# Network Coding and Random Binning for Multi-User Channels

Liang-Liang Xie
Department of Electrical and Computer Engineering
University of Waterloo, Waterloo, ON, Canada N2L 3G1
Email: llxie@ece.uwaterloo.ca

***Abstract*— With the technique of random binning, the idea of network coding is extended and applied to multiuser channel coding problems. Specially, a two-way relay channel and a three-way broadcast channel are considered in this paper, and the corresponding achievable rate regions are determined.**

## I. Introduction

Although a relatively new research topic, network coding [1] has attracted a lot of research interests in recent years, and is expected to bring fundamental changes to the basic principles of communication networks. Besides the wide variety of potential applications, the simplicity of the essential ideas of network coding also contributes to its success.

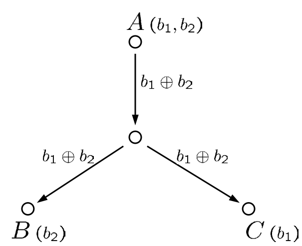


Fig. 1. The idea of network coding.

The basic idea of network coding can be explained with a simple network depicted in Fig. 1, where, node $A$ has two bits of information $b_1$ and $b_2$ to transmit to nodes $B$ and $C$ respectively. However, if node $B$ already knows $b_2$ and node $C$ already knows $b_1$, then instead of transmitting two bits $b_1$ and $b_2$ separately, node $A$ only needs to transmit one bit $b_1 \oplus b_2$ to nodes $B$ and $C$, since node $B$ can recover $b_1$ by computing $(b_1 \oplus b_2) \oplus b_2 = b_1$, and similarly, node $C$ can recover $b_2$ by computing $(b_1 \oplus b_2) \oplus b_1 = b_2$.

An interesting observation of the above network coding scheme is that although only one bit is transmitted by node $A$, two different bits can be recovered at nodes $B$ and $C$. Of course, this works only if nodes $B$ and $C$ have the appropriate side information. Hence, the success of network coding crucially depends on the availability of side information. Fortunately, in many communication networks, side information is a common phenomenon. For example, in networks with multiple routes, side information may come from other routes.

In this paper, we address a more general framework as depicted in Fig. 2, where, motivated by wireless communications, the channel dynamics at the physical layer is modelled as a broadcast channel $(\mathcal{X}_0, p(y_1, y_2|x_0), \mathcal{Y}_1 \times \mathcal{Y}_2)$, with one input $x_0$, transmitted by node $A$, and two outputs $y_1$ and $y_2$, received by nodes $B$ and $C$ respectively. Obviously, this includes the network in Fig. 1 as a special case by setting $y_1 = y_2 = x_0$.

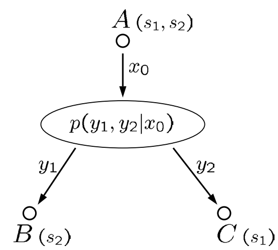


Fig. 2. A broadcast channel with side information at the receivers.

Consider the same problem where node $A$ has two independent messages $s_1$ and $s_2$ to send to nodes $B$ and $C$ respectively, while node $B$ already knows $s_2$ and node $C$ already knows $s_1$. Now, an immediate question is whether the idea of network coding can be applied to this more general framework, and what are the corresponding achievable rates.

It turns out that in this setting, the idea of network coding can be applied with random binning, a classical and fundamental technique in multiuser information theory, and the corresponding achievable rates are

$$\begin{cases} R_1 < I(X_0, Y_1) & (1) \\ R_2 < I(X_0, Y_2) & (2) \end{cases}$$

for any input distribution $p(x_0)$, where, $R_1$ is the rate of sending $s_1$ to node $B$, and $R_2$ is the rate of sending $s_2$ to node $C$.

Similarly, an interesting observation of (1)-(2) is that independent messages can be sent out to two receivers simultaneously at their respective link capacities by the same input. In addition, more generally than the network coding scheme used in Fig. 1, the rates $R_1$ and $R_2$ can be different.

