Neural Lyapunov Differentiable Predictive Control

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Abstract-We present a learning-based predictive control methodology using the differentiable programming framework with probabilistic Lyapunov-based stability guarantees. The neural Lyapunov differentiable predictive control (NLDPC) learns the policy by constructing a computational graph encompassing the system dynamics, state and input constraints, and the necessary Lyapunov certification constraints, and thereafter using the automatic differentiation to update the neural policy parameters. In conjunction, our approach jointly learns a Lyapunov function that certifies the regions of state-space with stable dynamics. We also provide a sampling-based statistical guarantee for the training of NLDPC from the distribution of initial conditions. Our offline training approach provides a computationally efficient and scalable alternative to classical explicit model predictive control solutions. We substantiate the advantages of the proposed approach with simulations to stabilize the double integrator model and on an example of controlling an aircraft model.

Index Terms—Differentiable predictive control, model predictive control, neural policy, Lyapunov guarantee.

I. INTRODUCTION

The optimization-based solution to the model predictive control (MPC) problems can become restrictive in some real-life applications due to the computational burden of computing implicit control policies by solving nonlinear optimization programs. Explicit MPC (eMPC) provides an alternative solution at the control inputs are pre-computed to avoid online optimization for certain prespecified parameters [1], [2]. However, the multiparametric optimization problem quickly becomes intractable for larger systems, thereby limiting its practical usability. Therefore, in the last decade, researchers have focused on reducing the computational complexity of the eMPC problems [3], [4]. More recent trends show promises in developing data-driven predictive control solutions for the underlying explicit MPC problem.

Learning-based MPC (LBMPC) [5] methods utilizes the learned system dynamic model from trajectory measurements via adaptive MPC frameworks [6], [7]. Learning-based approaches are also supplemented with various forms of robustness conditions and capture state-dependent uncertainties with work such as [8], with guarantees on bounded uncertainties in [9], to name a few. On the other hand, approximate MPC [10]–[14] utilizes a supervised learning framework to imitate the MPC trajectories, which, although efficient in implementation, is bottlenecked by its inherent dependency on the original MPC problem solutions. From the computer science perspective, researchers have focused on developing new data-driven model architectures inspired by control theory. Work such as [15] have explored MPCinspired neural architectures for learning neural control policies. The use of convex neural networks [16] for optimal predictive control problems is investigated in [17]. While [18] introduced implicit layers to solve constrained optimization problems within deep neural networks leading to works such as [19], [20]. Others [21], [22] have proposed the use of automatic differentiation to compute the sensitivities of the MPC problems for safe control in the context of imitation learning. To provide guarantees, [23] leverage linear matrix inequality (LMI) framework for enforcing Lyapunov stability conditions while learning neural control policies. A semidefinite programming-based safety verification and robustness of neural network control policies have been investigated in [24]. [25] presents adaptive safe learning with Lyapunov provable safety certificate. [26] presents a learner-falsifier-based Lyapunov guarantee for learning neural policies for optimal control of known dynamics. An alternative approach presented by [27], [28] provides safety guarantees for reinforcement learning (RL) algorithms by representing the control policy by an implicit differentiable MPC. [29] proposes to use neural Lyapunov functions to tune parameters of online implicit MPC. [30] presents a system identification methodology for nonlinear dynamics to jointly learn a controller and corresponding stable closedloop dynamics.

Inspired by these trends, in [31] we have introduced the differentiable predictive control (DPC) framework as a scalable data-driven alternative to analytical multiparametric programming solvers. DPC brings forth the idea of offline computation of explicit predictive control policies by leveraging automatic differentiation (AD) of the constrained optimization problem for direct computation of the policy gradients. In our previous work, we have demonstrated the scalability of the DPC framework on systems, including uncertainties and nonlinear constraints [32], [33].

Contributions. This paper brings forth the idea of jointly learning the Lyapunov function candidates to provide guarantees in the DPC policy learning framework. We parametrize the Lyapunov function using deep neural network architecture introduced in [34]. Then we solve the parametric optimal control problem with such Lyapunov certification conditions as a constraint via the DPC policy optimization algorithm [31]. The differentiable programming framework constructs a computational graph of the closed-loop parametrized system with neural predictive policy and learnable Lyapunov functions and then uses AD to backpropagate resultant loss using MPC objectives and soft constraint penalties to update policy and Lyapunov network parameters.

