu_i^*(\psi) = \int_{-\infty}^{\infty} {}^t\nu_i(y)\psi(y)dy$ *for* $1 \leqq i \leqq k$.

Lemma 6. *If* $|\lambda| > e^{-\alpha t}$ *the range of* $\Lambda_t - \lambda I$ *has codimension equal to the dimension of* $N^*_{\lambda,t}(N_{\lambda,t})$.

Proof of Lemma 3. We recall that if $U(y, t) = (\Lambda_t\psi)(y)(\tilde{U}(y, t) = (\Gamma_t\psi)(y))$, then $U = GU(\tilde{U} = H\tilde{U})$. From the definition of G and H we see that partial differentiation with respect to y of $E^0(\Lambda_t - \Gamma_t)\psi$ may be taken under the integral sign and that $E^0(\Lambda_t - \Gamma_t)\psi$ has for any $T > 0$ a uniformly bounded partial derivative in y for $0 \leqq t \leqq T$ and all y provided $||\psi||_\infty \leqq 1$. Thus the lemma follows as in the proof of the compactness of the operator $T_\lambda^{-1}P$ if it is shown that there is a uniform bound on the magnitude of $||((\Lambda_t - \Gamma_t)\psi)(y)||$, for $0 \leqq t \leqq T$ and all $||\psi||_\infty \leqq 1$ which tends to zero as $|y|$ tends to infinity. Such a bound follows easily from the iterative construction of $\Lambda_t\psi$ and $\Gamma_t\psi$.

For $\psi \in X$ with $||\psi||_\infty \leqq 1$ let $U_\psi(y, t) = \psi(y)$ for $t \geqq 0$ and all y so that $(\Lambda_t - \Gamma_t)\psi(y) = \lim_{m\to\infty} ((G^m - H^m)U_\psi)(y, t)$. For $m = 0, 1, 2, \cdots$ and fixed $T > 0$ let $M_m(y) = \sup_{||\psi||_\infty \leqq 1, 0 \leqq t \leqq T} ||((G^m - H^m)U_\psi)(y, t)||$. Then for $||\psi||_\infty \leqq 1$ we have

$$\begin{aligned} &((G^{m+1} - H^{m+1})U_\psi)^0(x, t) \\ &\qquad \leqq \int_0^t \int_{-\infty}^{\infty} F(x + t, y, t - s)[||A||\, M_m(y - s) + ||P(y - s)||\, e^{LT}]\, dy\, ds, \\ &((G^{m+1} - H^{m+1})U_\psi)^i(x, t) \\ &\qquad \leqq \int_0^t [||A||\, M_m(x + t - s) + ||P(x + t - s)||\, e^{LT}]\, ds, \quad i = 1, \cdots, n, \end{aligned} \tag{17}$$

and if $\lim_{x\to\pm\infty} M_m(x) = 0$, then $\lim_{x\to\pm\infty} M_{m+1}(x) = 0$. But $M_0(x) = 0$ for all x so the limit is 0 for $m = 0, 1, 2, \cdots$ and since $(G^m - H^m)U_\psi(x, t)$ converges uniformly to $((\Lambda_t - \Gamma_t)\psi(x))$ for $||\psi||_\infty \leqq 1$, $0 \leqq t \leqq T$ and all x we have the desired bound and the lemma is proved.

Proof of Lemma 4. Using the results of Lemma 3 we first establish that $(I - E^0)\Lambda_t$ can be expressed as the sum of a map with norm bounded by some $M_1^\sigma e^{-\sigma t}$ and a compact map. We derive a bound on $\Phi(x, t)$, the solution to $(\partial\Phi/\partial t)(x, t) = (B + P_{22}(x - t))\Phi(x, t)$ with $\Phi(x, 0) = I$ for all x. Recalling that $||e^{tB}|| \leqq M_1e^{-\alpha t}$ we choose for any $0 < \sigma < \alpha$ an $0 < \epsilon < (\alpha - \sigma)/4M_1$ and express $p_{22}(y)$ as $p^1(y) + p^2(y)$ where $||p^1(\cdot)||_\infty < \epsilon$ and where $p^2(y)$ is zero except on an interval $[j_0, j_1]$; we assume that p^1 and p^2 have been chosen with $||dp^1/dy||_\infty$, $||dp^2/dy||_\infty < \infty$. All this is possible since $||P(y)||$ tends to zero as $|y|$ tends to infinity. Now let $\Psi(x, t)$ satisfy $(\partial\Psi/\partial t)(x, t) = (B + p^1(x, t))\Psi(x, t)$ with $\Psi(x, 0) = I$ for all x. By a standard result given previously $||\Psi(x, t)\Psi^{-1}(x, s)|| \leqq 2M_1e^{-\sigma(t-s)}$ for $t \geqq s$ and all x. For fixed x we treat the case $x > j_1$. We have that