The achievability of (1)-(2) will be demonstrated in a more general setting discussed in Section II. As applications of this generalized idea of network coding, a two-way relay channel and a three-way broadcast channel will be discussed in Sections III and IV.

## II. Broadcast Channel with Side Information at the Receivers

Consider a discrete memoryless broadcast channel with one transmitter and $m$ receivers:

$$(\mathcal{X}_0,\, p(y_1,\ldots,y_m|x_0),\, \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m). \quad (3)$$

That is, at any time instant $t = 1, 2, \ldots$, the transmitter sends $X_0(t) \in \mathcal{X}_0$, and each receiver $i \in \{1, \ldots, m\}$ receives $Y_i(t) \in \mathcal{Y}_i$, according to $p(Y_1(t), \ldots, Y_m(t)|X_0(t))$.

Consider the problem where the transmitter wants to send independent messages to different receivers, while each receiver knows *a priori* the messages for the other receivers.

Due to the page limit, the standard definitions of codes and achievable rates are omitted, except a special note that each receiver $i \in \{1, \ldots, m\}$ decodes based on $(Y_i(1), \ldots, Y_i(T))$ and $W_{\{-i\}} = (W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_m)$, i.e., the messages for the other receivers.

*Theorem 2.1:* For the broadcast channel (3), with each receiver knowing *a priori* the messages for the other receivers, any rates $(R_1, R_2, \ldots, R_m)$ satisfying the following inequalities are simultaneously achievable:

$$R_i < I(X_0; Y_i), \quad i = 1, 2, \ldots, m, \quad (4)$$

for some $p(x_0)$.

*Proof:* For any fixed $p(x_0)$, choose any

$$R \geq \max_{1 \leq i \leq m} I(X_0; Y_i). \quad (5)$$

Codebook Generation: Independently generate $2^{TR}$ codewords $\mathbf{x}_0 = (x_{0,1}, \ldots, x_{0,T})$ according to $\prod_{t=1}^{T} p(x_{0,t})$, and index them by $\mathbf{x}_0(w)$, $w \in \{1, \ldots, 2^{TR}\}$.

Random Binning: Generate $2^{TR}$ bins, indexed by $B(k)$, with $k = 1, \ldots, 2^{TR}$. Independently throw $2^{T(R_1 + \cdots + R_m)}$ different message vectors $(w_1, \ldots, w_m)$ into the $2^{TR}$ bins according to the uniform distribution, where, each $w_i \in \{1, \ldots, 2^{TR_i}\}$ for $i = 1, \ldots, m$. Let $k(w, \ldots, w_m)$ be the index of the bin which contains the message vector $(w_1, \ldots, w_m)$.

Encoding: For any message vector $(w_1, \ldots, w_m)$, send the codeword $\mathbf{x}_0(k(w_1, \ldots, w_m))$.

Decoding: For each receiver $i$, based on the received vector $\mathbf{Y}_i = (Y_i(1), \ldots, Y_i(T))$ and the knowledge of $w_{\{-i\}}$, determine the unique vector $(w_1, \ldots, w_m)$ which has the same $w_{\{-i\}}$ and also satisfies the joint typicality check:

$$(\mathbf{x}_0(k(w_1, \ldots, w_m)), \mathbf{Y}_i) \in A_\epsilon^{(T)}(X_0, Y_i). \quad (6)$$

If there is none or more than one such vector, an error is declared.

