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Neutrino Mass, the Right-Handed Interaction and the Double Beta Decay. I

—— *Formalism* ——

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In order to shed light on the important question whether neutrinos are Dirac or Majorana particles, the double β decay is investigated within a general form of weak interaction Hamiltonian. The systematic study is made on the $0^+ \to J^+$ nuclear transitions for the two-neutrino and neutrinoless modes both in the two-nucleon- and N^*-mechanism. It is shown that for the neutrinoless mode, only the $0^+ \to 0^+$ transition in the two-nucleon mechanism is allowed if there is no right-handed interaction. When the right-handed interaction gives a sizable contribution, the role of the $0^+ \to 2^+$ transition becomes as important as the $0^+ \to 0^+$ transition. The comparison of our results with the previous ones is also presented.

§ 1. Introduction

The question whether neutrinos are massive or massless has become one of the recent important topics. This is motivated by the recent theoretical development of the grand unified theories where neutrinos are likely to be massive because leptons and quarks are treated on the equal basis. If neutrinos are massive, there arises an important question whether neutrinos are Dirac or Majorana particles. In models like $SO(10)$, neutrinos are assigned to Majorana particles to explain the small masses of the observed neutrinos.

In this paper, we investigate the double β decay which reveals directly the difference between Dirac and Majorana neutrinos. This seems to be the only experiment presently available for this purpose. There are two decay modes, i.e., the two-neutrino mode $(\beta\beta)_{2\nu}$,

$$N_A(A, Z) \to N_B(A, Z+2) + e^- + e^- + \bar{\nu}_e + \bar{\nu}_e\ , \tag{1·1}$$

and the neutrinoless mode $(\beta\beta)_{0\nu}$,

$$N_A(A, Z) \to N_B(A, Z+2) + e^- + e^-\ . \tag{1·2}$$

The $(\beta\beta)_{0\nu}$ mode is interesting because this takes place only when neutrinos are Majorana particles, while the $(\beta\beta)_{2\nu}$ mode occurs both for Dirac and Majorana neutrinos.

We analyze the $0^+ \to J^+$ nuclear transitions for these two modes in two mechanisms: One is the two-nucleon ($2n$)-mechanism where the successive transitions of two neutrons (n_1 and n_2) trigger the double β decay as shown in Fig. 1. The other is the N^*-mechanism[1] where the double β decay occurs through the transitions*) of the nuclei involving $\Delta(1232)$ as shown in Fig. 2. The detailed discussion for these mechanisms is given in Appendix A.

In the following, we use the weak interaction Hamiltonian,

$$H_W(x) = \frac{G_F}{\sqrt{2}}[j_{L\mu}^-(x) J_L^{+\mu}(x) + \lambda j_{R\mu}^-(x) J_R^{+\mu}(x)] + \text{h.c.}, \tag{1·3}$$

where $J_{L(R)}^{+\mu}(x)$ is the left (right)-handed hadronic current and the leptonic currents $j_{L\mu}^-(x)$ and $j_{R\mu}^-(x)$ are expressed as follows:

$$j_{L\mu}^-(x) = \bar{e}\gamma_\mu(1-\gamma_5)\nu_{eL}\,, \qquad j_{R\mu}^-(x) = \bar{e}\gamma_\mu(1+\gamma_5)\nu_{eR}'\,. \tag{1·4}$$

The current neutrinos ν_{eL} and ν_{eR}' are assumed to be the superposition of the mass eigenstate neutrino N_j's with the corresponding mass m_j's,[2]

$$\nu_{eL} = \sum_{j=1}^{2n} U_{ej} N_{jL}\,, \qquad \nu_{eR}' = \sum_{j=1}^{2n} V_{ej} N_{jR}\,, \tag{1·5}$$

where n is the number of generations and U_{ej} (V_{ej}) is the left (right)-mixing matrix.**)

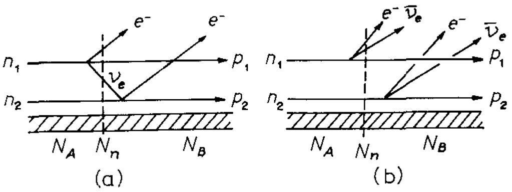

Fig. 1. The schematic diagrams for the $(\beta\beta)_{0\nu}$ mode (a) and for the $(\beta\beta)_{2\nu}$ mode (b) in *the 2n-mechanism.* The N_A, N_B and N_n are the parent, the daughter and the intermediate nucleus, respectively.

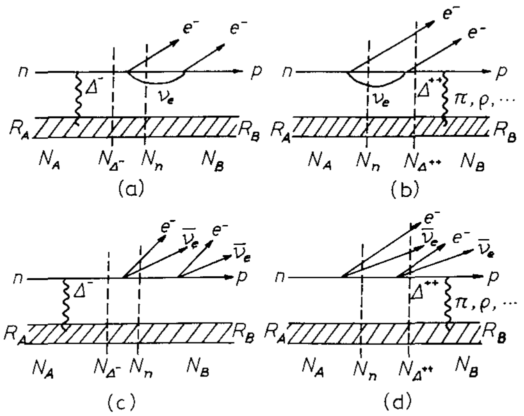

Fig. 2. The diagrams for the $(\beta\beta)_{0\nu}$ mode ((a) and (b)) and for the $(\beta\beta)_{2\nu}$ mode ((c) and (d)) in *the N^*-mechanism.* The N_{Δ^-} and $N_{\Delta^{++}}$ denote the intermediate nuclear states including Δ^- and Δ^{++}, respectively.

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*) Throughout this paper, we do not consider the radial and orbital excitations of $\Delta(1232)$.

**) If neutrinos are Dirac, U_{ej} and V_{ej} vanish for $j > n$.

In the $2n$-mechanism, the hadronic currents may be written as

$$J_L^{+\mu}(x)=\bar{\psi}_N\tau^+\gamma^\mu(g_V-g_A\gamma_5)\psi_N\,,\qquad J_R^{+\mu}(x)=\bar{\psi}_N\tau^+\gamma^\mu(g_V'+g_A'\gamma_5)\psi_N\,,\qquad (1\cdot6)$$

where $\psi_N{}^T=(p,\,n)$ and τ^+ is the isospin raising matrix. Here $g_V=\cos\theta$ and $g_V'=\cos\theta'$, where $\theta(\theta')$ is the left (right)-mixing angle between u and d quarks. We expect $\theta\simeq\theta_c$ where θ_c is the Cabbibo angle. Also,

$$g_A/g_V=g_A'/g_V'\simeq1.24 \qquad (1\cdot7)$$

is expected. This deviation from unity is due to the strong interaction renormalization.

In the N^*-mechanism, the hadronic currents are considered to act on quarks in a hadron and may be obtained by replacing $\psi_N{}^T$ with $\psi_q{}^T=(u,\,d)$ and taking $g_A/g_V=g_A'/g_V'=1$. The effect from the strong interaction will be taken into account by evaluating the matrix elements in the $SU(6)$ quark model.

Many works have been made on the double β decay.[1),3)~9)] However, the structure of H_W used here is somewhat different from the ones previously used. Our results will be compared with those obtained by others only in some special limits like zero neutrino mass ($m_\nu=0$) or $\lambda=0$. Since there are some disagreements between our results and the previous theoretical estimates, we shall show the derivations in some detail. Also, in Appendix C, the decay formulae are presented for the conventionally used Hamiltonian to show how different they are in two H_W's.

In § 2, the $(\beta\beta)_{0\nu}$ mode is investigated for the $0^+\to J^+$ transitions both in the $2n$- and N^*-mechanism. The $(\beta\beta)_{2\nu}$ mode is analyzed in § 3. The concluding remarks are given in § 4.

§ 2. The $(\beta\beta)_{0\nu}$ mode

In the $2n$-mechanism, the double β decay occurs through the transition of two neutrons in N_A to two protons in N_B as shown in Fig. 1(a). On the other hand, there are two possible processes in the N^*-mechanism: (i) The strong interaction first creates the nuclear state N_{Δ^-} which includes Δ^-. Subsequently N_{Δ^-} makes the weak decay as shown in Fig. 2(a). (ii) The parent nucleus N_A makes the weak decay to the nuclear state $N_{\Delta^{++}}$ which is converted into the daughter nucleus N_B as shown in Fig. 2(b). Since the strong interaction operates as internal force, $N_{\Delta^-}(N_{\Delta^{++}})$ has the same J^P as $N_A(N_B)$ has. Therefore, we are able to single out the second order weak interaction part from Figs. 2(a) and (b), and make a unified treatment both for the $2n$- and N^*-mechanism.

Let us consider the second order weak interaction parts in Figs. 1(a), 2(a) and (b) which are collectively expressed by the $N_\alpha\to N_\beta+2e^-$ transition. The weak interaction patterns of our Hamiltonian are given in Fig. 3. If two lepton

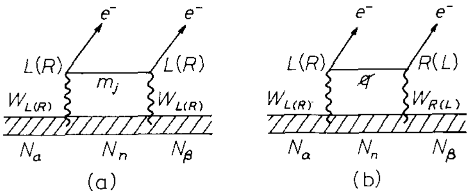

Fig. 3. The diagrams for the $(\beta\beta)_{0\nu}$ mode corresponding to the second order weak interactions. The wavy line represents the weak intermediate boson which controls the left- or right-handed weak interaction.

vertices are either combination of (L, L) or (R, R) as shown in Fig. 3(a), the contribution from this diagram is proportional to the mass m_j of the intermediate Majorana neutrino. When two vertices are (L, R) or (R, L) as in Fig. 3(b), the contribution is proportional to the neutrino four momentum q and the relative strength λ. This situation can be easily seen from the neutrino propagators given in Appendix B. Thus, the $(\beta\beta)_{0\nu}$ mode takes place only if neutrinos are Majorana and at least one of two parameters, m_j and λ, does not vanish.*)

From the above consideration, we can write the R-matrix element for $N_a \to N_\beta + e^-(p_1) + e^-(p_2)$,

$$R_W = \frac{1}{\sqrt{2}}\left(\frac{G_F}{\sqrt{2}}\right)^2 [(2\pi)^{-6}(p_1{}^0 p_2{}^0)^{-1} F(Z+2, p_1{}^0) F(Z+2, p_2{}^0)]^{1/2}$$
$$\times 2\sum_j \{ m_j [U_{ej}^2 t_{\mu\nu}^L K_{LL}^{\mu\nu} + \lambda^2 V_{ej}^2 t_{\mu\nu}^R K_{RR}^{\mu\nu}] + \lambda U_{ej} V_{ej} [u_{\mu\nu\rho}^L L_{LR}^{\mu\nu\rho} + u_{\mu\nu\rho}^R L_{RL}^{\mu\nu\rho}]\}, \tag{2·1}$$

where, by using $u^c = C\bar{u}^T$ with the charge conjugation matrix C,

$$t_{\mu\nu}^{L,R} = \bar{u}(p_1)\gamma_\mu(1\mp\gamma_5)\gamma_\nu u^C(p_2), \tag{2·2}$$

$$u_{\mu\nu\rho}^{L,R} = \bar{u}(p_1)\gamma_\mu(1\mp\gamma_5)\gamma_\rho\gamma_\nu u^C(p_2), \tag{2·3}$$

$$K_{ab}^{\mu\nu} = \int d\boldsymbol{x} d\boldsymbol{y} e^{-i(p_1 x + p_2 y)} \int \frac{d\boldsymbol{q}}{2(2\pi)^3 q^0} e^{iq(x-y)}$$
$$\times \langle N_\beta | \sum_n \left\{ \frac{J_a^{+\mu}(\boldsymbol{x})|N_n\rangle\langle N_n|J_b^{+\nu}(\boldsymbol{y})}{q^0 + E_n - E_a + p_2{}^0} + \frac{J_b^{+\nu}(\boldsymbol{y})|N_n\rangle\langle N_n|J_a^{+\mu}(\boldsymbol{x})}{q^0 + E_n - E_a + p_1{}^0} \right\} |N_a\rangle\,, \tag{2·4}$$

$$L_{ab}^{\mu\nu\rho} = \int d\boldsymbol{x} d\boldsymbol{y} e^{-i(p_1 x + p_2 y)} \int \frac{d\boldsymbol{q}}{2(2\pi)^3 q^0} q^\rho$$
$$\times \langle N_\beta | \sum_n \left\{ e^{iq(x-y)} \frac{J_a^{+\mu}(\boldsymbol{x})|N_n\rangle\langle N_n|J_b^{+\nu}(\boldsymbol{y})}{q^0 + E_n - E_a + p_2{}^0} \right.$$
$$\left. - e^{-iq(x-y)} \frac{J_b^{+\nu}(\boldsymbol{y})|N_n\rangle\langle N_n|J_a^{+\mu}(\boldsymbol{x})}{q^0 + E_n - E_a + p_1{}^0} \right\} |N_a\rangle\,. \tag{2·5}$$

*) In the $m_j = 0$ limit, R_W is proportional to $\lambda \sum U_{ej} V_{ej}$. If we take the gauge theories seriously, we should take $U_{e1} = V_{e,n+1} = 1$ and zero for others. Thus, the $(\beta\beta)_{0\nu}$ mode does not take place. However we keep the possibility $V_{e1} \neq 0$ on the phenomenological basis in order to compare our results with the previous works.