$$\Phi(x, t) = \begin{cases} \Psi(x, t), & 0 \leqq t < x - j_1, \\ \Phi(x, t)\Phi^{-1}(x, x - j_1)\Psi(x, x - j_1), & x - j_1 \leqq t \leqq x - j_0, \\ \Psi(x, t)\Psi^{-1}(x, x - j_0)\Phi(x, x - j_0)\Phi^{-1}(x, x - j_1)\Psi(x, x - j_1), & t > x - j_0 \end{cases}$$

since $p_{22}(y) = p^1(y)$ for $y \notin [j_0, j_1]$. Now

$$||\Phi(x, x - j_0)\Phi^{-1}(x, x - j_1)|| \leqq e^{(L+\epsilon)(j_1-j_0)}$$

since $||B + p_{22}(\cdot)||_\infty \leqq L + \epsilon$ so that

$$||\Phi(x, t)|| \leqq 4(M_1)^2 e^{(L+\epsilon)(j_1-j_0)} e^{\sigma(j_1-j_0)} e^{-\sigma t} \quad \text{for} \quad t \geqq 0.$$

It is easily seen that the bound holds for all x and more generally that if

$$M_2 = 4(M_1)^2 e^{(L+\sigma+\epsilon)(j_1-j_0)}, \quad \text{then} \quad ||\Phi(x, t)\Phi^{-1}(x, s)|| \leqq M_2 e^{-\sigma(t-s)} \quad \text{for} \quad t \geqq s.$$

We continue the proof in the coordinate system of (1). For $\psi \in X$ with $||\psi||_\infty \leqq 1$ let $W(x, t) = (\Lambda_t \psi)(x - t)$ and $\tilde{W}(x, t) = (\Gamma_t \psi)(x - t)$ and let $W^{1,n} = {}^t(W^1, \cdots, W^n)$. $W^{1,n}$ satisfies $(\partial W^{1,n}/\partial t)(x, t) = (c + p_{21}(x - t))W^0(x, t) + (B + p_{22}(x - t))W^{1,n}(x, t)$; so letting $W^0 = (W^0 - \tilde{W}^0) + \tilde{W}^0$ it follows that

$$\begin{aligned} W^{1,n}(x, t) &= \Phi(x, t)W^{1,n}(x, 0) \\ &+ \int_0^t \Phi(x, t)\Phi^{-1}(x, s)(c + p_{21}(x - s))(W^0(x, s) - \tilde{W}^0(y, s))\, ds \\ &+ \int_0^t \Phi(x, t)\Phi^{-1}(x, s)(c + p_{21}(x - s))\tilde{W}^0(x, s)\, ds. \end{aligned} \tag{17}$$

The first term on the right is dominated by $M_2 e^{-\sigma t}$ and the last term by $\int_0^t M_2 e^{-\sigma(t-s)}(L/(n + 1))Me^{-\alpha s}\, ds$ which is in turn less than $(M_2 ML/(n + 1)\cdot(\alpha - \sigma))e^{-\sigma t}$. Using the properties of $W^0 - \tilde{W}^0$ established in Lemma 3 it is seen that the second term satisfies the conditions stated in the compactness condition of Section 4.

Now combining the above results with Lemma 3 and the fact that $||\Gamma_t \psi|| \leqq Me^{-\alpha t}$ we see that the lemma is proved if we set $(K_t\psi)^0 = ((\Lambda_t - \Gamma_t)\psi)^0$ and define $(K_t\psi)^{1,n}$ by the second term on the right of (17) and of course set $Q_t = \Lambda_t - K_t$ and $M^\sigma = M + M_2 + M_2 ML/(n + 1)(\alpha - \sigma)$.

