Networked State Estimation over a Shared Communication Medium

M. Xia, V. Gupta and P. J. Antsaklis

Abstract— **This paper considers state estimation for multiple plants across a shared communication network. Each linear time-invariant plant transmits information through the common network according to either a time-triggered or an event-triggered rule. When the plants transmit according to an event triggered policy, every plant can access the network based on a priority mechanism. For the case when the plants access the network according to a time triggered algorithm, plant uses the network according to a round robin based off line scheduling policy. Performance in terms of the communication frequency and the estimation error covariance is analytically characterized. The main result is that for the same average communication rate, event-triggered schemes may perform worse than time-triggered schemes in terms of the resulting estimation error covariance when the effect of communication network is explicitly considered.**

I. Introduction

Event triggered sampling and transmission have emerged as exciting alternatives to more traditional periodic, or time-triggered, sampling and transmission. For the control and estimation of a scalar Weiner process, works such as [1], [2] showed that the average number of transmissions could be reduced significantly with event triggered schemes for the same state variance. Motivated by these results, event triggered schemes to ensure stability or passivity for arbitrary non-linear systems were designed in works such as [3], [4], [5]. However, performance analysis for such general cases (i.e., estimation/control performance metric for vector plants as a function of the communication rate) remains open. A stream of work to bypass this problem has been to design event triggered schemes for estimation for a guaranteed level of error covariance while satisfying constraints on the communication rate through a communication cost (see e.g., [6] and the references therein). However, an analytic trade-off between the covariance and communication rate by imposing a hard constraint on the number of communications typically leads to complicated event triggering rules that are difficult to compute and implement [7], [8].

Another direction in which the idea of event triggered control is being extended is by moving beyond the assumption that a single process needs to be estimated or controlled. If multiple processes are present, then events for various processes can trigger transmissions for more than one process at the same time. If the communication medium is shared, this can lead to congestion, and in turn, delays and packet losses. Realizing this fact, recent work has considered the interaction of control architecture and communication strategies in the setting of event triggered control.

Of particular interest to this paper is the work in [9] that considers a communication network being shared by a number of independent control loops. That work uses numerical methods to compute the control performance under various multiple access schemes such as TDMA (time division multiple access), FDMA (frequency division multiple access) and CSMA (carrier sense multiple access). The effect of packet loss due to contention of different loops using event triggered control and sharing a common medium is analyzed in [10]; however, the analysis is based on an assumption that the losses for different loops are independent, which does not hold in general [11]. Moreover, the analysis is limited to processes described by a single integrator driven by white noise. A simple ALOHA protocol is used for modeling the communication networks in [12]. Similar to [10], each loop is modeled by noisy integrator dynamics. The correlation among different loops is removed through a particular triggering rule and performance characterization is obtained. A more sophisticated strategy for conflict resolution when two plants wish to transmit simultaneously was considered in [11]. A Markov chain based model was introduced to characterize the probability of successful transmission for each plant in steady state. The key assumption (originating from [13]) was that the conditional probability of a busy channel for the attempting node to transmit is independent for each node. The correlations between various loops and the need for joint analysis between event trigger and CRM (collision resolution mechanism) were addressed; however, no performance analysis of the NCS was provided.

In this paper, we consider multiple plants transmitting information through a common network according to either a time triggered rule or an event triggered rule. To avoid collisions when multiple plants wish to transmit in the event triggered setting, we use CSMA for event trigger based on various priority rules as in [9]. For the case when the plants transmit according to a time triggered rule, no collisions are possible and we use a TDMA (round-robin) transmission schedule. Performance in terms of the communication rate and the estimation error covariance are analytically characterized under various medium access schemes. Our results demonstrate that simple time triggered scheme can outperform event triggered scheme when multiple loops share access to the network. This result may be of interest to designers while moving from implementing event triggered schemes for a single plant to a wider array of applications.

The rest of the paper is organized as follows. Section II presents the problem formulation. Preliminary results of a

The authors are with the department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA. (e-mail: mxia, vgupta2, antsaklis.1 @nd.edu). Research funded in part by NSF awards 0846631 and CNS-1035655.

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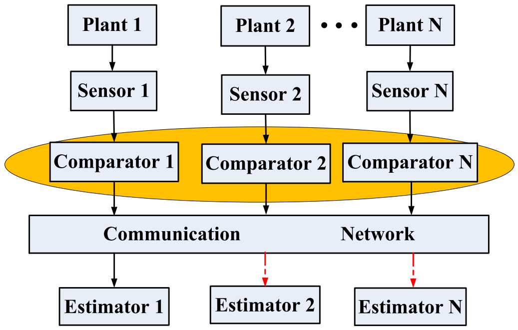

Fig. 1. Problem setup considered in this paper.

single plant in event trigger setting is presented in Section III. The main results are presented in Section IV. Numerical illustration is provided in Section V. This paper concludes with some avenues for future work in Section VI.

Notation: The n-dimensional real space is denoted by $\mathbb{R}^n$. The infinity norm of a vector x is denoted by $|x|$. For an m-dimensional multivariate Gaussian random variable X with mean vector μ and covariance R, we denote the generalization of the cumulative distribution function F function as $Pr(|X| \leq x) \triangleq F(m, \mu, R, x)$, where the inequality is interpreted element-wise. Also for the truncated multivariate Gaussian random variable obtained by truncating X between the vectors t_1 and t_2, define the variance by $\Sigma(X, t_1, t_2)$. As with the standard F functions and truncated Gaussian distributions, evaluation of these generalizations is done through Gaussian integrals (see, e.g., [14, Equation (16)] for formulas for the variance of truncated Gaussian distributions) and is a standard feature in most statistics packages.