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Additionally, we provide a theoretical framework for guaranteeing stability and constraint satisfaction during the training, where we sample initial conditions from a given distribution and characterize the system trajectories based on such initial conditions in a probabilistic manner. For such analysis, we utilize Hoeffding's inequality adapting from [35] which was described in the context of unsupervised learning of the constrained control policies. We perform detailed numerical experiments on stabilizing the double integrator model and controlling an aircraft model to bring out various intricacies of our design.

The paper is organized as follows. The problem formulation and necessary background is discussed in Section II. The proposed method is described in Section III with the necessary building blocks and algorithm. Section IV describes the probabilistic guarantees ensuring stability and constraint satisfaction with sampled initial conditions. Numerical experiments are presented in detail in Section V, and we provide concluding remarks in Section VI.

II. PROBLEM FORMULATION AND BACKGROUND

We consider the discrete-time non-linear dynamic system:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \ k \in \mathbb{N}_0^{N-1}, \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$ is the system state, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ is the control input at time k, and $\mathbb{N}_a^b = \{a, a + 1, \dots, b\}$ is a set of integers. We assume the dynamics f(.) is locally Lipschitz continuous and known. The system is also constrained by state and control input based constraints.

A. Parametric Optimal Control Problem

We intend to solve the following parametric optimal control problem:

$$\min_{\boldsymbol{\theta}} \sum_{k=0}^{N-1} \ell(\mathbf{x}_k, \mathbf{u}_k),$$
(2a)

s.t.
$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \ k \in \mathbb{N}_0^{N-1},$$
 (2b)

$$[\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}] = \boldsymbol{\pi}_{\theta}(\mathbf{x}_0), \qquad (2c)$$

$$h(\mathbf{x}_k) \le \mathbf{0},\tag{2d}$$

$$g(\mathbf{u}_k) \le \mathbf{0},\tag{2e}$$

$$\mathbf{x}_0 \sim P_{\mathbf{x}_0}.\tag{2f}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ are states and control actions, respectively, N defines finite time prediction horizon, and $\boldsymbol{\theta}$ are the parameters of the control policy (2c) to be optimized over a distribution of initial conditions (2f).

We consider the objective (2a) to be a differentiable function. We can consider quadratic objective functions with state and control energy minimization such as:

$$\ell(\mathbf{x}_k, \mathbf{u}_k, \mathbf{r}_k) = ||\mathbf{x}_k||_{Q_x}^2 + ||\mathbf{u}_k||_{Q_u}^2$$
(3)

with $\|\mathbf{a}\|_Q^2 = \mathbf{a}^T Q \mathbf{a}$ being weighted squared 2-norm. We consider the following assumptions:

Assumption 1: The nominal system dynamics model (1) is controllable, and known. The dynamics f(.) can be a learned model or a physics based model.

Assumption 2: The parametric optimal control objective $\ell(\mathbf{x}_k, \mathbf{u}_k)$ and constraints $h(\mathbf{x}_k)$, and $g(\mathbf{u}_k)$, respectively, are at least once differentiable.

B. Neural Predictive Control Policy

Without the specific parametric policy structure as in (2c), then the classical MPC solution provides an implicit solution for the control inputs utilizing nonlinear programming based solvers. On the contrary, we are interested in generating explicit model predictive (eMPC) control policies with a specific parametric control structure. For the system dynamic model (1), we want to compute parametric predictive control policies modeled by deep neural networks $\pi_{\theta}(\mathbf{x}_0) : \mathbb{R}^{n_x} \to \mathbb{R}^{N \times n_u}$ (4) that minimizes the parametric control objective function (2a) for prediction window of N steps, while satisfying the constraints. We consider the policy to be made of fully connected neural networks such as:

$$\mathbf{U} = \boldsymbol{\pi}_{\boldsymbol{\theta}}(\mathbf{x}_0) = \mathbf{W}_L \mathbf{z}_L + \mathbf{b}_L, \tag{4a}$$

$$\mathbf{z}_{l} = \boldsymbol{\sigma}(\mathbf{W}_{l-1}\mathbf{z}_{l-1} + \mathbf{b}_{l-1}), \qquad (4b)$$