Proof of Lemma 5. For $|\lambda| > M^\sigma e^{-\sigma t}$ there is an inverse to $(Q_t - \lambda I)$ and expressing $\Lambda_t - \lambda I$ as $(Q_t - \lambda I)(I + (Q_t - \lambda I)^{-1}K_t)$ we see that $N_{\lambda,t}$ is the null space of $I + (Q_t - \lambda I)^{-1}K_t$ and that $N^*_{\bar{\lambda},t}$ is the image under $((Q_t - \lambda I)^*)^{-1}$ of the null space of $(I + (Q_t - \lambda I)^{-1}K_t)^*$. This gives that the dimensions of $N_{\lambda,t}$ and of $N^*_{\bar{\lambda},t}$ are equal and that $\lambda \in \Sigma(\Lambda_t)$ if and only $N_{\lambda,t}$ is not the zero subspace. Now suppose that $N_{\lambda,t}$ has dimension k and let $\gamma_1, \cdots, \gamma_k$ be a basis. Since Λ_s sends $N_{\lambda,t}$ into $N_{\lambda,t}$ for all $s \geqq 0$ we have that $\Lambda_s\gamma_i = \sum_{j=1}^k r_{ij}(s)\gamma_j$ for $1 \leqq i \leqq k$ and $s \geqq 0$ where $(r_{ij}(s))$ is a continuous matrix valued function satisfying $(r_{ij}(s))(r_{ij}(u)) = (r_{ij}(s + u))$ for $s, u \geqq 0$ and $(r_{ij}(0)) = I$. Thus there is a matrix R such that $(r_{ij}(s)) = e^{sR}$ for $s \geqq 0$ and from the fact that $(r_{ij}(t)) = \lambda I$ we see that the Jordan canonical form of R must be diagonal and that with a suitable choice of basis $\psi_1, \cdots, \psi_k$ the action of Λ_s on $N_{\lambda,t}$ is given by $\Lambda_s\psi_i = e^{s\eta_i}\psi_i$, $1 \leqq i \leqq k$. Also since

$$(T + P - (L + 1))^{-1}\psi_i = -\int_0^\infty e^{-(L+1)s}\Lambda_s\psi_i\,ds$$

$$= -\int_0^\infty e^{\eta_i-(L+1)s}\psi_i\,ds = (n_i - (L+1))^{-1}\psi_i$$

we have that $\psi_i \in D$ and $(T + P)\psi_i = \eta_i\psi_i$, $1 \leqq i \leqq k$.

Because $e^{t\eta_i} = \lambda$ it follows that $\eta_i \in \mathcal{P}^+(T)$ and that N_{η_i} and $N^a_{\eta_i}$ are one dimensional for $1 \leqq i \leqq k$. Thus the $\eta_1, \cdots, \eta_k$ are distinct as claimed. Let $0 \neq \nu_i \in N^a_{\eta_i}$ and set $\nu^*_i(\psi) = \int_{-\infty}^{\infty} {}^t\nu_i(y)\psi(y)dy$. Then $(T^* + P^*)\nu^*_i = \eta_i\nu^*_i$ and for all $\psi \in D$, $(d/ds)(\Lambda^*_s\nu^*_i)(\psi) = (d/ds)\nu^*_i(\Lambda_s\psi) = \nu^*_i((T + P)\Lambda_s\psi) = \eta_i\nu^*_i(\Lambda_s\psi =)$ $\eta_i(\Lambda^*_s\nu^*_i)(\psi)$ while $(\Lambda^*_s\nu^*_i(\psi))_{s=0} = \nu^*_i(\psi)$. This shows that $(\Lambda^*_t\nu^*_i)(\psi) = e^{t\eta_i}\nu^*_i(\psi)$ for all $\psi \in D$ so that $\Lambda^*_t\nu^*_i = e^{t\eta_i}\nu^*_i$ for $1 \leqq i \leqq k$ and $\nu^*_1, \cdots, \nu^*_k$ is a basis for $N_{\lambda,t}$ as claimed. This completes the proof of the lemma.