*Analysis of Probability of Error:* First, with high probability, the true message vector $(w_1, \ldots, w_m)$ satisfies the typicality check (6). Second, note that there are $2^{TR_i} - 1$ wrong message vectors $(w_1, \ldots, w_m)$ with the same $w_{\{-i\}}$, and for each of them, error can result from two cases:

1) The wrong message vector lies in the same bin as the true one, thus with the same codeword assigned. Since the message vectors were thrown into the bins according to the uniform distribution, this happens with probability $2^{-TR}$.
2) The wrong message vector lies in a different bin, thus resulting in an independent codeword. According to the basic properties of typical sequences [2, Sec. 8.6], this wrong codeword satisfies (6) with probability upper bounded by:
$$2^{-T(I(X_0;Y_i)-3\epsilon)}$$

Hence, the total probability of error for each wrong message vector is upper bounded by:

$$2^{-TR} + 2^{-T(I(X_0;Y_i)-3\epsilon)}.$$

Since there are $2^{TR_i} - 1$ of them with $w_{\{-i\}}$ fixed, by the union bound, the total error probability is upper bounded by

$$\begin{aligned}&(2^{TR_i} - 1) \times (2^{-TR} + 2^{-T(I(X_0;Y_i)-3\epsilon)}) \\ &< 2^{-T(R-R_i)} + 2^{-T(I(X_0;Y_i)-R_i-3\epsilon)}\end{aligned}$$

which tends to 0 as $T \to \infty$, by (4)-(5) and by choosing $\epsilon$ small enough such that $I(X_0; Y_i) - R_i - 3\epsilon > 0$. □

*Remark 2.1:* It is interesting to note that in the proof above, the binning rate $R$ is flexible to choose according to (5).

*Remark 2.2:* By the standard technique of time-sharing [2, Sec. 14.3.3], the achievable rate region (4) can be expanded to

$$R_i < I(X_0; Y_i|Q), \quad i = 1, 2, \ldots, m, \quad (7)$$

for some $p(q)p(x_0|q)$.

Next, consider an extension to the case of correlated sources. Consider $m$ i.i.d. random processes $\{S_i(t),\, t = 1, 2, \ldots\}$, for $i = 1, \ldots, m$, with joint distribution $p(s_1, s_2, \ldots, s_m)$. Suppose each random process $\{S_i(t),\, t = 1, 2, \ldots\}$ is available to all the receivers except receiver $i$, for $i = 1, 2, \ldots, m$, while the transmitter knows all the $m$ random processes. The communication task is for the transmitter to send to each receiver $i$ the information about $\{S_i(t),\, t = 1, 2, \ldots, \}$. The following theorem characterizes the condition under which this can be done simultaneously for all the receivers.

*Theorem 2.2:* For the communication problem stated above, all the receivers can obtain their respective information through the broadcast channel simultaneously if for some $p(x_0)$,

$$H(S_i|S_{\{-i\}}) < I(X_0; Y_i), \quad i = 1, 2, \ldots, m, \quad (8)$$

where $S_{\{-i\}} := \{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_m\}$.

The proof is similar to that of Theorem 2.1 and is omitted. Similarly, the technique of time-sharing can also be applied.

## III. Two-Way Relay Channel

Consider a network of three nodes 1, 2, 3, with the input-output dynamics modelled by the discrete memoryless channel

$$(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3,\, p(y_1, y_2, y_3|x_1, x_2, x_3),\, \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3). \quad (9)$$

That is, at any time $t = 1, 2, \ldots$, the outputs $y_1(t)$, $y_2(t)$, $y_3(t)$ received by the three nodes respectively only depend on the inputs $x_1(t)$, $x_2(t)$, $x_3(t)$ transmitted by the three nodes at the same time according to $p(y_1(t), y_2(t), y_3(t)|x_1(t), x_2(t), x_3(t))$.

Consider the two-way relay problem where node 1 and node 2 communicate with each other at rates $R_1$ and $R_2$ respectively, with the help of the relay node 3.

We are interested in the simultaneously achievable rates $(R_1, R_2)$. Here, the standard definitions of codes and achievable rates are omitted, except a special note that at any time $t$, each node $i$ can choose its input $x_i(t)$ based on the past outputs $(y_i(t-1), y_i(t-2), \ldots, y_i(1))$ it has already received.

