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Neural synthesis of firing decisions in the brain

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Abstract

The brain generally miniaturizes its perceptions into what may be regarded as a model of what happens outside. We experience the world according to the capacity of our nervous system to register the stimuli we receive. In order to understand and control the environment there needs to be proportionality between the measurements represented in the miniaturized model that arises from the firings of our neurons, and the actual measurements in the real world. Thus our response to stimuli must satisfy the fundamental functional equation $F(ax) = bF(x)$. In other words, our interpretation of a stimulus as registered by the firing of our neurons is proportional to what it would be if it were not filtered through the brain. This equation is the homogeneous part of the inhomogeneous equation $F(ax) - bF(x) = G(x)$ with the forcing function $G(x)$. What interests us here is the mode of operation of the (firing) system that needs to always satisfy the homogeneous part.

Keywords: *Decision Making, Neural Firing and Synthesis, Fredholm's Equation, Neural Representation*

Beyindeki ateşleme kararlarının sinirsel sentezi

Özet

Beyin genellikle kendi algılarını dışarıda gerçekleşenlerin bir modeli olarak da kabul edilebilen şekilde küçültmektedir. Dış dünyayı sinir sistemimizin aldığımız uyarıcıları kayıt etme kapasitesi doğrultusunda deneyimleriz. Çevremizi anlamak ve kontrol edebilmek için, gerçek dünyadaki gerçek ölçümler ile sinir hücrelerinin ateşlemesinden meydana gelen küçültülmüş modelin temsil ettiği ölçümlerin orantılı olması gerekir. Bu nedenle uyarıcıya verdiğimiz tepki, $F(ax) = bF(x)$ temel işlevsel fonksiyonunu karşılamalıdır. Diğer bir ifadeyle, sinir hücrelerimizin ateşlemesi ile kayıt edilen bir uyarıcıya ait yorumlamamız, beyin tarafından filtre edilmediği durumdaki hali ile orantılıdır. Bu denklem $G(x)$ zorlayıcı denklemi ile birlikte, homojen olmayan $F(ax) - bF(x) = G(x)$ denkleminin homojen kısmıdır. Burada bizi ilgilendiren her zaman homojen kısmı tamamlamaya ihtiyaç duyan bu sistemin (ateşleme) çalışma biçimidir.

Anahtar Sözcükler: *Karar Alma, Sinirsel Ateşleme ve Sentez, Fredholm Denklemi, Sinirsel Betimleme*

1. Introduction

The brain is a network with several hierarchies as sub-networks. Connections in a hierarchy imply a special kind of integration and synthesis of signals. Integration means the addition and sequencing of signals in their temporal order. The signals are neural firings of a form that can be represented by Dirac delta functions.

The observation that the brain is a network means that it allows for feedback and cycling which also requires a synthesis of signals. Signals can have different strengths that in

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the brain are reflected in their frequency rather than in their amplitudes as the brain is a frequency modulating system. To combine different frequencies and obtain an outcome that has an underlying order requires a method of synthesis that maintains ratio and proportionality among these signals. We have used multicriteria decision-making methods to synthesize the many and varied signals in the brain. This paper is divided into four parts: 1) How the subconscious brain surrenders its hidden mode of operation through conscious discrete comparisons and synthesis in decision making; 2) How the discrete process generalizes to the continuous case embedded in which as a necessary condition for a solution is a proportionality functional equation whose solution is a damped periodic oscillation solution whose space-time Fourier transform takes the form of Dirac type distributions (firings of neurons that respond to stimuli); 3) How the synthesis of such distributions can serve as the basis for forming images and creating sound oscillations for example; and finally, 4) How the neural representation is made. See also the book [1].