Proof of Lemma 6. For $|\lambda| > e^{-\alpha t}$ choose $0 < \sigma < \alpha$ such that $|\lambda| > e^{-\sigma t}$. For sufficiently large m it follows that $|\lambda^m| > M^\sigma e^{-m\sigma t}$. Let $\psi_1, \cdots, \psi_k$ $(\nu^*_1, \cdots, \nu^*_k)$ be a basis for $N_{\lambda^m,mt}(N^*_{\lambda^m,mt})$ with $\Lambda_{mt}\psi_i(\Lambda^*_{mt}\nu^*_i)$ equal to $e^{mt\eta_i}\psi_i(e^{mt\eta_i}\nu^*_i)$ for $1 \leqq i \leqq k$ as in Lemma 5. Obviously $N_{\lambda,t} \subset N_{\lambda^m,mt}(N^*_{\lambda,t} \subset N^*_{\lambda^m,mt})$ and also the range of $\Lambda_t - \lambda I$ contains the range of $\Lambda_{mt} - \lambda^m I$. But the range of $\Lambda_{mt} - \lambda^m I$ is the image under $Q_{mt} - \lambda^m I$ of the range of $I + (Q_{mt} - \lambda^m I)^{-1}K_{mt}$ and is closed and of codimension k. This implies that the range of $\Lambda_t - \lambda I$ is closed and consists of those $\psi \in X$ such that $\nu^*(\psi) = 0$ for all $\nu^* \in N^*_{\lambda,t}$ so that the codimension of the range of $\Lambda_t - \lambda I$ is equal to the dimension of $N^*_{\lambda,t}$ as claimed. To complete the proof we need only show that the dimensions of $N_{\lambda,t}$ and $N^*_{\lambda,t}$ are equal.

If $\psi \in N_{\lambda,t}$, then $\psi = \sum_{i=1}^k a_i\psi_i$ and $\Lambda_t\psi = \lambda\sum_{i=1}^k a_i\psi_i = \sum_{i=1}^k a_ie^{t\eta_i}\psi_i$ so $a_i \neq 0$ implies that $e^{t\eta_i} = \lambda$ for $1 \leqq i \leqq k$. A similar statement for $\psi^* \in N^*_{\lambda,t}$ shows that the dimension of $N_{\lambda,t}$ and the dimension of $N^*_{\lambda,t}$ are both equal to the number of $1 \leqq i \leqq k$ with $e^{t\eta_i} = \lambda$. We return to the proof of the theorem which we give in the order (ii), (iii), (iv), (i), (v).

Proof of (ii). If $N_{\lambda,t}$ is not the zero subspace then obviously $\lambda \in \Sigma(\Lambda_t)$. If $N_{\lambda,t}$ is the zero subspace, then by Lemma 6 $\Lambda_t - \lambda I$ is one-one, onto and, by the open mapping theorem, $\lambda \in \mathcal{P}(\Lambda_t)$.

Proof of (iii). This follows immediately from the proof of Lemma 6.

Proof of (iv). This proof is given in the proof of Lemma 6.

Proof of (i). This follows from (ii) and (iii).

Proof of (v). Since $I + (Q_t - \lambda I)^{-1}K_t$ has an inverse for $|\lambda|$ sufficiently large the statement about holomorphic families of compact operators in Section 4 can be applied for $|\lambda| > M^\sigma e^{-\sigma t}$. For $0 < \sigma_1 < \sigma_2 < \alpha$ and $t > 0$ choose m so that $(M^{\sigma_2})^{1/m}e^{(\sigma_1-\sigma_2)t} < 1$. We have that the number of $\lambda \in \Sigma(\Lambda_{tm})$ with $|\lambda| > M^{\sigma_2}e^{-mt\sigma_2}$ is finite and so the number of $\lambda \in \Sigma(\Lambda_t)$ with $|\lambda| > (M^{\sigma_2})^{1/m}e^{-\sigma_2 t}$ is finite but $(M^{\sigma_2})^{1/m}e^{-\sigma_2 t} < e^{-\sigma_1 t}$ and so the number of $\lambda \in \Sigma(\Lambda_t)$ with $|\lambda| > e^{-\sigma_1 t}$ is finite. The number of $\eta \in \Sigma(T + P)$ with Re $\eta > -\sigma$ is of course equal to the sum of the dimension of $N_{\lambda,t}$ for $|\lambda| > e^{-\sigma t}$ with $\lambda \in \Sigma(\Lambda_t)$ for any fixed $t > 0$.