$$\mathbf{z}_0 = \mathbf{x}_0,\tag{4c}$$

where $\mathbf{U} = [\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}] \in \mathbb{R}^{N \times n_u}$ represents the generated control sequence over N time steps. Vector \mathbf{x}_0 represents full state initial condition feedback, and subsequently the vector \mathbf{x}_k provides the instantaneous state vector at time step k. Considering the architecture of the policy class, \mathbf{z}_i represents hidden states, \mathbf{W}_i , and \mathbf{b}_i are the weights and biases of the *i*-th layer, respectively. These policy parameters are compactly represented as $\boldsymbol{\theta}$ which we intend to optimize. The nonlinearity $\boldsymbol{\sigma} : \mathbb{R}^{n_{z_l}} \to \mathbb{R}^{n_{z_l}}$ is constructed by executing differentiable activation function $\boldsymbol{\sigma} : \mathbb{R} \to \mathbb{R}$ in an element-wise manner.

C. Stability via Neural Lyapunov Functions

We intend to provide Lyapunov stability certificate during the training of the explicit neural predictive control policy $\pi_{\theta}(\mathbf{x}_0)$. The closed-loop control at time step k can be computed by taking the first control action of the predictive sequence denoted by $\mathbf{u}_k = [\pi_{\theta}(\mathbf{x}_k)]_0$, however with slight abuse of notation we continue to use the notation $\pi_{\theta}(\mathbf{x}_k)$ as our control input \mathbf{u}_k . The Lyapunov function $V(\mathbf{x}_k)$ can characterize the stability of the closed-loop dynamics as follows.

Lemma 1: (Lyapunov Characterization) : Using the neural predictive control policy $\mathbf{u}_k = \pi_{\boldsymbol{\theta}}(\mathbf{x}_k)$ for discrete time steps k, with the Liptsitz continuous closed-loop $f(\mathbf{x}_k, \mathbf{u}_k)$, for the equilibrium $\mathbf{x}_{\mathcal{O}} = \mathbf{0}$, if there exists a set $\mathcal{D} \subset \mathbb{R}^n$ on which the function V(.) is always positive-definite, and satisfy,

$$V(f(\mathbf{x}_k, \mathbf{u}_k)) - V(\mathbf{x}_k) < 0, \forall \mathbf{x}_k \in \mathcal{D} - \mathbf{0},$$
(5)

then $\mathbf{x}_{\mathcal{O}} = \mathbf{0}$ is an asymptotically stable equilibrium, and V(.) is a control Lyapunov function (CLF) for the closed-loop dynamics.

We consider a parametrized closed-loop Lyapunov function, the CLF, with deep neural network parameters ϕ , denoted as $V_{\phi}(\mathbf{x}_k)$. As proposed in [36], we can consider the following structure for the Lyapunov function candidate based on input-convex neural network (ICNN):

$$\mathbf{z}_1 = \sigma_0 \left(\mathbf{W}_0 \mathbf{x} + \mathbf{b}_0 \right),\tag{6}$$

$$\mathbf{z}_{i+1} = \sigma_i \left(\mathbf{U}_i \mathbf{z}_i + \mathbf{W}_i \mathbf{x} + \mathbf{b}_i \right), i = 1, \dots, k-1, \quad (7)$$

$$\mathbf{g}(\mathbf{x}) \equiv \mathbf{z}_{\mathbf{k}},\tag{8}$$

which makes the function g(x) convex in x. The activations $\sigma_k(.)$ are are monotonically non decreasing convex nonlinear scalar functions, such as the ReLU, U_i are positive weight mappings, and other weights and biases are real-valued. To make the candidate Lyapunov function positive definite another compositional operation is performed as:

$$V(\mathbf{x}) = \sigma_{k+1}(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{0})) + \epsilon \|\mathbf{x}\|_2^2,$$
(9)

where $\sigma_k(.)$ is a positive convex non-decreasing function with $\sigma_k(0) = 0$ and ϵ is a small constant. To make the Lyapunov function continuously differentiable we can consider smooth ReLU activation as mentioned in [36].

At this point, we state our problem statement as:

P. Learn the neural predictive control policy $\mathbf{u}_{\mathbf{k}} = \pi_{\theta}(\mathbf{x}_{\mathbf{k}})$ for the parametric optimal control problem (2) along with simultaneous Lyapunov certification using the parametric control Lyapunov function $V_{\phi}(\mathbf{x}_k)$ following Lemma 1 such that the closed-loop $f(\mathbf{x}_k, \mathbf{u}_k)$ dynamics remain Lyapunovstable along with solving for the desired optimal control performance.

