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NETWORK DESIGN PROBLEM WITH CONGESTION EFFECTS: A CASE OF BILEVEL PROGRAMMING

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Recently much attention has been focused on multilevel programming, a branch of mathematical programming that can be viewed either as a generalization of min-max problems or as a particular class of Stackelberg games with continuous variables. The network design problem with continuous decision variables representing link capacities can be cast into such a framework. We first give a formal description of the problem and then develop various suboptimal procedures to solve it. Worst-case behaviour results concerning the heuristics, as well as numerical results on a small network, are presented.

Key words: Network Design, Bilevel Programming, Variational Inequalities, Stackelberg Games.

1. Introduction

There are many situations where different levels of decisions are involved. Most of the large-scale mathematical programming literature that deals with decentralized management of resources covers models of such problems where the various decision makers act in a cooperative manner. Models of noncooperative decision making are found in the game theory literature and are usually restricted to a small number of players or variables. Recently some authors have investigated situations where decision makers at various levels act in a hierarchical manner: at a given level, decision makers are bound by the decisions of the lower levels and maximize their own profit accordingly, taking into account the reactions of the lower levels; for a complete and formal description see Bard and Falk [4]. If x and y represent the decision vectors associated with the upper and lower level respectively, the bilevel programming problem considered may be formulated as:

$$\begin{aligned}&\min_{x\in X} \quad F(x,y)\\ &\text{subject to} \quad y\in\left\{\operatorname*{argmin}_{z\in Y(x)} G(x,z)\right\}\end{aligned} \tag{1}$$

where X is the feasible set of the x-variables, and $Y(x)$ the feasible set, possibly dependent on x, of the y-variables. Algorithms have been developed for linear cost

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functions (F and G linear, X and Y polyhedral) by Bialas and Karwan [6], Bard and Falk [4], Papavassilopoulos [22], Candler and Townsley [8], and for the general case by Bard [3] and Papavassilopoulos [23].

In this paper we consider a generalization of problem (1) where the y-variables are the solution of an equilibrium problem formulated as a variational inequality. Then the problem becomes:

$$\begin{aligned} &\min_{x\in X} \quad F(x,y) \\ &\text{subject to} \quad (y-z)^{\mathrm{T}}H(x,y)\leq 0 \ \ \forall z\in Y(x), \\ &\qquad\qquad\quad y\in Y(x). \end{aligned} \tag{2}$$

The network design problem that we consider has already been formulated and studied, in particular, by Dafermos [9], Abdulaal and Leblanc [2], Dantzig et al. [12] and Gershwin et al. [26]; it consists in optimally balancing the transportation, investment and maintenance costs of a network subject to congestion, where users behave according to Wardrop's first principle of traffic equilibrium (user-optimum, see Wardrop [27]) which is akin to the Nash–Cournot principle of the theory of games (Haurie and Marcotte [15]). The x-vector represents the improvements made to the network, X the set of feasible improvements, y the multicommodity flow variables and $Y(x)$ the polyhedron of feasible flows, which is independent of x. In particular, the x-variables will reflect the capacities of the network links; for a given link flow, the corresponding link traversal time will be a decreasing function of its capacity. In other words: the higher the capacity, the lower the congestion.

The remainder of the paper is structured as follows: first we give a mathematical formulation of the problem, then we prove some theoretical results about the problem; finally we propose solution algorithms of a heuristic nature which will be analyzed theoretically and tested numerically.

2. Mathematical formulation

Throughout this paper, the subscript 'a' will refer to a link of the network. Denote

N: set of nodes.
A: set of one-way links.
Ω: set of origins.
Δ: set of destinations.
$v^k=(v_a^k)_{a\in A}$: flow vector of commodity k (trips originating from origin k).
$v_a=\sum_{k\in\Omega} v_a^k$: total flow on link a.
$v=(v_a)_{a\in A}$: total flow vector.
z_a: capacity of link a.
g_{kl}: trip demand between origin $k\in\Omega$ and destination $l\in\Delta$.

In the remainder of the paper, we will call a set of flow vectors *feasible* if it satisfies the following flow conservation and non-negativity constraints:

$$\forall i \in N\colon \sum_{j\in N} v_{ij}^k - \sum_{j\in N} v_{ji}^k = \begin{cases} \sum_{l\in\Delta} g_{kl} & \text{if } i = k \in \Omega, \\ -g_{ki} & \text{if } i \in \Delta, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$v_{ij}^k \geq 0.$$

Let Φ be the set of multicommodity flow vectors satisfying (3). For the sake of brevity we will write $v \in \Phi$ although, formally, we should write $(v^k)_{k\in\Omega} \in \Phi$. Let $S(v, z)$ be a vector-valued function giving the traversal times of all links as a function of the total flow vector v and the capacity vector $z = (z_a)_{a\in A}$. It may be shown (Smith [24], Dafermos [10]) that a feasible flow vector v^* constitutes a user equilibrium for a given capacity vector z, if and only if it satisfies the following variational inequality:

$$(v^* - v^{(i)})^{\mathrm{T}} S(v^*, z) \leq 0, \quad i = 1, 2, \ldots, n, \tag{4}$$

for all extremal flows $v^{(i)}$ of Φ.

Whenever the Jacobian matrix $\nabla_v S(v, z)$ is symmetric for fixed z, (4) is equivalent to solving the convex programming problem:

$$\operatorname*{Min}_{v\in\Phi} \int_0^v S(t, z)\,\mathrm{d}t \tag{5}$$

where the line integral does not depend on the path of integration. The solution of (5) may be efficiently found by using primal methods of feasible directions for solving convex flow problems (see Nguyen [21], Leblanc [17], Daganzo [11], Bertsekas and Gafni [5], Florian [14]).

Assumption 1. For z fixed, the function $S(v, z)$ is continuously differentiable, positive and strictly monotone for all nonnegative v.

Under Assumption 1, a solution to (4) exists and is unique (Aashtiani and Magnanti [1]).