II. Problem Formulation

Consider the problem setup as shown in Fig. 1, where N ($N \geq 1$) plants transmit information over a shared network with the following associated assumptions.

Plant and Sensor: The i-th plant (denoted by $\mathcal{S}_i$) is described by the following discrete linear time-invariant evolution:

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + w_i(k), \\ y_i(k) &= x_i(k), \end{aligned} \quad (1)$$

where $x_i(k) \in \mathbb{R}^n$ denotes the state vector, $y_i(k) \in \mathbb{R}^m$ is the output vector, $w_i(k)$ is the process noise assumed to be white and Gaussian with zero mean and covariance $R_{w_i} > 0$. For the analytical results in the paper, we will consider $n = m = 1$, although the arguments can be easily generalized at the expense of more notation. The initial condition of the process $x_i(0)$ is assumed to be a Gaussian random vector with zero mean and covariance $R_i(0)$. The process noise $\{w_i(k)\}$ and the initial condition $x_i(0)$ are assumed to be mutually independent.

Estimator: At every time k, the ith estimator generates a minimum mean squared error (MMSE) estimate for the state $x_i(k)$ based on whatever information is available to it. In a time-triggered architecture, this information is the set of measurements until time k that have been transmitted across the network according to a pre-designed periodic schedule. In an event-triggered architecture, the estimate is calculated based on any information transmitted by the comparator, as well as the time steps at which information transmission occurs. Denote the estimate for state $x_i(k)$ held by the ith estimator as $\hat{x}_i^{dec}(k)$. Since we assume that the sensors can observe the states directly, at the ith estimator, we have

$$\hat{x}_i^{dec}(k) = \begin{cases} x_i(k), & \text{if the } i\text{th packet received,} \\ A_i \hat{x}_i^{dec}(k-1), & \text{otherwise.} \end{cases}$$

where $A_i \hat{x}_i^{dec}(k-1)$ is the optimal estimate at the estimator if the estimator did not receive any information at time k [15].

Comparator: The event-triggered algorithm is implemented at the comparator. We consider a simple level based scheme. The local event for the ith plant is defined as

$$| e_i^{comp}(k) | > \varepsilon_i, \quad (2)$$

where $e_i^{comp}(k) \triangleq x_i(k) - A_i \hat{x}_i^{dec}(k-1)$, and the threshold ε_i is a given constant.

Communication Network: The communication network is modeled as satisfying the following assumptions.

- A1: The network does not permit simultaneous transmissions and the transmission delay is less than one time step according to the process evolution [11], [12].
- A2: The plant sends information according to an offline scheduling for time-triggered schemes or whenever an event is generated for event-triggered schemes.
- A3: When two or more plants send information simultaneously, the network will transmit the packet received from the plant with highest priority as determined by a pre-assigned priority mechanism [9] and the rest of the packets will be discarded. In this paper, we consider three priority assignment mechanisms described later.
- A4: In both the time-triggered and event-triggered setups, the network allows every plant to transmit at least once every T time steps to guard against the practical concerns of synchronization, dying sensors and so on.

We are interested in the problem of analyzing the performance of the system as measured by the following two metrics: (1) The communication rate P, which is defined as the average probability for the network to transmit information at any time step; (2) The quality of estimate for the NCS, which is measured by the aggregate error covariance,

$$J = \sum_{i=1}^{N} \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t} \mathbb{E}\left\{ e_i^{dec}(k) [e_i^{dec}(k)]^T \right\},$$

with $e_i^{dec}(k) \triangleq x_i(k) - \hat{x}_i^{dec}(k)$ as the estimation error for $\mathcal{S}_i$.

III. Preliminary Results: Single Plant across a Dedicated Network

In this section, we present results for event triggered estimation of a single plant and the results are generalized to multiple plants in the next section.

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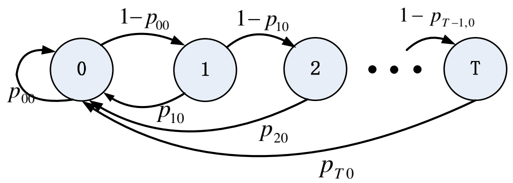

Fig. 2. Transition graph of the Markov Chain defined for a single plant.

We drop the subscript i in this section. The information can be successfully transmitted through the network whenever $|e^{comp}(k)| > \varepsilon$ since there is no contention to access the network. As shown in Fig. 2, we can define a discrete-time discrete-state Markov chain $\mathcal{M}$ with $T+1$ modes, the state $\{X(k)\}_{k\geq 0}$ and the transition probabilities

$$p_{ij} = Pr(X(k+1) = j \big| X(k) = i),$$

such that $X(k) = j$ implies that at time k, the last transmission occurred at time $k-j$.

The communication frequency and the estimation error covariance are characterized by this Markov chain. To this end, define the random variables

$$Z_i(k) = \sum_{j=0}^{i} A^j w(k+i-j), \qquad 0 \leq i \leq T. \tag{3}$$

Since the noise $w(k)$ is white, the probability density function of the variables $Z_i(k)$ is independent of k. In the sequel, we will simply write Z_i to denote the random variables. Clearly, for any i, the vector random variable $M_i = \left[\begin{array}{c} Z_0^T, Z_1^T, \cdots, Z_i^T \end{array}\right]^T$ has a multi-variate normal distribution with mean 0 and covariance matrix R_i as

$$\begin{bmatrix} R_w & R_w A^T & \cdots & R_w (A^T)^i \\ AR_w & AR_w A^T + R_w & AR_w (A^T)^2 + R_w A^T & \cdots \\ \vdots & & \ddots & \\ A^i R_w & \cdots & & \end{bmatrix}.$$

Now for $1 \leq i \leq T$, define the events

$$N_i = (|\ Z_0 \ | < \varepsilon) \cap (|\ Z_1 \ | < \varepsilon) \cap \cdots \cap (|\ Z_{i-1} \ | < \varepsilon), \tag{4}$$

with the convention that N_0 is the sure event. The following result is immediate:

$$Pr(N_i) = F(ni, 0, R_i, \varepsilon\underline{1}), \tag{5}$$

with $Pr(N_0) = 1$.