Here the first $1/\sqrt{2}$ in Eq. (2·1) is the statistical factor for the emitted two electrons, $a(b)$ takes L and R, and N_n is the intermediate nuclear state with the energy E_n. The Fermi factor for the emitted electrons is approximated by

$$F(Z,\ p^0)=(p^0/|\boldsymbol{p}|)2\pi\alpha Z[1-\exp(-2\pi\alpha Z)]^{-1}\ . \tag{2·6}$$

It should be noted that in the $2n$-mechanism the R-matrix is obtained by taking $N_\alpha=N_A$ and $N_\beta=N_B$ in the R_W-matrix. In the N^*-mechanism, the R_W-matrix which corresponds to the 2nd order perturbation of H_W is a part of the R-matrix, as given in Eq. (A·3) of Appendix A.

Now we adopt the following approximations: (i) The energy of the intermediate nucleus E_n is replaced by the average value $\langle E_n\rangle$. (ii) The non-relativistic impulse approximation is used for the hadronic currents $J_L^{+\mu}(\boldsymbol{x})$ and $J_R^{+\mu}(\boldsymbol{x})$. (iii) The first two terms of the multipole expansion for the lepton wave function are kept; $\exp[-i(\boldsymbol{p}_1\boldsymbol{x}+\boldsymbol{p}_2\boldsymbol{y})]\simeq 1-i(\boldsymbol{p}_1\boldsymbol{x}+\boldsymbol{p}_2\boldsymbol{y})$.

Under the approximation (i), the intermediate nuclear states can be summed by closure. By the approximation (ii), the hadronic currents may be expressed as follows:

$$J_L^{+\mu}(\boldsymbol{x})=\sum_n \tau_n{}^+(g_V g^{\mu 0}\mathbf{1}_n+g_A g^{\mu j}\sigma_n{}^j)\delta(\boldsymbol{x}-\boldsymbol{r}_n), \tag{2·7}$$

where the subscript n implies that the operators act on the n-th nucleon in the $2n$-mechanism or the n-th quark in the N^*-mechanism. Note that for the parity conserving $0^+\to J^+$ transitions, the first term in the multipole expansion contributes to $K_{ab}^{\mu\nu}$ and $L_{ab}^{\mu\nu 0}$ terms, while the dipole term to $L_{ab}^{\mu\nu k}$.

Within these approximations, the q-integrations in $K_{ab}^{\mu\nu}$ and $L_{ab}^{\mu\nu\rho}$ can be formally performed and the results are

$$\begin{aligned}K_{LL}^{\mu\nu}=&\frac{1}{8\pi}[\langle H_1(r,m_j)\rangle+\langle H_2(r,\ m_j)\rangle]\\&\times\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+(g_V g^{\mu 0}+g_A\sigma_n{}^j g^{\mu j})(g_V g^{\nu 0}+g_A\sigma_m{}^k g^{\nu k})|N_\alpha\rangle\ ,\end{aligned} \tag{2·8}$$

$$\begin{aligned}L_{LR}^{\mu\nu 0}=&\frac{1}{8\pi}[A_1\langle H_1(r,\ m_j)\rangle-A_2\langle H_2(r,\ m_j)\rangle]\\&\times\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+(g_V g^{\mu 0}+g_A\sigma_n{}^j g^{\mu j})(g_V{}'g^{\nu 0}-g_A{}'\sigma_m{}^k g^{\nu k})|N_\alpha\rangle\ ,\end{aligned} \tag{2·9}$$

$$\begin{aligned}L_{LR}^{\mu\nu k}=&\frac{-1}{8\pi}[\langle rH_1{}'(r,\ m_j)\rangle+\langle rH_2{}'(r,\ m_j)\rangle]\\&\times\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+\frac{1}{2}[(p_1-p_2)^l\hat{r}_{nm}^l+(p_1+p_2)^l\hat{r}_{+nm}^l]\hat{r}_{nm}^k\\&\qquad\times(g_V g^{\mu 0}+g_A\sigma_n{}^p g^{\mu p})(g_V{}'g^{\nu 0}-g_A{}'\sigma_m{}^q g^{\nu q})|N_\alpha\rangle\\&-\frac{i}{8\pi}[\langle H_1{}'(r,\ m_j)\rangle+\langle H_2{}'(r,\ m_j)\rangle]\\&\times\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+(g_V g^{\mu p}D_n{}^p+g_A g^{\mu 0}C_n)(g_V{}'g^{\nu q}D_m{}^q-g_A{}'g^{\nu 0}C_m)|N_\alpha\rangle\ .\end{aligned} \tag{2·10a}$$

We note that only the first term of the multipole expansion is taken into account for $K_{LL}^{\mu\nu}$ and $L_{LR}^{\mu\nu 0}$, and the dipole term is used to obtain the first term of $L_{LR}^{\mu\nu k}$. In order to maintain the consistency of the approximation, the relativistic correction of the hadronic current should be included. The second term of $L_{LR}^{\mu\nu k}$ (2·10a) is due to this correction. Here C_n and $\boldsymbol{D}_n$ are defined by

$$C_n=\boldsymbol{\sigma}_n\cdot(\boldsymbol{q}_n-2\boldsymbol{P}_n)/2M\ , \qquad (2\cdot 10\text{b})$$

$$\boldsymbol{D}_n=[(\boldsymbol{q}_n-2\boldsymbol{P}_n)+i(\boldsymbol{\sigma}_n\times\boldsymbol{q}_n)]/2M\ , \qquad (2\cdot 10\text{c})$$

where $\boldsymbol{P}$ and $\boldsymbol{q}$ are the momentum and the momentum transfer of the nucleon, respectively.[11] In Eqs. (2·8)~(2·10), $H_i(r, m_j)$ is the potential-like term due to the exchange of neutrino and is defined as

$$H_i(|\boldsymbol{r}_n-\boldsymbol{r}_m|, m_j)=\int\frac{d\boldsymbol{q}}{2\pi^2}\frac{e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_n-\boldsymbol{r}_m)}}{q^0(q^0+A_i)}\ , \qquad (2\cdot 11)$$

where $q^0=\sqrt{|\boldsymbol{q}|^2+m_j^2}$ and $A_i=\langle E_n\rangle-M_a+p_i^0$. Also, $H_i'=dH_i/dr$, $\boldsymbol{r}_{nm}=\boldsymbol{r}_n-\boldsymbol{r}_m$, $\hat{r}_{nm}=\boldsymbol{r}_{nm}/|\boldsymbol{r}_{nm}|$, and $\hat{r}_{+nm}=(\boldsymbol{r}_n+\boldsymbol{r}_m)/|\boldsymbol{r}_{nm}|$. The terms like $\langle H_i\rangle$ and $\langle rH_i'\rangle$ represent the average values of "potentials" with the weight of nuclear tensor operators.*) Note that the potential $H_i(r, m_j)$ behaves like $1/r$ for $m_j\lesssim O(\text{MeV})$ and e^{-m_jr}/r for $m_j\gtrsim O(\text{GeV})$. The replacement E_n by $\langle E_n\rangle$ (the approximation (i)) is not crucial because the main contribution to the potentials comes from $|\boldsymbol{q}|\geq 20$ MeV which is much larger than A_i ($\simeq$a few MeV). The other terms, $K_{RR}^{\mu\nu}$, $L_{RL}^{\mu\nu 0}$ and $L_{RL}^{\mu\nu k}$, are obtained by taking the interchanges($g_V\leftrightarrow g_V'$)and ($g_A\leftrightarrow -g_A'$)in the expressions of $K_{LL}^{\mu\nu}$, $L_{LR}^{\mu\nu 0}$ and $L_{LR}^{\mu\nu k}$, respectively.

The product of the leptonic and hadronic parts can be easily calculated and the results are as follows:

$$\left\{\begin{matrix} t_{\mu\nu}^L & K_{LL}^{\mu\nu} \\ t_{\mu\nu}^R & K_{RR}^{\mu\nu}\end{matrix}\right\}=\frac{1}{8\pi}[\langle H_1\rangle+\langle H_2\rangle]\bar{u}(p_1)(1\pm\gamma_5)u^C(p_2)\left\{\begin{matrix} g_A^2 \\ g_A'^2\end{matrix}\right\}\left[\left(\frac{g_V}{g_A}\right)^2 M_F-M_{GT}\right], \qquad (2\cdot 12)$$

$$u_{\mu\nu 0}^L L_{LR}^{\mu\nu 0}+u_{\mu\nu 0}^R L_{RL}^{\mu\nu 0}=\frac{1}{8\pi}[A_1\langle H_1\rangle-A_2\langle H_2\rangle]$$
$$\times\bar{u}(p_1)\gamma^0 u^C(p_2)g_Ag_A'[(g_V/g_A)^2M_F-M_{GT}], \qquad (2\cdot 13)$$

$$u_{\mu\nu k}^L L_{LR}^{\mu\nu k}+u_{\mu\nu k}^R L_{RL}^{\mu\nu k}=\frac{-1}{8\pi}[\langle rH_1'\rangle+\langle rH_2'\rangle]$$
$$\times g_Ag_A'\bar{u}(p_1)[\gamma^k(p_1-p_2)^l P^{lk}+\gamma_5\gamma^0(p_1+p_2)^l Q^l]u^C(p_2)$$
$$+\frac{i}{2\pi}[\langle H_1'\rangle+\langle H_2'\rangle]g_Ag_A'\bar{u}(p_1)\gamma_5\gamma^k u^C(p_2)R^k\ , \qquad (2\cdot 14)$$

*) The average is defined as $\langle f\rangle=\langle N_B|\Sigma_{n,m}fO_{nm}|N_A\rangle/\langle N_B|\Sigma_{n,m}O_{nm}|N_A\rangle$, where O_{nm} is the nuclear tensor operator.

where

$$M_F=\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+|N_\alpha\rangle\ , \tag{2·15}$$

$$M_{GT}=\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_m|N_\alpha\rangle\ , \tag{2·16}$$

$$Q^l=\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+\hat{r}_{+nm}^l\left[2\left(\frac{g_V}{g_A}\right)(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_n)-i\hat{r}_{nm}\cdot(\boldsymbol{\sigma}_n\times\boldsymbol{\sigma}_m)\right]|N_\alpha\rangle\ , \tag{2·17}$$

$$P^{lk}=\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+\hat{r}_{nm}^l\{\hat{r}_{nm}^k[(g_V/g_A)^2+\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_m]$$
$$+2i(g_V/g_A)(\hat{r}_{nm}\times\boldsymbol{\sigma}_n)^k-2\sigma_m{}^k(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_n)\}|N_\alpha\rangle\ , \tag{2·18a}$$

$$R^k=\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+\left\{i(\hat{r}_{nm}\times\boldsymbol{\sigma}_m)^kC_n+i\left(\frac{g_V}{g_A}\right)^2(\hat{r}_{nm}\times\boldsymbol{D}_n)^k\right.$$
$$\left.+\left(\frac{g_V}{g_A}\right)[\hat{r}_{nm}^k(C_n-\boldsymbol{\sigma}_n\cdot\boldsymbol{D}_m)+\sigma_n{}^k(\hat{r}_{nm}\cdot\boldsymbol{D}_m)+(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_n)D_m{}^k\right\}|N_\alpha\rangle\ . \tag{2·18b}$$

It is clear from the nuclear matrix elements given above that the $0^+\to J^+$ ($J\geqq 3$) transitions are forbidden within our approximations.