Using the previous notation, the bilevel program (2) may be written as:

$$\begin{aligned} &\operatorname*{Min}_{\substack{z\geq 0\\ v\in\Phi}} \quad v^{\mathrm{T}} S(v, z) + g(z) \\ &\text{subject to} \quad (v - v^{(i)})^{\mathrm{T}} S(v, z) \leq 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{6}$$

where $g(z)$ represents the capital investment and operating costs of building and operating a network with capacities z.[1]

[1] Assuming a discount factor $\rho > 0$, we have: $g(z) = [\rho/(1+\rho)]c(z) + m(z)$ where $c(z)$ is the actual cost of building the network and $m(z)$ is the operating cost.

Formulation (6) suggests a finitely convergent constraint accumulation algorithm (see Marcotte [20]). This can be contrasted with a somewhat similar result (Corollary 2.1) in Blankenship and Falk [7]) where $\bar{S}(v) \triangleq \int_0^v S(t, z)\,\mathrm{d}t$ is *concave* rather than convex.

Assumption 2. The function $g(z)$ is link-separable, i.e.,

$$g(z) = \sum_{a \in A} g_a(z_a) \quad \text{with each function } g_a \text{ nonnegative}$$

increasing and continuously differentiable. Also: $g(0) = 0$.

The bilevel problem (6) may be rewritten as:

$$\operatorname*{Min}_{z \geqslant 0} \quad v(z)^{\mathrm{T}} S(v(z), z) + g(z) \tag{7}$$

where $v(z)$ is the unique solution to variational inequality (4).

Some attempts have been made to solve (7) directly, using descent methods (see [2]). Difficulties stem from the fact that the objective function, although continuous, is not in general convex or differentiable as a function of the argument z (Dafermos [9], Marcotte [19]). Also, each evaluation of the objective in (7) requires the solution of a user equilibrium problem, which is prohibitive.

We will now introduce a related problem that is easier to solve and can be used to provide both lower bounds and upper bounds on the true optimum.

A system-optimal problem is obtained by removing the variational constraint on v; problem (6) then becomes:

$$\operatorname*{Min}_{\substack{v \in \Phi \\ z \geqslant 0}} \quad F(v, z) = v^{\mathrm{T}} S(v, z) + g(z). \tag{8}$$

Let (v^*, z^*) and $(\bar{v}, \bar{z})$ be the solutions to (6) and (8) respectively. Since $v(z)$, the user-optimum flow corresponding to capacity vector z, is a feasible solution for problem (6), we have the following obvious relations:

$$F(\bar{v}, \bar{z}) \leqslant F(v(z^*), z^*) = F(v^*, z^*) \leqslant F(v(\bar{z}), \bar{z}).$$

Assumption 3. The function $S(v, z)$ is link-separable and has the form: $S(v, z) = (S_a(v_a/z_a))_{a \in A}$, where S_a are positive, increasing, continuously differentiable and convex functions. By convention we set $S_a(v_a/z_a) = 0$ whenever $v_a = z_a = 0$.

Under Assumption 3, the function $v^{\mathrm{T}} \cdot S(v, z)$ can easily be shown to be convex. Furthermore, if the function g is convex, then it follows that (8) is a convex programming problem in the variables v and z.

It can be further simplified (Los [18], Marcotte [20]) by first performing the minimization with respect to the z_a-variables, for each link-flow v_a fixed:

$$\operatorname*{Min}_{z_a \geq 0} v_a S_a\left(\frac{v_a}{z_a}\right) + g_a(z_a) \tag{9}$$

whose unique solution will be denoted $z_a(v_a)$. This yields the problem:

$$\operatorname*{Min}_{v \in \Phi} \quad \sum_{a \in A} v_a S_a\left(\frac{v_a}{z_a(v_a)}\right) + g_a(z_a(v_a)). \tag{10}$$

This last formulation is that of a standard convex flow assignment problem.

3. Linear or concave objective

When the cost function g is linear, both the system-optimum and user-optimum problem possess interesting properties. We first recall a result of Los [18]:

Proposition 3. *If Assumptions* 1, 2, 3 *hold and the cost function g is linear, i.e., $g_a(z_a) = l_a z_a \; \forall a \in A$, then the objective function in problem* (10) *is linear.*

Proof. Replacing $g_a(z_a)$ by $l_a z_a$ in (9) we obtain the one dimensional minimization problems:

$$\operatorname*{Min}_{z_a \geq 0} \quad v_a S_a\left(\frac{v_a}{z_a}\right) + l_a z_a. \tag{11}$$

By differentiating with respect to z_a, we obtain the necessary and sufficient condition for a minimum:

$$\left(\frac{v_a}{z_a}\right)^2 S'_a\left(\frac{v_a}{z_a}\right) = l_a. \tag{12}$$

Let u_a be the unique solution to the equation $x^2 S'_a(x) = l_a$. Then u_a is positive and assigns a value to the ratio v_a/z_a, including the case where $v_a = z_a = 0$.[2]

Replacing z_a by v_a/u_a in the objective (11) yields:

$$F(v, z) = \sum_{a \in A} \left(\frac{l_a}{u_a} + S_a(u_a)\right) v_a$$

which is linear in v. □

Proposition 3 shows that the system-optimum problem (11) can be solved by performing shortest path calculations.

The next proposition states that, under slightly more restrictive assumptions, the system-optimum and user-optimum problems are equivalent, regardless of the topology of the maximal network or the values of the entries of the trip matrix.

[2] This will also be true of all heuristics considered in this paper.

Assumption 4. The cost function is linear and each of its components is proportional to the corresponding arc length and capacity, i.e.,

$$g_a(z_a) = ld_a z_a$$

where d_a represents the physical length of arc a, and l is a constant.

Assumption 4 implies that the investment and maintenance costs are homogeneous throughout the network.

Assumption 5. The link traversal times are nonnegative and homogeneous throughout the network and proportional to the arc lengths, i.e.,

$$S_a\left(\frac{v_a}{z_a}\right) = d_a\sigma\left(\frac{v_a}{z_a}\right) \quad \text{for some function } \sigma.$$

Proposition 4. *Under Assumptions* 1, 2, 3, 4 *and* 5, *the solution*(s) *to the system-optimum problem* (8) *is feasible and thus optimal for the user-optimal problem* (6).