Lemma 1: Consider the Markov chain $\mathcal{M}$ as defined above. The transition probabilities p_{ij} are given by

$$p_{ij} = \begin{cases} 1 - \frac{F(n(i+1), 0, R_{i+1}, \varepsilon\underline{1})}{F(ni, 0, R_i, \varepsilon\underline{1})} & 0 \leq i \leq T-1, j = 0 \\ 1 & i = T, j = 0 \\ 1 - p_{i0} & 0 \leq i \leq T-1, j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Proof: We concentrate on the case when $0 \leq i \leq T-1, j = 0$ since the other expressions are obvious from the structure of the Markov chain shown in Fig. 2. Consider the transition probability p_{00}. Since $X(k) = 0$ is equivalent to $e^{dec}(k) = 0$, we have

$$\begin{aligned} p_{00} &= Pr(X(k+1) = 0 \big| X(k) = 0) \\ &= Pr(|w(k)| > \varepsilon \big| e^{dec}(k) = 0) \\ &\overset{(a)}{=} Pr(|w(k)| > \varepsilon) = Pr(|\ Z_0 \ | > \varepsilon), \end{aligned}$$

where (a) holds because $e^{dec}(k)$ is independent of the process noise at time step k. Similarly, for any i such that $0 \leq i \leq T-1$, the probability

$$\begin{aligned} p_{i0} =& Pr(X(k+1) = 0 \big| X(k) = i) \\ \overset{(b)}{=}& Pr\left(|\ Z_i \ | > \varepsilon \big| N_i, e^{dec}(k-i) = 0\right) \\ \overset{(c)}{=}& Pr\left(|\ Z_i \ | > \varepsilon \big| \ |\ Z_{i-1} \ | < \varepsilon, \cdots, |\ Z_0 \ | < \varepsilon\right) \\ =& \frac{Pr(|\ Z_i \ | > \varepsilon, N_i)}{Pr(N_i)} = 1 - \frac{Pr(N_{i+1})}{Pr(N_i)}, \end{aligned}$$

where (b) follows the Markovian property and the definiations in (3), and (c) holds because $e^{dec}(k-i)$ is independent of the process noise after time step $k-i$ and in particular, Z_i. Now the result follows from (5), which can be evaluated using Gaussian integrals and the fact that $p_{T0} = 1$. ■

Theorem 1: The average communication rate for the event triggered algorithm described above is given by $\frac{1}{1+\sum_{j=1}^{T}\prod_{i=0}^{j-1}(1-p_{i0})}$, which can be calculated using (6).

Proof: The average communication rate for the system is given by $\lim_{k\to\infty} Pr(X(k) = 0)$. From the fact that p_{i0}'s are time-invariant and using the structure of the Markov chain from Fig. 2, the probability for each mode j $(j \geq 1)$ can be computed as

$$\begin{aligned} Pr(X(k) = j) &= (1 - p_{j-1,0}) Pr(X(k) = j-1) \\ &= \prod_{i=0}^{j-1} (1 - p_{i0}) Pr(X(k) = 0). \end{aligned} \tag{7}$$

Thus, the balance equation for the Markov chain yields

$$\begin{aligned} 1 &= \sum_{j=0}^{T} Pr(X(k) = j) \\ &= Pr(X(k) = 0) + \sum_{j=1}^{T}\prod_{i=0}^{j-1} (1 - p_{i0}) Pr(X(k) = 0) \\ &= (1 + \sum_{j=1}^{T}\prod_{i=0}^{j-1} (1 - p_{i0})) Pr(X(k) = 0). \end{aligned}$$

The required probability $Pr(X(k) = 0)$ can now be calculated as $Pr(X(k) = 0) = \frac{1}{1+\sum_{j=1}^{T}\prod_{i=0}^{j-1}(1-p_{i0})}$. ■

The other performance metric is the covariance of estimation error $\Pi(k) = \mathbb{E}[e^{dec}(k)(e^{dec}(k))^T]$ which is given by the following relation.

Theorem 2: The steady state average error covariance $\Pi = \lim_{k\to\infty} \Pi(k)$ for the event triggered algorithm described above is given by

$$\Pi = \sum_{j=1}^{T}\prod_{t=0}^{j-1} (1 - p_{t0}) Pr(X(k) = 0) \Sigma_{M,j}(j,j), \tag{8}$$

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where $\Sigma_{M,j} = \Sigma(M_j, -\varepsilon\underline{1}, \varepsilon\underline{1})$.

Proof: We use the relation $\Pi(k) = \sum_{j=0}^{T} Pr(X(k) = j)\mathbb{E}[e^{dec}(k)(e^{dec}(k))^T \mid X(k) = j]$.