*Theorem 3.1:* For the two-way relay problem defined above, any rates $(R_1, R_2)$ satisfying the following inequalities are simultaneously achievable:

$$R_1 < I(X_1; Y_3|X_2, X_3) \tag{10}$$

$$R_2 < I(X_2; Y_3|X_1, X_3) \tag{11}$$

$$R_1 + R_2 < I(X_1, X_2; Y_3|X_3) \tag{12}$$

and

$$R_1 < I(X_1, X_3; Y_2|X_2) \tag{13}$$

$$R_2 < I(X_2, X_3; Y_1|X_1) \tag{14}$$

for some $p(x_1)p(x_2)p(x_3)$.

*Proof:* For any fixed $p(x_1)p(x_2)p(x_3)$, choose any

$$R \geq \max\{I(X_3; Y_2|X_2), I(X_3; Y_1|X_1)\}. \tag{15}$$

We use Markov block coding argument. Consider $B$ blocks of transmission, each of $T$ transmission slots. A sequence of $B-1$ indices, $w_1(b) \in \{1, \ldots, 2^{TR_1}\}, b = 1, 2, \ldots, B-1$ will be sent over from node 1 to node 2 in $TB$ transmission slots, and at the same time, another sequence of $B-1$ indices, $w_2(b) \in \{1, \ldots, 2^{TR_2}\}, b = 1, 2, \ldots, B-1$ will be sent over from node 2 to node 1.

Codebook Generation:

1) For node 1, independently generate $2^{TR_1}$ i.i.d. $T$-sequences $\mathbf{x}_1 = (x_{1,1}, \ldots, x_{1,T})$ in $\mathcal{X}_1^T$ according to $p(x_1)$. Index them as $\mathbf{x}_1(w_1)$, $w_1 \in \{1, 2, \ldots, 2^{TR_1}\}$.
2) For node 2, independently generate $2^{TR_2}$ i.i.d. $T$-sequences $\mathbf{x}_2 = (x_{2,1}, \ldots, x_{2,T})$ in $\mathcal{X}_2^T$ according to $p(x_2)$. Index them as $\mathbf{x}_2(w_2)$, $w_2 \in \{1, 2, \ldots, 2^{TR_2}\}$.
3) For node 3, independently generate $2^{TR}$ i.i.d. $T$-sequences $\mathbf{x}_3 = (x_{3,1}, \ldots, x_{3,T})$ in $\mathcal{X}_3^T$ according to $p(x_3)$. Index them as $\mathbf{x}_3(w_3)$, $w_3 \in \{1, 2, \ldots, 2^{TR}\}$.

Random Binning: Generate $2^{TR}$ bins, indexed by $B(k)$, with $k = 1, \ldots, 2^{TR}$. Independently throw each index pairs $(w_1, w_2)$, $w_1 \in \{1, 2, \ldots, 2^{TR_1}\}$, $w_2 \in \{1, 2, \ldots, 2^{TR_2}\}$ into the $2^{TR}$ bins according to the uniform distribution. Let $k(w_1, w_2)$ be the index of the bin which contains the index pair $(w_1, w_2)$.

Encoding: In each block $b = 1, 2, \ldots, B$, node 1 sends the $T$-sequence $\mathbf{x}_1(w_1(b))$, and node 2 sends the $T$-sequence $\mathbf{x}_2(w_2(b))$, where $w_1(B)$ and $w_2(B)$ are set to be 1.

Node 3 sends $\mathbf{x}_3(1)$ in block 1. At the end of each block $b = 1, \ldots, B-1$, node 3 has an estimate $(\hat{w}_1(b), \hat{w}_2(b))$ of $(w_1(b), w_2(b))$ based on the $T$-sequence $\mathbf{Y}_3(b)$ it received during the block $b$, and sends the $T$-sequence $\mathbf{x}_3(k(\hat{w}_1(b), \hat{w}_2(b)))$ in the next block $b+1$.