III. NEURAL LYAPUNOV DIFFERENTIABLE PREDICTIVE CONTROL (NLDPC) FRAMEWORK

In differentiable predictive control, we train the explicit predictive control policy by utilizing the system dynamics f(.) to generate predictive trajectories in a self-supervised way. We assume that the system's state dynamics can be obtained by performing system identification or mathematical modeling of the underlying physics. The system dynamics and the neural predictive control architecture are combined in a computational graph to create a differentiable closedloop representation. The parameters of the control policy are updated via stochastic gradients descent by backpropagating the gradients of the optimal control objective function and constraints penalties. In this work, we expand the DPC framework to allow for joint learning of neural predictive control policies and neural Lyapunov functions enforcing closed-loop stability of the controlled system. We first discuss the penalty-based approach in dealing with the state, control constraints, and the Lyapunov-based certification criterion.

A. Soft state, control, and Lyapunov constraints

The state and action constraints $h(\mathbf{x}_k), g(\mathbf{x}_k)$, and the control Lyapunov certification criterion $V_{\phi}(\mathbf{x}_k)$ as shown in Lemma 1 are modeled as soft constraints using the ReLU activation function represented by $p_x(\mathbf{x}_k), p_u(\mathbf{u}_k)$,

and $p_V(\mathbf{x}_{k+1}, \mathbf{x}_k)$, respectively as follows:

$$p_x(\mathbf{x}_k) = ||\mathsf{ReLU}(h(\mathbf{x}_k))||_2 \tag{10a}$$

$$p_u(\mathbf{u}_k) = ||\mathsf{ReLU}(g(\mathbf{u}_k))||_2 \tag{10b}$$

$$p_V(\mathbf{x}_{k+1}, \mathbf{x}_k) = ||\text{ReLU}(V_\phi(\mathbf{x}_{k+1}) - V_\phi(\mathbf{x}_k))||_2, \quad (10c)$$

with ReLU standing for rectifier linear unit function. This structural representation will help us to formulate the neural Lyapunov DPC problem as described in the next sub-section.

B. Neural Lyapunov DPC Problem

To formulate the learning-based predictive control problem, we utilize the batched training strategy where we generate multiple batches of randomly sampled initial condition that will be used for forward pass roll outs of the DPC computational graph. We sample the initial conditions following user-defined distribution \mathcal{D} :

$$\mathbf{x}_0 \sim \mathcal{D}, \ \mathbf{x}_0 \in \mathbb{X} \subseteq \mathbb{R}^{n_x}.$$
 (11)

Putting all together, we formulate the neural Lyapunov DPC as following sampled parametric optimal control problem:

$$\min_{\theta,\phi} \frac{1}{mN} \sum_{i=1}^{m} \sum_{k=0}^{N-1} \left(\ell(\mathbf{x}_{k}^{i}, \mathbf{u}_{k}^{i}) + Q_{V} p_{V}(\mathbf{x}_{k+1}^{i}, \mathbf{x}_{k}^{i}) + \right)$$
(12a)

$$Q_x p_x(\mathbf{x}_k^i) + Q_u p_u(\mathbf{u}_k^i)),$$

s.t.
$$\mathbf{x}_{k+1}^i = f(\mathbf{x}_k^i, \mathbf{u}_k^i), \ k \in \mathbb{N}_0^{N-1},$$
 (12b)

$$[\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}]^{\iota} = \boldsymbol{\pi}_{\theta}(\mathbf{x}_k^{\iota}), \qquad (12c)$$

$$\mathbf{x}_0^i \in \mathbb{X} \subseteq \mathbb{R}^{n_x} \sim \mathcal{D}. \tag{12d}$$

here the index *i* denotes the batch sample. The DPC loss function is made of multiple terms, i.e., the optimal control objectibe $\ell(\mathbf{x}_k, \mathbf{u}_k) : \mathbb{R}^{n_x+n_u} \to \mathbb{R}$, and penalties of parametric constraints $p_x(h(\mathbf{x}_k)) : \mathbb{R}^{n_h} \to \mathbb{R}$, $p_u(g(\mathbf{u}_k)) :$ $\mathbb{R}^{n_g} \to \mathbb{R}$, and $p_V(\mathbf{x}_{k+1}, \mathbf{x}_k) : \mathbb{R} \to \mathbb{R}$. Q_V, Q_x and Q_u represent the scalar penalty weights.

To utilize differentiable programming, we keep the following assumption.