In the following, a further simplification is made by replacing A_i in the potentials with their average value, i.e.,

$$\mu_0=(A_1+A_2)/2m_e=[\langle E_n\rangle-(M_A+M_B)/2]/m_e\ . \tag{2·19}$$

Table I. Allowed transitions and relative order of magnitudes. Each parenthesis under the interaction indicates the possible transition, and m_ν and λ represent the typical neutrino mass and the relative strength of the right-handed interaction, respectively.

$(\beta\beta)_{0\nu}$ mode					
interaction patterns	2n-mechanism		N^*-mechanism		Feynman diagrams
	$m_\nu\lesssim O(\text{ev})$	$m_\nu\gtrsim O(\text{GeV})$	$m_\nu\lesssim O(\text{eV})$	$m_\nu\gtrsim O(\text{GeV})$	
$m_\nu K_{LL}^{\mu\nu}$ $(0^+\to 0^+)$	$m_\nu\left\langle\frac{1}{r}\right\rangle$	$m_\nu\left\langle\frac{e^{-m_\nu r}}{r}\right\rangle$	no contribution		Fig. 3(a)
$\lambda^2 m_\nu K_{RR}^{\mu\nu}$ $(0^+\to 0^+)$	$\lambda^2 m_\nu\left\langle\frac{1}{r}\right\rangle$	$\lambda^2 m_\nu\left\langle\frac{e^{-m_\nu r}}{r}\right\rangle$	no contribution		
$\lambda L_{ab}^{\mu\nu 0}$ $(0^+\to 0^+)$	$\lambda(p_1{}^0-p_2{}^0)\left\langle\frac{1}{r}\right\rangle$	$\lambda(p_1{}^0-p_2{}^0)\left\langle\frac{e^{-m_\nu r}}{r}\right\rangle$	no contribution		Fig. 3(b)
$\lambda L_{ab}^{\mu\nu k}$ $(0^+\to 0^+, 1^+, 2^+)$	$\lambda\lvert\boldsymbol{p}_1\pm\boldsymbol{p}_2\rvert\left\langle\frac{1}{r}\right\rangle$	$\lambda\lvert\boldsymbol{p}_1\pm\boldsymbol{p}_2\rvert\left\langle\frac{e^{-m_\nu r}}{r}\right\rangle$	$\lambda\lvert\boldsymbol{p}_1\pm\boldsymbol{p}_2\rvert\left\langle\frac{1}{r}\right\rangle_\Delta$	$\lambda\lvert\boldsymbol{p}_1\pm\boldsymbol{p}_2\rvert\left\langle\frac{e^{-m_\nu r}}{r}\right\rangle_\Delta$	

As a consequence, we can write H_i's in the single form,

$$H_1(r, m_j) = H_2(r, m_j) \equiv H(r, m_j, \mu_0). \qquad (2\cdot20)$$

(2-a) *The 2n-mechanism*

The R-matrix is obtained from R_W in Eq. (2·1) by replacing N_α and N_β with N_A and N_B, respectively. It should be noted that the M_F and M_{GT} terms are of rank 0 with respect to the angular momentum, and the Q^l term of rank 1. On the other hand, the P^{lk} term consists of irreducible tensor operators of ranks 0, 1 and 2. Consequently, the terms $t^L_{\mu\nu}K^{\mu\nu}_{LL}$, $t^R_{\mu\nu}K^{\mu\nu}_{RR}$ and $u^L_{\mu\nu 0}L^{\mu\nu 0}_{LR}+u^R_{\mu\nu 0}L^{\mu\nu 0}_{RL}$ contribute only to the $0^+\to0^+$ transition. While, $u^L_{\mu\nu k}L^{\mu\nu k}_{LR}+u^R_{\mu\nu k}L^{\mu\nu k}_{RL}$ contributes to the $0^+\to0^+$, 1^+, and 2^+ transitions. These features and the relative order of magnitudes are listed in Table I for various types of terms.

(i) The $0^+\to0^+$ transition

The nuclear matrix elements M_F, M_{GT} and $P^{ii}/3$ contribute to this transition. We obtain

$$\begin{aligned} d\Gamma^{2n}_{0\nu}(0^+\to0^+) &= (a_{0\nu}/m_e{}^7)\, g_A{}^4 \\ &\times\{(p_1{}^0 p_2{}^0 - \boldsymbol{p}_1\cdot\boldsymbol{p}_2)[|X_1+X_4|^2+|X_2+X_4|^2] \\ &+\frac{1}{2m_e{}^2}(p_1{}^0-p_2{}^0)^2(p_1{}^0p_2{}^0+\boldsymbol{p}_1\cdot\boldsymbol{p}_2-m_e{}^2)|X_3-X_4|^2 \\ &-2m_e{}^2\,\mathrm{Re}(X_1+X_4)(X_2+X_4)^* \\ &-(p_1{}^0-p_2{}^0)^2\,\mathrm{Re}(X_1+X_2+2X_4)(X_3-X_4)^*\} \\ &\times p_1{}^0p_2{}^0\delta(p_1{}^0+p_2{}^0+M_B-M_A)\,d\cos\theta\, dp_1{}^0\, dp_2{}^0\,, \end{aligned} \qquad (2\cdot21)$$

where θ is the angle between two emitted electrons,

$$a_{0\nu}=\frac{G_F{}^2 m_e{}^9}{2(2\pi)^5}\left\{\frac{2\pi\alpha(Z+2)}{1-\exp[-2\pi\alpha(Z+2)]}\right\}^2, \qquad (2\cdot22)$$

$$X_1=\sum_j (m_j/m_e)\,U^2_{ej}\langle H\rangle[(g_V/g_A)^2M_F-M_{GT}], \qquad (2\cdot23)$$

$$X_2=\lambda^2\sum_j (m_j/m_e)\,V^2_{ej}\langle H\rangle(g_V'/g_V)^2[(g_V/g_A)^2M_F-M_{GT}], \qquad (2\cdot24)$$

$$X_3=\lambda\sum_j U_{ej}V_{ej}\langle H\rangle(g_V'/g_V)[(g_V/g_A)^2M_F-M_{GT}], \qquad (2\cdot25)$$

$$X_4=\lambda\sum_j U_{ej}V_{ej}\langle rH'\rangle(g_V'/3g_V)\left[(g_V/g_A)^2M_F+\frac{1}{3}M_{GT}-2M_T\right]. \qquad (2\cdot26)$$

Here M_F and M_{GT} are defined in Eqs. (2·15) and (2·16), and

$$M_T=\langle N_B|\sum_{n,m}\tau_n{}^+\tau_m{}^+\left[(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_n)(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_m)-\frac{1}{3}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_m\right]|N_A\rangle\,. \qquad (2\cdot27)$$

The decay rate for the $0^+\to 0^+$ transition in the $2n$-mechanism is

$$\Gamma_{0\nu}^{2n}(0^+\to 0^+)=(a_{0\nu}/m_e^2)g_A^4\{G_{01}(T)[|X_1+X_4|^2+|X_2+X_4|^2] + G_{02}(T)|X_3-X_4|^2-G_{03}(T)\mathrm{Re}(X_1+X_4)(X_2+X_4)^* - G_{04}(T)\mathrm{Re}(X_1+X_2+2X_4)(X_3-X_4)^*\}, \qquad (2\cdot 28)$$

where

$$G_{01}(T)=\frac{1}{15}T(T^4+10T^3+40T^2+60T+30), \qquad (2\cdot 29)$$

$$G_{02}(T)=\frac{1}{210}T^4(T^3+14T^2+77T+70), \qquad (2\cdot 30)$$

$$G_{03}(T)=\frac{2}{3}T(T^2+6T+6), \qquad (2\cdot 31)$$

$$G_{04}(T)=\frac{1}{15}T^3(T^2+10T+10). \qquad (2\cdot 32)$$

Here T is the maximum kinetic energy release,

$$T=(M_A-M_B-2m_e)/m_e\,. \qquad (2\cdot 33)$$

Let us compare our results with those obtained previously. In the limit of $\lambda=0$ and $U_{ej}=\delta_{j1}$, we found that the overall normalization by Greuling and Whitten[5] is twice as large as ours.*) In the other limit of $m_j=0$, our results can be compared with those by Primakoff and Rosen[4] who used the quite general form for H_W.**) They assumed $M_T=0$ and their result is twice as large as ours.*)

(ii) The $0^+\to 2^+$ transition

Only the rank 2 part of the P^{lk} term contributes to this transition. The final result is

$$d\Gamma_{0\nu}^{2n}(0^+\to 2^+)=(a_{0\nu}/m_e^9)(g_A^2g_A'^2/30)|\lambda\sum_j U_{ej}V_{ej}\langle rH'\rangle|^2(N_2^{pq},N_2^{pq}) \times\{3(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2-\boldsymbol{p}_1\cdot\boldsymbol{p}_2[10(p_1^0p_2^0+m_e^2)+|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2] +5(p_1^0p_2^0+m_e^2)(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2)-|\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2\} \times p_1^0p_2^0\delta(p_1^0+p_2^0+M_B-M_A)d\cos\theta dp_1^0dp_2^0\,, \qquad (2\cdot 34)$$

where $a_{0\nu}$ is defined in Eq. (2·22), $(N_2^{pq},N_2^{pq})\equiv\sum_{J_z}\sum_{p,q=1}^3 N_2^{pq*}N_2^{pq}$ with

*) It seems that they did not take account of the statistical factor. See, e.g., Eq. (20) of Ref. 5) and Eq. (33) of Ref. 4).