Decoding: At the end of each block $b = 1, \ldots, B-1$, node 3 determines the unique index pair $(w_1(b), w_2(b))$ which satisfies the joint typicality check:

$$(\mathbf{x}_1(w_1(b)), \mathbf{x}_2(w_2(b)), \mathbf{X}_3(b), \mathbf{Y}_3(b)) \in A_\epsilon^{(T)}(X_1, X_2, X_3, Y_3)$$

where $\mathbf{X}_3(b)$ denotes the $T$-sequence sent by node 3 during block $b$. If there is none or more than one such pair, an error is declared.

At the end of each block $b = 2, \ldots, B$, node 2 determines the unique index $w_1(b-1)$ which, in block $b$, satisfies the joint typicality check:

$$(\mathbf{x}_3(k(w_1(b-1), w_2(b-1))), \mathbf{X}_2(b), \mathbf{Y}_2(b)) \in A_\epsilon^{(T)}(X_3, X_2, Y_2)$$

and also in block $b-1$, satisfies the joint typicality check:

$$\begin{aligned}(\mathbf{x}_1(w_1(b-1)), \mathbf{x}_3(k(\check{w}_1(b-2), w_2(b-2))), \mathbf{X}_2(b-1), \\ \mathbf{Y}_2(b-1)) \in A_\epsilon^{(T)}(X_1, X_3, X_2, Y_2)\end{aligned}$$

where, $\check{w}_1(b-2)$ is the estimate of $w_1(b-2)$ node 2 made at the end of block $b-1$. If there is none or more than one such $w_1(b-1)$, an error is declared.

Similarly, node 1 decodes $w_2(b-1)$ according to the following two joint typicality checks:

$$(\mathbf{x}_3(k(w_1(b-1), w_2(b-1))), \mathbf{X}_1(b), \mathbf{Y}_1(b)) \in A_\epsilon^{(T)}(X_3, X_1, Y_1)$$

$$\begin{aligned}(\mathbf{x}_2(w_2(b-1)), \mathbf{x}_3(k(w_1(b-2), \check{w}_2(b-2))), \mathbf{X}_1(b-1), \\ \mathbf{Y}_1(b-1)) \in A_\epsilon^{(T)}(X_2, X_3, X_1, Y_1)\end{aligned}$$

where, $\check{w}_2(b-2)$ is the estimate of $w_2(b-2)$ node 1 made at the end of block $b-1$.

*Analysis of Probability of Error:* First, according to the capacity region of the multiple access channel, the bounds (10)-(12) make sure that node 3 can decode $(w_1(b), w_2(b))$ with arbitrarily small probability of error. Second, node 2 can decode $w_1(b-1)$ with arbitrarily small probability of error if

$$R_1 < I(X_3; Y_2|X_2) + I(X_1; Y_2|X_3, X_2)$$

where the two mutual informations follow from the two typicality checks respectively, and their combination leads to (13). Similarly, for node 1, (14) is needed. □

## IV. Three-Way Broadcast Channel

Consider the same three-node network as described in the previous section with the channel model (9). Now, instead of the two-way relay, consider a three-way broadcast problem: Each node $i$ wants to send the same information to both the other two nodes at the same rate $R_i$, for $i = 1, 2, 3$.

*Theorem 4.1:* For the three-way broadcast problem described above, any rates $(R_1, R_2, R_3)$ satisfying the following inequalities are simultaneously achievable:

$$R_1 < \max\{I(X_1; Y_2|X_3, X_2), I(X_1; Y_3|X_2, X_3)\} \tag{16}$$

$$R_2 < \max\{I(X_2; Y_3|X_1, X_3), I(X_2; Y_1|X_3, X_1)\} \tag{17}$$

$$R_3 < \max\{I(X_3; Y_1|X_2, X_1), I(X_3; Y_2|X_1, X_2)\} \tag{18}$$

and

$$R_1 + R_2 < I(X_1, X_2; Y_3|X_3) \quad (19)$$
$$R_2 + R_3 < I(X_2, X_3; Y_1|X_1) \quad (20)$$
$$R_3 + R_1 < I(X_3, X_1; Y_2|X_2) \quad (21)$$

for some $p(x_1)p(x_2)p(x_r)$.