10. The projection associated with the eigenvalue 1. The fact that $\Lambda_t(d\phi/dy) = d\phi/dy$ for $t \geqq 0$ so that 1 is an eigenvalue of Λ_t has significance in the study of the exponential stability of (3) at $d\phi/dy$. If there is a $\psi \in D$ such that $(T + P)\psi = d\phi/dy$, then $\Lambda_t\psi = \psi + t(d\phi/dy)$ and (3) is not exponentially stable at $d\phi/dy$. For such a $\psi \in D$ if C is a small circle about 1 it is easily seen that

$$\frac{1}{2\pi i}\int_C (z - \Lambda_t)^{-1}\psi\, dz = \frac{1}{2\pi i}\int_C \left(\frac{1}{z - 1}\psi + \frac{t}{(z - 1)^2}\frac{d\phi}{dy}\right) dz = \psi.$$

In this section we give conditions which insure that the spectral projection $(1/2\pi i) \int_C (z - \Lambda_t)^{-1} dz$ is a projection onto a multiple of $d\phi/dy$. Recall that N_0^a is one dimensional and let $0 \neq \gamma \in N_0^a$ and set $\gamma^*(\psi) = \int_{-\infty}^{\infty} {}^t\gamma(y)\psi(y)dy$ for $\psi \in X$. If $\gamma^*(d\phi/dy) \neq 0$ we assume that γ has been normalized so that $\gamma^*(d\phi/dy) = 1$ and we define the projection S on X by $S\psi = \gamma^*(\psi)(d\phi/dy)$ we have the following theorem.

Theorem 5. *If $\gamma^*(d\phi/dy) = 0$, then*

(i) *there is a $\psi \in D$ such that $(T + P)\psi = d\phi/dy$ and for this ψ*
(ii) *$(1/2\pi i) \int_C (zI - T - P)^{-1}\psi\, dz = \psi$ for C a circle about 0 and*
(iii) *$(1/2\pi i) \int_C (zI - \Lambda_t)^{-1}\psi\, dz = \psi$ for C a circle about 1.*

If $\gamma^*(d\phi/dy) \neq 0$, then

(iv) *there is no $\psi \in D$ such that $(T + P)\psi = d\phi/dy$ and*
(v) *$(1/2\pi i) \int_C (zI - T + P)^{-1} dz = S$ for C a sufficiently small circle about 0 and*
(vi) *$(1/2\pi i) \int_C (zI - \Lambda_t)^{-1} dz = S$ for C a sufficiently small circle about 1 provided there is no $0 \neq \eta \in \Sigma(T + P)$ with $e^{t\eta} = 1$ for this $t > 0$.*

Proof. We have seen in the introduction to this section that (i) implies (iii) and in a similar manner it is seen that (i) implies (ii). We now prove (i) and (iv) and then (v), (vi).

Proof of (i) *and* (iv). Recall that there is a solution $\psi \in D$ to $(T + P)\psi = d\phi/dy$ if and only if $T^{-1}d\phi/dy$ is in the range of $I + T^{-1}P$ and that $T^{-1}d\phi/dy$ is in the range of $I + T^{-1}P$ if and only if $T^*\gamma^*(T^{-1}d\phi/dy) = 0$ since $T^*\gamma^* \neq 0$ is in the one dimensional null space of $(I + T^{-1}P)^*$. But $T^*\gamma^*(T^{-1}d\phi/dy) = \gamma^*(d\phi/dy)$.

Proof of (v). We show that $S = (1/2\pi i) \int_c (z - T - P)^{-1} dz$ by showing that $(zI - T - P)^{-1} = (zI - T - P - S)^{-1} + (1/z(1 - z))S$ for $z \in \mathcal{P}(T + P)$ and $z \neq 1$ and that $T + P + S$ has an inverse. We note that $S^*\gamma^* = \gamma^*(d\phi/dy)\gamma^*$ so that $0 = S^*\psi^*((T + P)(\psi)) = \psi^*(S(T + P)\psi)$ for all $\psi \in D$ and $\psi^* \in X^*$.

