Network planning using two-stage programming under uncertainty

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Definition of the problem

The underlying deterministic problem is the following:

$$\min(c'u + d'v)$$
$$Au \geq b$$
$$Tu + Mv \geq \xi,$$

where ξ is random variable.

First stage problem:

$$\begin{aligned} &\min(c'u + \mathrm{E}(\mu(u,\xi))) \\ &Au \geq b, \\ &u \in K, \end{aligned}$$

where $\mu(u,\xi)$ is the optimal value of the **Second stage problem**

$$\begin{aligned} &\min(d'v) \\ &Tu + Mv \geq \xi, \end{aligned}$$

and $K = \{u; \forall\xi\ \exists v : Tu + Mv \geq \xi\}$.

How to solve?

- Solve the first stage problem (function to be minimalized depends only on u)
- Realize ξ
- Decide on v

Properties

- K is a convex polyhedron.
- For every fixed ξ the second stage optimal value $\mu(u,\xi)$ is a convex function of the variable u.
- If the set of possible values of ξ is convex, then $\mu(u,\xi)$ is convex in ξ.
- If $\mathrm{E}(\xi)$ exists and $\mu(u,\xi)$ is finite for every u and ξ, then $\mathrm{E}\,\mu(u,\xi)$ exists and is a convex function of u.

Up to now we assumed that there exists $\mu(u,\xi)$ for every ξ. Because such assumption is not always valid, we will require the solvability of the second stage problem only by probability:

First stage problem:

$$\begin{aligned} \min(c'u + \mathrm{E}(\mu(u,\xi))) \\ Au \geq b, \\ P(u \in K) \geq p, \end{aligned}$$

where p is prescribed and $\mu(u,\xi)$ is the optimal value of the new **Second stage problem**

$$\begin{aligned} \min(d'v + t'z) \\ Tu + Mv + z \geq \xi, \end{aligned}$$

where $t'z$ is the cost of infeasibility. Sometimes non-linear function $\sum t_j(z_j)$ will be used.

Assumptions

- The power systems form a network.
- The system must have a reliability level $1 - \text{LOLP} \geq p$, where LOLP is the probability that demand will be higher than supply.
- We may increase the power generating and the tie line capacities.
- No distinction will be made between capacity and reserve capacity.
- Each power system can help others. The tie line between systems a and b is not necessarily the same as that between b and a.
- Power dispatching and outage cost money.

Notations

N	Number of power systems
x_j	Generating capacity of the j^{th} system, to be determined
$x_j^{(l)}, x_j^{(u)}$	Prescribed lower and upper bound for x_j
y_{hk}	Capacity of the line between system h and system k, to be determined
$y_{hk}^{(l)}, x_{hk}^{(u)}$	Prescribed lower and upper bound for y_{hk}
i_j	Total power dispatched from j^{th} system
i_{hk}	Power dispatched for system h to system k
p	Prescribed lower level for $1 - \text{LOLP}$
ξ_j	Random demand for electric power at the j^{th} system
$c_j(.)$	Cost function of the generating capacity at the j^{th} system
$c_{hk}(.)$	Cost function of the interconnection capacity between systems h and k
d_{hk}	Cost of dispatching one unit of power from system h to system k
$t_j(.)$	Cost of outage of magnitude z at system j

Let $A(x, y, \xi)$ denote the event that for fixed x and y the total demand in the pool can be met by a suitable power dispatch.

The problem now looks like the following:

$$\min(\sum_j c_j(x_j) + \sum_h \sum_k c_{hk}(y_{hk}) + \mathrm{E}(\mu))$$

$$P(A(x, y, \mu)) \geq p,$$
$$x_j^{(l)} \leq x_j \leq x_j^{(u)},$$
$$y_{hk}^{(l)} \leq y_{hk} \leq y_{hk}^{(u)}$$

A **flow** i on a network is defined as a function on $N \times N$ satisfying for all $h, k \in N$ the relations

$$\begin{aligned} i(h,k) + i(k,h) &= 0, \\ i(h,k) &\leq y_{hk}. \end{aligned}$$

Further define **network demand** $q : N \to \mathbb{R}$ as $q_j = x_j - \xi_j$ and for $S, T \subset N$:

$$\begin{aligned} q(S) &= \sum_{h \in S} q(h), \\ i(S,T) &= \sum_{h \in S} \sum_{k \in T} i(h,k). \end{aligned}$$

Properties of q and i
For all S_1, S_2, T such as $S_1 \cap S_2 = \emptyset$ we have

$$\begin{aligned} q(S_1 \cup S_2) &= q(S_1) + q(S_2), \\ i(S_1 \cup S_2, T) &= i(S_1, T) + i(S_2, T), \\ i(T, S_1 \cup S_2) &= i(T, S_1) + i(T, S_2), \\ i(S, S) &= 0, \\ i(S, T) &\le y_{ST}, \end{aligned}$$

where

$$y(S, T) = \sum_{h \in S} \sum_{k \in T} y_{ST}$$

The network demand is called **feasible** if there exists a flow i such that for every $a \in N$ we have

$$i(N, a) \geq q(a)$$

Theorem (Feasibility theorem)

The network demand q is feasible if and only if for every $S \subset N$ we have

$$q(\overline{S}) \leq y(S, \overline{S}).$$

Example 1

Let us consider the pool constisting only of two points. Denote $y = y_{12} = y_{21}$. The feasibility theorem gives

$$\begin{aligned} \xi_1 - x_1 + \xi_2 - x_2 &\le 0, \\ \xi_1 - x_1 &\le y, \\ \xi_2 - x_2 &\le y. \end{aligned}$$