2. Generalization to the Continuous Case

In a previous paper [2] we dealt with the principal eigenvalue problem and the principal eigenvector as derived from a positive reciprocal matrix. The problem leads to solving the homogeneous system of equations,

$$\sum_{j=1}^{n} a_{ij}\, w_j = \lambda_{\max}\, w_i \qquad (1)$$

$$\sum_{i=1}^{n} w_i = 1 \qquad (2)$$

with $a_{ji}=1/a_{ij}$ or $a_{ij}\, a_{ji}=1$ (the reciprocal property), $a_{ij} > 0$. This system and normalization condition generalizes to the continuous case through Fredholm's integral equation of the second kind (which we have also derived directly from first principles to describe the response of a neuron to stimuli):

$$\int_a^b K(s,t)\, w(t)\, dt = \lambda_{\max}\, w(s) \qquad (3)$$

$$\int_a^b w(s) ds = 1 \qquad (4)$$

where instead of the matrix A we have a positive kernel, $K(s,t) > 0$. A solution $w(s)$ of this equation is a right eigenfunction.

The standard way in which (3) is written is to move the eigenvalue to the left hand side which gives it the reciprocal form

$$\lambda \int_a^b K(s,t) w(t) dt = w(s) \qquad (5)$$

An eigenfunction of this equation is determined to within a multiplicative constant. If $w(t)$ is an eigenfunction corresponding to the characteristic value λ and if C is an arbitrary constant, we see by substituting in the equation that $Cw(t)$ is also an eigenfunction corresponding to the same λ. The value λ=0 is not a characteristic value because we have the corresponding solution $w(t)$=0 for every value of t, which is the trivial case, excluded in our discussion. Here also, we have the reciprocal property

$$K(s,t)\ K(t,s) = 1 \tag{6}$$

so that $K(s,t)$ is not only positive, but also reciprocal. An example of this type of kernel is $K(s,t) = e^{s-t} = e^s / e^t$. As in the finite case, the kernel $K(s,t)$ is consistent if it satisfies the relation

$$K(s,t)\ K(t,u)= K(s,u), \text{ for all } s, t, \text{ and } u \tag{7}$$

It follows by putting $s = t = u$, that $K(s,s)$=1 for all s which is analogous to having ones down the diagonal of the matrix in the discrete case.

The most important part of what follows is the derivation of the fundamental equation, a functional equation whose solution is an eigenfunction of our basic Fredholm equation.

Theorem 1 K(s,t) is consistent if and only if it is separable of the form:

$$K(s,t)=k(s)/k(t) \tag{8}$$

Theorem 2 If K(s,t) is consistent, the solution of (5) is given by

$$w(s) = \frac{k(s)}{\int_S k(s)ds} \tag{9}$$

We note that this formulation is general and applies to all situations where a continuous ratio scale is needed. It applies equally to the derivation or justification of ratio scales in the study of scientific phenomena. We now determine the form of $k(s)$ and also of $w(s)$.

In the discrete case, the normalized eigenvector was independent of whether all the elements of the pairwise comparison matrix A are multiplied by the same constant a or not, and thus we can replace A by aA and obtain the same eigenvector. Generalizing this result we have:

$$K(as, at)=aK(s,t)=k(as)/k(at)=a\ k(s)/k(t) \tag{10}$$

which means that K is a homogeneous function of order one.

Theorem 3 A necessary and sufficient condition for $w(s)$ to be an eigenfunction solution of Fredholm's equation of the second kind, with a consistent kernel that is homogeneous of order one is that it satisfy the functional equation

$$w(as)=bw(s) \tag{11}$$

where $b = \alpha a$ for $\alpha > 0$.

It is clear that whatever aspect of the real world we consider, sight, sound, touch, taste, smell, heat and cold, at each instant, their corresponding stimuli impact our senses numerous times. A stimulus S of magnitude s, is received as a similarity transformation as, $a > 0$ referred to as a dilation of s. It is a *stretching* if a > 1, and a *contraction* if a < 1. When relating response to a dilated stimulus of magnitude as to response to an unaltered stimulus whose magnitude is s, we have the proportionality relation we just wrote down:

$w(as)/w(s) = b$ We refer to equation (11) as: The Functional Equation of Ratio Scales. Because of its wider implications in science, we may call it: *The Fundamental Equation of Proportionality and Order.*

If we substitute $s=a^u$ in (11) we have (see Aczél and Kuczma [3]):

$$w(a^{u-1}) - bw(a^u) = 0$$

Again if we write $w(a^u)=b^u p(u)$, we obtain: $p(u+1) - p(u)=0$.