Assumption 3: The optimal parametric objective function $\ell(\mathbf{x}_k, \mathbf{u}_k)$, and state, input and Lyapunov constraint penalties $p_x(\mathbf{x}_k)$, $p_u(\mathbf{u}_k)$, and $p_V(\mathbf{x}_{k+1}, \mathbf{x}_k)$ are differentiable at almost every point in their domain.

C. Neural Lyapunov DPC Computational Graph

The formulation (12) is being implemented in the form of a differentiable program for optimizing the neural parameters θ and ϕ for neural predictive policy and Lyapunov function respectively. We construct the computation graph for the differentiable program which optimizes the neural parameters by differentiating the neural Lyapunov DPC loss function (12a) and backpropagating the gradients through the parametrized closed-loop dynamics given by the system model (12b) and neural control policy (12c). Fig. 1 shows the data flow in the computational graph of the presented neural Lyapunov DPC method. We have developed a generic framework of output feedback structure where we can use an



Fig. 1: Neural Lyapunov DPC methodology.

estimator parametrized by deep neural networks to estimate the initial condition from a past output and input trajectories. However, for the simplicity of the exposition in this paper, we consider a full state feedback. The policy component reads the current states and generates predictive control trajectories over the prediction horizon. Thereafter the initial state and control trajectories are used to generate a *N*-step ahead rollout of the system dynamics obtaining state trajectories. The generated trajectories are then passed through penaltybased constraints including Lyapunov constraint (5).

D. Neural Lyapunov DPC Gradients

Finally, all these individual components described above lead to the neural NLDPC loss function $\mathcal{L}_{\text{NLDPC}}$ (12a). One can use stochastic gradient descent and its variants to optimize the loss function $\mathcal{L}_{\text{NLDPC}}$ over the sampled distribution of initial conditions (12d). The differentiable closed-loop dynamics with the learnable policy, optimal control objective, and constraints allow us to use automatic differentiation [37] to directly compute policy gradients. As described before, the problem (12) is represented as a computational graph, and therefore, using chain rule, one can compute the gradients of $\mathcal{L}_{\text{NLDPC}}$ w.r.t. the policy parameters θ as follows:

$$\nabla_{\theta} \mathcal{L}_{\text{NLDPC}} = \frac{\partial \ell(\mathbf{x}, \mathbf{u})}{\partial \theta} + \frac{\partial p_{V}(\mathbf{x})}{\partial \theta} + \frac{\partial p_{x}(\mathbf{x})}{\partial \theta} + \frac{\partial p_{x}(\mathbf{x})}{\partial \theta} + \frac{\partial p_{u}(\mathbf{u})}{\partial \theta} = \frac{\partial \ell(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \frac{\partial \mathbf{u}}{\partial \theta} + \frac{\partial \ell(\mathbf{x}, \mathbf{u})}{\partial \theta} \frac{\partial \mathbf{u}}{\partial \theta} + \frac{\partial p_{V}(\mathbf{x})}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \theta} + \frac{\partial p_{u}(\mathbf{u})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \theta},$$
(13)

where $\frac{\partial \mathbf{u}}{\partial \theta}$ represents partial derivatives of the neural policy with respect to its weights, computed using backpropagation. Similarly, we can compute $\nabla_{\phi} \mathcal{L}_{\text{NLDPC}}$, i.e., gradients with respect to neural Lyapunov function parameters. The parametric optimal control problem (12) now can be solved by using scalable stochastic gradient optimization algorithms such as AdamW [38]. This provides us with a way to optimize the policy offline. In practice, we use automatic differentiation frameworks such as Pytorch [39] to compute these gradients in the NLDPC computational graph.

E. Simultaneous Learning of Constrained Neural Control Policy and Lyapunov Function

We now describe how to learn both a control policy and a neural Lyapunov function together, so that the Lyapunov conditions can be rigorously verified to ensure the stability of the system. $\nabla_{\theta} \mathcal{L}_{\text{NLDPC}}$, and $\nabla_{\phi} \mathcal{L}_{\text{NLDPC}}$ help us to train these policies and neural Lyapunov function simultaneously with batches of state trajectory data for different samples of initial conditions. The NLDPC computational graph has been used along with reverse-mode automatic differentiation framework to implement the algorithm. Algorithm 1 presents the steps of the proposed framework.