Proof. Under the above assumptions, the system-optimum problem is

$$\min_{\substack{v\in\Phi \\ z\geq 0}} \sum_{a\in A} v_a d_a \sigma\left(\frac{v_a}{z_a}\right) + ld_a z_a. \tag{13}$$

At the optimum one must have:

$$\left(\frac{v_a}{z_a}\right)^2 \sigma'\left(\frac{v_a}{z_a}\right) = l.$$

Let u be the unique solution to the equation

$$x^2\sigma'(x) = l.$$

We have: $z_a = v_a/u$. After substitution in (13) we get:

$$\min_{v\in\Phi} \sum_{a\in A} d_a\left[\sigma(u) + \frac{l}{u}\right] v_a$$

which is equivalent to:

$$\min_{v\in\Phi} \sum_{a\in A} d_a v_a. \tag{14}$$

Let v^* be a (not necessarily unique) optimal solution to (14), and let $z^* = v^*/u$. We must prove that v^* is the user equilibrium corresponding to the capacity vector z^*, i.e. that v^* satisfies the variational inequality:

$$\sum_{a\in A} (v_a^* - v_a^{(i)}) d_a \sigma\left(\frac{v_a^*}{z_a^*}\right) \leq 0, \quad i = 1, 2, \ldots n. \tag{15}$$

v^* being solution of (14), we have:

$$\sum_{a \in A} (v_a^* - v_a^{(i)}) d_a \leqslant 0, \quad i = 1, 2, \ldots n.$$

Multiplying by the constant $\sigma(v_a^*/z_a^*) = \sigma(u)$, we get the desired result. □

In general however, when Assumptions 4 and 5 are not satisfied, characterization of the optimal solution for the user-optimum problem (6) is difficult. The next proposition gives a property possessed by at least one solution flow vector v^*.

Proposition 5. *If g is linear and assumptions* 1, 2, 3 *hold, there exists a solution vector* (v^*, z^*) *to* (6) *for which* v^* *is an extremal flow, i.e.,* $v^* = v^{(i)}$ *for a certain index i.*

Proof. We make the change of variable:

$$k_a = \frac{v_a}{z_a}.$$

This is equivalent to using the congestion levels on the links of the maximal network as the primary decision variables. The user-optimum problem (6) then becomes:

$$\begin{aligned} &\min_{k \geqslant 0} \quad F(v, k) = \sum_{a \in A} v_a S_a(k_a) + \frac{l_a v_a}{k_a} \\ &\text{subject to} \quad v \in \left\{ \arg\min_{w \in \Phi} \sum_{a \in A} w_a S_a(k_a) \right\}. \end{aligned} \tag{16}$$

Let (v^*, k^*) be a solution to the above problem (16). Since Φ is convex and may be assumed bounded without loss of generality, we can express v^* as a convex combination of extremal points of Φ:

$$v^* = \sum_{i \in I} \lambda_i v^{(i)}, \quad \lambda_i > 0, \ \sum_{i \in I} \lambda_i = 1,$$

where I is a subset of $(1, 2, \ldots, n)$. Now, v^* is feasible for (16) and must therefore be a minimum for the expression

$$\sum_{a \in A} w_a S_a(k_a^*),$$

i.e.,

$$\sum_{a \in A} \sum_{i \in I} \lambda_i v_a^{(i)} S_a(k_a^*) \leqslant \sum_{a \in A} w_a S_a(k_a^*) \quad \forall w \in \Phi,$$

$$\sum_{i \in I} \lambda_i \sum_{a \in A} v_a^{(i)} S_a(k_a^*) \leqslant \sum_{a \in A} w_a S_a(k_a^*) \quad \forall w \in \Phi.$$

The λ_i being positive, we conclude that

$$\sum_{a \in A} v_a^{(i)} S_a(k_a^*) \leqslant \sum_{a \in A} w_a S_a(k_a^*) \quad \forall w \in \Phi \ \forall i \in I$$

and the $v^{(i)}$'s are feasible for (6), with $k=k^*$. Now

$$F(v^*, k^*) = \sum_{a\in A}\left(\sum_{i\in I}\lambda_i v_a^{(i)}\right)\left(S_a(k_a^*)+\frac{l_a}{k_a^*}\right)$$

$$= \sum_{i\in I}\lambda_i\left[\sum_{a\in A} v_a^{(i)}\left(S_a(k_a^*)+\frac{l_a}{k_a^*}\right)\right] = \sum_{i\in I}\lambda_i F(v^{(i)}, k^*).$$

Thus $F(v^{(i)}, k^*) = F(v^*, k^*)$ for all i in I by the optimality of (v^*, k^*), and the $v^{(i)}$'s are optimal. This completes the proof. □

Corollary 6. *If g is concave and Assumptions* 1, 2, 3 *hold, there exists an optimal solution* (v^*, z^*) *to* (6) *where v^* is an extremal flow of Φ.*

Proof. Using the notations in the proof of Proposition 5 we prove, following the same line of reasoning, the feasibility of the extremal flows $v^{(i)}$ for problem (6). Also,

$$F(v^*, k^*) = \sum_{a\in A}\left\{\left[\sum_{i\in I}(\lambda_i v_a^{(i)})S_a(k_a^*)\right] + g_a\left(\frac{\sum_{i\in I}\lambda_i v_a^{(i)}}{k_a^*}\right)\right\}$$

implies

$$F(v^*, k^*) \geqslant \left[\sum_{i\in I}\lambda_i \sum_{a\in A} v_a^{(i)} S_a(k_a^*)\right] + \left[\sum_{i\in I}\lambda_i \sum_{a\in A} g_a\left(\frac{v_a^{(i)}}{k_a^*}\right)\right] \tag{17}$$

by concavity of g_a.