This is a periodic function of period one in the variable u (such as cos $u/2\pi$). Note that if the parameters a and s are real, then so is u which may be negative even if a and s are both assumed to be positive. Finally we have

$$w(s) = b^{\log s/\log a} p(\frac{\log s}{\log a})$$

The right hand side of this equation is a damped periodic function.

By dividing its variable by its period, any periodic function can be reduced to a periodic function of period one. Thus, whatever is known about periodic functions applies to periodic functions of period one and conversely. If P is periodic of period T, i.e. $P(x+T)=P(x)$, then $p(x)=P(Tx)$ will be periodic of period 1:

$$p(x+1)=P(T(x+1))=P(Tx+T)=P(Tx)=p(x);$$

the converse operation is obvious.

If in the last equation $p(0)$ is not equal to 0, we can introduce $C=p(0)$ and $P(u)=p(u)/C$, we have for the general response function $w(s)$,

$$w(s) = Ce^{\log b \frac{\log s}{\log a}} P\left(\frac{\log s}{\log a}\right) \tag{12}$$

where P is also periodic of period 1 and $P(0)=1$. Note that $C > 0$ only if $p(0)$ is positive. Otherwise, if $p(0) < 0$, $C < 0$.

Analogously, the general complex solution of our functional equation is given by:

$$w(z) = Cb^{[\log |z| / \log |a|]} g(z)$$

where $C > 0$. The [ ] in the above expression denotes the "closest integer from below" function, and g is an arbitrary solution of $g(az)=g(z)$.

Near zero, the exponential factor which is equal to $s^{\log b/\log a}$, "slims" $w(s)$ if $\log b/\log a > 0$ and "spreads" $w(s)$ if $\log b/\log a < 0$. Because s is the magnitude of a stimulus and cannot be negative, we do not have a problem with complex variables here so long as both a and b are real and both positive. Our solution in the complex domain has the form:

$$w(z)=z^{\ln b/\ln a} P(\ln z/\ln a) \tag{13}$$

Here $P(u)$ with $u = \ln z/\ln a$, is an arbitrary multivalued periodic function in u of period 1. Even without the multivaluedness of P, the function $w(z)$ could be multivalued because ln b/ln a is generally a complex number. If P is single-valued and ln b/ln a turns out to be an integer or a rational number, then $w(z)$ is a single-valued or finitely multivalued function, respectively. This generally multivalued solution is obtained in a way analogous to the real case.

3. This Solution Leads to the Weber-Fechner Law

Note in (12) that the periodic function $P(u)$ is bounded and the negative exponential leads to an alternating series. Thus, to a first order approximation one obtains the Weber-Fechner law for response to a stimulus s:

$A \log s + B$. We assume that $B=0$, and hence the response belongs to a ratio scale.

In 1846 Ernst Heinrich Weber [4] found that the concept that a just-noticeable difference in a stimulus is proportional to the magnitude of the original stimulus. For example, people while holding in their hand different weights, could distinguish between a weight of 20 g and a weight of 21 g, but could not if the second weight is only 20.5 g. On the other hand, while they could not distinguish between 40 g and 41 g, they could between 40g and 42g, and so on at higher levels. We need to increase a stimulus s by a minimum amount $\Delta\Delta s$ to reach a point where our senses can first discriminate between s and $s + \Delta s$. Δs is called the just noticeable difference (jnd). The ratio $r = \Delta s/s$ does not depend on s. Weber's law states that change in sensation is noticed when the stimulus is increased by a constant percentage of the stimulus itself. This law holds in ranges where Δs is small when compared with s, and hence in practice it fails to hold when s is either too small or too large. Aggregating or decomposing stimuli as needed into clusters or hierarchy levels is an effective way for extending the uses of this law.