For $j = 0$, since the estimation error $e^{dec}(k) = 0$, we obtain $\mathbb{E}[e^{dec}(k)(e^{dec}(k))^T \mid X(k) = j] = 0$. For $j > 0$, we use the fact that the error covariance $e^{dec}(k)$ under the event $X(k) = j$ is simply $\sum_{i=0}^{j} A^i w(k-i)$. However, since the process noise $w(j)$ is white and has a time-invariant probability distribution function, we can alternatively write

$$\mathbb{E}[e^{dec}(k)(e^{dec}(k))^T \mid X(k) = j] = \text{var}[Z_{j-1} \mid N_j],$$

where var(X) is the variance of the random variable X and N_i was defined in (4). The variance of Z_{j-1} is given by the (j,j)-th element of the variance matrix of M_j; however, as calculated under the truncation imposed by N_j, i.e., all the elements $Z_0, \cdots, Z_{j-1}$ being bounded between $-\varepsilon\underline{1}$ and $\varepsilon\underline{1}$. This variance is given by $\Sigma_{M,j}(j,j)$. Together with (7), this yields the desired expression. ■

Together, these two results provide analytic expressions for the communication frequency and average error covariance given any level ε.

IV. Main Results: Multiple Plants Sharing the Network

In this section, we present the main results of this paper.

A. Markov Model for Multiple Plants

When $N \geq 2$ plants transmit information over a common network, similar to the single plant case, we can define a discrete-time discrete-state Markov chain $\mathcal{M}$ with $N_s = (T+1)T\cdots(T-N+2)$ states $\{X(k)\}_{k\geq 0} \in \mathbb{R}^N$ and the transition probabilities

$$Pr[X(k+1) = \underline{\text{m}} \big| X(k) = \underline{\text{n}}] \triangleq p(m_1, \cdots, m_N \big| n_1, \cdots, n_N),$$

such that $X(k) = \underline{\text{m}}$ implies that at time k, the last transmission for the ith plant occurred at time $k - m_i$. Note that $m_i \neq m_j$ for all $i \neq j$ since the network does not permit simultaneous transmissions. Performance of event triggered algorithms can be characterized by this Markov chain. In the following analysis, we concentrate on the case when $N = 2$ (and the arguments can be easily generalized to $N > 2$).

In this case, at every time step, there are 3 possibilities of information transmission:

- The network transmits information from $\mathcal{S}_1$.
- The network transmits information from $\mathcal{S}_2$.
- The network does not transmit any information.

This corresponds to the structure of the Markov chain. In particular, for any mode $\{i_1, i_2\}$ when $i_1, i_2 < T$, it can go to the following modes correspondingly,

$$\begin{bmatrix} 0 \\ i_2+1 \end{bmatrix}, \begin{bmatrix} i_1+1 \\ 0 \end{bmatrix}, \begin{bmatrix} i_1+1 \\ i_2+1 \end{bmatrix},$$

whose transition probabilities are determined by the scheduling policies. For the modes with $i_1 = T$ or $i_2 = T$, the network transits information for $\mathcal{S}_1$ or $\mathcal{S}_2$, respectively. Thus, for any scheduling policy, we have the following transitions

$$\begin{bmatrix} T \\ i_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ i_2+1 \end{bmatrix}, \begin{bmatrix} i_1 \\ T \end{bmatrix} \to \begin{bmatrix} i_1+1 \\ 0 \end{bmatrix}$$

occurring with probability 1. To clarify this, consider the following example.

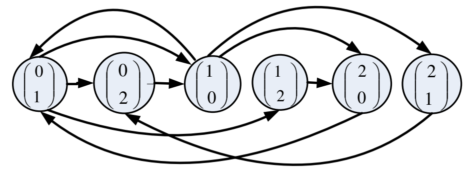

Fig. 3. Illustrating example for the Markov model with $T = 2$.

Example 3: Consider a NCS with $N = 2$ plants over a shared medium. Assume the maximum delay that each plant can tolerate is $T = 2$. We can define a Markov chain with the following $N_s = 6$ modes as shown in Fig. 3. The communication rate for $\mathcal{S}_1$ and $\mathcal{S}_2$ are given as

$$P_1 = Pr(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) + Pr(\begin{bmatrix} 0 \\ 2 \end{bmatrix}),$$
$$P_2 = Pr(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + Pr(\begin{bmatrix} 2 \\ 0 \end{bmatrix}),$$

respectively. The communication rate for the network is then given by $P_0 = P_1 + P_2$. From the mode $\{1, 0\}$ and $\{0, 1\}$, there are three possible transitions and the following transitions are with probability 1.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

To characterize the system performance, we need to calculate the probability of each Markov mode. To this end, define $\underline{\text{P}} \in \mathbb{R}^{N_s}$ as the vector for probability of each mode and define $\underline{\text{b}} = [1, 0, \cdots] \in \mathbb{R}^{N_s}$. The relations of the modes are given through the following equation

$$\Delta\underline{\text{P}} = \underline{\text{b}}, \tag{9}$$

where $\Delta \in \mathbb{R}^{N_s \times N_s}$ with the first row $[1, 1, \cdots, 1]$ given by the balance equation and the rest elements can be determined from the structure of the Markov model. We can verify that the matrix Δ always has full rank. This guarantees that the above equation (9) has a unique solution.

Remark 1: The matrix Δ may not be unique since the relations between the Markov modes can be expressed in various manners, however, all these choices will give the same probability of each mode in the end.

We next are characterize the matrix Δ and evaluate the performance of event triggered algorithms with static, random and dynamic schedulers through the Markov model defined above.

B. Event Trigger with Static Scheduler

We begin our analysis with the case when every plant has been given a fixed priority to access the network. We assume that the ith plant has the ith priority without loss of generality. Thus, we assume $\mathcal{S}_1$ wins the arbitration to access the network whenever it contends with $\mathcal{S}_2$.