Algorithm 1 Neural Lyapunov DPC (NLDPC) 1: **Input** identified or physics based system dynamics f(.)2: Input candidate Lyapunov function $V_{\phi}(\mathbf{x})$ 3: Input optimal control loss $\ell(\mathbf{x}, \mathbf{u})$ and constraints $h(\mathbf{x})$, and $g(\mathbf{u})$ 4: **Input** optimizer \mathbb{O} 5: **Construct** NLDPC computational graph based on (12) 6: for epochs=1: N_{epoch} do **Sample** initial conditions \mathbf{x}_0 from distribution \mathcal{D} 7: Evaluate forward pass of the NLDPC computational 8: graph and compute $\mathcal{L}_{\text{NLDPC}}$ 9: Differentiate NLDPC computational graph w.r.t. 10: policy and Lyapunov candidate parameters to obtain 11: gradients $\nabla_{\theta} \mathcal{L}_{\text{NLDPC}}$ and $\nabla_{\phi} \mathcal{L}_{\text{NLDPC}}$ 12: **Update** control parameters $\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}_{\text{NLDPC}}$ 13: 14:

- 14: **Update** Lyapunov parameters $\phi \leftarrow \phi + \alpha \nabla_{\phi} \mathcal{L}_{\text{NLDPC}}$ 15: **end for**
- 16: **Return**: trained policy π_{θ} and Lyapunov function V_{ϕ}

IV. STABILITY GUARANTEES

Probabilistic considerations have been discussed in works [35], [40], [41] for learning-based MPC setting. We use sampling-based guarantees for the neural Lyapunov DPC framework extending the approach of [35]. The parametric optimization problem (12) has been solved using Algorithm 1 where we created multiple batches with sampled initial conditions $\mathbf{x}_0^i \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ from a distribution \mathcal{D} . The

variations in the initial condition are shown with superscript i, and we create m such scenarios.

We denote the set of closed-loop state rollouts by \mathcal{T}^i , and the control trajectories provided by the learned NLDPC policy by $[\mathbf{u}_0^i, \ldots, \mathbf{u}_{N-1}^i] = \pi_{\theta}(\mathbf{x}_0^i)$, for $i \in \mathbb{N}_1^m$, i.e.,

$$\mathcal{T}^{i}: \begin{cases} \{\mathbf{x}_{k}^{i}\}, \ \forall k \in \mathbb{N}_{0}^{N-1}, \\ \mathbf{x}_{k+1}^{i} = \mathbf{A}\mathbf{x}_{k}^{i} + \mathbf{B}\mathbf{u}_{k}^{i} \end{cases}$$

Considering the constraints, we can create a probabilistic metric for the sampled roll-outs such as,

$$\mathbf{P}^{i} = \operatorname{True}: \begin{cases} h(\mathbf{x}_{k}^{i}) \leq \mathbf{0}, \ \forall k \in \mathbb{N}_{0}^{N-1}, \\ g(\mathbf{u}_{k}^{i}) \leq \mathbf{0}, \ \forall k \in \mathbb{N}_{0}^{N-1}, \\ V_{\phi}(f(\mathbf{x}_{k}^{i}, \mathbf{u}_{k}^{i})) - V(\mathbf{x}_{k}^{i}) < 0, \end{cases}$$
(14)

along with the indicator:

$$\mathcal{I}(\mathcal{T}^i) := \begin{cases} 1 & \text{if } \mathbf{P}^i = \text{True.} \\ 0, & \text{otherwise,} \end{cases}$$
(15)

which indicates if the learning based policy $\pi_{\theta}(\mathbf{x}_0^i)$ satisfies the required constrained during the sampled roll-out.

Theorem 2: Consider sampled initial conditions for solving the parametric optimal control problem (12) with assumptions 1–3, and the predictive control policy using Algorithm 1. Selecting constraint violation probability $1 - \kappa$, and level of confidence parameter δ , if the empirical risk on the indicator function (15), denoted as $\tilde{\sigma}$, with sufficiently large number of sample trajectories m satisfy $\kappa \leq \tilde{\sigma} - \sqrt{-\frac{\ln \frac{\delta}{2}}{2m}}$, then the learned NLDPC policy $\pi_{\theta}(\mathbf{x}_k)$ will constraint satisfaction and closed-loop stability in probabilistic sense.