In 1860 Gustav Theodor Fechner [5] considered a sequence of just noticeable increasing stimuli. He denotes the first one by s_0. The next just noticeable stimulus is given by $s_1 = s_0 + \Delta s_0 = s_0 + \frac{\Delta s_0}{s_0} s_0 = s_0(1+r)$ based on Weber's law. Similarly

$$s_2 = s_1 + \Delta s_1 = s_1(1+r) = s_0(1+r)^2 \equiv s_0\alpha^2$$

In general

$$s_n = s_{n-1}\alpha = s_0\alpha^n \ (n = 0,1,2,...)$$

Thus stimuli of noticeable differences follow sequentially in a geometric progression. Fechner noted that the corresponding sensations should follow each other in an arithmetic sequence at the discrete points at which just noticeable differences occur. But the latter are obtained when we solve for n. We have

$$n = \frac{(\log s_n - \log s_0)}{\log \alpha}$$

and sensation is a linear function of the logarithm of the stimulus. Thus if M denotes the sensation and s the stimulus, the psychophysical law of Weber-Fechner is given by

$$M = a \log s + b, \quad a \neq 0$$

We consider the responses to the stimuli to be measured on a ratio scale (b=0). A typical response has the form $M_i = a \log \alpha^i$, $i = 1,\ldots,n$, or one after another they have the form:

$$M_1 = a \log \alpha, M_2 = 2a \log \alpha, \ldots, \ M_n = na \log \alpha$$

We take the ratios M_i/M_1, i=1,…,n of these responses in which the first is the smallest and serves as the unit of comparison, thus obtaining the *integer* values 1, 2, …, n of the fundamental scale of the AHP. It appears that numbers are intrinsic to our ability to make comparisons, and were not invented by our primitive ancestors. We must be grateful to them for the discovery of the symbolism.

Stimuli received by the brain from nature are transformed to chemical and electrical neural activities that result in summation and synthesis. This is transformed to awareness of nature by converting the electrical synthesis (vibrations caused by a pattern) to a space-time representation. The way the brain goes back and forth from a pattern of stimuli to its electro-chemical synthesis and then to a representation of its response to that spacio-temporal pattern is by applying the Fourier transform to the stimulus and the inverse Fourier transform to form its response. What we have been doing so far is concerned with the inverse Fourier transform. We now need to take its inverse to develop expressions for the response.

We now show that the space-time Fourier transform of (13) is a combination of Dirac distributions. Our solution of Fredholm's equation here is given as the Fourier transform,

$$f(\omega) = \int_{-\infty}^{+\infty} F(x)\, e^{-2\pi i\omega x}\, dx = Ce^{\beta\omega}\, P(\omega) \tag{14}$$

whose inverse transform is given by:

$$\left(1/_{2\pi}\right) \log a \sum_{-\infty}^{\infty} a_n \left[\frac{2\pi n+\theta(b)-x}{\log a\,|b|+(2\pi n+\theta(b)-x)}\right] \delta(2\pi n+\theta(b)-x) \tag{15}$$

where $\delta(2\pi n+\theta(b)-x)$ is the Dirac delta function. This is supporting evidence in favor of our ratio scale model.

4. The Formation of Images and Sounds with Dirac Distributions

Complex valued functions cannot be drawn as one does ordinary functions of three real variables. The reason is that complex functions contain an imaginary part. Nevertheless, one can make a plot of the modulus or absolute value of such a function. The basic assumption we made to represent the response to a sequence of individual stimuli is that all the layers in a network of neurons are identical, and each stimulus value is represented by the firing of a neuron in each layer. A shortcoming of this representation is that it is not invariant with respect to the order in which the stimuli are fed into the network. It is known in the case of vision that the eyes do not scan pictures symmetrically if they are not symmetric, and hence our representation must satisfy some order invariant principle. Taking into account this principle would allow us to represent images independently of the form in which stimuli are input into the network. For example, we recognize an image even if it is subjected to a rotation, or to some sort of deformation. Thus, the invariance principle must include affine and similarity transformations. This invariance would allow the network to recognize images even when they are not identical to the ones from which it recorded a given concept, e.g., a bird. The next step would be to use the network representation given here with additional conditions to uniquely represent patterns from images, sounds and perhaps other sources of stimuli such as smell. Our representation focuses on the real part of the magnitude rather than the phase of the Fourier transform. Tests have been made to see the effect of phase and of magnitude on the outcome of a representation of a complex valued function. There is much more blurring due to change in magnitude than there is to change in phase. Thus we focus on representing responses in terms of Dirac functions, sums of such functions, and on approximations to them without regard to the coefficients in (15) [6-8].