Therefore

$$F(v^*, k^*) \geqslant \sum_{i\in I}\lambda_i\left[\sum_{a\in A} v_a^{(i)} S_a(k_a^*) + g_a\left(\frac{v_a^{(i)}}{k_a^*}\right)\right] = \sum_{i\in I}\lambda_i F(v^{(i)}, k^*)$$

Now (v^*, k^*) optimal implies that

$$F(v^{(i)}, k^*) = F(v^*, k^*) \quad \text{for all } i \text{ in } I$$

and the $v^{(i)}$'s are optimal for (6). □

Corollary 7. *If g is strictly concave and Assumptions* 1, 2, 3 *hold, any solution* (v^*, z^*) *to* (6) *is such that v^* is an extremal point of Φ.*

Proof. In the preceding proof, the inequality (17) is strict[3] unless I is composed of a single element i^*, and

$$\lambda_{i^*} = 1, \qquad v^* = v^{(i^*)}. \qquad □$$

[3] We rule out the rather uninteresting case where all v_a^k's are zero for some origin k.

The last two corollaries are interesting because they do not rest in any way on the concavity (or strict concavity) of the objective function $F(v, z)$. They can be contrasted with the results of Bialas and Karwan [6] pertaining to the linear bilevel programming problem. Actually, F may be nonconcave even with the S_a's convex and the g_a's concave.

Proposition 5 and its two corollaries, although providing a partial characterization of the optimal solution, cannot be used directly to design a search algorithm. There still remains the problem of optimally allocating an extremal flow $v^{(i)}$ to the network asociated with the capacity vector z.

4. Heuristic procedures

The formulation (6) suggests (see Marcotte [20]) the use of a restriction procedure for solving problem (6). Unfortunately the resulting restricted subproblems are nonconvex.

In this section we propose four heuristic procedures whose theoretical and numerical behavior will be analyzed in the next two sections. In the remainder of the paper the congestion functions will be of the BPR-type, i.e.,

$$S_a\left(\frac{v_a}{z_a}\right) \equiv \alpha_a + \beta_a\left(\frac{v_a}{z_a}\right)^p \quad \text{with } p \text{ positive,} \tag{18}$$

and the cost functions will be power functions: $g_a(z_a) = l_a z_a^m$ with $m > 0$, thus satisfying Assumptions 1, 2 and 3 previously stated.

One heuristic (namely H3) relies on these particular functional forms. The others can be readily applied to the general case.

4.1. Heuristic H0

This heuristic determines a feasible suboptimal solution $(v(\bar{z}), \bar{z})$, where $\bar{z}$ is the vector of capacities on the network that are optimal in the system-optimal problem (8), and $v(\bar{z})$ is the used equilibrium flow vector corresponding to the network having $\bar{z}$ as its capacity vector. That is, we let $(\bar{v}, \bar{z})$ be solution of the system-optimum problem (8) and let $(v(\bar{z}), \bar{z})$ be the feasible suboptimal solution.

In view of the difficulty of solving the user-optimum problem, this procedure has been proposed, among others, by Dantzig et al. [12].

4.2. Heuristic H2

This procedure has been applied by Tan et al. [26] to the Traffic Signals Setting Problem. One iteration of the algorithm consists of solving sequentially an optimization problem involving the capacity variables, with the flow variables fixed, and a user-optimized equiibrium problem corresponding to this new capacity vector. After several iterations of this process, one hopes that convergence toward a 'good' solution will be obtained.

Algorithm H2

Step 1 (Initialization). Choose any positive capacity vector $\bar{z}$.

Step 2. Solve the equilibrium problem (equilibrium phase)

$$\operatorname*{Min}_{v \in \Phi} \sum_{a \in A} \int_0^{v_a} S_a\left(\frac{t}{\bar{z}_a}\right) \mathrm{d}t \tag{19}$$

whose solution is $\bar{v}$.

Step 3. Solve (optimization phase)

$$\operatorname*{Min}_{z \geq 0} \sum_{a \in A} \bar{v}_a S_a\left(\frac{\bar{v}_a}{z_a}\right) + g_a(z_a) \tag{20}$$

whose solution is z^*; set $\bar{z} \leftarrow z^*$.

If $\|\bar{z} - z^*\| < \varepsilon$, STOP; otherwise return to step 2. $\triangle$

The optimization phase at step 3 simply consists of finding the roots of the equations:

$$\left(\frac{v_a}{z_a}\right)^2 S_a'\left(\frac{v_a}{z_a}\right) = g_a'(z_a), \quad a \in A.$$

It is shown in Marcotte [19, 20] that heuristic H2 actually solves the following convex optimization problem:

$$\operatorname*{Min}_{v \in \Phi} \sum_{a \in A} \int_0^{v_a} S_a\left(\frac{t}{z_a(t)}\right) \mathrm{d}t \tag{21}$$

where $z_a(v_a)$ is solution of (20).

Under our assumptions about the functional form of S_a and g_a we have:

$$z_a(v_a) = \left(\frac{p\beta_a}{ml_a}\right)^{1/(m+p)} \cdot v_a^{(p+1)/(p+m)}. \tag{22}$$

4.3. Heuristic H3

The idea consists of finding a capacity vector for which the system-optimal flow vector $\bar{v}$ is an equilibrium flow.

Putting the value of $z_a(v_a)$ given by expression (22) into the objective $F(v, z)$ we find

$$\bar{v} \in \left\{ \operatorname*{arg\,min}_{v \in \Phi} \sum_{a \in A} v_a\left(\alpha_a + d_a v_a{}^{p(m-1)/(m+p)}\right)\right\} \tag{23}$$

where

$$d_a = (p+m)\left(\frac{\beta_a}{m}\right)^{m/(m+p)} \left(\frac{l_a}{p}\right)^{p/(m+p)} = \frac{\beta_a}{m}(p+m)\left(\frac{ml_a}{p\beta_a}\right)^{p/(p+m)}.$$

In order that the solution be an equilibrium, it must satisfy the variational inequality

$$\sum_{a\in A}(v_a - v_a^{(i)})\left(\alpha_a + \beta_a\left(\frac{v_a}{z_a}\right)^p\right) \leq 0, \quad i=1,\dots,n. \tag{24}$$

Now, (24) corresponds to the mathematical program:

$$\operatorname*{Min}_{v\in\Phi} \quad \sum_{a\in A}\int_0^{v_a}\left[\alpha_a + \beta_a\left(\frac{t}{\zeta_a(t)}\right)^p\right]\mathrm{d}t \tag{25}$$

where $\zeta_a(v_a)$ is a function giving the desired value of z_a, and depends on the corresponding link-flow v_a. The problems defined by (23) and (25) will possess identical solutions if:

$$\int_0^{v_a}\left[\left(\alpha_a + \beta_a\left(\frac{t}{\zeta_a(t)}\right)^p\right]\mathrm{d}t \equiv v_a(\alpha_a + d_a v_a^{p(m-1)/(m+p)})$$

for all $v_a \geq 0$.