If we denote $i = i(1,2) = -i(2,1)$, we also receive following inequalities

$$\begin{aligned} x_1 - i &\ge \xi_1, \\ x_2 + i &\ge \xi_2, \\ -y \le i &\le y. \end{aligned}$$

Example 2
In this example the pool will be formed by three noded conncted together. Denote $y_1 = y_{12}, y_2 = y_{23}$ and $y_3 = y_{31}$. Using the feasibility theorem

$$\begin{aligned}
\xi_1 - x_1 + \xi_2 - x_2 + \xi_3 - x_3 &\leq 0, \\
\xi_1 - x_1 &\leq y_1 + y_3, \\
\xi_2 - x_2 &\leq y_1 + y_2, \\
\xi_3 - x_3 &\leq y_2 + y_3, \\
\xi_1 - x_1 + \xi_2 - x_2 &\leq y_2 + y_3, \\
\xi_2 - x_2 + \xi_3 - x_3 &\leq y_1 + y_3, \\
\xi_1 - x_1 + \xi_3 - x_3 &\leq y_1 + y_3.
\end{aligned}$$

The next result will be also similar to that one received in the previous example

$$
\begin{aligned}
i_{21} + i_{31} &\geq \xi_1 - x_1, \\
i_{12} + i_{32} &\geq \xi_2 - x_2, \\
i_{13} + i_{23} &\geq \xi_3 - x_3, \\
i_{12} + i_{21} &= 0, \\
i_{23} + i_{32} &= 0, \\
i_{13} + i_{31} &= 0, \\
-y_{12} \leq y_1 &\leq y_{12}, \\
-y_{23} \leq y_2 &\leq y_{23}, \\
-y_{13} \leq y_3 &\leq y_{13},
\end{aligned}
$$

Example 3

The power system is the same as in example 1 but this time we will use the model presented in the introduction.

The first stage problem is the following:

$$\min(c_1(x_1) + c_2(x_2) + c(y) + \mathrm{E}(\mu(x_1, x_2, y, \xi_1, \xi_2)))$$

$$P\begin{pmatrix} \xi_1 + \xi_2 \le x_1 + x_2 \\ \xi_1 \le x_1 + y \\ \xi_2 \le x_2 + y \end{pmatrix} \ge p$$

$$x_j^{(l)} \le x_j \le x_j^{(u)}$$

$$y^l \le y \le y^{(u)}$$

and the second stage problem

$$
\begin{aligned}
min(d_{12}i^{+} + d_{21}i^{-} + t_1(z_1) + t_2(z_2))\\
x_1 - i^{+} + i^{-} + z_1 \geq \xi_1,\\
x_2 + i^{+} - i^{-} + z_2 \geq \xi_2,\\
0 \leq i^{+}, i^{-} \leq y,\\
i^{+}, i^{-}, z_1, z_2 \geq 0.
\end{aligned}
$$

The main task of this example is to compute $E(\mu)$. But firstly we have to consider four possible cases:

(i) if $\xi_1 - x_1 \le 0$ and $\xi_2 - x_2 \le 0$ then $\mu = 0$.

(ii) if $\xi_1 - x_1 \le 0$ and $\xi_2 - x_2 > 0$ then
$i^+ = \min(x_1 - \xi_1, \xi_2 - x_2, y),\ i^- = 0,\ z_1 = 0,\ z_2 = \xi_2 - x_2 - i^+$ and
$\mu = d_{12} i^+ + t_2(z_2)$.

(iii) if $\xi_1 - x_1 > 0$ and $\xi_2 - x_2 \le 0$ then
$i^+ = 0,\ i^- = \min(\xi_1 - x_1, x_2 - \xi_2, y),\ z_1 = \xi_1 - x_1 - i^-,\ z_2 = 0$
and $\mu = d_{21} i^- + t_1(z_1)$.

(iv) if $\xi_1 - x_1 > 0$ and $\xi_2 - x_2 > 0$ then
$i^+ = 0,\ i^- = 0,\ z_1 = \xi_1 - x_1,\ z_2 = \xi_2 - x_2$ and $\mu = t_1(z_1) + t_2(z_2)$.

And finally

$$
\begin{aligned}
E(\mu) = &\int_{x_2}^{\infty}\int_{-\infty}^{x_1} (d_{12}\min(x_1 - v_1, v_2 - x_2, y) + \\
&+ t_2(v_2 - x_2 - \min(x_1 - v_1, v_2 - x_2, y)))f(v_1, v_2)dv_1dv_2 + \\
&+ \int_{-\infty}^{x_2}\int_{x_1}^{\infty} (d_{21}\min(v_1 - x_1, x_2 - v_2, y) + \\
&+ t_1(v_1 - x_1 - \min(v_1 - x_1, x_2 - v_2, y)))f(v_1, v_2)dv_1dv_2 + \\
&+ \int_{x_2}^{\infty}\int_{x_1}^{\infty} (t_1(v_1 - x_1) + t_2(v_2 - x_2))f(v_1, v_2)dv_1dv_2
\end{aligned}
$$

where $f(v_1, v_2)$ is joint probability density function of continuous joint distributions ξ_1 and ξ_2.

Than you for your patience.