The functions $\{t^{\alpha} e^{-\beta t},\ \alpha,\ \beta \geq 0\}$ result from modeling the neural firing as a pairwise comparison process in time. It is assumed that a neuron compares neurotransmitter-generated charges in increments of time. This leads to the continuous counterpart of a reciprocal matrix known as a reciprocal kernel. A reciprocal kernel ***K*** is an integral operator that satisfies the condition ***K***(*s,t*)***K***(*t,s*) = 1, for all *s* and *t*. The response function *w*(*s*) of the neuron in spontaneous activity results from solving the homogeneous equation (4). If

$$\lim_{\xi \to 0} K(\xi s, \xi t)$$

exists, where $\boldsymbol{K}$ is a compact integral operator defined on the space $\boldsymbol{L}_2[0,b]$ of Lebesgue square integrable functions. If the reciprocal kernel $\boldsymbol{K}(s,t)$ is Lebesgue square integrable and continuously differentiable, then

$$w(t) = t^{\alpha} e^{g(t)} / \int_0^b t^{\alpha} e^{g(t)} dt$$

satisfies (4) for some choice of $g(t)$. Because finite linear combinations of the functions $\{t^{\alpha} e^{-\beta t}, \alpha, \beta \geq 0\}$ are dense in the space of bounded continuous functions **C**[0,*b*] we can approximate $t^{\alpha} e^{g(t)}$ by linear combinations of $t^{\alpha} e^{-\beta t}$ and hence we substitute $g(t) = -\beta t, \beta \geq 0$, in the eigenfunction $w(t)$. The density of neural firing is not completely analogous to the density of the rational numbers in the real number system. The rationals are countably infinite, the number of neurons is finite but large. In speaking of density here we may think of making a sufficiently close approximation (within some prescribed bound rather than arbitrarily close).

We use the functions:

$$\{t^{\alpha} e^{-\beta t}, \alpha, \beta \geq 0\}$$

to represent images and sounds.

Before we describe how the network can be used to represent images and sound, we summarize the mathematical model on which the neural density representation is based.

Neural responses are impulsive and hence the brain is a discrete firing system. It follows that the spontaneous activity of a neuron during a very short period of time in which the neuron fires is given by:

$$w(t) = \sum_{k=1}^{R} \gamma_k (t - \tau_k)^{\alpha} e^{-\beta(t-\tau_k)}$$

if the neuron fires at the random times τ_k , k=1, 2 ,..., R. The empirical findings of Poggio and Mountcastle [8] support the assumption that R and the times τ_k , k=1, 2 ,..., R are probabilistic. However, as observed by Brinley [6] the parameters α and β vary from neuron to neuron, but are constant for the firings of each neuron. Non-spontaneous activity can be characterized as a perturbation of background activity. To derive the response function when neurons are stimulated from external sources, we consider an inhomogeneous equation to represent stimuli acting on the neuron in addition to existing spontaneous activity. Thus, we solve the inhomogenous Fredholm equation of the 2nd kind given by:

$$w(s) - \lambda_0 \int_0^b K(s,t) w(t) dt = f(s)$$

This equation has a solution in the Sobolev $W_p^k(\Omega)$ space

of distributions (in the sense of Schwartz) in $L_p(W)$ whose derivatives of order k also belong to the space $L_p(W)$, where W is an open subset of R^n.