This identity is realized if we take

$$\zeta_a(v_a) = b_a v_a^n \quad \text{with } n = \frac{p+1}{p+m} \text{ and } b_a = \left(\frac{1}{p+1}\right)^{1/p}\left(\frac{p\beta_a}{ml_a}\right)^{1/(p+m)} \tag{26}$$

Thus the solution vector $(\bar{v}_a, b_a\bar{v}_a^n)_{a\in A}$ is feasible for (6).

4.4. *Heuristic H5*

Heuristics H2 and H3 can be viewed as particular instances of a general heuristic procedure which consists of solving a convex network optimization problem in v and z whose solution satisfies the variational constraints (6). Indeed, consider the parameterized problem:

$$\operatorname*{Min}_{v,z} \quad J(v,z) \triangleq \sum_{a\in A}\int_0^{v_a}\left[\alpha_a + \beta_a\left(\frac{t}{z_a}\right)^p\right]\mathrm{d}t + \xi_a l_a z_a^m. \tag{27}$$

There exists a vector $\xi = (\xi_a)_{a\in A}$ such that the optimal solution to (27) is optimal for (6). However the choice of such an optimal ξ is of the same degree of difficulty as solving the original bilevel problem. The rational behind heuristic H5 is to set all ξ_a's to some predetermined value ξ, yielding the following convex mathematical programming problem:

$$\operatorname*{Min}_{v,z} \sum_{a\in A} v_a\left[\alpha_a + \frac{\beta_a}{p+1}\left(\frac{v_a}{z_a}\right)^p\right] + \xi l_a z_a^m. \tag{28}$$

This corresponds to minimizing the weighted sum of user-perceived travelling times and of the investment and maintenance costs of the network.

If ξ is set to the value $1/(p+1)$, then the optimal solution to (28) consists of an equilibrium flow vector $\tilde{v}$ associated with a capacity vector $\tilde{z}$ satisfying the system-optimal relation (22). Since this corresponds to the characterization of the solutions obtained from heuristic H2, it can be concluded that heuristic H2 is subsumed by heuristic H5. Similarly, it is easily shown that heuristic H3 corresponds to setting ξ to the value $p+1$.

5. Worst-case analysis of the heuristics

In this section we develop upper bounds on the ratio of the values of heuristic and exact solutions. The analysis does not depend on the network structure.

Definition

$$R_m^p(H) = \operatorname*{Sup}_{\substack{\alpha,\beta,l\\ N,A}} \frac{\text{Cost of heuristic solution}}{\text{Cost of optimal solution}} = \operatorname*{Sup}_{\substack{\alpha,\beta,l\\ N,A}} \frac{F(v(z^H), z^H)}{F(v^*, z^*)}$$

where z^H is the capacity vector obtained from heuristic H (H = H0, H2, H3 or H5).[4]

5.1. Heuristic $H0$

Proposition 8. $\lim_{p\to\infty} R_1^p(H0) \geq 2$.

Proof. Consider the network of Figure 1.
We set

$$S_1(x) = p, \qquad S_2(x) = S_3(x) = \tfrac{1}{2}(1 + x^{p^2}),$$

$$g_1(x) = 0, \qquad g_2(x) = g_3(x) = \tfrac{1}{2}px.$$

Since links 2 and 3 play similar roles, we can replace them by a fictitious link 4 that must carry a flow of at least p units, and with congestion and cost functions given by

$$S_4(x) = 1 + x^{p^2}, \qquad g_4(x) = px.$$

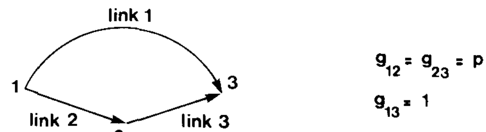

Fig. 1. Worst-case network for heuristic H0.

[4] Heuristics $H1$ and $H4$, based on nonconvex network optimization programs, are analyzed in [20].

The system-optimum problem is

$$\operatorname*{Min}_{\substack{v\in\Phi\\ z\geqslant 0}} \quad pv_1+v_4\left[1+\left(\frac{v_4}{z_4}\right)^{p^2}\right]+pz_4.$$

The optimal z_4 must satisfy

$$p^2\left(\frac{v_4}{z_4}\right)^{p^2+1}=p$$

and the problem becomes

$$\operatorname*{Min}_{v\in\Phi} \quad pv_1+v_4\left[1+\left(\frac{1}{p}\right)^{p^2/(p^2+2)}+p^{(p^2+2)/(p^2+1)}\right].$$

The optimal solution is

$$\bar{v}_1=1, \qquad \bar{v}_4=p, \qquad \bar{z}_4=p^{(p^2+1)/(p^2+1)}.$$

The cost of this solution is $F(\bar{v},\bar{z})=p^2+\mathrm{O}(p)$, where $\lim_{p\to\infty}\mathrm{O}(p)/p<\infty$.

Now the equilibrium is not reached since travel time on link 4 is less than the corresponding travel time on link 1. A fraction of travellers will therefore switch from link 1 to link 4, until travel times on both links are equal to p. This equilibrium occurs before v_1 gets to zero, since we have:

$$1+\left(\frac{\bar{v}_4+1}{\bar{z}_4}\right)^{p^2}=1+\left(\frac{p+1}{p^{(p^2+2)/(p^2+1)}}\right)^{p^2}>p \text{ for } p \text{ sufficiently large.}$$

The cost of this solution is

$$F(v(\bar{z}),\bar{z})=2p^2+\mathrm{O}(p).$$

The optimal solution consists of slightly increasing the capacity on arc 4 to allow for all $(p+1)$ travellers; the corresponding cost is

$$F(v^*,z^*)=p^2+\mathrm{O}(p)$$

Thus $\lim_{p\to\infty}F(v(\bar{z}),\bar{z})/F(v^*,z^*)\geqslant 2$. □

5.2. Heuristic H2

Proposition 9. $R_1^p(H2)=p+1$.