*Proof (outline):* As the standard, random codebooks generated according to $p(x_i)$ for each node $i = 1, 2, 3$, and the Markov block coding argument are used.

We need different coding schemes for different situations that may arise from (16)-(18). By symmetry, we only need to consider the following seven cases.

*Case 1.* $(R_1, R_2, R_3)$ satisfy

$$R_1 < \min\{I(X_1; Y_2|X_3, X_2), I(X_1; Y_3|X_2, X_3)\}$$
$$R_2 < \min\{I(X_2; Y_3|X_1, X_3), I(X_2; Y_1|X_3, X_1)\}$$
$$R_3 < \min\{I(X_3; Y_1|X_2, X_1), I(X_3; Y_2|X_1, X_2)\}$$

In this case, actually, Markov block coding is not needed. Each node $i$ simply sends its own message $w_i$, and every node $j$ can decode $\{w_i, i \neq j\}$ according to the multiple access constraints.

*Case 2.* $(R_1, R_2, R_3)$ satisfy

$$I(X_1; Y_2|X_3, X_2) \leq R_1 < I(X_1; Y_3|X_2, X_3) \quad (22)$$
$$R_2 < \min\{I(X_2; Y_3|X_1, X_3), I(X_2; Y_1|X_3, X_1)\} \quad (23)$$
$$R_3 < \min\{I(X_3; Y_1|X_2, X_1), I(X_3; Y_2|X_1, X_2)\} \quad (24)$$

In each block $b = 1, \ldots, B$, node 1 sends $w_1(b)$; node 2 sends $w_2(b)$; and node 3 sends $(w_3(b), w_1(b-1))$ by random binning, where, set $w_1(0) = 1$. At the end of block $b$, node 1 can decode $w_2(b)$ and $w_3(b)$, and node 3 can decode $w_1(b)$ and $w_2(b)$, according to the multiple access constraints; while node 2 can jointly decode $w_3(b)$ and $w_1(b-1)$ based on blocks $b$ and $b-1$, due to (21) and $R_3 < I(X_3; Y_2|X_2)$, which follows from (21) and the first part of (22).

*Case 3.* $(R_1, R_2, R_3)$ satisfy

$$I(X_1; Y_2|X_3, X_2) \leq R_1 < I(X_1; Y_3|X_2, X_3) \quad (25)$$
$$I(X_2; Y_3|X_1, X_3) \leq R_2 < I(X_2; Y_1|X_3, X_1) \quad (26)$$
$$R_3 < \min\{I(X_3; Y_1|X_2, X_1), I(X_3; Y_2|X_1, X_2)\} \quad (27)$$

In each block $b = 1, \ldots, B$, node 1 sends $(w_1(b), w_2(b-1))$ by random binning; node 2 sends $w_2(b)$; and node 3 sends $(w_3(b), w_1(b-1))$ by random binning, where, set $w_2(0) = w_3(0) = 1$. At the end of block $b$, node 1 can decode $w_2(b)$ and $w_3(b)$ according to the multiple access constraints; while node 2 can jointly decode $w_3(b)$ and $w_1(b-1)$ based on blocks $b$ and $b-1$, due to (21) and $R_3 < I(X_3; Y_2|X_2)$, which follows from (21) and the first part of (25); and node 3 can decode $w_1(b)$ and $w_2(b-1)$ based on blocks $b$ and $b-1$, due to (19) and $R_1 < I(X_1; Y_3|X_3)$, which follows from (19) and the first part of (26).