Proof: We generate m number of sampled initial conditions which lead to m number of trajectories. The initial conditions are sampled in an i.i.d. manner, resulting in i.i.d. \mathcal{T}^i , and $\mathcal{I}(\mathcal{T}^i)$. The empirical risk for all the trajectories is given by,

$$\tilde{\sigma} = \frac{1}{m} \sum_{i=1}^{m} \mathcal{I}(\mathcal{T}^i).$$
(16)

We can guarantee stability and constraint satisfaction if for $\mathbf{x}_0^i \sim \mathcal{D}$, we have, $\mathcal{I}(\mathcal{T}^i) = 1$, i.e. $h(\mathbf{x}_k^i) \leq \mathbf{0}$, $g(\mathbf{x}_k^i) \leq \mathbf{0}$, $V_{\phi}(f(\mathbf{x}_k^i, \mathbf{u}_k^i)) - V(\mathbf{x}_k^i) < 0$, $\mathbf{x}_N^i \in \mathcal{X}_T$, $\forall i \in \mathbb{N}_1^m$.

Let us denote $\sigma := \mathbf{Pr}(\mathcal{I}(\mathcal{T}^i) = 1)$. Recalling Hoeffding's Inequality [35] to estimate σ from $\tilde{\sigma}$ we can write:

$$\mathbf{Pr}(|\tilde{\sigma} - \sigma| \ge \alpha) \le 2\exp(-2r\alpha^2) \quad \forall \alpha > 0.$$
 (17)

Thereafter, using $\delta := 2\exp(-2r\alpha^2)$, with confidence $1-\delta$ one can have,

$$\mathbf{Pr}(\mathcal{I}(\mathcal{T}^i) = 1) = \sigma \ge \tilde{\sigma} - \alpha.$$
(18)

Therefore with a chosen confidence δ and risk lower bound $\kappa \leq \Pr(\mathcal{I}(\mathcal{T}^i)) = 1$, the empirical risk can be bounded as:

$$\kappa \le \tilde{\sigma} - \alpha = \tilde{\sigma} - \sqrt{-\frac{\ln\frac{\delta}{2}}{2m}} \tag{19}$$

In a nutshell, once we fix the confidence δ and risk lower bound κ , the empirical risk $\tilde{\sigma}$ and α is evaluated for an experimental value of m. Therefore when policies trained via Algorithm 1 satisfies (19), then with confidence at least $1 - \delta$, at least a fraction of κ trajectories \mathcal{T}^i will satisfy $\mathbf{P}^i =$ True. Therefore, state, action and Lyapunov constraints are satisfied with the confidence $1 - \delta$. This concludes the proof.

V. NUMERICAL CASE STUDIES

A. Example 1: Stabilizing Double Integrator

We start with the task of learning a stabilizing neural feedback policy for an unstable system. The unstable double integrator model is given as:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1.2 & 1.0\\ 0.0 & 1.0 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 1.0\\ 0.5 \end{bmatrix} \mathbf{u}_k.$$
 (20)

The system is subject to state and input constraints given as $\mathbf{x}_k \in \mathcal{X} := {\mathbf{x} \mid -10 \le x \le 10}$, and $\mathbf{u}_k \in \mathcal{U} := {\mathbf{u} \mid -1 \le u \le 1}$, respectively. We consider the following quadratic objectives,

$$\mathcal{L}_{\text{MPC}} = \sum_{k=0}^{N-1} \left(||\mathbf{x}_k||_{Q_x}^2 + ||\mathbf{u}_k||_{Q_u}^2 \right).$$
(21)

For learning neural Lyapunov DPC, we intend to learn a full state feedback neural policy $\pi_{\theta}(\mathbf{x}_k)$ using Algorithm 1. We also impose terminal set constraints $\mathbf{x}_N \in \mathcal{X}_f$:= $\{\mathbf{x} \mid -0.1 \leq x \leq 0.1\}$. We also impose the Lyapunov constraints given in (12). The weights are considered to be $Q_V = 2, Q_h = 10, Q_g = 100, Q_{\chi_f} = 1$, while for the control objective (21) we consider prediction horizon N = 1, and weights $Q_x = 5$, $Q_u = 0.5$. The neural policy (4) $\pi_{\theta}(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}$ is trained with 4 layers, 20 hidden states, with bias, and ReLU activation functions. For the Lyapunov function $V_{\phi}(\mathbf{x})$ we use 8 layers with 40 hidden states and ReLU activations. The training set \mathcal{X}^{train} consists of 3333 normally sampled initial conditions \mathbf{x}_0^i fully covering the admissible set \mathcal{X} . The training has been performed for 300 epochs. Fig. 2 shows the phase portrait of the two states which has been stabilized and the level curves of the learned Lyapunov function has also been superimposed. The learned Lyapunov function is shown in Fig. 3, and the closedloop state trajectories in Fig. 5 which shows the closedloop stabilizing performance of NLDPC. Fig. 4 shows the finite differences of the learned Lyapunov function for two consecutive time steps of the closed-loop dynamics in the left panel, and compared with the quadratic Lyapunov $\mathbf{x}^{\top}\mathbf{x}$ in the right panel. The figure shows that the quadratic Lyapunov can give a conservative safe region of operations whereas the learned version using NLDPC is less conservative and tend to justify the actual learned closed-loop behaviour.