Lemma 2: By using static scheduler, for any $0 \le i < T$,

$$p(0, i+1 \big| T, i) = 1; p(i+1, 0 \big| i, T) = 1. \tag{10}$$

Furthermore, for $0 < i < T$, we have

$$p(0, i+1 \big| 0, i) = p_{0,0}^{(1)}, \tag{11}$$

where $p_{0,0}^{(1)}$ can be calculated through (6) using $\{A_1, w_1\}$.

Proof: The equality (10) holds since the network transmits information for each plant at least once every T time steps. (11) holds because $\mathcal{S}_1$ has higher priority and thus information transmission is delayed for $\mathcal{S}_2$ when local event for $\mathcal{S}_1$ is generated and $i < T$, i.e.

$$\begin{aligned} p(0, i+1 \big| 0, i) &= Pr\left(\begin{bmatrix} 0 \\ i+1 \end{bmatrix} \mid \begin{bmatrix} 0 \\ i \end{bmatrix}\right) \\ &= Pr(X_1(k+1) = 0 \mid X_1(k) = 0) \\ &= Pr(|w_1(k)| > \varepsilon) \triangleq p_{0,0}^{(1)}. \end{aligned}$$

■

To illustrate the application of this result, let us consider Example 3 again. We have the following relation

$$Pr\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{01}^{(1)} \bar{p}_{10}^{(2)} + Pr\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right), \tag{12}$$

where $p_{01}^{(1)} = 1 - p_{00}^{(1)}$ and $\bar{p}_{10}^{(2)}$ given by

$$\bar{p}_{10}^{(2)} = Pr(|A_2 w_2(k-1) + w_2(k)| > \varepsilon).$$

One step further, we have the following transition,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and from these transitions we have

$$\begin{aligned} Pr\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) &= Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{01}^{(1)} \bar{p}_{10}^{(2)} p_{12}^{(1)} p_{01}^{(2)} \\ &+ Pr\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \bar{p}_{12}^{(1)} p_{01}^{(2)}, \end{aligned} \tag{13}$$

where $p_{01}^{(2)} = 1 - p_{00}^{(2)}$ and

$$p_{12}^{(1)} = Pr(|A_1 w_1(k-1) + w_1(k)| < \varepsilon \big| |w_1(k-1)| < \varepsilon)$$

can be calculated through (6) by using $\{A_2, w_2\}$ and respectively $\{A_1, w_1\}$. $\bar{p}_{12}^{(1)}$ is given by

$$\bar{p}_{12}^{(1)} = Pr(|A_1 w_1(k-1) + w_1(k)| < \varepsilon).$$

We can obtain the following relations in a similar manner,

$$Pr\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{01}^{(1)} \bar{p}_{12}^{(2)}, \tag{14}$$

$$\begin{aligned} Pr\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) &= Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{01}^{(1)} \bar{p}_{10}^{(2)} p_{12}^{(1)} p_{00}^{(2)} \\ &+ Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{01}^{(1)} \bar{p}_{12}^{(2)} + Pr\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \bar{p}_{12}^{(1)} p_{00}^{(2)}, \end{aligned} \tag{15}$$

$$\begin{aligned} Pr\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) &= Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{01}^{(1)} \bar{p}_{10}^{(2)} p_{12}^{(1)} p_{01}^{(2)} \\ &+ Pr\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) p_{00}^{(1)} + Pr\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \bar{p}_{12}^{(1)} p_{01}^{(2)}, \end{aligned} \tag{16}$$

where $\bar{p}_{12}^{(2)} = 1 - \bar{p}_{10}^{(2)}$.

In such a way, we represent the probabilities of all modes through the relations with mode $\{0, 1\}$ and $\{0, 2\}$ as in (12)-(16). Then from the balance equation that the sum of all probabilities equal to 1, we can solve for probability of each mode. More compactly, define

$$a = p_{01}^{(1)} \bar{p}_{10}^{(2)} p_{12}^{(1)} p_{00}^{(2)} + p_{01}^{(1)} \bar{p}_{12}^{(2)}, b = p_{01}^{(1)} \bar{p}_{10}^{(2)} p_{12}^{(1)} p_{01}^{(2)},$$

and we obtain the probability for every individual mode from equation (9) with Δ given as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ p_{01}^{(1)} \bar{p}_{10}^{(2)} & 1 & -1 & 0 & 0 & 0 \\ p_{01}^{(1)} \bar{p}_{12}^{(2)} & 0 & 0 & -1 & 0 & 0 \\ a & \bar{p}_{12}^{(1)} p_{00}^{(2)} & 0 & 0 & -1 & 0 \\ b & \bar{p}_{12}^{(1)} p_{01}^{(2)} & 0 & 0 & 0 & -1 \\ p_{00}^{(1)} + c & -1 + \bar{p}_{12}^{(1)} p_{01}^{(2)} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Remark 2: Notice $\bar{p}_{12}^{(1)} \neq p_{12}^{(1)}$, since in transitions such as

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$X_1(k) = 1$ is caused by $t_2 = T$ independent of the error $|w_1(k-1)|$ which yields $\bar{p}_{12}^{(1)}$. However, in transitions such as

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$X_1(k) = 1$ is caused by $|w_1(k-1)| < \varepsilon$ and this yields $p_{12}^{(1)}$. Similarly, we have $\bar{p}_{10}^{(2)} \neq p_{10}^{(2)}$.

Remark 3: For single plant case, we can easily obtain the relations between the modes from the structure of the Markov model. Particularly, the matrix Δ for single plant is given as

$$\begin{bmatrix} 1 & 1 & & & & \cdots 1 \\ p_{01} & -1 & & & & \\ & p_{12} & -1 & & & \\ & & p_{23} & -1 & & \\ & & & \ddots & \ddots & \\ & & & & p_{T-1,T} & -1 \end{bmatrix},$$

and the transition probabilities are given in Lemma 1. For the multiple case, however, it is more complicated because of coupling of the two Markov states in one mode.