Thus $S(T + P) = (T + P)S = 0$. Now $T^{-1}P$ is compact and $T + P + S$ has an inverse if and only $I + T^{-1}P + T^{-1}S$ has an inverse. While $I + T^{-1}P + T^{-1}S$ has an inverse if and only it has a zero null space and thus if and only $T + P + S$ has a zero null space. But if $(T + P + S)\psi = 0$, then $(T + P)\psi = -\gamma^*(\psi)d\phi/dy$ and from (iv) this implies that $\gamma^*(\psi) = 0$ so $(T + P)\psi = 0$. But this in turn implies that $\psi = \alpha d\phi/dy$ for some complex α and since $\gamma^*(\psi) = \gamma^*(\alpha d\phi/dy) = \alpha$ we must have $\psi = 0$. Thus $T + P + S$ has an inverse.

Also since $(zI - T - P)S\psi = zS\psi$ for all $\psi \in X$ we have that $(zI - T - P)^{-1}S = (1/z)S$ for $z \in \mathcal{P}(T + P)$ so that $(zI - T - P - S)[(zI - T - P)^{-1} - (1/z(1 - z))S] = [(zI - T - P)^{-1} - (1/z(1 - z))S](zI - T - P - S) = I - (z - T - P)^{-1}S\ (1/z)S = I$ and $(zI - T - P - S)^{-1} = (zI - T - P)^{-1} - (1/z(z - 1))S$ for $z \in \mathcal{P}(T + P)$ and $z \neq 1$ as desired.

Proof of (vi). Just as in the proof of (v) we see that $S\Lambda_t = \Lambda_t S$ and that under the conditions given $(\Lambda_t - I + S)$ has an inverse and that $(zI - \Lambda_t)^{-1} = (zI - \Lambda_t - S)^{-1} - (1/z(z - 1))S$.

11. Proof of the main results. We are now in a position to prove the main results. The proof of Theorem 1 is a routine use of the spectral theory of operators while Theorem 2 follows from some additional properties of γ and $d\phi/dy$.

Proof of Theorem 1. If the conditions stated in the theorem on (4) and (6) do not hold, solutions which violate exponential stability are easily constructed. On the other hand these conditions are equivalent to the statement that there is a $0 < \sigma < \alpha$ such that the only point in $\Sigma(\Lambda_t)$ outside the circle of radius $e^{-\sigma t}$ about the origin is the point 1 and that $(1/2\pi i) \int_C (zI - \Lambda_t)dz = S$ for C a sufficiently small circle about 1 for all $t > 0$. The remainder of the proof follows easily from the fact that

$$\Lambda_m = \frac{1}{2\pi i} \int_{|z| = e^{-\sigma}} z^m(z - \Lambda_1)^{-1}\, dz + S$$

for $m = 1, 2, \cdots$.

Proof of Theorem 2. We see from Theorem 5 that there is a bounded solution to (6) if and only if $\gamma^*(d\phi/dy) = \int_{-\infty}^{\infty} {}^t\gamma(y)(d\phi/dy)(y)dy = 0$. If we set $\bar{\gamma} = {}^t(\gamma^0, \gamma^0 - d\gamma^0/dy, \gamma^1, \cdots, \gamma^m)$, then $-d\bar{\gamma}/dy + {}^t(\bar{A}_\lambda + \bar{P})\gamma = 0$. It follows that $(d/dy)({}^t\bar{\gamma}\overline{d\phi}/dy) = 0$ where $\overline{d\phi}/dy = {}^t(d^2\phi^0/dy^2, d\phi^0/dy, \cdots, d\phi^n/dy)$ satisfies $(d/dy)(\overline{d\phi}/dy) + (\bar{A}_\lambda + \bar{P})\overline{d\phi}/dy = 0$. But since both $(d\phi/dy)(y)$ and $\gamma(y)$ tend to zero as $|y|$ tends to infinity we must have ${}^t\bar{\gamma}(y)(\overline{d\phi}/dy)(y) = 0$ for all y. Using this equality

$$\int_{-\infty}^{\infty} {}^t\gamma(y)\, \frac{d\phi}{dy}(y)\, dy = \int_{-\infty}^{\infty} \left(\frac{d\gamma^0}{dy}(y)\, \frac{d\phi^0}{dy}(y) - \frac{d^2\phi^0}{dy^2}(y)\gamma^0(y)\right) dy$$
$$= 2\int_{-\infty}^{\infty} \frac{d\gamma^0}{dy}(y)\, \frac{d\phi^0}{dy}(y)\, dy = -2\int_{-\infty}^{\infty} \frac{d^2\phi^0}{dy^2}(y)\gamma^0(y)\, dy$$

on integration by parts and the theorem is proved.

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