**) The correspondence between our notation and theirs in Ref. 4) is as follows: $C_a=D_a=G_F(g_a+\lambda g_a')/2$, $C_a\delta_a=D_a\delta_a=-G_F(g_a-\lambda g_a')/2$, for $a=V$ and A, and $\Gamma_V=\gamma_\mu$, $\Gamma_A=\gamma_\mu\gamma_5$, $U_{ej}=V_{ej}=\delta_{j1}$. It is also reminded that the result by Molina and Pascual[9] is four times as large as ours.

$$N_2^{pq} = \langle N_B(2^+)|\sum_{n,m} \tau_n{}^+ \tau_m{}^+ \hat{r}_{nm}^p \{ \hat{r}_{nm}^q [(g_V/g_A)^2 + \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_m]$$
$$+ 2i(g_V/g_A)(\hat{r}_{nm} \times \boldsymbol{\sigma}_n)^q - 2\sigma_m{}^q(\hat{r}_{nm} \cdot \boldsymbol{\sigma}_n)\}|N_A(0^+)\rangle . \qquad (2\cdot35)$$

The decay rate is

$$\Gamma_{0\nu}^{2n}(0^+ \to 2^+) = (a_{0\nu}/m_e{}^2)(g_A g_A{}')^2 |\lambda \sum_j U_{ej} V_{ej} \langle rH' \rangle|^2 (N_2^{pq}, N_2^{pq}) G_{21}(T), \qquad (2\cdot36)$$

where

$$G_{21}(T) = \frac{2^3}{7!} T^2 (4T^5 + 56T^4 + 343T^3 + 1050T^2 + 1540T + 840). \qquad (2\cdot37)$$

(iii) The $0^+ \to 1^+$ transition

The terms Q^l, $\varepsilon_{lkj}P^{kj}$ and R^l contribute to this transition. After small calculations, the decay formula is

$$d\Gamma_{0\nu}^{2n}(0^+ \to 1^+) = (a_{0\nu}/m_e{}^9)(g_A{}^2 g_A'^2/6)|\lambda \sum_j U_{ej} V_{ej}|^2$$
$$\times [\{(\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 (p_1{}^0 p_2{}^0 + \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + m_e{}^2)(N_1{}^q, N_1{}^q)$$
$$+ 2[(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2 (p_1{}^0 p_2{}^0 + m_e{}^2) + (\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 - |\boldsymbol{p}_1|^2)(\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 - |\boldsymbol{p}_2|^2)](N_2{}^q, N_2{}^q)$$
$$+ 4(\boldsymbol{p}_1 \times \boldsymbol{p}_2)^2 \, \mathrm{Im}(N_1{}^q, N_2{}^q)\} \langle rH' \rangle^2$$
$$- 8\{2[(p_1{}^0 + p_2{}^0)\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 - (p_1{}^0 |\boldsymbol{p}_2|^2 + p_2{}^0 |\boldsymbol{p}_1|^2)\mathrm{Re}(N_2{}^q, R^q)$$
$$- [(p_1{}^0 + p_2{}^0)\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + (p_1{}^0 |\boldsymbol{p}_2|^2 + p_2{}^0 |\boldsymbol{p}_1|^2)\mathrm{Im}(N_1{}^q, R^q)\} \langle H' \rangle \langle rH' \rangle$$
$$+ 16\{3(p_1{}^0 p_2{}^0 - m_e{}^2) - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2\}(R^q, R^q) \langle H' \rangle^2]$$
$$\times p_1{}^0 p_2{}^0 \delta(p_1{}^0 + p_2{}^0 + M_B - M_A) d\cos\theta \, dp_1{}^0 \, dp_2{}^0 \, , \qquad (2\cdot38)$$

where $(N_i{}^q, N_j{}^q) \equiv \sum_{J_z} \sum_{q=1}^3 N_i^{q^*} N_j{}^q$ with

$$N_1{}^q = \langle N_B(1^+)|\sum_{n,m} \tau_n{}^+ \tau_m{}^+ [2(g_V/g_A)(\hat{r}_{nm} \cdot \boldsymbol{\sigma}_n) - i\hat{r}_{nm} \cdot (\boldsymbol{\sigma}_n \times \boldsymbol{\sigma}_m)] \hat{r}_{+nm}^q |N_A(0^+)\rangle \, , \qquad (2\cdot39)$$

$$N_2{}^q = \langle N_B(1^+)|\sum_{n,m} \tau_n{}^+ \tau_m{}^+$$
$$\times \{i(g_V/g_A)[\hat{r}_{nm} \times (\hat{r}_{nm} \times \boldsymbol{\sigma}_n)]^q - (\hat{r}_{nm} \cdot \boldsymbol{\sigma}_n)(\hat{r}_{nm} \times \boldsymbol{\sigma}_m)^q\}|N_A(0^+)\rangle \, . \qquad (2\cdot40)$$

The terms $N_1{}^q$ and $N_2{}^q$ come from Q^q and $\varepsilon_{qlk}P^{lk}$, respectively.

The decay rate is

$$\Gamma_{0\nu}^{2n}(0^+ \to 1^+) = (a_{0\nu}/m_e{}^2)(g_A g_A{}')^2 |\lambda \sum_j U_{ej} V_{ej}|^2$$
$$\times [\{G_{11}(T)[(N_1{}^q, N_1{}^q) + 4(N_2{}^q, N_2{}^q) + 4\,\mathrm{Im}(N_1{}^q, N_2{}^q)]$$

$$+G_{12}(T)[(N_1{}^q, N_1{}^q)+2(N_2{}^q, N_2{}^q)]\}\langle rH'\rangle^2$$
$$+G_{13}(T)(R^q, R^q)\langle H'\rangle^2/m_e{}^2$$
$$+G_{14}(T)[2\,\mathrm{Re}(N_2{}^q, R^q)+\mathrm{Im}(N_1{}^q, R^q)]\langle rH'\rangle\langle H'\rangle/m_e], \tag{2·41}$$

where

$$G_{11}(T)=\frac{2^3}{3\cdot 7!}T^3(3T^4+42T^3+210T^2+420T+280), \tag{2·42}$$

$$G_{12}(T)=\frac{2^3}{7!}T^2(4T^5+56T^4+343T^3+1050T^2+1540T+840), \tag{2·43a}$$

$$G_{13}(T)=\frac{2^6}{5!}T^2(T^3+10T^2+35T+30), \tag{2·43b}$$

$$G_{14}(T)=\frac{2^5}{3\cdot 5!}T^2(T^4+12T^3+55T^2+100T+60). \tag{2·43c}$$

(2-b) *The N^*-mechanism*

The R-matrix for the N^*-mechanism is obtained by substituting the R_W-matrices corresponding to the $N_{\Delta^-}\to N_B+2e^-$ and $N_A\to N_{\Delta^{++}}+2e^-$ transitions into Eq. (A·3) in Appendix A.

The nuclear matrix elements in the R_W-matrix are given by Eqs. (2·15) ~(2·18). Since there is no standard method to treat the relativistic correction in the quark model, this correction (the second term in $L_{LR}^{\mu\nu k}$) is discarded. It should be noted that the operators*) $\tau^+_{n(m)}$, $\sigma^j_{n(m)}$, $\hat{r}'_{nm}$ and $\hat{r}'_{+nm}$ act on quarks and change the intrinsic part of the hadron. The nucleon $N(\frac{1}{2}^+)$ and $\Delta(\frac{3}{2}^+)$ are assigned in the $SU(6)$ quark model to the states $l=0$, i.e., the zero orbital angular momentum states around the center of hadron. From these considerations, we conclude that M_F, M_{GT} and Q^l defined in Eqs. (2·15)~(2·17) turn out to be zero, and only the term P^{lk} in Eq. (2·18) contributes. (See Appendix A for the detailed discussion.) Consequently, the term $u^L_{\mu\nu k}L^{\mu\nu k}_{LR}+u^R_{\mu\nu k}L^{\mu\nu k}_{RL}$ contributes to the $(\beta\beta)_{0\nu}$ mode in the N^*-mechanism, while $t^L_{\mu\nu}K^{\mu\nu}_{LL}$, $t^R_{\mu\nu}K^{\mu\nu}_{RR}$ and $u^L_{\mu\nu 0}L^{\mu\nu 0}_{LR}+u^R_{\mu\nu 0}L^{\mu\nu 0}_{RL}$ vanish.

In summarizing the above discussion, the $(\beta\beta)_{0\nu}$ mode in the N^*-mechanism takes place only when $\lambda\neq 0$, whether neutrinos are massive or massless. These results are listed in Table I. Halprin et al.[7] have derived the bounds of the neutrino mass both in the $2n$- and N^*-mechanism. However, the bounds they obtained in the N^*-mechanism seem to be meaningless, because there is no contribution from the m_ν-term within the N^*-mechanism adopted in the present paper. We would like to emphasize here that the above discussions are independent of the "factorization hypothesis" which will be used later.

*) $\hat{r}'_{nm}$ and $\hat{r}'_{+nm}$ are defined in terms of the position operators of quarks measured from the center of the hadron.

The non-vanishing product of the leptonic and hadronic parts for the $N_{\Delta^-}\to N_B+2e^-$ transition is

$$u^L_{\mu\nu k}L^{\mu\nu k}_{LR}+u^R_{\mu\nu k}L^{\mu\nu k}_{RL}=\frac{1}{6\pi}\langle rH'\rangle_\Delta(g_A g_A')$$

$$\times(p_1-p_2)^l\bar{u}(p_1)\gamma^k u^C(p_2)[i\epsilon_{lkj}(g_V/g_A)M^j+M^{lk}], \qquad (2\cdot44)$$

where

$$M^j=\langle N_B|\sum_{n,m}\tau_n{}^+\tau_m{}^+\sigma_n{}^j|N_{\Delta^-}\rangle\,, \qquad (2\cdot45)$$

$$M^{jk}=\langle N_B|\sum_{n,m}\tau_n{}^+\tau_m{}^+\sigma_n{}^j\sigma_m{}^k|N_{\Delta^-}\rangle\,. \qquad (2\cdot46)$$

The nuclear tensor operators in Eqs. (2·45) and (2·46) only change the spin and isospin of Δ^- and leave the remainder unchanged. Therefore, M^j and M^{jk} represent essentially the matrix elements between Δ^- and p. Also, the expectation value of the "potential" $\langle rH'\rangle$ should be taken between Δ^- and p. Note that $g_A=g_V$ and $g_A'=g_V'$ should be taken in the N^*-mechanism as explained in § 1.

Now we use the "factorization hypothesis"[2)] (see Eq. (A·10)) and evaluate the decay formula,*)

$$d\Gamma^{N^*}_{0\nu}=(a_{0\nu}/m_e{}^9)(2^6/3^3)(g_V g_V')^2$$

$$\times|\lambda\sum_j U_{ej}V_{ej}\langle rH'\rangle_\Delta|^2 P(\Delta)|\langle\Phi_f|\Phi_i\rangle|^2$$

$$\times\{2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2-\boldsymbol{p}_1\cdot\boldsymbol{p}_2[(p_1{}^0+p_2{}^0)^2+4(p_1{}^0p_2{}^0+m_e{}^2)]$$

$$+3(p_1{}^0p_2{}^0+m_e{}^2)(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2)\}$$

$$\times p_1{}^0p_2{}^0\delta(p_1{}^0+p_2{}^0+M_B-M_A)d\cos\theta dp_1{}^0dp_2{}^0\,. \qquad (2\cdot47)$$

Here $P(\Delta)$ is the probability of producing Δ per nucleon inside the nucleus, and $\langle\Phi_f|\Phi_i\rangle$ represents the overlap between the initial and final nuclei. The $N_A\to N_{\Delta^{++}}+2e^-$ transition is also included in the above formula. We refer to Appendix A for the detailed discussion.

The decay rate is

$$\Gamma^{N^*}_{0\nu}=(a_{0\nu}/m_e{}^2)(g_V g_V')^2$$

$$\times|\lambda\sum_j U_{ej}V_{ej}\langle rH'\rangle_\Delta|^2 P(\Delta)|\langle\Phi_f|\Phi_i\rangle|^2 G^{N^*}(T), \qquad (2\cdot48)$$

*) Here we have used the following results: $\sum\tilde{M}^{j*}\tilde{M}^k=(16/3)\delta_{jk}$, $\sum\tilde{M}^{j*}\tilde{M}^{kl}=0$, $\sum\tilde{M}^{jk*}\tilde{M}^{lm}=16(\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}-\frac{2}{3}\delta_{jk}\delta_{lm})$, where $\tilde{M}^j$ and $\tilde{M}^{jk}$ are defined from M^j and M^{jk} by replacing N_{Δ^-} and N_B with Δ^- and p, respectively. The same relations hold for the matrix elements between n and Δ^{++}.

where

$$G^{N^*}(T)=\frac{2^{10}}{3^3\cdot 7\,!}T^2[39T^5+546T^4+3297T^3+9870T^2+14140T+7560]. \tag{2·49}$$

The above decay formula is applicable to all $0^+\to 0^+$, 1^+ and 2^+ transitions by the appropriate choice of $P(\Delta)|\langle\Phi_f|\Phi_i\rangle|^2$.