By solving (9), we obtain the probability of each Markov mode. The following result is immediate.

Theorem 4: For $T=2$, the average communication rate for $\mathcal{S}_1$ under event triggered algorithm described above is given by $P_1 = Pr(\{0,1\}) + Pr(\{0,2\})$, and $P_2 = Pr(\{1,0\}) + Pr(\{2,0\})$ for $\mathcal{S}_2$ through $\underline{\mathrm{P}} = \Delta^{-1}\underline{\mathrm{b}}$ with Δ given in (17). Furthermore, the average communication rate for the network is given by $P_0 = P_1 + P_2$.

The other performance metric is the covariance of the estimation error $\Pi_i(k) = \mathbb{E}[e_i^{dec}(k)(e_i^{dec}(k))^T]$, which is given by the following result.

Theorem 5: For $T=2$, the steady state average error covariance for the rth plant, $\Pi_r = \lim_{k\to\infty}\Pi_r(k)$, under the event triggered algorithm described above is given by $\Pi_r(k) = \sum_{j=1}^{N_s}\Pi_r(j)$ from (18-24). Furthermore, the average error covariance for the NCS is given by $\Pi = \Pi_1 + \Pi_2$.

Proof: To calculate Π_1, we use the relation $\Pi_1 = \sum_{j=0}^{N_s}\Pi_1(j)$, where $\Pi_1(j)$ corresponds to the error covariance under the Markov mode j as defined above. We have

$$\Pi_1(1) = 0, \Pi_1(2) = 0, \tag{18}$$

since the estimation error $e_1^{dec}(k) = 0$. Under the Markov mode $\{1,0\}$, we have

$$\begin{aligned}\Pi_1(3) = &Pr(\{0,1\})\Delta(1,3)\mathrm{var}\{w_1(k) \mid |w_1(k)| < \varepsilon\}\\ &+ Pr(\{0,2\})\mathrm{var}\{w_1(k)\}.\end{aligned}$$

As for single plant case, $\mathrm{var}\{w_1(k) \mid |w_1(k)| < \varepsilon\}$ is given by $\Sigma_{M,1}^{(1)}(1,1)$. Thus, we have

$$\Pi_1(3) = Pr(\{0,1\})\Delta(1,3)\Sigma_{M,1}^{(1)}(1,1) + Pr(\{0,2\})R_{w_1}. \tag{19}$$

Under the Markov mode $\{1,2\}$, we have

$$\Pi_1(4) = Pr(\{0,1\})\Delta(1,4)\Sigma_{M,1}^{(1)}(1,1). \tag{20}$$

We can also obtain the error covariance under mode $\{2,0\}$,

$$\begin{aligned}\Pi_1(5) = &Pr(\{0,2\})\bar{p}_{12}^{(1)}p_{00}^{(2)}\Xi_1\\ &+ Pr(\{0,1\})p_{01}^{(1)}\bar{p}_{10}^{(2)}p_{12}^{(1)}p_{00}^{(2)}\Xi_2\\ &+ Pr(\{0,1\})p_{01}^{(1)}\bar{p}_{12}^{(2)}\Xi_3,\end{aligned} \tag{21}$$

where $\Xi_1 = \Sigma(A_1w_1(k-1)+w_1(k), -\varepsilon, \varepsilon)$ can be evaluated through Gaussian integrals, $\Xi_2 = \Sigma_{M,2}^{(1)}(2,2)$, and

$$\begin{aligned}\Xi_3 &= \mathrm{var}\{A_1w_1(k-1) + w_1(k) \mid |w_1(k-1)| < \varepsilon\}\\ &= A_1\Sigma_{M,1}^{(1)}(1,1)A_1^T + R_{w_1}.\end{aligned}$$

Also, the error covariance under the mode $\{2,1\}$ is given by

$$\begin{aligned}\Pi_1(6) = &Pr(\{0,2\})\bar{p}_{12}^{(1)}p_{01}^{(2)}\Xi_1\\ &+ Pr(\{0,1\})p_{01}^{(1)}\bar{p}_{10}^{(2)}p_{12}^{(1)}p_{01}^{(2)}\Xi_2.\end{aligned} \tag{22}$$

To calculate Π_2, similar to calculation of Π_1, we use the relation $\Pi_2 = \sum_{j=0}^{N_s}\Pi_2(j)$ with

$$\Pi_2(3) = 0, \Pi_2(5) = 0, \tag{23}$$

since the estimation error $e_2^{dec}(k) = 0$. We can also have the following relations

$$\begin{aligned}\Pi_2(1) &= R_{w_2},\\ \Pi_2(2) &= Pr(\{0,1\})p_{00}^{(1)}\Xi_4 + Pr(\{2,1\})\Xi_5,\\ \Pi_2(4) &= Pr(\{1,2\})\Xi_4,\\ \Pi_2(6) &= Pr(\{2,1\})\Sigma_{M,1}^{(2)}(1,1),\end{aligned} \tag{24}$$

where $\Xi_4 = \Sigma(A_2w_2(k-1) + w_2(k), -\varepsilon, \varepsilon)$ and $\Xi_5 = A_2\Sigma_{M,1}^{(2)}(1,1)A_2^T + R_{w_2}$. Together with the probabilities from the previous theorem, this yields the desired expressions. ■

Remark 4: For single plant case, $e^{dec}(k) = 0$ for $X(k) = 0$ and $X(k) = j > 0$ implies the estimation error in previous steps all less than ε. As a result, the error covariance under the mode $X(k) = j > 0$ is simply

$$\begin{aligned}\Pi(j) &\triangleq Pr(X(k) = j)\mathbb{E}[e^{dec}(k)(e^{dec}(k))^T \mid X(k) = j]\\ &= Pr(X(k) = j)\Sigma_{M,j}(j,j)\end{aligned}$$

and the average estimation error covariance can be calculated as $\sum_{j=1}^{T}\Pi(j)$. For the multiple case, however, we have to identify how it comes to the current mode, caused by local events or network constraints, which yields different expressions for the error covariance.