B. Example 2: Stabilizing PVTOL aircraft model

For the next example, we considered a planar vertical take-off and landing (PVTOL) aircraft model as appeared in [42]. We have $\mathbf{x}_k \in \mathbb{R}^6$ representing positions (x, y), velocities (\dot{x}, \dot{y}) , and orientation of the center of mass θ , and its derivative $\dot{\theta}$. The control inputs $\mathbf{u} \in \mathbb{R}^2$ are the forces from the aircraft thrusters. We use a discrete-time



Fig. 2: Closed-loop phase portrait with overlaid Lyapunov contours for double integrator.



Fig. 4: Stability regions via discrete-time Lyapunov stability condition (5) for double integrator.

PVTOL model with sampling time of 0.2 seconds. Here, the quadratic control objective uses weights $Q_x = 3$, $Q_u = 0.1$ with prediction horizon N = 10, and subject to state and input constraints $\mathbf{x}_k \in \mathcal{X} := \{\mathbf{x} \mid -5 \leq x \leq 5\}$, and $\mathbf{u}_k \in \mathcal{U} := {\mathbf{u} \mid -5 \leq u \leq 5}.$ The penalty weights are considered to be, $Q_V = 3, Q_h = 2, Q_g = 2$, respectively. The neural policy (4) $\pi_{\mathbf{W}}(\mathbf{x}) : \mathbb{R}^6 \to \mathbb{R}^{N \times 2}$ is similarly considered with 4 layers, 20 hidden states, with bias, and ReLU activation functions. For the Lyapunov function $V_{\phi}(\mathbf{x})$ we use 8 layers with 40 hidden states and ReLU activations, as before. We generate 9000 normally distributed initial conditions with 3000 each for training, validation and testing sets. Fig. 6 shows the phase portrait sliced only with the aircraft velocities, and the overlaid learned Lyapunov level curves. The learned Lyapunov function with only the above mentioned dimensions are shown in Fig. 7 showing the positive definiteness. Fig. 9 shows all the states and control actions in the closed-loop which satisfies all the required constraints. Fig. 8 shows the finite differences of the Learned and the quadratic Lyapunov functions for two time-steps of the closed-loop where again the learned Lyapunov function guarantees a larger region of safety than the quadratic one.



Fig. 3: Learned Lyapunov function for double integrator.



Fig. 5: Closed-loop state and control trajectories for double integrator.

VI. CONCLUSIONS

This paper presented a differentiable programming-based framework for joint learning of explicit neural predictive control policies and neural Lyapunov functions. We bring forth the idea of constraining the parametric optimal program with learnable Lyapunov-based stability certifications. We supplement our training framework with initial condition sampling-based probabilistic guarantees using Hoeffding's inequality. Numerical experiments on standard dynamic system examples of double integrator and PVTOL aircraft model substantiate the algorithmic and theoretical results. Future research will look into more exhaustive experiments with nonlinear system models and using learned identificationbased models in the NLDPC framework including different architectures of neural Lyapunov function candidates. On the theoretical side, we intend to expand the framework to support time-varying Lyapunov functions and dissipative system conditions used in economic MPC to deal with plant model mismatch and measurement noise.

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Fig. 6: Closed-loop phase portrait with overlaid Lyapunov contours for PVTOL aircraft. Visualized on 2D slice of the 6D state space.



Fig. 8: Stability regions via discrete-time Lyapunov stability condition (5) for PVTOL aircraft. Visualized on 2D slice of the 6D state space.

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Fig. 7: Learned Lyapunov function for PVTOL aircraft. Visualized on 2D slice of the 6D state space.



Fig. 9: Closed-loop state and control trajectories for PVTOL aircraft.

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