§ 3. The $(\beta\beta)_{2\nu}$ mode

In a similar way to the case of the $(\beta\beta)_{0\nu}$ mode, the $0^+\to J^+$ transitions are investigated. In our Hamiltonian in Eq. (1·3), the $(\beta\beta)_{2\nu}$ mode takes place through the process,

$$N_A(p_A)\to N_B(p_B)+e^-(p_1)+e^-(p_2)+\bar{N}_i(k_1)+\bar{N}_j(k_2). \tag{3·1}$$

The contribution from the right-handed interaction is suppressed by λ ($\lambda\ll 1$) so that this is neglected here.

The R_W-matrix due to the $V-A$ interaction for the $N_\alpha\to N_\beta+2e^-+\bar{N}_i+\bar{N}_j$ transition is expressed by

$$\begin{aligned}R_{Wij}=&\frac{\epsilon_{ij}}{\sqrt{2}}\left(\frac{G_F}{\sqrt{2}}\right)^2 U_{ei}U_{ej}\\&\times[(2\pi)^{-12}(p_1{}^0p_2{}^0k_1{}^0k_2{}^0)^{-1}F(Z+2,\,p_1{}^0)F(Z+2,\,p_2{}^0)]^{1/2}\\&\times[E_{\mu\nu}J^{\mu\nu}-(p_1\leftrightarrow p_2)],\end{aligned} \tag{3·2}$$

where

$$E_{\mu\nu}=\bar{u}(p_1)\gamma_\mu(1-\gamma_5)u^C(k_1)\bar{u}(p_2)\gamma_\nu(1-\gamma_5)u^C(k_2), \tag{3·3}$$

$$\begin{aligned}J^{\mu\nu}=&\int d\boldsymbol{x}d\boldsymbol{y}e^{-i[(p_1+k_1)\boldsymbol{x}+(p_2+k_2)\boldsymbol{y}]}\\&\times\langle N_\beta|\sum_n\left\{\frac{J_L^{+\mu}(\boldsymbol{x})|N_n\rangle\langle N_n|J_L^{+\nu}(\boldsymbol{y})}{E_n-E_\alpha+p_2{}^0+k_2{}^0}+\frac{J_L^{+\nu}(\boldsymbol{y})|N_n\rangle\langle N_n|J_L^{+\mu}(\boldsymbol{x})}{E_n-E_\alpha+p_1{}^0+k_1{}^0}\right\}|N_\alpha\rangle\,.\end{aligned} \tag{3·4}$$

Here the term $\varepsilon_{ij}/\sqrt{2}$ is the statistical factor for the final two electrons and two neutrinos, i.e., $\varepsilon_{ij}=1/\sqrt{2}$ for $i=j$ and $=1$ for $i\neq j$. The full R-matrix for the $(\beta\beta)_{2\nu}$ mode can be readily obtained in the same manner as for the $(\beta\beta)_{0\nu}$ mode.

Now we use the approximations (i), (ii) and (iii) introduced in § 2. Under these assumptions, the nuclear part $J^{\mu\nu}$ can be simplified. Note that the $0^+\to J^+$ ($J\geqq 3$) transitions are forbidden.

(3-a) *The 2n-mechanism*

The R-matrix is obtained from the R_W-matrix by the replacements, $N_\alpha \to N_A$ and $N_\beta \to N_B$.

(i) The $0^+ \to 0^+$ transition

After straightforward calculations, we obtain

$$d\Gamma_{2\nu}^{2n}(0^+ \to 0^+) = (a_{2\nu}/m_e^9)\frac{1}{8}[p_1^0 p_2^0 C - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 D]$$
$$\times (k_1^0 k_2^0)^2 p_1^0 p_2^0 \delta(p_1^0 + p_2^0 + k_1^0 + k_2^0 + M_B - M_A)$$
$$\times d\cos\theta dp_1^0 dp_2^0 dk_1^0 dk_2^0 , \tag{3·5}$$

where

$$a_{2\nu} = (2\sum_{i\geq j}{}' \epsilon_{ij}^2 |U_{ei} U_{ej}|^2)\frac{1}{4}\frac{G_F^4 m_e^9}{2\pi^7}\left\{\frac{2\pi\alpha(Z+2)}{1-\exp[-2\pi\alpha(Z+2)]}\right\}^2 , \tag{3·6}$$

$$C = g_V^4(K^2 - KL + L^2)|M_F|^2$$
$$-2g_V^2 g_A^2 KL\,\mathrm{Re}(M_F M_{GT}^*) + \frac{1}{3}g_A^4(K^2 + KL + L^2)|M_{GT}|^2 , \tag{3·7}$$

$$D = g_V^4 KL|M_F|^4 - \frac{2}{3}g_V^2 g_A^2(K^2 + KL + L^2)\mathrm{Re}(M_F M_{GT}^*)$$
$$+\frac{1}{9}g_A^4(2K^2 + 5KL + 2L^2)|M_{GT}|^2 . \tag{3·8}$$

Here M_F and M_{GT} are defined in Eqs. (2·15) and (2·16), and

$$K = [\langle E_n \rangle - M_A + p_1^0 + k_1^0]^{-1} + [\langle E_n \rangle - M_A + p_2^0 + k_2^0]^{-1} , \tag{3·9}$$
$$L = [\langle E_n \rangle - M_A + p_2^0 + k_1^0]^{-1} + [\langle E_n \rangle - M_A + p_1^0 + k_2^0]^{-1} . \tag{3·10}$$

The primed sum in Eq. (3·6) should extend over all energetically allowed neutrinos in the final state. Rigorously speaking, the neutrino masses m_j in k_1^0 and k_2^0 should be taken into account in this primed sum. If all neutrinos are allowed to contribute and the replacement of k_i^0 by $|\boldsymbol{k}_i|$ is permissible, then $(2\sum_{i\leq j}' \epsilon_{ij}^2 |U_{ei}|^2 |U_{ej}|^2) = 1$. The factor 1/4 in $a_{2\nu}$ is to represent the statistical factor for the case $U_{ej} = \delta_{j1}$.

To perform the phase space integration, we neglect the masses of neutrinos and assume the following replacement (within a few % errors): $p_i^0 + k_j^0 \to \langle p_i^0 + k_j^0 \rangle = (M_A - M_B)/2$. Then $K \simeq L \simeq 2(\mu_0 m_e)^{-1}$.

Now the straightforward calculations lead to

$$d\Gamma_{2\nu}^{2n}(0^+\to 0^+)=(a_{2\nu}/m_e^{11})(g_A{}^4/60)|(g_V/g_A)^2M_F-M_{GT}|^2\mu_0^{-2}$$
$$\times p_1{}^0p_2{}^0(p_1{}^0p_2{}^0-\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(M_A-M_B-p_1{}^0-p_2{}^0)^5$$
$$\times d\cos\theta dp_1{}^0dp_2{}^0\,. \tag{3·11}$$

The decay rate is

$$\Gamma_{2\nu}^{2n}(0^+\to 0^+)=a_{2\nu}g_A{}^4|(g_V/g_A)^2M_F-M_{GT}|^2\mu_0^{-2}F_0(T), \tag{3·12}$$

where

$$F_0(T)=\frac{2^4}{11!}T^7(T^4+22T^3+220T^2+990T+1980). \tag{3·13}$$

Primakoff and Rosen have derived Eq. (3·5) in the limit of $U_{ej}=\delta_{j1}$. Note that their result has a few misprints and also is four times as large as ours.*) Concerning this overall normalization, our result in Eq. (3·11) agrees in this limit with that by Konopinski.[6)]

(ii) The $0^+\to 2^+$ transition

For this transition, we have

$$d\Gamma_{2\nu}^{2n}(0^+\to 2^+)=(a_{2\nu}/m_e{}^9)(g_A{}^4/8)$$
$$\times(M_2^{pq},M_2^{pq})(K-L)^2(p_1{}^0p_2{}^0+\frac{1}{3}\boldsymbol{p}_1\cdot\boldsymbol{p}_2)$$
$$\times(k_1{}^0k_2{}^0)^2p_1{}^0p_2{}^0\delta(p_1{}^0+p_2{}^0+k_1{}^0+k_2{}^0+M_B-M_A)$$
$$\times d\cos\theta dp_1{}^0dp_2{}^0dk_1{}^0dk_2{}^0\,, \tag{3·14}$$

where $(M_2^{pq},M_2^{pq})\equiv\sum_{J_z}\sum_{p,q=1}^3M_2^{pq*}M_2^{pq}$ with

$$M_2^{pq}=\langle N_B(2^+)|\sum_{n,m}\tau_n{}^+\tau_m{}^+\sigma_n{}^p\sigma_m{}^q|N_A(0^+)\rangle\,. \tag{3·15}$$

To simplify the term $(K-L)^2$, we use the approximation $p_i{}^0+k_j{}^0\simeq\langle p_i{}^0+k_j{}^0\rangle=(M_A-M_B)/2$ only in the denominator of $K-L$ and obtain $K-L=2(p_1{}^0-p_2{}^0)(k_1{}^0-k_2{}^0)(\mu_0m_e)^{-3}$. This approximation is valid within several % errors for $\mu_0\geq 4$. After the phase space integration, we get

$$d\Gamma_{2\nu}^{2n}(0^+\to 2^+)=(a_{2\nu}/m_e^{15})(g_A{}^4/420)(M_2^{pq},M_2^{pq})\mu_0^{-6}p_1{}^0p_2{}^0(p_1{}^0-p_2{}^0)^2$$
$$\times\left(p_1{}^0p_2{}^0+\frac{1}{3}\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right)(M_A-M_B-p_1{}^0-p_2{}^0)^7d\cos\theta dp_1{}^0dp_2{}^0\,. \tag{3·16}$$

*) See the footnotes on page 1747 and Eq. (60) of Ref. 4). Greuling and Whitten[5)] give results four times as large as ours.

The decay rate is given by

$$\Gamma_{2\nu}^{2n}(0^+\to 2^+)=a_{2\nu}g_A{}^4(M_2^{pq},\ M_2^{pq})\mu_0^{-6}F_2(T), \tag{3·17}$$

where

$$F_2(T)=\frac{2^6 3^2}{15!}T^{11}(T^4+30T^3+420T^2+1820T+2730). \tag{3·18}$$

(iii) The $0^+\to 1^+$ transition

Similarly, we obtain

$$d\Gamma_{2\nu}^{2n}(0^+\to 1^+)=(a_{2\nu}/m_e{}^9)(g_A{}^2 g_V{}^2/4)$$
$$\times(M_1{}^p,\ M_1{}^p)(K-L)^2\left(p_1{}^0 p_2{}^0+\frac{1}{3}\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right)$$
$$\times(k_1{}^0 k_2{}^0)^2 p_1{}^0 p_2{}^0\delta(p_1{}^0+p_2{}^0+k_1{}^0+k_2{}^0+M_B-M_A)$$
$$\times d\cos\theta dp_1{}^0 dp_2{}^0 dk_1{}^0 dk_2{}^0\ , \tag{3·19}$$

where $(M_1{}^p,\ M_1{}^p)\equiv\sum_{J_z}\sum_{p=1}^3 M_1^{p^*}M_1{}^p$ with

$$M_1{}^p=\langle N_B(1^+)|\sum_{n,m}\tau_n{}^+\tau_m{}^+\sigma_n{}^p|N_A(0^+)\rangle\ . \tag{3·20}$$

Note that the transition formula given above is exactly the same as the one for the $0^+\to 2^+$ transition given in Eq. (3·14), aside from the overall normalization. Therefore, the decay rate can be read off from the one for the $0^+\to 2^+$ transition.

Before closing this subsection, we would like to mention the work by Molina and Pascual[9] who estimated the $0^+\to J^+$ transitions. We found several errors in their formulae: (i) They showed that $\Gamma_{2\nu}^{2n}(0^+\to 1^+)=0$, while we get the non-vanishing rate as given in Eq. (3·19). (ii) Their decay rates of the $0^+\to 0^+$, 2^+ transitions for the $(\beta\beta)_{2\nu}$ mode are twice as large as ours.