Remark 5: For $T > 2$, a similar Markov chain can be defined by considering two more variables for each mode indicating how long has each plant signaled it wants to transmit and basically the issue is that to calculate the transition probabilities, one has to track the past states as well. This will result in too many Markov states and the analysis is more involved.

C. Event Trigger with Random Scheduler

With random scheduler, both plants have the chance to win the arbitration when contention occurs. Denote P_α as the probability for $\mathcal{S}_1$ to win, and $1 - P_\alpha$ for $\mathcal{S}_2$. The access probability P_α is provided by the network [11]. When there is no contention, the plant can transmit information successfully whenever its local event is generated.

Consider the Markov model shown in Fig. 3. As mentioned earlier, one has to track the past states to calculate the transition probabilities. As an example, consider the transition from mode $\{0,1\}$ to $\{1,0\}$. The transition probability for $X_2(k) = 1 \to 0$ is not given by $\bar{p}_{10}^{(2)}$ (as for static scheduler). The reason is that $X_2(k) = 1$ in the mode $\{0,1\}$ depends on the error $w_2(k-1)$ in the previous step. Similarly, in transition $\{0,1\} \to \{1,0\} \to \{2,0\}$, the transition probability for $X_1(k) = 1 \to 2$ is not given by $\bar{p}_{12}^{(1)}$ since $X_1(k) = 0 \to 1$ might be caused by $|w_1(k-1)| > \varepsilon$ as well. However, the approximation of ignoring this past and calculating transition probability only with the current state is close, which is verified through simulations. Through such approximations, the matrix Δ for a random scheduler is given as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \Delta_{21} & -1 & 0 & 0 & 0 & 1 \\ \Delta_{31} & 1 & -1 & 0 & 0 & 0 \\ p_{01}^{(1)}(1-\bar{p}_{10}^{(2)}) & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \Delta_{53} & 1 & -1 & 0 \\ 0 & 0 & \bar{p}_{12}^{(1)}(1-p_{00}^{(2)}) & 0 & 0 & -1 \end{bmatrix}, \quad (25)$$

where

$$\Delta_{31} = [p_{01}^{(1)} + p_{00}^{(1)}(1-P_\alpha)]\bar{p}_{10}^{(2)},$$

$$\Delta_{53} = p_{00}^{(2)}[\bar{p}_{12}^{(1)} + (1-\bar{p}_{12}^{(1)})(1-P_\alpha)],$$

$$\Delta_{21} = (1-p_{01}^{(1)})(\bar{p}_{10}^{(2)}P_\alpha + 1 - \bar{p}_{10}^{(2)}).$$

By solving equation (9) with Δ given in (25), we can get the probability for each mode. The approximate results calculated in this way match closely to the Monte Carlo simulations as demonstrated in Section V. We can therefore characterize the communication rate and error covariance from this Markov model along the same lines as for static scheduler.

D. Event Trigger with Dynamic Scheduler

With dynamic scheduler, when two local events are generated simultaneously, the network grants the one with maximum error $|e_i^{comp}(k)|$ to access the network first. As a result, the network transmits information for $\mathcal{S}_1$ if it has a larger error when both local events are generated, i.e.

$$|e_1^{comp}(k)| > |e_2^{comp}(k)|, |e_1^{comp}(k)| > \varepsilon, |e_2^{comp}(k)| > \varepsilon,$$

or the following events occur

$$|e_1^{comp}(k)| > \varepsilon, |e_2^{comp}(k)| < \varepsilon.$$

Define the conditional probability P_d as follows,

$$P_d \triangleq Pr(|e_1^{comp}| > |e_2^{comp}| \,|\, |e_1^{comp}| > \varepsilon, |e_2^{comp}| > \varepsilon),$$

where the dependence of the errors on time k is omitted for notational convenience. It is worthwhile to point out that for random scheduler case, when both errors exceed the predefined threshold, the probability of the network to transmit information for $\mathcal{S}_1$ is actually

$$P_\alpha Pr(|e_1^{comp}| > \varepsilon, |e_2^{comp}| > \varepsilon).$$

For the dynamic case, unlike P_α defined above, P_d depends on the magnitudes of the errors of both plants and hence the interference between the plants and the shared medium becomes more complicated. P_d can be exactly evaluated through Gaussian integrals because the errors are Gaussian random variables as defined in (3). However, for simplicity, we can use

$$\lambda \triangleq Pr(|e_1^{comp}| > |e_2^{comp}|)$$

as an approximation of the conditional probability P_d. In fact, we have $\lambda = 1/2$ based on the following arguments. From the fact that

$$\begin{aligned} &Pr(|e_1^{comp}| > |e_2^{comp}|) = Pr(|e_1^{comp}|^2 > |e_2^{comp}|^2) \\ &= Pr(e_1^{comp} + e_2^{comp} > 0, e_1^{comp} - e_2^{comp} > 0) \\ &+ Pr(e_1^{comp} + e_2^{comp} < 0, e_1^{comp} - e_2^{comp} < 0) \\ &\overset{(e)}{=} Pr(e_1^{comp} + e_2^{comp} > 0)Pr(e_1^{comp} - e_2^{comp} > 0) \\ &+ Pr(e_1^{comp} + e_2^{comp} < 0)Pr(e_1^{comp} - e_2^{comp} < 0), \end{aligned}$$

and (e) holds because $e_1^{comp} + e_2^{comp}$ and $e_1^{comp} - e_2^{comp}$ are Gaussian random variables and mutually independent. Since e_1^{comp} and e_2^{comp} are zero mean, we have

$$\begin{aligned} Pr(e_1^{comp} + e_2^{comp} < 0) = 1/2, \\ Pr(e_1^{comp} - e_2^{comp} < 0) = 1/2. \end{aligned}$$

This yields the desired result. Therefore, the communication rate can be calculated as a special case of random access by setting $P_\alpha = \lambda = 1/2$. The results given by this approximation match the Monte Carlo experiments very closely as demonstrated in Section V.