5. Neural Representation

We created a 2-dimensional network of neurons consisting of layers [9,10]. For illustrative purposes, we assume that there is one layer of neurons corresponding to each of the stimulus values. Thus, if the list of stimuli consists of *n* numerical values, we created *n* layers with a specific number of neurons in each layer. With the assumption that each numerical stimulus is represented by the firing of one and only one neuron, each layer of the network must also consist of *n* neurons with thresholds varying between the largest and the smallest values of the list of stimuli. We also assumed that the firing threshold of each neuron had the same width. Thus, if the perceptual range of a stimulus varies between two values θ_1 and θ_2, and each layer of the network has *n* neurons, then a neuron in the ith position of the layer will fire if the stimulus value falls between

$$\theta_1 + (i-1)\frac{\theta_2 - \theta_1}{n-1} \quad \text{and} \quad \theta_1 + i\,\frac{\theta_2 - \theta_1}{n-1}$$

Picture Experiment: In the graphics experiment the bird and rose pictures required 124 and 248 data points, respectively, whereas the sound experiment required 1000 times more data points. Once the (*x,y*) coordinates of the points were obtained, the x-coordinate was used to represent time and the *y*-coordinate to represent response to a stimulus. The numerical values associated with the drawings in Figure 2 and were tabulated and the numbers provided the input to the neurons in the networks built to represent the bird and the rose.

Sound Experiment: In the sound experiment we first recorded with the aid of Mathematica the first few seconds of Haydn's symphony no.102 in B-flat major and Mozart's symphony no. 40 in G minor. The result is a set of numerical amplitudes between -1 and 1. Each of these amplitudes was used to make neurons fire when the amplitude falls within a prescribed threshold range. Under the assumption that each neuron fires in response to one stimulus, we would need the same number of neurons as the sample size, i.e., 117,247 in Haydn's symphony and 144,532 in Mozart's symphony. Our objective was to approximate the amplitude using one neuron for each amplitude value, and then use the resulting values in Mathematica to play back the music. A small sample of the numerical data for Mozart's symphony is displayed in Figure 3.

This task is computationally demanding even for such simple geometric figures as the bird and the flower shown in Figures 1 and 2. For example, for the bird picture, the stimuli list consists of 124 values, and we would need 124^2=15376 neurons, arranged in 124 layers of 124 neurons each The network and the data sampled to form the picture given in Figure 1, were used to create a 124x124 network of neurons consisting of 124 layers with 124 neurons in each layer. Each dot in the figures is generated by the firing of a neuron in response to a stimulus falling within the neuron's lower and upper thresholds.

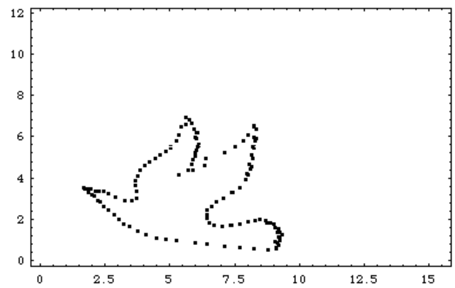

Figure 1 Bird

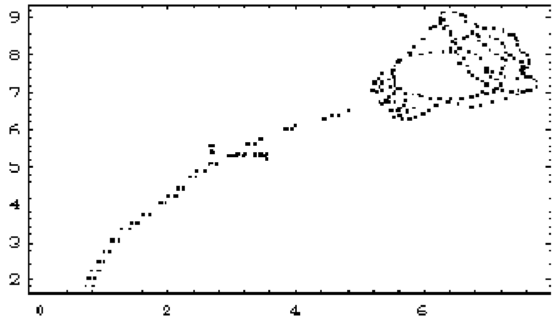

Figure 2 Rose

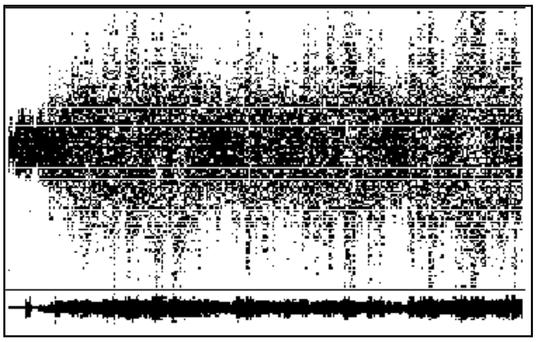

Figure 3 Mozart's Symphony No. 40.

Needed are advances in technology that makes it easier and faster to draw figures, and sound waves and perhaps other sense responses represented with mathematically dense functions in order to advance the results of this research to more detailed brain activities.

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