Proof. For $m=1$ and $S_a(x)=\alpha_a+\beta_a x^p$, problem (21) takes the form

$$\operatorname*{Min}_{v\in\Phi} \quad \sum_{a\in A} v_a\left[\alpha_a+\beta_a\left(\frac{l_a}{p\beta_a}\right)^{p/(p+1)}\right]. \tag{29}$$

Let $\tilde{v}$ be solution of (29) and $\tilde{z}_a = z_a(\tilde{v}) = (p\beta_a/l_a)^{1/(p+1)}\tilde{v}_a$. We have

$$\begin{aligned}
F(\tilde{v}, \tilde{z}) &= \sum_{a\in A} \tilde{v}_a\left[\alpha_a + \beta_a\left(\frac{\tilde{v}_a}{\tilde{z}_a}\right)^p\right] + l_a\tilde{z}_a \\
&= \sum_{a\in A} \tilde{v}_a\left[\alpha_a + (p+1)\beta_a\left(\frac{l_a}{p\beta_a}\right)^{p/(p+1)}\right] \\
&\leqslant (p+1)\sum_{a\in A} \tilde{v}_a\left[\alpha_a + \beta_a\left(\frac{l_a}{p\beta_a}\right)^{p/(p+1)}\right] \\
&\leqslant (p+1)\sum_{a\in A} \bar{v}_a\left[\alpha_a + \beta_a\left(\frac{l_a}{p\beta_a}\right)^{p/(p+1)}\right] = (p+1)F(\bar{v}, \bar{z})
\end{aligned}$$

where $\bar{v}$ is solution of the system-optimum problem.

Finally,

$$F(\tilde{v}, \tilde{z}) \leqslant (p+1)F(\bar{v}, \bar{z}) \leqslant (p+1)F(v^*, z^*).$$

This proves that $R_1^p(H2) \leqslant p+1$. To prove the reverse inequality, consider the network of Figure 2.

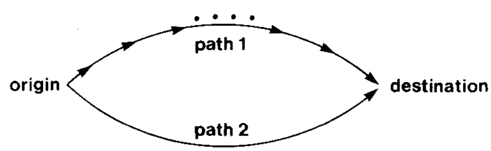

Fig. 2. Worst-case network for heuristic H2.

Path 1 is composed of n identical arcs for which $\alpha_a = \beta_a = 1$ and $l_a = p(1/n)^{(p+1)/p}$.
Path 2 comprises a unique arc with $\alpha_a = \beta_a = 1$ and $l_a = p(n-1)^{(p+1)/p}$.
One unit of flow must travel from the origin to the destination.
Heuristic H2 yields the path flows:

$$\tilde{v}_1 = 0, \qquad \tilde{v}_2 = 1,$$

with total cost

$$F(\tilde{v}, \tilde{z}) = n(p+1) - p.$$

The optimal solution consists in routing all the flow on path 1, with cost: $F(v^*, z^*) = n + p + 1$.

Therefore

$$\lim_{n\to\infty} \frac{F(\tilde{v}, \tilde{z})}{F(v^*, z^*)} = \lim_{n\to\infty} \frac{n(p+1)-p}{n+p+1} = p+1. \qquad \square$$

5.3. Heuristic H3

Proposition 10

$$R_m^p(H3) = \frac{m(p+1)}{m+p} + \frac{p}{(m+p)(p+1)^{m/p}}.$$

Proof. Let $(\bar{\bar{v}}, \bar{\bar{z}})$ be the solution given by heuristic H3. From (26) we find its cost:

$$F(\bar{\bar{v}}, \bar{\bar{z}}) = \sum_{a\in A} \bar{\bar{v}}_a \left\{ \alpha_a + \left[\frac{m(p+1)}{p+m} d_a + l_a \left(\frac{\beta_a}{d_a} \cdot \frac{m+p}{m(p+1)} \right)^{m/p} \right] \bar{\bar{v}}_a^{p(m-1)/(p+m)} \right\}.$$

The error is maximum when all α_a are zero, which yields:

$$R_m^p(H3) = \operatorname*{Sup}_{\beta_a, l_a} \frac{m(p+1)}{m+p} + \left[\frac{m+p}{m(p+1)} \right]^{m/p} \frac{\beta_a^{m/p} l_a}{d_a^{(m+p)/p}}$$

$$= \frac{m(p+1)}{m+p} + \frac{p}{(m+p)(p+1)^{m/p}}. \quad \square$$

For $m = 1$ we get

$$R_1^p(H3) = 1 + \frac{p}{(p+1)^{1+1/p}} \leqslant p + 1 = R_1^p(H2).$$

Corollary 11. $\lim_{m\to 0, m>0} R_m^p(H3) = 1$ *and* $\lim_{p\to\infty} R_1^p(H3) = 2$.

Proof. The proof is straightforward. $\square$

The first statement of Corollary 11 indicates that, in the important particular case where the investment costs are strongly concave (economies of scale exist), the solutions to the user-optimized and system-optimized problems will be almost similar. Concavity of the g_a's leads however to a concave optimization problem for which no efficient solution procedures exist yet. The limit case $m = 0$ corresponds to the discrete network design problem (without congestion) and has been treated extensively in the literature: Magnanti and Wong [28], Hoang [16], Dionne [13], etc.

5.4. Heuristic H5

The following lemma will be used to find bounds for $R_1^p(H5)$.