(3-b) *The N^*-mechanism*

The R-matrix element for the N^*-mechanism is obtained from Eq. (A·3) in Appendix A by substituting the R_W-matrices corresponding to the $N_{\Delta^-}\to N_B$ and $N_A\to N_{\Delta^{++}}$ transitions shown in Figs. 2(c) and (d). The hadronic part of the amplitude for $N_{\Delta^-}\to N_B+2e^-+\bar{N}_i+\bar{N}_j$ is expressed as follows:

$$(J^{\mu\nu})_{N^*}=K\left\{g_V g_A(g^{\mu 0}g^{\nu j}+g^{\mu j}g^{\nu 0})M^j+\frac{1}{2}g_A{}^2(g^{\mu j}g^{\nu k}+g^{\mu k}g^{\nu j})M^{jk}\right\}, \tag{3·21}$$

where K, M^j and M^{jk} are defined in Eqs. (A·9), (2·45) and (2·46). In order to express the contributions from the M^j and M^{jk} terms clearly, we retain g_V and g_A explicitly although $g_A=g_V$ should be taken in the N^*-mechanism.

By using the factorization hypothesis, the decay formula is obtained in the following form:

$$d\Gamma_{2\nu}^{N^*}=(a_{2\nu}/m_e{}^9)g_V{}^4 2^2\cdot 3P(\Delta)|\langle\Phi_f|\Phi_i\rangle|^2(K-L)^2\left(p_1{}^0 p_2{}^0+\frac{1}{3}\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right)$$
$$\times(k_1{}^0 k_2{}^0)^2 p_1{}^0 p_2{}^0\delta(p_1{}^0+p_2{}^0+k_1{}^0+k_2{}^0+M_B-M_A)$$
$$\times d\cos\theta dp_1{}^0 dp_2{}^0 dk_1{}^0 dk_2{}^0\,, \tag{3·22}$$

where both the $N_{\Delta^-}\to N_B$ and $N_A\to N_{\Delta^{++}}$ transitions are included according to the argument given in Appendix A. It is amusing to observe that the above formula is exactly the same as the one for the $0^+\to 2^+$ transition in the $2n$-mechanism aside from the overall normalization.

The decay rate is

$$\Gamma_{2\nu}^{N^*}=a_{2\nu}g_V{}^4 2^5\cdot 3P(\Delta)|\langle\Phi_f|\Phi_i\rangle|^2\mu_0^{-6}F_2(T), \tag{3·23}$$

where $F_2(T)$ is defined in Eq. (3·18) and $a_{2\nu}$ is in Eq. (3·6).

We would like to note that this formula is completely different from the one obtained by Picciotto.[8] This is due to the fact that he used a crucial approximation*) for the R-matrix, instead of taking the spin sum explicitly. However, his approximation cannot be regarded as reasonable. Note also that he has neglected the M^{jk} term ($g_A{}^4$ term) in Eq. (3·22) which turns out to be dominant for the $(\beta\beta)_{2\nu}$ mode.

§ 4. Concluding remarks

In this paper, we have presented the formulae for parity conserving $0^+\to J^+$ transitions within the weak interaction Hamiltonian in Eq. (1·3) which is motivated by the grand unified theories. The emphasis has been made on the transitions to the excited states which become important in analyzing the data on the half-life, especially the data obtained by the geological method.

We have found several interesting "selection rules" as given in Table I: (i) If $\lambda=0$ (no right-handed interaction), the N^*-mechanism does not contribute to the $(\beta\beta)_{0\nu}$ mode, whether neutrinos are massive or massless. (ii) If $\lambda=0$, the $0^+\to 1^+$ and 2^+ transitions of the $(\beta\beta)_{0\nu}$ mode are forbidden and the $0^+\to 0^+$ transition is only allowed. We emphasize that these selection rules for the N^*-mechanism do not depend on the factorization hypothesis. The selection rule (i) seems to nullify the neutrino mass bounds derived by Halprin et al. in the N^*-mechanism.[7]

It may be worth while to mention that some care is necessary to use the previous theoretical estimates numerically, because there are various errors as discussed in § 3 for the $(\beta\beta)_{2\nu}$ mode and in § 2 and Appendix C for the $(\beta\beta)_{0\nu}$ mode.

*) Picciotto[8] used the approximation $\{E_{\mu\nu}J^{\mu\nu}-(p_1\leftrightarrow p_2)\}\sim E_{\mu\nu}g_V g_A(g^{\mu0}g^{\nu k}+g^{\mu k}g^{\nu 0})M^k\mu_0^{-1}$. We have evaluated the spin sum exactly.

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Appendix A

—— Brief Description of the 2n- and N^*-Mechanisms ——

(a) *The general description*

In order to deal with the N^*-mechanism, the following effective Hamiltonian is considered,

$$H_{\rm int}=H_W+H_S\ , \qquad (\text{A}\cdot 1)$$

where H_S represents the effective interaction for the transition $N+N\leftrightarrow\Delta+N$ by the exchange of π, ρ, $\cdots$.

In the $2n$-mechanism, the double β decay takes place through the 2nd order perturbation in H_W and the 0th order in H_S as shown in Fig. 1 and the R-matrix is

$$R^{2n}=\langle N_B,\ l|R_W|N_A\rangle\ , \qquad (\text{A}\cdot 2)$$

where l stands for either $2e^-$ or $2e^-+2\bar{\nu}_e$ depending on the $(\beta\beta)_{0\nu}$ or $(\beta\beta)_{2\nu}$ mode, respectively. The R_W represents the R-matrix due to the 2nd order weak interaction.

In the N^*-mechanism, the double β decay occurs through the 2nd order in H_W and the 1st order in H_S as shown in Fig. 2.*) Then the R-matrix element for the N^*-mechanism may be expressed in the following forms:

$$R^{N^*}=\sum_{N_\Delta}\Big\{\langle N_B,\ l|R_W|N_{\Delta^-}\rangle\frac{1}{M_A-E_{\Delta^-}}\langle N_{\Delta^-}|H_S|N_A\rangle + \langle N_B|H_S|N_{\Delta^{++}}\rangle\frac{1}{M_B-E_{\Delta^{++}}}\langle N_{\Delta^{++}},\ l|R_W|N_A\rangle\Big\}, \qquad (\text{A}\cdot 3)$$

where M_A and M_B are the masses of the parent (N_A) and daughter (N_B) nuclei, respectively, and E_Δ is the energy of the intermediate nucleus which includes $\Delta(1232)$. Note that the nuclear state $N_{\Delta^-}(N_{\Delta^{++}})$ has the same J^P as $N_A(N_B)$ has.

Let us consider, for definiteness, the $N_{\Delta^-}\to N_B+2e^-$ transition. The nuclear states N_{Δ^-} and N_B are expressed in the following forms:

$$|N_{\Delta^-}\rangle=|\Delta^-\rangle_S\otimes|\Delta^-\rangle_L\otimes|R_{\Delta^-}\rangle\ , \qquad (\text{A}\cdot 4)$$

*) There may be the third possible combination of H_W and H_S such as the sequence H_W-H_S-H_W in contrast to the H_W-H_W-H_S in Fig. 2(a). Since this contribution is expected to be small, this is not considered in this paper.

$$|N_B\rangle=|p\rangle_s\otimes|p\rangle_L\otimes|R_B\rangle\ , \qquad (\mathrm{A}\cdot5)$$

where s and L in the hadronic states stand for the intrinsic (spin and isospin) part and the orbital angular momentum part with respect to the center of the nucleus. Here, $|R_{\Delta^-}\rangle$ and $|R_B\rangle$ represent the remainders of the nuclear states. It should be understood that there is some appropriate sum with respect to the angular momenta. The nuclear matrix elements in Eqs. (2·15)~(2·18) can be written in this notation as follows:

$$\langle N_B|O|N_{\Delta^-}\rangle=\langle R_B|\otimes_L\langle p|\otimes_s\langle p|\ O|\Delta^-\rangle_s\otimes|\Delta^-\rangle_L\otimes|R_{\Delta^-}\rangle\ , \qquad (\mathrm{A}\cdot6)$$

where O represents one of the nuclear tensor operators appearing in M_F, M_{GT}, Q^l and P^{lk}.

Let us discuss what kinds of nuclear tensor operators change quark states inside the hadron. Obviously, the operators $\tau^+_{n(m)}$ and $\sigma^j_{n(m)}$ act on quarks. As for $\hat{r}_{nm}$ and $\hat{r}_{+nm}$, some caution is necessary. Consider the following decomposition of the position operator for the n-th quark; $\boldsymbol{r}_n=\boldsymbol{r}_G+\boldsymbol{r}_n'$ where $\boldsymbol{r}_G$ is the position operator of Δ^- measured from the center of N_{Δ^-}. The relative coordinate $\boldsymbol{r}_n'$ changes the orbital angular momentum of quarks around the center of the hadron. Thus we conclude that the relevant operators for quarks in Δ^- are $\tau^+_{n(m)}$, $\sigma^j_{n(m)}$, $\boldsymbol{r}'_{nm}$ and $\boldsymbol{r}'_{+nm}$. With this caution, the nuclear matrix elements are calculated in the $SU(6)$ quark model where $\Delta(\frac{3}{2}^+)$ and the nucleon $N(\frac{1}{2}^+)$ are assigned to $l=0$. The nuclear tensor operators contributing to the transition $\Delta(\frac{3}{2}^+)\to N(\frac{1}{2}^+)$ should be of rank 0 with respect to $\boldsymbol{r}'_{nm}$ and $\boldsymbol{r}'_{+nm}$ and of rank 1 or 2 with respect to the spin part. We conclude from Eqs. (2·15)~(2·18) that M_F and M_{GT} do not contribute to this transition. The Q^l and P^{lk} take the following forms:

$$Q^l=\frac{1}{3}{}_s\langle p|\sum_{n,m}\tau_n{}^+\tau_m{}^+ \times[(\sigma_n{}^l-\sigma_m{}^l)-i(\boldsymbol{\sigma}_n\times\boldsymbol{\sigma}_m)^l](\hat{r}'_{nm}\cdot\hat{r}'_{+nm})|\Delta^-\rangle_s\cdot{}_L\langle p|\Delta^-\rangle_L\langle R_B|R_{\Delta^-}\rangle\ , \qquad (\mathrm{A}\cdot7)$$

$$P^{lk}=-\frac{2}{3}{}_s\langle p|\sum_{n,m}\tau_n{}^+\tau_m{}^+[i\epsilon_{lkj}\sigma_n{}^j+\sigma_n{}^k\sigma_m{}^l]|\Delta^-\rangle_s\cdot{}_L\langle p|\Delta^-\rangle_L\langle R_B|R_{\Delta^-}\rangle\ . \qquad (\mathrm{A}\cdot8)$$

In the $SU(6)$ quark model, we get $Q^l=0$ which may be understood from the following argument. Note that the spin operators in Q^l are antisymmetric under the interchange of quarks so that it is expected that the spin part of ${}_s\langle p|\sum_{nm}\tau_n{}^+\tau_m{}^+[(\sigma_n{}^l-\sigma_m{}^l)-i(\boldsymbol{\sigma}_n\times\boldsymbol{\sigma}_m)^l]$ is also antisymmetric. Since the spin wave function of $|\Delta^-\rangle_s$ is symmetric, $Q^l=0$ is concluded. Therefore, only P^{lk} contributes to the $(\beta\beta)_{0\nu}$ mode. The same argument also holds for the $N_A\to N_{\Delta^{++}}+2e^-$ transition.

A similar argument applies to the $(\beta\beta)_{2\nu}$ mode. Of course, it should be noted that K and L in Eq. (3·22) should be modified as follows:

$$K=[\langle E_n\rangle-E_{\Delta^-}+p_1{}^0+k_1{}^0]^{-1}+[\langle E_n\rangle-E_{\Delta^-}+p_2{}^0+k_2{}^0]^{-1}\,, \tag{A·9}$$

and similarly for L.