Remark 6: $\lambda \neq 1/2$ for $N \geq 2$, although λ can be evaluated through Gaussian integrals for the general case.

Remark 7: The error covariance is different from random scheduler case (with $P_\alpha = 1/2$) since for dynamic scheduler there exists additional condition on the magnitudes of e_1^{comp} and e_2^{comp}. However, the error covariance can be evaluated through Gaussian integrals as well.

E. Time Triggered Algorithm

In this section, we evaluate the performance of time triggered scheme with TDMA. Since we do not consider the cost of using the network, we assume the network transmits information at every time step. For $N = 2$, there exist two possible schedules: $S_1 = \{1, 2, 1, 2 \cdots\}$ and $S_2 = \{2, 1, 2, 1 \cdots\}$. If $A_1 = A_2$ and $R_{w_1} = R_{w_2}$, it can be verified that the two round robin schedules S_1 and S_2 are both optimal. Otherwise, one can find an optimal schedule by evaluating the cost function for every possible schedule [9]. Therefore, for $N = 2$, both schedules are optimal and the system performance can be calculated as

$$J = \frac{1}{2}(R_{w_2} + R_{w_1}).$$

V. Simulation Results

In this section, we present numerical examples to illustrate our main results. The system model is given by (1) with $A_1 = 0.8$ and $A_2 = 0.5$ and we assume $w_i, x_i(0)$ $(i = 1, 2)$ are zero-mean Gaussian random variables with unit covariance. We set $T = 2$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$. For various values of ε from 0 to 4, we evaluated system performance for static, random and dynamic schedulers. We compared the analytic results to Monte Carlo simulations of the system.

The comparison of using static scheduler is shown in Fig. 4 for the communication rate in the top plot and in the bottom one for the error covariance. It can be seen that

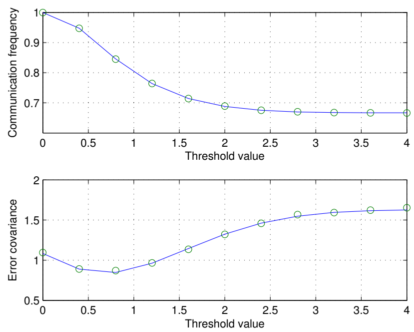

Fig. 4. Performance metrics for the NCS obtained from derived analytic expressions and Monte Carlo simulations.

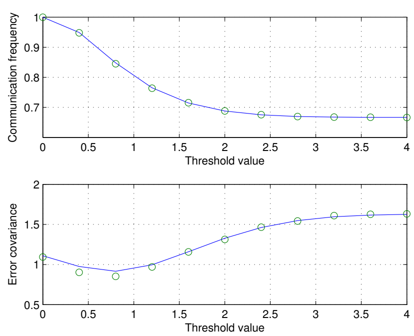

Fig. 5. Performance metrics for the NCS using random scheduler obtained from approximate expressions and Monte Carlo simulations.

the analytic results match the Monte Carlo simulations very closely. From the bottom plot in Fig. 4, we can see that for $\varepsilon \in [0.2, 1.2]$, the error covariance for event trigger is less than time trigger; however, for other values of $\varepsilon \in [0, 4]$, time triggered algorithm performs better. This implies that there is a probability of 75% for event-triggered algorithm to perform worse than time-triggered algorithm if we choose the threshold randomly.

The system performance by using approximate models for random scheduler in terms of the communication rate and the error covariance is provided in Fig. 5 with $P_\alpha = 0.7$. The results for communication rates by using dynamic scheduler in Fig. 6 by setting $P_\alpha = 0.5$. It can be seen that the results obtained from approximate models for both cases match the Monte Carlo simulations very closely.

VI. Final Remarks

This paper studies state estimation for a NCS with multiple plants over a shared communication network. Each plant transmits information through the common network according to either a time-triggered or an event-triggered rule. For a time-triggered algorithm combined with TDMA, each plant uses the network according to an off-line scheduling. For an event-triggered algorithm with CSMA, each plant is assumed to access the network based on one of the following scheduling strategies: static, random or dynamic schedulers.

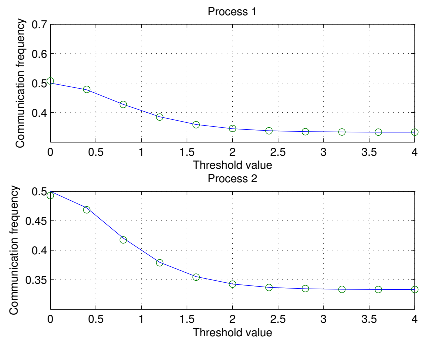

Fig. 6. Communication rates for each plant using dynamic scheduler.

Performance in terms of the communication rate and estimation error covariance is analytically characterized. Our results demonstrate that event-triggered schemes may preform worse than time-triggered schemes when considering the effect of communication strategies.

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