*Case 4.* $(R_1, R_2, R_3)$ satisfy

$$I(X_1; Y_2|X_3, X_2) \leq R_1 < I(X_1; Y_3|X_2, X_3) \quad (28)$$
$$I(X_2; Y_3|X_1, X_3) > R_2 \geq I(X_2; Y_1|X_3, X_1) \quad (29)$$
$$R_3 < \min\{I(X_3; Y_1|X_2, X_1), I(X_3; Y_2|X_1, X_2)\} \quad (30)$$

In each block $b = 1, \ldots, B$, node 1 sends $w_1(b)$; node 2 sends $w_2(b)$; and node 3 sends $(w_3(b), w_1(b-1), w_2(b-1))$ by random binning, where, set $w_1(0) = w_2(0) = 1$. At the end of block $b$, node 3 can decode $w_1(b)$ and $w_2(b)$ according to the multiple access constraints; while node 1 can decode $w_3(b)$ and $w_2(b-1)$ based on blocks $b$ and $b-1$, due to (20) and $R_3 < I(X_3; Y_1|X_1)$, which follows from (20) and the second part of (29); and node 2 can decode $w_3(b)$ and $w_1(b-1)$ based on blocks $b$ and $b-1$, due to (21) and $R_3 < I(X_3; Y_2|X_2)$, which follows from (21) and the first part of (28).

*Case 5.* $(R_1, R_2, R_3)$ satisfy

$$I(X_1; Y_2|X_3, X_2) > R_1 \geq I(X_1; Y_3|X_2, X_3) \quad (31)$$
$$I(X_2; Y_3|X_1, X_3) \leq R_2 < I(X_2; Y_1|X_3, X_1) \quad (32)$$
$$R_3 < \min\{I(X_3; Y_1|X_2, X_1), I(X_3; Y_2|X_1, X_2)\} \quad (33)$$

This case cannot happen, since (19) and the second part of (31) imply that $R_2 < I(X_2; Y_3|X_3)$, which contradicts to the first part of (32).

*Case 6.* $(R_1, R_2, R_3)$ satisfy

$$I(X_1; Y_2|X_3, X_2) \leq R_1 < I(X_1; Y_3|X_2, X_3) \quad (34)$$
$$I(X_2; Y_3|X_1, X_3) \leq R_2 < I(X_2; Y_1|X_3, X_1) \quad (35)$$
$$I(X_3; Y_1|X_2, X_1) \leq R_3 < I(X_3; Y_2|X_1, X_2) \quad (36)$$

In each block $b = 1, \ldots, B$, node 1 sends $(w_1(b), w_2(b-1))$ by random binning; node 2 sends $(w_2(b), w_3(b-1))$ by random binning; and node 3 sends $(w_3(b), w_1(b-1))$ by random binning, where, set $w_1(0) = w_2(0) = w_3(0) = 1$. At the end of block $b$, node 1 can jointly decode $w_2(b)$ and $w_3(b-1)$ based on blocks $b$ and $b-1$, due to (20) and $R_2 < I(X_2; Y_1|X_1)$, which follows from (20) and the first part of (36); node 2 can jointly decode $w_3(b)$ and $w_1(b-1)$ based on blocks $b$ and $b-1$, due to (21) and $R_3 < I(X_3; Y_2|X_2)$, which follows from (21) and the first part of (34); and node 3 can decode $w_1(b)$ and $w_2(b-1)$ based on blocks $b$ and $b-1$, due to (19) and $R_1 < I(X_1; Y_3|X_3)$, which follows from (19) and the first part of (35).

*Case 7.* $(R_1, R_2, R_3)$ satisfy

$$I(X_1; Y_2|X_3, X_2) \leq R_1 < I(X_1; Y_3|X_2, X_3) \quad (37)$$
$$I(X_2; Y_3|X_1, X_3) \leq R_2 < I(X_2; Y_1|X_3, X_1) \quad (38)$$
$$I(X_3; Y_1|X_2, X_1) > R_3 \geq I(X_3; Y_2|X_1, X_2) \quad (39)$$

This case cannot happen, since (21) and the first part of (37) imply that $R_3 < I(X_3; Y_2|X_2)$, which contradicts to the second part of (39). □

*Remark 4.1:* It is easy to check that Theorem 4.1 implies Theorem 3.1 by setting $R_3 = 0$.

## REFERENCES


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