Lemma 12. *Let* $\alpha = (\alpha_a)_{a\in A}$ *and* $\beta = (\beta_a)_{a\in A}$ *be two arc length vectors of non-negative numbers and* $k > 0$. *Let*:

$$\bar{v} \in \left\{ \operatorname*{arg\,min}_{v\in\Phi} \bar{F}(v) \triangleq (\alpha+\beta)^{\mathrm{T}} v \right\}, \tag{30}$$

$$\bar{\bar{v}} \in \left\{ \operatorname*{arg\,min}_{v\in\Phi} \bar{\bar{F}}(v) \triangleq (\alpha+k\beta)^{\mathrm{T}} v \right\}. \tag{31}$$

Then we have

$$\operatorname*{Sup}_{\alpha,\beta,\Phi} \frac{\bar{\bar{F}}(\bar{v})}{\bar{\bar{F}}(\bar{\bar{v}})} = \operatorname{Max}\left\{ k, \frac{1}{k} \right\}.$$

Proof. Suppose k is greater than 1 and consider a network with two parallel arcs linking an origin and a destination. Let $\alpha=(n,1)$, $\beta=(0, n-1-\frac{1}{n})$ for n large. We have: $\bar v=(0,1)$ since

$$\alpha_1+\beta_1=n>1+n-1-\frac{1}{n}=\alpha_2+\beta_2.$$

Also: $\bar{\bar v}=(1,0)$ since

$$\alpha_1+k\beta_1=n<1+k\left(n-1-\frac{1}{n}\right)=\alpha_2+k\beta_2,$$

for n sufficiently large. Hence

$$\frac{\bar{\bar F}(\bar v)}{\bar{\bar F}(\bar{\bar v})}=\frac{1+k(n-1-1/n)}{n}\xrightarrow[n\to\infty]{} k$$

implies

$$\operatorname*{Sup}_{\alpha,\beta,\Phi}\frac{\bar{\bar F}(\bar v)}{\bar{\bar F}(\bar{\bar v})}\geq k.$$

But

$$\bar{\bar F}(\bar v)\leq k\bar F(\bar v)\leq k\bar F(\bar{\bar v})\leq k\bar{\bar F}(\bar{\bar v}) \text{ always holds, and}$$

$$\operatorname*{Sup}_{\alpha,\beta,\Phi}\frac{\bar{\bar F}(\bar v)}{\bar{\bar F}(\bar{\bar v})}\leq k.$$

Therefore

$$\operatorname*{Sup}_{\alpha,\beta,\Phi}\frac{\bar{\bar F}(\bar v)}{\bar{\bar F}(\bar{\bar v})}= k.$$

If k is less than 1 we replace problem (31) by the equivalent problem: $\mathrm{Min}_{v\in\Phi}\,(\alpha/k+\beta)^{\mathrm{T}}v$. Reasoning exactly as before, exchanging the roles of α and β, we find

$$\operatorname*{Sup}_{\alpha,\beta,\Phi}\frac{\bar{\bar F}(\bar v)}{\bar{\bar F}(\bar{\bar v})}=\frac{1}{k}.$$

Removing the condition on k yields the desired result. □

Proposition 13

$$1+\frac{p}{\xi(p+1)}\leq R_1^p(H5)\leq\frac{\xi^{p/(p+1)}}{(p+1)^{1/(p+1)}}\left[1+\frac{p}{\xi(p+1)}\right]^2.$$

Proof. After substituting z_a by its value in the objective of (27) and setting m to 1, we get the problem:

$$\operatorname*{Min}_{v\in\Phi}\quad\sum_{a\in A}v_a\left[\alpha_a+\frac{\xi^{p(p+1)}}{(p+1)^{1/(p+1)}}(p+1)\beta_a^{1/(p+1)}\left(\frac{l_a}{p}\right)^{p/(p+1)}\right]. \tag{32}$$

The real cost of the solution $v^{(\xi)}$ to (27) is:

$$F(v^{(\xi)}, z^{(\xi)}) = \sum_{a\in A} v_a^{(\xi)}\left[\alpha_a + \beta_a\left(\frac{v_a^{(\xi)}}{z_a^{(\xi)}}\right)^p\right] + l_a z_a^{(\xi)}$$

where $z_a^{(\xi)} = [p\beta_a/(p+1)\xi l_a]^{1/(p+1)} v_a^{(\xi)}$, for all a in A. After replacing $z_a^{(\xi)}$ by its value we obtain

$$F(v^{(\xi)}, z^{(\xi)}) = \sum_{a\in A} v_a^{(\xi)}\left[\alpha_a + \frac{1}{p+1}\left(\xi^{p/(p+1)}(p+1)^{p/(p+1)} + \frac{p}{\xi^{1/(p+1)}(p+1)^{1/(p+1)}}\right)e_a\right] \tag{33}$$

with

$$e_a = (p+1)\beta_a^{1/(p+1)}\left(\frac{l_a}{p}\right)^{p/(p+1)}.$$

Applying Lemma 12 to expressions (32) and (33) we get:

$$\begin{aligned} R_1^p(\mathrm{H5}) &\geqslant \frac{1}{p+1}\left[\xi^{p/(p+1)}(p+1)^{p/(p+1)} + \frac{p}{\xi^{(1/(p+1)}(p+1)^{1/(p+1)}}\right]\cdot\frac{(p+1)^{1/(p+1)}}{\xi^{p/(p+1)}} \\ &= \frac{1}{p+1}\left(p+1+\frac{p}{\xi}\right) = 1 + \frac{p}{\xi(p+1)} > 1. \end{aligned}$$

This proves the first inequality. Now consider the system-optimal cost function, obtained after replacing z by its value:

$$F(v, z(v)) = \sum_{a\in A} v_a(\alpha_a + e_a). \tag{34}$$

Applying again Lemma 12 to (32) and (34), the second inequality will hold if the quantity

$$\sigma(\xi) = \frac{\xi^{p/(p+1)}}{(p+1)^{1/(p+1)}}\left(1 + \frac{p}{\xi(p+1)}\right)$$

is greater or equal to 1. Letting $p/(p+1) = q$ we have

$$\sigma(\xi) = \frac{1}{(1/(1-q))^{1-q}}\xi^q\left(1+\frac{q}{\xi}\right).$$

The minimum value of $\sigma(\xi)$ is obtained when $q\xi^{q-1} + q(q-1)\xi^{q-2} = 0$ or $\xi = 1-q$ and

$$\begin{aligned} \sigma(\xi) &\geqslant (1-q)^{1-q}(1-q)^q\left(1+\frac{q}{1-q}\right) \\ &= (1-q)^{1-q}\frac{(1-q)^q}{1-q} = (1-q)^{1-q}(1-q)^{q-1} = 1. \qquad \square \end{aligned}$$