(b) *The factorization hypothesis*

As we have seen in the previous subsection (a), the R_W-matrix may be written in the following form:

$$\langle N_B,\ l|R_W|N_{\Delta^-}\rangle={}_s\langle p,\ l|R_W|\Delta^-\rangle_s\ {}_L\langle p|\Delta^-\rangle_L\langle R_B|R_{\Delta^-}\rangle\,. \tag{A·10}$$

This is valid under the approximations (i), (ii) and (iii) introduced in § 2 and in the framework of the $SU(6)$ quark model. The "factorization hypothesis" means the approximation that the amplitude ${}_s\langle p,\ l|R_W|\Delta^-\rangle_s$ is modified by the following replacement,

$${}_s\langle p,\ l|R_W|\Delta^-\rangle_s\Rightarrow e^{i\varphi}[\sum_{S_{\Delta z},S_{pz}}{}' |{}_s\langle p,\ l|R_W|\Delta^-\rangle_s|^2]^{1/2}\,, \tag{A·11}$$

where the primed sum means the spin average with respect to Δ. Under this replacement, the R-matrix is rewritten as follows:

$$R^{N^*}=[\sum_{S_{\Delta z},S_{pz}}{}' |{}_s\langle p,\ l|R_W|\Delta^-\rangle_s|^2]^{1/2}\sqrt{2}P(\Delta)^{1/2}\langle\Phi_f|\Phi_i\rangle\,, \tag{A·12}$$

where

$$\sqrt{2}P(\Delta)^{1/2}\langle\Phi_f|\Phi_i\rangle={}_L\langle p|\Delta^-\rangle_L\langle R_B|R_{\Delta^-}\rangle e^{i\varphi}\frac{\langle N_{\Delta^-}|H_S|N_A\rangle}{M_A-E_{\Delta^-}}$$
$$+\frac{\langle N_B|H_S|N_{\Delta^{++}}\rangle}{M_B-E_{\Delta^{++}}}e^{i\varphi'}\langle R_{\Delta^{++}}|R_A\rangle_L\langle\Delta^{++}|n\rangle_L\,. \tag{A·13}$$

Here we have used the relation

$$\sum_{S_{\Delta z},S_{pz}}{}' |{}_s\langle p,\ l|R_W|\Delta^-\rangle_s|^2=\sum_{S_{\Delta z},S_{nz}}{}' |{}_s\langle \Delta^{++},\ l|R_W|n\rangle_s|^2\,. \tag{A·14}$$

The factors $P(\Delta)$ and $\langle\Phi_f|\Phi_i\rangle$ are introduced to give some physical image of the N^*-mechanism. Let us assume the decomposition

$$\langle N_{\Delta^-}|H_S|N_A\rangle\simeq\langle\Delta^-|H_S|n\rangle\langle R_{\Delta^-}|R_A\rangle\,. \tag{A·15}$$

Now the probability admixture $P(\Delta)$ may be defined as

$$P(\Delta)=\frac{1}{N_n}\sum_{n,\Delta^-}\left|\frac{\langle\Delta^-|H_S|n\rangle}{E_{\Delta^-}-M_A}\right|^2, \tag{A·16}$$

where N_n is the number of neutrons which actively participate in the double β decay and the sum of n extends over all those neutrons. In other words, $P(\Delta)$

is the probability to make Δ^- per neutron. Now $\langle\Phi_f|\Phi_i\rangle$ may be written

$$\langle\Phi_f|\Phi_i\rangle\simeq{}_L\langle p|\Delta^-\rangle_L\langle R_B|R_{\Delta}\text{-}\rangle\langle R_{\Delta}\text{-}|R_A\rangle\simeq{}_L\langle p|\Delta^-\rangle_L\langle R_B|R_A\rangle\,. \qquad (\mathrm{A\cdot17})$$

Here we have used $\langle R_{\Delta}\text{-}|R_A\rangle\simeq\delta_{R_{\Delta}\text{-}R_A}$. In this way, the $\langle\Phi_f|\Phi_i\rangle$ may be interpreted as the overlap between the initial and final nuclear wave functions.

Appendix B

—— Majorana Neutrino Propagators ——

The quantization of the Majorana field is rather complicated. It can be done straightforwardly by decomposing Majorana field into two-component field. The quantized form of the Majorana field is derived by substituting the quantized two-component field into it.[10] In this way, we obtain

$$N(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\boldsymbol{p}}{\sqrt{p^0}}\sum_s\{a(p,s)u(p,s)e^{-ipx}+a^+(p,s)u^C(p,s)e^{ipx}\}, \qquad (\mathrm{B\cdot1})$$

where $u^C=C\bar{u}^T$. The creation and annihilation operators $a^+(p,s)$ and $a(p,s)$ satisfy the canonical commutation relation,

$$[a(p,s),a^+(p',s')]_+=\delta_{ss'}\delta(\boldsymbol{p}-\boldsymbol{p}'). \qquad (\mathrm{B\cdot2})$$

The Majorana propagator can be calculated straightforwardly from Eqs. (B·1) and (B·2),

$$\langle 0|T[N(x)\bar{N}(y)]|0\rangle=iS_F(x-y). \qquad (\mathrm{B\cdot3})$$

Note that we also obtain

$$\langle 0|T[N(x)N^T(y)]|0\rangle=iS_F(x-y)C^T\,. \qquad (\mathrm{B\cdot4})$$

This is only possible for the (self-conjugate) Majorana field and makes the $(\beta\beta)_{0\nu}$ mode possible.

As defined in Eq. (1·5), current neutrinos ν_{eL} and ν'_{eR} are the superpositions of massive Majorana neutrinos N_j. We obtain from Eqs. (1·5) and (B·4),

$$\begin{aligned}\langle 0|T[\nu_{eL}(x)\nu_{eL}^T(y)]|0\rangle &= i\sum_j U_{ej}^2\left(\frac{1-\gamma_5}{2}\right)S_F(x-y)C^T\left(\frac{1-\gamma_5{}^T}{2}\right)\\ &= i\sum_j m_j U_{ej}^2\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x-y)}}{q^2-m_j{}^2+i\epsilon}\left(\frac{1-\gamma_5}{2}\right)C^T\,,\end{aligned} \qquad (\mathrm{B\cdot5})$$

$$\begin{aligned}\langle 0|T[\nu_{eL}(x)\nu_{eR}'^T(y)]|0\rangle &= i\sum_j U_{ej}V_{ej}\left(\frac{1-\gamma_5}{2}\right)S_F(x-y)C^T\left(\frac{1+\gamma_5{}^T}{2}\right)\\ &= i\sum_j U_{ej}V_{ej}\int\frac{d^4q}{(2\pi)^4}\frac{\not{q}e^{-iq(x-y)}}{q^2-m_j{}^2+i\epsilon}\left(\frac{1+\gamma_5}{2}\right)C^T\,.\end{aligned} \qquad (\mathrm{B\cdot6})$$

The other propagators are similarly obtained.

Appendix C

—— *The Double β Decay in the Weak Interaction Hamiltonian Previously Used* ——

The weak interaction Hamiltonian used by many others[1),5),7)~9)] is

$$H_W=\left(\frac{G_F}{\sqrt{2}}\right)\bar{e}\gamma_\mu[(1-\gamma_5)+\eta(1+\gamma_5)]\nu_e\bar{p}\gamma^\mu(g_V-g_A\gamma_5)n+\text{h.c.} \tag{C·1}$$

It should be emphasized that the parameters η in Eq. (C·1) and λ in Eq. (1·3) have different physical meanings. The parameter η represents the admixture of $V+A$ interaction in the leptonic current, while λ is the relative strength of the right-handed to the left-handed weak interactions. For the $(\beta\beta)_{2\nu}$ mode, the decay formulae are the same as the ones given in § 4 because the contribution due to the η term can be neglected. Here we only present the results for the $(\beta\beta)_{0\nu}$ mode.

We obtain the following R_W-matrix corresponding to Eqs. (2·1), (2·12) ~(2·18),

$$R_W=\frac{1}{\sqrt{2}}\left(\frac{G_F}{\sqrt{2}}\right)^2[(2\pi)^{-6}(p_1{}^0p_2{}^0)^{-1}F(Z+2,\,p_1{}^0)F(Z+2,\,p_2{}^0)]^{-1/2}$$
$$\times 2\{m_\nu(t^L_{\mu\nu}+\eta^2t^R_{\mu\nu})K^{\mu\nu}_{LL}+\eta u^{L+R}_{\mu\nu\rho}L^{\mu\nu\rho}_{LL}\}, \tag{C·2}$$

where $u^{L+R}_{\mu\nu\rho}\equiv u^L_{\mu\nu\rho}+u^R_{\mu\nu\rho}$ and*)

$$(t^L_{\mu\nu}+\eta^2t^R_{\mu\nu})K^{\mu\nu}_{LL}=\frac{1}{4\pi}\langle H\rangle g_A{}^2$$
$$\times\,\bar{u}(p_1)[(1+\gamma_5)+\eta^2(1-\gamma_5)]u^C(p_2)[(g_V/g_A)^2M_F-M_{GT}], \tag{C·3}$$

$$u^{L+R}_{\mu\nu0}L^{\mu\nu0}_{LL}=\frac{1}{4\pi}\langle H\rangle g_A{}^2(p_1{}^0-p_2{}^0)$$
$$\times\,\bar{u}(p_1)\{\gamma^0[(g_V/g_A)^2M_F+M_{GT}]+2(g_V/g_A)\gamma^j\mathcal{R}^j\}u^C(p_2), \tag{C·4}$$

$$u^{L+R}_{\mu\nu k}L^{\mu\nu k}_{LL}=\frac{-1}{4\pi}\langle rH'\rangle g_A{}^2\bar{u}(p_1)\{(p_1-p_2)^l[\gamma^k\mathcal{P}^{lk}+\gamma^0\mathcal{P}^l]$$
$$+(p_1+p_2)^l[\gamma_5\gamma^0Q^l+\gamma_5\gamma^kQ^{lk}]\}u^C(p_2). \tag{C·5}$$

Here M_F and M_{GT} are defined in Eqs. (2·15) and (2·16), and

*) The relativistic correction terms (v/c) in the hadron current are not taken into account in this appendix.

$$\mathcal{R}^l=\langle\sigma_n{}^l\rangle,\quad \mathcal{P}^l=2(g_V/g_A)\langle\hat{r}_{nm}^l(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_m)\rangle\,,$$

$$\mathcal{P}^{lk}=\langle\hat{r}_{nm}^l\{\hat{r}_{nm}^k[(g_V/g_A)^2-(\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_m)]+2\sigma_m{}^k(\hat{r}_{nm}\cdot\boldsymbol{\sigma}_n)\}\rangle\,,$$

$$Q^l=i(g_V/g_A)^2\langle\hat{r}_{+nm}^l\hat{r}_{nm}\cdot(\boldsymbol{\sigma}_n\times\boldsymbol{\sigma}_m)\rangle,\quad Q^{lk}=-2i(g_V/g_A)\langle\hat{r}_{+nm}^l(\hat{r}_{nm}\times\boldsymbol{\sigma}_m)^k\rangle\,,\tag{C·6}$$

where abbreviation $\langle O\rangle\equiv\langle N_\beta|\sum_{n,m}\tau_n{}^+\tau_m{}^+O|N_\alpha\rangle$ is used.

The $m_\nu K_{LL}^{\mu\nu}$ term contributes only to the $0^+\to0^+$ transition, while $\eta L_{LL}^{\mu\nu0}$ contributes to $0^+\to0^+$, 1^+ and $\eta L_{LL}^{\mu\nu k}$ to $0^+\to0^+$, 1^+, 2^+, cf., Table I.