Corollary 14. *For $\xi=1$ we have $2 \leqslant \lim_{p\to\infty} R_1^p(\text{H5})=4$.*

Proof. Replace ξ by 1 in Proposition 13; we have

$$\lim_{p\to\infty} 1+\frac{p}{p+1}=2 \quad \text{and} \quad \lim_{p\to\infty} \frac{1}{(p+1)^{1/(p+1)}}\left(1+\frac{p}{p+1}\right)^2=4. \qquad \square$$

In order to obtain strong bounds for heuristic H5, we can choose $\xi=\xi(p)$ such as to minimize the term $U(\xi)=\sigma(\xi)(1+p/\xi(p+1))$. We make the change of variable: $p/(p+1)=q$, and we rewrite:

$$U(\xi)=(1-q)^{1-q}\xi^q\left(1+\frac{q}{\xi}\right)^2.$$

We want to minimize the term $V(\xi)=\xi^q(1+q/\xi)^2=\xi^q+2q\xi^{q-1}+q^2\xi^{q-2}$ with respect to the variable ξ.

The minimum value is obtained at the positive zero of the derivative:

$$V'(\xi)=q\xi^{q-3}(\xi^2+2(q-1)\xi+q(q-2)).$$

Thus

$$\xi=2-q=2-\frac{p}{p+1}. \qquad \square$$

Remarks. For $\xi=1/(p+1)$ (heuristic H2), we have $\sigma(\xi)=1$ and $R_1^p(\text{H2})=p+1$, which corresponds to the statement of Proposition 9.

For $\xi=p+1$ (heuristic H3) we have $R_1^p(\text{H5}) \geqslant 1+p/(p+1)^2$ and equality holds for $p=1$. For p less or equal to 1, it provides a valid *upper* bound; for p greater than 1 it is not valid any more, but the upper bound is much too large, being $O(p)$. In general the upper bound on $R_1^p(\text{H5})$ will be quite pessimistic.

Following the same line of reasoning, it is possible to derive bounds for $R_m^p(\text{H5})$. The proof requires the equivalent of Lemma 12 for nonlinear cost functions and is technical. Furthermore the upper bounds tend to be loose.

6. Computational results

The numerical tests have been performed on a network taken from Steenbrink [25] and reproduced in Figure 3. It is characterized by a large number of paths between pairs of nodes, and these paths share many common arcs. The first data set utilizes Steenbrink's congestion functions. The second data set utilizes a much denser trip matrix; parameters have been chosen in such a way as to induce high congestion levels, hoping that there would be significant differences between the system-optimum and user-optimum solutions. However, in one instance the H2-solution was actually better than the system-optimal solution! This result can be explained by numerical inaccuracies arising while solving convex flow problems. These were solved using a modification by Florian [14] of the Frank–Wolfe procedure

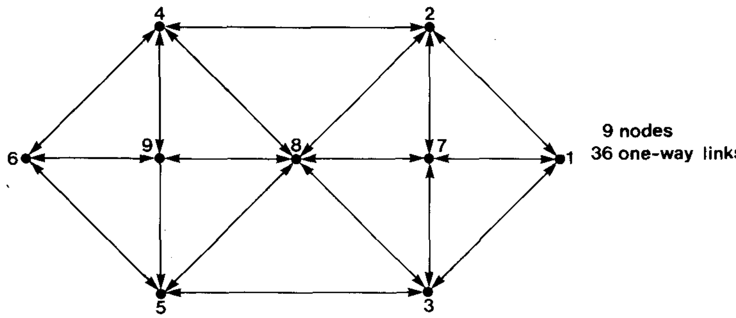

Fig. 3. Steenbrink's network: 9 nodes, 36 links.

requiring more computer memory, but with much better convergence properties. Furher details on these tests may be found in Marcotte [20].

Tables 1 and 2 give the ratios of the heuristic objective to the system-optimal objective, for data sets 1 and 2 respectively.

In the first experience with data set 1, the values of the H0 and H2 solutions are identical, although the flows and capacities are quite different. This suggests the existence of distinct optima.

Table 1
Numerical results with first data set

	p	m	H0	H2	H3	H5 ($\xi = 1$)
1	1	1	1.003	1.003	1.080	1.023
2	1	2	1.002	1.001	1.202	1.025
3	1	3	1.018	1.034	1.270	1.038
4	2	1	1.000	1.000	1.202	1.082
5	2	2	1.002	1.001	1.583	1.134
6	4	1	1.000	1.000	1.379	1.216
7	4	2	1.000	0.999*	1.953	1.358

* Due to imperfect convergence.

Table 2
Numerical results with second data set

	p	m	H0	H2	H3	H5 ($\xi = 1$)
1	1	1	1.000	1.019	1.193	1.068
2	1	2	1.002	1.005	1.367	1.046
3	2	1	1.003	1.006	1.320	1.139
4	2	2	1.001	1.003	1.627	1.146
5	4	1	1.002	1.007	1.449	1.256
6	4	2	1.000	1.001	1.936	1.350

Solutions given by heuristic H0 are consistently the best, and so close to the system-optimum value, that they are, to all extent, optimal. This good performance of H0 has been previously reported in Los [18]. One should note finally that, unlike heuristics H3 and H5, the quality of the H0 and H2 solutions does not seem to be much influenced by various of the parameters p and m.

8. Conclusion

In this paper we have conducted a fairly extensive study of a particular kind of continuous, nonlinear network design problem, which can be modelled as a bilevel programming problem. Although there is not much hope of developing exact solution algorithms for large or even medium-size networks, the numerical experiments tend to show that heuristics H0 and H2 yield near-optimal solutions.

Theoretical attempts have been made to locally improve on the solution obtained from H0, based on cutting-plane ideas derived from the nonconvex formulation (6). This resulted in two algorithms, H1 and H4, that have been described and analyzed in [20]. However the overall complexity of these algorithms more than offsets the slight improvements they could bring to the suboptimal solutions provided by either H0 or H2.

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