(a) *The 2n-mechanism*

For the $0^+\to0^+$ transition, the decay formula is expressed as follows:

$$\begin{aligned}d\Gamma_{0\nu}^{2n}(0^+\to0^+)=&(a_{0\nu}/m_e{}^7)g_A{}^4\\&\times\{(p_1{}^0p_2{}^0-\boldsymbol{p}_1\cdot\boldsymbol{p}_2)[|Y_1+Y_4-Y_5|^2+|Y_2+Y_4+Y_5|^2]\\&+\frac{1}{2m_e{}^2}(p_1{}^0-p_2{}^0)^2(\alpha_-+\boldsymbol{p}_1\cdot\boldsymbol{p}_2)|Y_3-Y_4|^2\\&+\frac{1}{2m_e{}^2}(p_1{}^0+p_2{}^0)^2(\alpha_++\boldsymbol{p}_1\cdot\boldsymbol{p}_2)|Y_5|^2\\&-2m_e{}^2\,\mathrm{Re}(Y_1+Y_4-Y_5)(Y_2+Y_4+Y_5)^*\\&-(p_1{}^0-p_2{}^0)^2\,\mathrm{Re}(Y_1+Y_2+2Y_4)(Y_3-Y_4)^*\\&+(p_1{}^0+p_2{}^0)^2\,\mathrm{Re}(Y_1-Y_2-2Y_5)Y_5{}^*\}\\&\times p_1{}^0p_2{}^0\,\delta(p_1{}^0+p_2{}^0+M_B-M_A)d\cos\theta dp_1{}^0dp_2{}^0\,,\end{aligned}\tag{C·7}$$

where

$$\alpha_\pm=p_1{}^0p_2{}^0\pm m_e{}^2\tag{C·8}$$

and

$$\begin{aligned}&Y_1=(m_\nu/m_e)\langle H\rangle[(g_V/g_A)^2M_F-M_{GT}],\quad Y_2=\eta^2Y_1\,,\\&Y_3=\eta\langle H\rangle[(g_V/g_A)^2M_F+M_{GT}],\\&Y_4=(\eta/3)\langle rH'\rangle[(g_V/g_A)^2M_F-M_{GT}/3+2M_T],\\&Y_5=(\eta/3)(-2ig_V/g_A)\langle rH'\rangle\langle\hat{r}_{+nm}\cdot(\hat{r}_{nm}\times\boldsymbol{\sigma}_m)\rangle\,.\end{aligned}\tag{C·9}$$

Note that if $\eta=0$, the above formula agrees with Eq. (2·21) with $\lambda=0$. If $\eta\neq0$, there arise the following differences: (i) A new nuclear matrix element Y_5 appears. (ii) There are some sign differences in Y_3 and Y_4 in comparison with X_3 and X_4. (iii) If Y_5, M_F and M_T are discarded, both decay formulae are similar because only the difference comes from $Y_3 \simeq -X_3$ and $Y_4 \simeq -X_4$ whose contributions are minor. Thus, it is rather difficult to distinguish these two H_W's.

In the formula derived by Primakoff and Rosen,[1] the terms Y_5 as well as M_T are not included. Also, Greuling and Whitten[5] did not include Y_5 either. Note that their formulae are two times as large as ours.

For the $0^+ \to 2^+$ transition, the decay formula is

$$d\Gamma_{0\nu}^{2n}(0^+ \to 2^+) = (a_{0\nu}/m_e{}^9)(g_A{}^4/30)|\eta\langle rH'\rangle|^2$$
$$\times [\chi_+(\mathcal{N}_1^{pq}, \mathcal{N}_1^{pq}) + \chi_-(\mathcal{N}_2^{pq}, \mathcal{N}_2^{pq})]$$
$$\times p_1{}^0 p_2{}^0 \delta(p_1{}^0 + p_2{}^0 + M_B - M_A) d\cos\theta dp_1{}^0 dp_2{}^0\ , \qquad (\mathrm{C}\cdot 10)$$

where

$$\chi_\pm = 3(\boldsymbol{p}_1 \cdot \boldsymbol{p}_2)^2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_1(10\alpha_\pm + |\boldsymbol{p}_1|^2 + |\boldsymbol{p}_2|^2)$$
$$+ 5\alpha_\pm(|\boldsymbol{p}_1|^2 + |\boldsymbol{p}_2|^2) - |\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2\ ,$$
$$\mathcal{N}_1^{pq} = \langle \hat{r}_{nm}^p\{\hat{r}_{nm}^q[(g_V/g_A)^2 - (\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_m)] + 2\sigma_m{}^q(\hat{r}_{nm} \cdot \boldsymbol{\sigma}_n)\}\rangle\ ,$$
$$\mathcal{N}_2^{pq} = -2i(g_V/g_A)\langle \hat{r}_{+nm}^p(\hat{r}_{nm} \times \boldsymbol{\sigma}_m)^q\rangle\ . \qquad (\mathrm{C}\cdot 11)$$

In a similar way to Eq. (2·34), the $0^+ \to 2^+$ transition occurs only if $\eta \neq 0$. In the place of the tensor operator $\hat{r}_{nm}^p(\hat{r}_{nm} \times \boldsymbol{\sigma}_m)^q$, a new operator $\hat{r}_{+nm}^p(\hat{r}_{nm} \times \boldsymbol{\sigma}_m)^q$ enters here. It is interesting to note that if $m_e = 0$, both decay formulae agree with each other except for the overall normalization.

For the $0^+ \to 1^+$ transition, we obtain

$$d\Gamma_{0\nu}^{2n}(0^+ \to 1^+) = (a_{0\nu}/m_e{}^9)(g_A{}^4/6)\eta^2 \sum_{a,b=1}^{5} D_{ab}(\mathcal{N}_a{}^p, \mathcal{N}_b{}^p)$$
$$\times p_1{}^0 p_2{}^0 \delta(p_1{}^0 + p_2{}^0 + M_B - M_A) d\cos\theta dp_1{}^0 dp_2{}^0\ , \qquad (\mathrm{C}\cdot 12)$$

where

$$\mathcal{N}_1{}^j = (g_V/g_A)\langle H\rangle\langle\sigma_n{}^j\rangle\ , \quad \mathcal{N}_2{}^j = (g_V/g_A)\langle rH'\rangle\langle \hat{r}_{nm}^j(\hat{r}_{nm} \cdot \boldsymbol{\sigma}_m)\rangle\ ,$$
$$\mathcal{N}_3{}^j = \langle rH'\rangle\langle \hat{r}_{+nm}^j \hat{r}_{nm} \cdot (\boldsymbol{\sigma}_n \times \boldsymbol{\sigma}_m)\rangle\ , \quad \mathcal{N}_4{}^j = \langle rH'\rangle\langle(\hat{r}_{nm} \times \boldsymbol{\sigma}_m)^j(\hat{r}_{nm} \cdot \boldsymbol{\sigma}_n)\rangle\ ,$$
$$\mathcal{N}_5{}^j = (g_V/g_A)\langle rH'\rangle\langle[\hat{r}_{+nm} \times (\hat{r}_{nm} \times \boldsymbol{\sigma}_m)]^j\rangle\ . \qquad (\mathrm{C}\cdot 13)$$

The non-zero coefficients $D_{ab} = D_{ba}$ are as follows:

$$D_{11} = 4(p_1{}^0 - p_2{}^0)^2(3\alpha_+ - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2), \quad D_{12} = -4(p_1{}^0 - p_2{}^0)^2(\alpha_+ + \boldsymbol{p}_1 \cdot \boldsymbol{p}_2),$$
$$D_{15} = 4(p_1{}^0 - p_2{}^0)^2(\alpha_+ - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2), \quad D_{22} = 4(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2(\alpha_- + \boldsymbol{p}_1 \cdot \boldsymbol{p}_2),$$
$$D_{25} = -2D_{34} = 8[|\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2 - (\boldsymbol{p}_1 \cdot \boldsymbol{p}_2)^2],$$
$$D_{33} = (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2(\alpha_+ + \boldsymbol{p}_1 \cdot \boldsymbol{p}_2),$$

$$D_{44}=2[(\boldsymbol{p}_1\times\boldsymbol{p}_2)^2+(\boldsymbol{p}_1-\boldsymbol{p}_2)^2(\alpha_+-\boldsymbol{p}_1\cdot\boldsymbol{p}_2)],$$

$$D_{55}=-2[(\boldsymbol{p}_1\times\boldsymbol{p}_2)^2-(\boldsymbol{p}_1+\boldsymbol{p}_2)^2(\alpha_--\boldsymbol{p}_1\cdot\boldsymbol{p}_2)]. \tag{C·14}$$

In contrast to Eq. (2·38), the above formula is complicated. Instead of two first terms in $N_1{}^q$ and $N_2{}^q$ in Eqs. (2·39) and (2·40), the new terms $\mathcal{N}_1{}^j$, $\mathcal{N}_4{}^j$ and $\mathcal{N}_5{}^j$ now enter into the decay formula. Morina and Pascual[9] have also evaluated the $0^+\to 2^+$ and 1^+ transitions. Their results are four times as large as ours in addition to some errors.

(b) *N*-mechanism*

We have found that both $\eta L_{LL}^{\mu\nu 0}$ and $\eta L_{LL}^{\mu\nu k}$ terms in Eqs. (C·4) and (C·5) contribute to the $(\beta\beta)_{0\nu}$ mode in contrast to the case in Table I, while the term $m_\nu K_{LL}^{\mu\nu}$ in Eq. (C·3) does not again. That is, whether neutrinos are massive or massless, the N^*-mechanism has no contribution to the $(\beta\beta)_{0\nu}$ mode if $\eta=0$.

By using the factorization hypothesis explained in Appendix A, we obtain

$$d\Gamma_{0\nu}^{N^*}=(a_{0\nu}/m_e{}^9)(2^5g_A{}^2/3^3)\eta^2P(\Delta)|\langle\Phi_f|\Phi_i\rangle|^2$$
$$\times[(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2(3g_A{}^2-g_V{}^2)\langle rH'\rangle^2-\boldsymbol{p}_1\cdot\boldsymbol{p}_2\beta_1+\beta_2]$$
$$\times p_1{}^0p_2{}^0\delta(p_1{}^0+p_2{}^0+M_B-M_A)d\cos\theta dp_1{}^0dp_2{}^0\,, \tag{C·15}$$

where

$$\beta_1=\left\{g_A{}^2(10\alpha_++|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2)+g_V{}^2\left[\alpha_--\frac{1}{2}(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2)\right]\right\}\langle rH'\rangle_\Delta{}^2$$
$$+(9g_V{}^2/2)(p_1{}^0-p_2{}^0)^2\langle H\rangle_\Delta{}^2+3g_V{}^2(p_1{}^0-p_2{}^0)^2\langle rH'\rangle_\Delta\langle H\rangle_\Delta\,, \tag{C·16}$$

$$\beta_2=\left\{g_A{}^2[5\alpha_+(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2)-|\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2]-\frac{1}{2}g_V{}^2\alpha_-(|\boldsymbol{p}_1|^2+|\boldsymbol{p}_2|^2)\right\}\langle rH'\rangle_\Delta{}^2$$
$$+(27g_V{}^2/2)\alpha_+(p_1{}^0-p_2{}^0)^2\langle H\rangle_\Delta{}^2-3g_V{}^2\alpha_+(p_1{}^0-p_2{}^0)^2\langle rH'\rangle_\Delta\langle H\rangle_\Delta\,. \tag{C·17}$$

Here $g_A=g_V$ is understood. New terms proportional to $\langle H\rangle_\Delta$ which do not exist in Eq. (2·47) come from the $\eta L_{LL}^{\mu\nu 0}$ term. The $g_A{}^4$ term in (C·15) gives exactly the same behavior as the one in (2·47) but the $g_A{}^2g_V{}^2$ term differs. However, since the $g_A{}^4$ term is expected to contribute dominantly, both decay formulae give similar results. Primakoff and Rosen[1] have worked out the decay formula by taking only the $g_A{}^2g_V{}^2$ term into account.

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